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The Linguistic Concept's Reduction Methods under Symmetric Linguistic-Evaluation Information

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Abstract: Knowledge reduction is a crucial topic in formal concept analysis. There always exists uncertain, symmetric linguistic-evaluation information in social life, which leads to high complexity in the process of knowledge representation. In order to overcome this problem, we are focused on studying the linguistic-concept-reduction methods in an uncertain environment with fuzzy linguistic information. Based on three-way decisions and an attribute-oriented concept lattice, we construct a fuzzy-object-induced three-way attribute-oriented linguistic (FOEAL) concept lattice, which provides complementary conceptual structures of a three-way concept lattice with symmetric linguistic-evaluation information. Through the granular concept of the FOEAL lattice, we present the corresponding linguistic concept granular consistent set and granular reduction. Then, we further employ the linguistic concept discernibility matrix and discernibility function to calculate the granular reduction set. A similar issue on information entropy is investigated to introduce a method of entropy reduction for the FOEAL lattice, and the relation between the linguistic concept granular reduction and entropy reduction is discussed. The efficiency of the proposed method is depicted by some examples and comparative analysis.

Keywords: concept lattice; fuzzy formal context; granular reduction; three-way concept lattice; linguistic term set; symmetry and asymmetry; machine learning

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1. Introduction

Formal concept analysis, which was originally proposed by Wille [1], has proven to be a powerful tool for data analysis and knowledge processing. Its fundamental concept is to study the relationships between objects and attributes in a formal context, and to represent the implicit relationships between conceptual knowledge through the creation of a visual lattice structure. The concept lattice, as the core data structure of formal concept analysis, has been widely utilized in various fields with significant potential applications, including software engineering [2], data mining [3], knowledge discovery [4], and others [5–7].

The extension of formal concept analysis and its integration with other theories have seen significant research. Duntsch and Gediga [8] employed modal-style operators to propose an attribute-oriented concept lattice, and then Yao [9] gave an object-oriented concept lattice. Aiming at the data structure in a fuzzy environment, fuzzy concept lattices were also proposed [10,11]. In view of the advantage of three-way decisions that can give decisions with delayed decision making in the context of incomplete information, Qi and Yao et al. [12,13] proposed a three-way concept lattice that can express not only the case where objects share attributes, but also the case where objects do not share attributes. Based on that, researchers have achieved many goals. Qian et al. [14] investigated the connections between the object-oriented concept lattice, property oriented concept lattice, three-way object-oriented concept lattice, and three-way property oriented concept lattice. Hu and Wang et al. [15,16] proposed an algorithm for constructing a multi-granularity concept lattice and discussed the relationship between types of operators under multi-view data based on three-way concept analysis. Singh [17] discussed the incompleteness of fuzzy attribute set to generate a wise three-way fuzzy concept with a neutrosophic graph and a neutrosophic lattice. Xu et al. [18] designed a dynamic learning mechanism for processing continuous fuzzy data using a three-way concept lattice, which can update the concept information faster and effectively improve classification accuracy.

Knowledge reduction is an indispensable technique for improving the efficiency and effectiveness of concept lattice construction. The current main types of reduction can be divided into concept-lattice reduction, attribute reduction, and object reduction, whose primary purpose is to eliminate redundant information while keeping a condition unchanged. Making use of a discernibility matrix to obtain reduction is widespread due to the high compatibility of discernibility functions [19–21]. Meanwhile, types of attribute reduction methods were discussed by preserving the lattice structure, join(meet)-irreducible elements, and granular consistent set, respectively [22–24]. In addition, some scholars have extended attribute reduction methods to the formal decision context [25–27]. Chen et al. [28,29] investigated granular reduction and attribute reduction to overcome the limitations of large-scale data sets in terms of excessive spatial complexity. Qin et al. [30] used the equivalence relation to explore an attribute reduction of attribute (object)-oriented concept lattices. However, there always exists fuzzy and uncertain information in real life. How to deal with fuzzy data has always been an essential issue in machine learning [31–35]. Thus, studying knowledge reduction in a fuzzy formal context is necessary. In order to avoid distorting the knowledge structure during the reduction process, Zhai et al. [36] presented fuzzy-attribute reduction based on Lukasiewicz adjoint operators and hedge. Singh et al. [37] calculated fuzzy concept weights based on Shannon entropy to propose a concept-reduction method with less complexity. Further studies [38,39] on matrix-based fuzzy attribute reduction were put forward in consistent formal fuzzy decision contexts, which provide a new direction for fuzzy three-way concept analysis.

In real-life applications, it is often challenging to provide precise values for evaluation due to the presence of uncertainty and complexity in the environment. Instead, evaluation information in natural language is often more relevant to human cognition. As a result, the decision problem of using symmetric and asymmetric linguistic variables for qualitative assessment has become a topic of significant interest [40–45]. Furthermore, taking advantage of natural language and formal concept analysis, Yang et al. [46] and Zou et al. [47,48] analyzed formal contexts with symmetric linguistic-evaluation information. However, from the analysis mentioned above, we can observe that there are still some challenges in linguistic formal concept analysis.

- The formal concepts in classical formal concept analysis only express whether objects share attributes or whether attributes are shared by objects. In real life, it is not enough to study only these two relations between objects and attributes. Likewise, formal contexts with linguistic information face the same problem. Thus, other relationships that may exist between objects and linguistic concepts need to be discussed as well.
- The construction of conceptual knowledge is inherently a complex challenge. There is easily a large amount of redundant information in the process of knowledge processing, resulting in a high amount of computational complexity. Therefore, there is an urgent need to propose a reduction method that can reduce the complexity of linguistic concept knowledge.
- Scholars have achieved many results in fuzzy formal concept analysis. A large amount of fuzzy information can exist in a linguistic environment, so studying the fuzzy linguistic concept formal context is necessary based on the challenges presented above.

In order to overcome the challenges, this paper investigates two linguistic concept reduction methods in a fuzzy linguistic concept formal context, and the main contributions are listed below.

- Based on three-way concept lattice and modal operators with possibilities and necessity, a fuzzy-object-induced three-way attribute-oriented linguistic (FOEAL) concept lattice is proposed to express more information in a fuzzy linguistic concept formal context.

- A novel linguistic-concept granular-reduction method based on the FOEAL lattice is designed to preserve granular concept information, which reduces the scale of conceptual knowledge in a linguistic environment.
- In order to highlight the importance of linguistic-concept information, an entropy-reduction method based on the FOEAL lattice is also presented. We further verified that the set of this entropy reduction is consistent with those of the granular reduction.
- The examples of the student-debate competition can confirm the rationality, and the comparative analysis provides strong evidence for the effectiveness of the proposed method.

The remainder of this paper is structured as follows. In Section 2, we review some basic notions of concept lattices and linguistic-term sets. In Section 3, we present a fuzzy-object-induced three-way attribute-oriented linguistic concept lattice and study the linguistic-concept granular-reduction method based on it. In Section 4, an entropy reduction method of a fuzzy-object-induced three-way attribute-oriented linguistic-concept lattice is presented, and its relationship to granular reduction is discussed. Section 5 provides a comparative analysis that demonstrates the superiority of our proposed method over other techniques. Finally, this paper is completed with a conclusion in Section 6.

2. Preliminaries

This section gives some basic notions on concept lattices and linguistic-term sets, then introduces some related work on linguistic-concept reduction.

2.1. Basic Notions on Concept Lattice

Definition 1. *[9]. Let $K = (G, M, I)$ be a formal context, where $G = \{x_i | i \in 1, 2, \ldots, n\}$ is a set of objects and $M = \{a_j | j \in 1, 2, \ldots, m\}$ is a set of attributes. I is a subset of Cartesian products $G \times M$, which represents the binary relation between G and M. $I(x, a) = 1$ denotes that the object has the attribute, and $I(x, a) = 0$ denotes that the object does not have the attribute.*

Definition 2. *[12]. Let $K = (G, M, I)$ be a formal context. Given $X \subseteq G$ and $B \subseteq M$, a pair of operators, $* : P(G) \to P(M)$ and $* : P(M) \to P(G)$, are defined by:*

$$X^* = \{a \in M | \forall x \in G, xIa\}, \tag{1}$$
$$B^* = \{x \in G | \forall a \in M, xIa\}. \tag{2}$$

If $X^* = B, B^* = X$, then we call (X, B) a formal concept. Here, X is called an extent, and B is called an intent of the concept (X, B).

The set of all the formal concepts forms a complete lattice called the concept lattice is which denoted by $L(G, M, I)$. For any $(X_1, B_1), (X_2, B_2) \in L(G, M, I)$, the partial order relation "$\leq$", infimum and supremum are given as follows:

$$(X_1, B_1) \leq (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow B_2 \subseteq B_1), \tag{3}$$
$$(X_1, B_1) \wedge (X_2, B_2) = (X_1 \cap X_2, (B_1 \cup B_2)^{**}), \tag{4}$$
$$(X_1, B_1) \vee (X_2, B_2) = ((X_1 \cup X_2)^{**}, B_1 \cap B_2). \tag{5}$$

Definition 3. *[12]. Let $K = (G, M, I)$ be a formal context. Given $X \subseteq G$ and $B \subseteq M$, a pair of negative operators, $\overline{*} : P(G) \to P(M)$ and $\overline{*} : P(M) \to P(G)$, are defined by:*

$$X^{\overline{*}} = \{a \in M | \forall x \in X, \neg(xIa)\} = \{a \in M | \forall x \in X, xI^c a\}, \tag{6}$$
$$B^{\overline{*}} = \{x \in G | \forall a \in B, \neg(xIa)\} = \{x \in G | \forall a \in B, xI^c a\}. \tag{7}$$

Here, $I^c = (G \times M) - I$. If $X^{\overline{*}} = B, B^{\overline{*}} = X$, then we call (X, B) an N-concept.

$NL(G, M, I)$ denotes a complete lattice formed by all the N-concepts, in which the partial order relations, infimum and supremum are analogous to the $L(G, M, I)$.

The attribute-oriented concept lattice was proposed by employing modal-style operators as follows [8].

Definition 4. *[9,49]. Let $K = (G, M, I)$ be a formal context, $X \subseteq G$, $B \subseteq M$. Operators $\Box : P(M) \to P(G)$ and $\diamond : P(G) \to P(M)$ are defined as follows:*

$$X^{\diamond} = \{a \in M | a^* \cap X \neq \emptyset\}, \tag{8}$$
$$B^{\Box} = \{x \in G | x^* \subseteq B\}. \tag{9}$$

Definition 5. *[9,49]. Let $K = (G, M, I)$ be a formal context. For $(X \subseteq G, B \subseteq M)$, (X, B) is called an attribute-oriented concept if $X^{\diamond} = B$ and $X = B^{\Box}$. X and B are called the extent and intent of the attribute-oriented concept, respectively.*

The set of all the attribute-oriented concepts, denoted by $L_A(G, M, I)$, forms a complete lattice and is referred to as an attribute-oriented concept lattice. The meet "$\wedge$", join "$\vee$" and partial order relation "$\leq$" are given by:

$$(X_1, B_1) \wedge (X_2, B_2) = (X_1 \cap X_2, (B_1 \cap B_2)^{\Box\diamond}), \tag{10}$$
$$(X_1, B_1) \vee (X_2, B_2) = ((X_1 \cup X_2)^{\diamond\Box}, B_1 \cup B_2), \tag{11}$$
$$(X_1, B_1) \leq (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow B_1 \subseteq B_2). \tag{12}$$

Motivated by the idea of three-way decision, Qi and Yao et al. [12] put forward three-way concept lattices from the two aspects of objects and attributes. The following is an object-induced three-way concept lattice.

Definition 6. *[12]. Let $K = (G, M, I)$ be a formal context. Given $X \subseteq G$ and $B \subseteq M$, a pair of object-induced three-way operators, $\lessdot : P(G) \to DP(M)$ and $\gtrdot : DP(M) \to P(G)$, are defined by: $X^{\lessdot} = (X^*, X^{\bar{*}})$, $(B, C)^{\gtrdot} = \{x \in G | x \in B^* and x \in C^{\bar{*}}\} = B^* \cap C^{\bar{*}}$.*

Definition 7. *[12]. Let $K = (G, M, I)$ be a formal context. A pair $(X, (B, C))$ of an object subset $X \subseteq G$ and two attribute subsets $B, C \subseteq M$ is called an object-induced three-way concept, or an OE-concept of (G, M, I), if and only if $X^{\lessdot} = (B, C)$ and $(B, C)^{\gtrdot} = X$. X is called the extent, and (B, C) is called the intent of the OE-concept, respectively.*

The OE-concepts $(X_1, (B_1, C_1))$ and $(X_2, (B_2, C_2))$ are ordered by:

$$(X_1, (B_1, C_1)) \leq (X_2, (B_2, C_2)) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow (B_1, C_1) \supseteq (B_2, C_2) \Leftrightarrow B_1 \supseteq B_2, C_1 \supseteq C_2).$$

All OE-concepts are denoted by $OEL(G, M, I)$; then a complete lattice called a object-induced three-way concept lattice(OE-lattice) is formed. The infimum and supremum on $OEL(G, M, I)$ are given by:

$$(X_1, (B_1, C_1)) \wedge (X_2, (B_2, C_2)) = (X_1 \cap X_2, ((B_1, C_1) \cup (B_2, C_2)))^{\gtrdot\lessdot}), \tag{13}$$
$$(X_1, (B_1, C_1)) \vee (X_2, (B_2, C_2)) = ((X_1 \cup X_2)^{\lessdot\gtrdot}, (B_1, C_1) \cap (B_2, C_2))). \tag{14}$$

2.2. Linguistic Term Set

In complex environments, due to the fact that some problems are difficult to be concretely expressed by fuzzy sets, Zadeh [50] proposed linguistic variables with symmetric evaluation information such as "good", "very good", "bad" and "very bad". Furthermore, Herrera [51] proposed a linguistic-term set $S = \{s_\alpha | \alpha = 0, 1, ..., g\}$ to describe all the discrete linguistic variables, where s_α denotes a possible value for a linguistic variable, and it also satisfies the following characteristics:

(1) order relation: $s_\alpha \geq s_\beta$, if $\alpha \geq \beta$,
(2) negation operator: $Neg(s_\alpha) = s_\beta$, where $\beta = g - \alpha$,
(3) maximization operator: $\max\{s_\alpha, s_\beta\} = s_\alpha$, if $\alpha \geq \beta$,
(4) minimization operator: $\min\{s_\alpha, s_\beta\} = s_\beta$, if $\alpha \geq \beta$.

It can be seen from the above that the linguistic-term set has an odd number of linguistic terms. If the middle evaluation value represents "medium", then the rest of the linguistic labels are replaced symmetrically around it. For example, a linguistic-term set with five terms can be defined as $S = \{s_0 =$ very bad, $s_1 =$ bad, $s_2 =$ medium, $s_3 =$ good, $s_4 =$ very good$\}$.

2.3. Linguistic Concept Lattice

Zou et al. [48] presented a linguistic concept lattice by introducing symmetric linguistic-evaluation information into formal concept analysis as follows.

Let $S = \{s_\alpha | \alpha \in 0, 1, 2, \ldots, g\}$ be a linguistic-term set and $L = \{l^j | j \in 1, 2, \ldots, m\}$ be an attribute set. Then, we can get the set $L_{s_\alpha} = \{l^j_{s_k} | j \in 1, 2, \ldots, m, k \in 0, 1, 2, \ldots, g\}$ defined on S and L, which is called the linguistic concept set. For example, a linguistic concept l_{s_0} means that attribute l is evaluated as a linguistic term s_0. Based on that, we give a linguistic concept formal context.

Definition 8. *[48]. A linguistic concept formal context is defined as a triple $(G, L_{s_\alpha}, \Im)$, where $G = \{x_i | i \in 1, 2, \ldots, n\}$ is a non-empty finite object set and $L_{s_\alpha} = \{l^j_{s_k} | j \in 1, 2, \ldots, m, k \in 0, 1, 2, \ldots, g\}$ is a non-empty finite linguistic concept set. $\Im$ is the binary relationship from G to L_{s_α}; i.e., $\Im \subseteq G \times L_{s_\alpha}$. $(x, l_{s_k}) \in \Im$ indicates that the object can be described by a linguistic concept l_{s_k}. $(x, l_{s_k}) \notin \Im$ indicates that the object x cannot be described by a linguistic concept l_{s_k}.*

Let $(G, L_{s_\alpha}, \Im)$ be a linguistic concept formal context. For $X \subseteq G$ and $B_{s_\alpha} \subseteq L_{s_\alpha}$, we define a pair of operators on "$'$" as follows:

$$X' = \{l_{s_k} \in L_{s_\alpha} | \forall x \in X, (x, l_{s_k}) \in \Im\}, \quad (15)$$
$$B'_{s_\alpha} = \{x \in G | \forall l_{s_k} \in B_{s_\alpha}, (x, l_{s_k}) \in \Im\}. \quad (16)$$

If $X' = B_{s_\alpha}$ and $B'_{s_\alpha} = X$, then a pair (X, B_{s_α}) is called a linguistic concept knowledge. The partial order relation "$\leq$" for linguistic concept knowledge $(X_1, B^1_{s_\alpha})$ and $(X_2, B^2_{s_\alpha})$ is defined as follows:

$$(X_1, B^1_{s_\alpha}) \leq (X_2, B^2_{s_\alpha}) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow B^2_{s_\alpha} \subseteq B^1_{s_\alpha}).$$

The family of all the linguistic concept knowledge of $(G, L_{s_\alpha}, \Im)$ forms a complete lattice, which is referred to as a linguistic concept lattice [48]. The infimum and supremum are given by:

$$(X_1, B^1_{s_\alpha}) \wedge (X_2, B^2_{s_\alpha}) = (X_1 \cap X_2, (B^1_{s_\alpha} \cup B^2_{s_\alpha})''), \quad (17)$$
$$(X_1, B^1_{s_\alpha}) \vee (X_2, B^2_{s_\alpha}) = ((X_1 \cup X_2)'', B^1_{s_\alpha} \cap B^2_{s_\alpha}). \quad (18)$$

2.4. Linguistic Concept Reduction

Knowledge reduction can reduce the complexity of concept generation in concept lattice theory. Aiming at different formal contexts, there are many reduction models, most of which are based on a classical concept lattice [52,53], fuzzy concept lattice [54], rough concept lattice [55], three-way concept lattice [56,57], etc. [58,59]. However, most of these models are based on either the binary or fuzzy relationships between objects and attributes. In real-life situations, people often prefer to use linguistic values to express these relationships, which cannot be accommodated by the existing conceptual models. Suppose now that there is the following practical problem.

As the country's economy and society continue to develop, people are paying more and more attention to education. Education at school is one of the essential components of a person's life, so parents are cautious when choosing schools. In order to evaluate the following five schools, several experts were invited to assess the three aspects of teaching facilities, teaching quality, and school environment, which can be formulated in mathematical language as five objects, $X = \{x_1, x_2, x_3, x_4, x_5\}$; and three attributes, $B = \{a =$ teaching facilities, $b =$ teaching quality, $c =$ school environment$\}$, respectively. Obviously, the binary relation of whether the object has attributes or whether the attributes can be possessed by the object cannot describe the factual background of the instance in detail. Moreover, since experts sometimes prefer to use natural language such as "well", "bad" and "somewhat good" to evaluate things, information loss will inevitably occur when fuzzy sets are used to deal with the relationship between objects and attributes, which highlights the advantage of linguistic variables that can handle discrete linguistic information. Zou et al. [48] embedded linguistic-term sets with symmetric linguistic variables into formal contexts and used linguistic information to describe attributes directly; e.g., if the semantic evaluation base for school evaluations by experts is the linguistic-term set $S = \{s_0 =$ extremely poor, $s_1 =$ very poor, $s_2 =$ medium, $s_3 =$ very good, $s_4 =$ extremely good$\}$, then the teaching facility is very good, which can be represented by the linguistic concept a_{s_3}, and each attribute value can generate the corresponding five linguistic concepts. Therefore, instead of using the traditional formal concept analysis model, we can represent the relationship between objects and attributes using a binary table between objects and linguistic concepts, as demonstrated in Table 1, where $I(x_1, a_{s_2}) = 1$ indicates that the teaching facilities of school x_1 are rated as medium.

Table 1. Linguistic concept formal context (G, L_{s_α}, I).

	$\boldsymbol{a}$					$\boldsymbol{b}$					$\boldsymbol{c}$				
	$\boldsymbol{a_{s_0}}$	$\boldsymbol{a_{s_1}}$	$\boldsymbol{a_{s_2}}$	$\boldsymbol{a_{s_3}}$	$\boldsymbol{a_{s_4}}$	$\boldsymbol{b_{s_0}}$	$\boldsymbol{b_{s_1}}$	$\boldsymbol{b_{s_2}}$	$\boldsymbol{b_{s_3}}$	$\boldsymbol{b_{s_4}}$	$\boldsymbol{c_{s_0}}$	$\boldsymbol{c_{s_1}}$	$\boldsymbol{c_{s_2}}$	$\boldsymbol{c_{s_3}}$	$\boldsymbol{c_{s_4}}$
x_1	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0
x_2	0	0	0	1	0	0	0	0	0	1	0	0	1	0	0
x_3	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0
x_4	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1
x_5	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0

Generating concept lattices is a computationally complex problem that falls under the category of NP-hard problems, and therefore, reduction techniques are essential. It is also important to investigate reduction methods that are specifically designed for symmetric linguistic information. As a crucial component of the linguistic concept formal context, reducing linguistic concepts holds great potential for practical applications. However, previous research, such as that described in reference [48], has only focused on studying reduction methods for linguistic concept formal contexts and incomplete linguistic concept formal contexts. When making decisions based on multiple expert evaluations, it is common for opinions to differ, which can result in vague and uncertain information. To address this issue, we decided to investigate the method of reducing linguistic concepts while taking ambiguity into consideration.

3. Fuzzy-Object-Induced Three-Way Attribute-Oriented Linguistic Concept Lattice

In this section, from the perspective of linguistic concepts, we first construct a fuzzy-object-induced three-way linguistic concept lattice. Subsequently, a reduction method for linguistic concepts is studied in the fuzzy linguistic concept formal context, based on the principles of granular computing.

3.1. The Construction of a Fuzzy-Object-Induced Three-Way Attribute-Oriented Linguistic Concept Lattice

Definition 9. *Let $(G, L_{s_\alpha}, \widetilde{I})$ be a fuzzy linguistic concept formal context, where $G = \{x_i | i \in 1, 2, \ldots, n\}$ is a non-empty finite object set and $L_{s_\alpha} = \{l_{s_k}^j | j \in 1, 2, \ldots, m, k \in 0, 1, \ldots, g\}$ is a non-empty finite linguistic concept set. For $\mu_{\widetilde{I}}(x, l_{s_k}) : G \times L_{s_\alpha} \to [0, 1]$, $\widetilde{I} = \{< (x, l_{s_k}), \mu_{\widetilde{I}}(x, l_{s_k}) > |(x, l_{s_k}) \in G \times L_{s_\alpha}\}$ is the fuzzy relationship from G to L_{s_α}. $\widetilde{I}(x, l_{s_k}) = \gamma$ denotes that the object x has the linguistic concept l_{s_k} to degree γ, where $\gamma \in [0, 1]$.*

Definition 10. *Let $(G, L_{s_\alpha}, \widetilde{I})$ be a fuzzy linguistic concept formal context and $\lambda \in [0, 1]$ be a fuzzy credibility threshold. For $X \subseteq G$ and $\tilde{B}_{s_\alpha} \subseteq F(L_{s_\alpha})$, the operators are defined as follows:*

$$\tilde{x}' = \{< l_{s_k}, \mu_X(l_{s_k}) > |\forall x \in G, \widetilde{I}(x, l_{s_k}) \geq \lambda\} \tag{19}$$

$$l'_{s_k} = \{x \in G | \forall l_{s_k} \in L_{s_\alpha}, \widetilde{I}(x, l_{s_k}) \geq \lambda\} \tag{20}$$

$$X^{\widetilde{\diamond}} = \{< l_{s_k}, \mu_X(l_{s_k}) > | l'_{s_k} \cap X \neq \emptyset\} \tag{21}$$

$$\tilde{B}_{s_\alpha}^{\widetilde{\Box}} = \{x \in G | \tilde{x}' \subseteq \tilde{B}_{s_\alpha}\}. \tag{22}$$

where $\mu_X(l_{s_k}) = \bigvee_{x \in X} \widetilde{I}(x, l_{s_k})$, $\tilde{x}' \subseteq \tilde{B}_{s_\alpha}$ means that the set of linguistic concepts x' is contained in B_{s_α} under a fuzzy credibility threshold λ, and the membership degree of linguistic concepts within x' is less than or equal to the corresponding membership degree of linguistic concepts within B_{s_α}.

Definition 11. *Let $(G, L_{s_\alpha}, \widetilde{I})$ be a fuzzy linguistic concept formal context; $I^c = 1 - G \times L_{s_\alpha}$. For $X \subseteq G$ and $\tilde{B}_{s_\alpha} \subseteq F(L_{s_\alpha})$, the operators are defined as follows:*

$$\tilde{x}^{\bar{\prime}} = \{< l_{s_k}, \mu_X^c(l_{s_k}) > |\forall x \in G, \widetilde{I^c}(x, l_{s_k}) \geq \lambda\} \tag{23}$$

$$l_{s_k}^{\bar{\prime}} = \{x \in G | \forall l_{s_k} \in L_{s_\alpha}, \widetilde{I^c}(x, l_{s_k}) \geq \lambda\} \tag{24}$$

$$X^{\overline{\widetilde{\diamond}}} = \{< l_{s_k}, \mu_X^c(l_{s_k}) > | l_{s_k}^{\bar{\prime}} \cap X \neq \emptyset\} \tag{25}$$

$$\tilde{B}_{s_\alpha}^{\overline{\widetilde{\Box}}} = \{x \in G | \tilde{x}^{\bar{\prime}} \subseteq \tilde{B}_{s_\alpha}\} \tag{26}$$

where $\mu_X^c(l_{s_k}) = \bigvee_{x \in X} \widetilde{I^c}(x, l_{s_k})$.

Definition 12. *Let $(G, L_{s_\alpha}, \widetilde{I})$ be a fuzzy linguistic concept formal context. For $X \subseteq G$ and $\tilde{B}_{s_\alpha}, \tilde{C}_{s_\alpha} \subseteq F(L_{s_\alpha})$, the fuzzy-object-induced three-way attribute-oriented linguistic operators, or FOEAL-operators, are defined by $X^{\preceq} = (X^{\widetilde{\diamond}}, X^{\overline{\widetilde{\diamond}}})$ and $(\tilde{B}_{s_\alpha}, \tilde{C}_{s_\alpha})^{\succeq} = \tilde{B}_{s_\alpha}^{\widetilde{\Box}} \cap \tilde{C}_{s_\alpha}^{\overline{\widetilde{\Box}}}$, respectively.*

Definition 13. *Let $(G, L_{s_\alpha}, \widetilde{I})$ be a fuzzy linguistic concept formal context. For $X \subseteq G$ and $\tilde{B}_{s_\alpha}, \tilde{C}_{s_\alpha} \subseteq F(L_{s_\alpha})$, a pair $(X, (\tilde{B}_{s_\alpha}, \tilde{C}_{s_\alpha}))$ is called a fuzzy-object-induced three-way attribute-oriented linguistic concept, or a FOEAL-concept of $(G, L_{s_\alpha}, \widetilde{I})$ if $X = (\tilde{B}_{s_\alpha}, \tilde{C}_{s_\alpha})^{\succeq}$ and $X^{\preceq} = (\tilde{B}_{s_\alpha}, \tilde{C}_{s_\alpha})$ are satisfied, where X is the extent and $(\tilde{B}_{s_\alpha}, \tilde{C}_{s_\alpha})$ is the intent of the FOEAL-concept.*

For two FOEAL-concepts $(X_1, (\tilde{B}_{s_\alpha}^1, \tilde{C}_{s_\alpha}^1))$ and $(X_2, (\tilde{B}_{s_\alpha}^2, \tilde{C}_{s_\alpha}^2))$, we define the partial order relation as follows:

$$(X_1, (\tilde{B}_{s_\alpha}^1, \tilde{C}_{s_\alpha}^1)) \leq (X_2, (\tilde{B}_{s_\alpha}^2, \tilde{C}_{s_\alpha}^2)) \Leftrightarrow X_1 \subseteq X_2 \Leftrightarrow (\tilde{B}_{s_\alpha}^1, \tilde{C}_{s_\alpha}^1) \subseteq (\tilde{B}_{s_\alpha}^2, \tilde{C}_{s_\alpha}^2),$$

where $(\tilde{B}_{s_\alpha}^1, \tilde{C}_{s_\alpha}^1) \subseteq (\tilde{B}_{s_\alpha}^2, \tilde{C}_{s_\alpha}^2) \Leftrightarrow \tilde{B}_{s_\alpha}^1 \subseteq \tilde{B}_{s_\alpha}^2 (\Leftrightarrow \tilde{C}_{s_\alpha}^1 \subseteq \tilde{C}_{s_\alpha}^2, \tilde{B}_{s_\alpha}^1 \subseteq \tilde{B}_{s_\alpha}^2)$ means that $B_{s_\alpha}^1 \subseteq B_{s_\alpha}^2$ and $\mu_{X_1}(B_{s_\alpha}^1) \leq \mu_{X_2}(B_{s_\alpha}^2)$, so $(X_1, (\tilde{B}_{s_\alpha}^1, \tilde{C}_{s_\alpha}^1))$ is called a sub-concept of $(X_2, (\tilde{B}_{s_\alpha}^2, \tilde{C}_{s_\alpha}^2))$, and $(X_2, (\tilde{B}_{s_\alpha}^2, \tilde{C}_{s_\alpha}^2))$ is called a super-concept of $(X_1, (\tilde{B}_{s_\alpha}^1, \tilde{C}_{s_\alpha}^1))$.

The set of all FOEAL-concepts, denoted by $FOEAL(G, L_{s_\alpha}, \widetilde{I})$, forms a complete lattice which is called a fuzzy-object-induced three-way attribute-oriented concept lattice (FOEAL-lattice). The infimum and supremum are given by:

$$(X_1, (\tilde{B}^1_{s_\alpha}, \tilde{C}^1_{s_\alpha})) \wedge (X_2, (\tilde{B}^2_{s_\alpha}, \tilde{C}^2_{s_\alpha})) = (X_1 \cap X_2, ((\tilde{B}^1_{s_\alpha}, \tilde{C}^1_{s_\alpha}) \cap (\tilde{B}^2_{s_\alpha}, \tilde{C}^2_{s_\alpha}))^{\succeq\preceq}), \tag{27}$$

$$(X_1, (\tilde{B}^1_{s_\alpha}, \tilde{C}^1_{s_\alpha})) \vee (X_2, (\tilde{B}^2_{s_\alpha}, \tilde{C}^2_{s_\alpha})) = ((X_1 \cup X_2)^{\preceq\succeq}, (\tilde{B}^1_{s_\alpha}, \tilde{C}^1_{s_\alpha}) \cup (\tilde{B}^2_{s_\alpha}, \tilde{C}^2_{s_\alpha})). \tag{28}$$

Example 1. *Suppose that Table 2 is a fuzzy linguistic concept formal context, denoted by $(G, L_{s_\alpha}, \widetilde{I})$, which describes the level of student debate in a classroom setting. The object set G consists of five students, represented as x_1, x_2, x_3, x_4 and x_5. The attribute set L includes three skills: language skills (a), debating skills (b) and personal image (c). We use the linguistic-term set $S = \{s_0 = low, s_1 = medium, s_2 = high\}$ to evaluate the attributes a, b, and c. Thus, the generated linguistic concept set is $L_{s_\alpha} = \{a_{s_0}, a_{s_1}, a_{s_2}, b_{s_0}, b_{s_1}, b_{s_2}, c_{s_0}, c_{s_1}, c_{s_2}\}$. For instance, $\widetilde{I}(x_1, a_{s_2}) = 0.7$ implies that the membership degree of student x_1 in the high language skills category is 0.7.*

Table 2. Fuzzy linguistic concept formal context $(G, L_{s_\alpha}, \widetilde{I})$.

G/M	***a***			***b***			***c***		
	a_{s_0}	a_{s_1}	a_{s_2}	b_{s_0}	b_{s_1}	b_{s_2}	c_{s_0}	c_{s_1}	c_{s_2}
x_1	0.1	0.2	0.7	0.2	0.2	0.6	0.7	0.2	0.1
x_2	0.7	0.1	0.2	0.2	0.2	0.6	0.4	0	0.6
x_3	0.3	0.2	0.5	0.4	0.5	0.1	0.1	0.6	0.3
x_4	0.1	0.2	0.7	0.3	0.2	0.5	0.8	0	0.2

We can obtain the complementary formal context of $(G, L_{s_\alpha}, \widetilde{I})$ in Table 3 and generate the FOEAL-concepts in Table 4 with a fuzzy credibility threshold of $\lambda = 0.6$. Taking the FOEAL-concept $(\{x_1, x_3\}, (\{(a_{s_2}, 0.7), (b_{s_2}, 0.6), (c_{s_0}, 0.7), (c_{s_1}, 0.6)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (b_{s_2}, 0.9), (c_{s_0}, 0.9), (c_{s_1}, 0.8), (c_{s_2}, 0.9)\}))$ as an example, this FOEAL-concept indicates that the highest level of $\{x_1, x_3\}$ as a whole can reach a membership degree with high language skills for 0.7, a membership degree with high debating skills for 0.6, a membership degree with low personal image for 0.7 and a membership degree with medium personal image for 0.6 from positive information. At the same time, it also shows that except for these four indicator levels, none of the other indicator levels can reach 0.6. From the perspective of negative information, it expresses two possibilities, one of which is that at least one of these two objects $\{x_1, x_3\}$ can reach 0.6 for linguistic concept a_{s_2}. The other is that the membership degrees for a_{s_2} of the two objects $\{x_1, x_3\}$ do not reach 0.6 in both the original and complementary fuzzy linguistic concept formal contexts. We can determine which of these two cases the concept belongs to is directly from the positive information of the FOEAL-concept. In this case, it belongs to the first possibility.

Table 3. The complementary fuzzy linguistic concept formal context $(G, L_{s_\alpha}, \widetilde{I}^c)$.

G/M	***a***			***b***			***c***		
	a_{s_0}	a_{s_1}	a_{s_2}	b_{s_0}	b_{s_1}	b_{s_2}	c_{s_0}	c_{s_1}	c_{s_2}
x_1	0.9	0.8	0.3	0.8	0.8	0.4	0.3	0.8	0.9
x_2	0.3	0.9	0.8	0.8	0.8	0.4	0.6	1	0.4
x_3	0.7	0.8	0.5	0.6	0.5	0.9	0.9	0.4	0.7
x_4	0.9	0.8	0.3	0.7	0.8	0.5	0.2	1	0.8

3.2. The Granular Reduction of Fuzzy Linguistic Concept Formal Context

In order to reduce the scale of the fuzzy linguistic concept formal context and the complexity of constructing the FOEAL lattice, we present a linguistic-concept granular reduction that preserves the granular concept information of the FOEAL lattice.

Definition 14. *Let* $(G, L_{s_\alpha}, \widetilde{I})$ *be a fuzzy linguistic concept formal context,* $\forall x \in G$*; the granular concept of FOEAL-concept is defined by* $(x^{\widetilde{\diamond}\widetilde{\Box}} \cap x^{\overline{\widetilde{\diamond}}\overline{\widetilde{\Box}}}, (x^{\widetilde{\diamond}}, x^{\overline{\widetilde{\diamond}}}))$.

Definition 15. *Let* $(G, L_{s_\alpha}, \widetilde{I})$ *be a fuzzy linguistic concept formal context,* $\forall x \in G$*. A linguistic concept subset* $D_{s_\alpha} \subseteq L_{s_\alpha}$ *is called a fuzzy-object-induced three-way attribute-oriented linguistic granular (FOEALG)-consistent set if* $x^{\preceq L_{s_\alpha} \succeq L_{s_\alpha}} = x^{\preceq D_{s_\alpha} \succeq D_{s_\alpha}}$*. If there is no proper linguistic concept subset* $E_{s_\alpha} \subset D_{s_\alpha}$ *such that* E_{s_α} *is a FOEALG-consistent set, then* D_{s_α} *is called a FOEALG reduction of* $(G, L_{s_\alpha}, \widetilde{I})$.

Table 4. FOEAL concepts of Table 2.

Extent	Intent
$\{x_1\}$	$(\{(a_{s_2}, 0.7), (b_{s_2}, 0.6), (c_{s_0}, 0.7)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (c_{s_1}, 0.8), (c_{s_2}, 0.9)\})$
$\{x_2\}$	$(\{(a_{s_0}, 0.7), (b_{s_2}, 0.6), (c_{s_0}, 0.6)\}, \{(a_{s_1}, 0.9), (a_{s_2}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (c_{s_0}, 0.6), (c_{s_1}, 1)\})$
$\{x_3\}$	$(\{(c_{s_1}, 0.6)\}, \{(a_{s_0}, 0.7), (a_{s_1}, 0.8), (b_{s_0}, 0.6), (b_{s_2}, 0.9), (c_{s_0}, 0.9), (c_{s_2}, 0.7)\})$
$\{x_4\}$	$(\{(a_{s_2}, 0.7), (c_{s_0}, 0.8)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.8), (b_{s_0}, 0.7), (b_{s_1}, 0.8), (c_{s_1}, 1), (c_{s_2}, 0.8)\})$
$\{x_1, x_2\}$	$(\{(a_{s_0}, 0.7), (a_{s_2}, 0.7), (b_{s_2}, 0.6), (c_{s_0}, 0.7)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.9), (a_{s_2}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (c_{s_0}, 0.6), (c_{s_1}, 1), (c_{s_2}, 0.9)\})$
$\{x_1, x_3\}$	$(\{(a_{s_2}, 0.7), (b_{s_2}, 0.6), (c_{s_0}, 0.7), (c_{s_1}, 0.6)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (b_{s_2}, 0.9), (c_{s_0}, 0.9), (c_{s_1}, 0.8), (c_{s_2}, 0.9)\})$
$\{x_1, x_4\}$	$(\{(a_{s_2}, 0.7), (b_{s_2}, 0.6), (c_{s_0}, 0.8)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (c_{s_1}, 1), (c_{s_2}, 0.9)\})$
$\{x_2, x_3\}$	$(\{(a_{s_0}, 0.7), (b_{s_2}, 0.6), (c_{s_0}, 0.6), (c_{s_1}, 0.6)\}, \{(a_{s_0}, 0.7), (a_{s_1}, 0.9), (a_{s_2}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (b_{s_2}, 0.9), (c_{s_0}, 0.9), (c_{s_1}, 1), (c_{s_2}, 0.7)\})$
$\{x_2, x_4\}$	$(\{(a_{s_0}, 0.7), (a_{s_2}, 0.7), (b_{s_2}, 0.6), (c_{s_0}, 0.8)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.9), (a_{s_2}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (c_{s_0}, 0.6), (c_{s_1}, 1), (c_{s_2}, 0.8)\})$
$\{x_3, x_4\}$	$(\{(a_{s_2}, 0.7), (c_{s_0}, 0.8), (c_{s_1}, 0.6)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.8), (b_{s_0}, 0.7), (b_{s_1}, 0.8), (b_{s_2}, 0.9), (c_{s_0}, 0.9), (c_{s_1}, 1), (c_{s_2}, 0.8)\})$
$\{x_1, x_2, x_3\}$	$(\{(a_{s_0}, 0.7), (a_{s_2}, 0.7), (b_{s_2}, 0.6), (c_{s_0}, 0.7), (c_{s_1}, 0.6)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.9), (a_{s_2}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (b_{s_2}, 0.9), (c_{s_0}, 0.9), (c_{s_1}, 1), (c_{s_2}, 0.9)\})$
$\{x_1, x_2, x_4\}$	$(\{(a_{s_0}, 0.7), (a_{s_2}, 0.7), (b_{s_2}, 0.6), (c_{s_0}, 0.8)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.9), (a_{s_2}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (c_{s_0}, 0.6), (c_{s_1}, 1), (c_{s_2}, 0.9)\})$
$\{x_1, x_3, x_4\}$	$(\{(a_{s_2}, 0.7), (b_{s_2}, 0.7), (c_{s_0}, 0.8), (c_{s_1}, 0.6)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (b_{s_2}, 0.9), (c_{s_0}, 0.9), (c_{s_1}, 1), (c_{s_2}, 0.9)\})$
$\{x_2, x_3, x_4\}$	$(\{(a_{s_0}, 0.7), (a_{s_2}, 0.7), (b_{s_2}, 0.6), (c_{s_0}, 0.8), (c_{s_1}, 0.6)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.9), (a_{s_2}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (b_{s_2}, 0.9), (c_{s_0}, 0.9), (c_{s_1}, 1), (c_{s_2}, 0.8)\})$
$\{x_1, x_2, x_3, x_4\}$	$(\{(a_{s_0}, 0.7), (a_{s_2}, 0.7), (b_{s_2}, 0.6), (c_{s_0}, 0.8), (c_{s_1}, 0.6)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.9), (a_{s_2}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (b_{s_2}, 0.9), (c_{s_0}, 0.9), (c_{s_1}, 1), (c_{s_2}, 0.9)\})$
$\emptyset$	$(\emptyset, \emptyset)$

Theorem 1. *Given a fuzzy linguistic concept formal context* $(G, L_{s_\alpha}, \widetilde{I})$, $\forall x \in G$, $D_{s_\alpha} \subseteq L_{s_\alpha}$, $(x^{\widetilde{\diamond}\widetilde{\Box}} \cap x^{\overline{\widetilde{\diamond}}\overline{\widetilde{\Box}}}) = (x^{\widetilde{\diamond}D_{s_\alpha}\widetilde{\Box}D_{s_\alpha}} \cap x^{\overline{\widetilde{\diamond}}D_{s_\alpha}\overline{\widetilde{\Box}}D_{s_\alpha}})$ *if and only if* D_{s_α} *is a FOEALG-consistent set.*

Proof. Definition 15 provides immediate proof for this theorem. □

Given a fuzzy linguistic concept formal context $\mathcal{K} = (G, L_{s_\alpha}, \widetilde{I})$, denote all FOEALG reductions by $Red(\mathcal{K}_{AG})$. Then, the linguistic concept set can be divided as follows:

(1) core linguistic concept set C_r: $C_r = \cap Red(\mathcal{K}_{AG})$,
(2) relatively necessary linguistic concept set K_r: $K_r = \cup Red(\mathcal{K}_{AG}) - \cap Red(\mathcal{K}_{AG})$,

(3) unnecessary linguistic concept set I_r: $I_r = G - \cup Red(\mathcal{K}_{AG})$.

Definition 16. *Given a fuzzy linguistic concept formal context $\mathcal{K} = (G, L_{s_\alpha}, \widetilde{I})$, $\forall x, y \in G$, we define*

$$DIS_{FOEALG}(x,y) = \{l_{s_k} \in L_{s_\alpha} | \widetilde{I}_\lambda(x, l_{s_k}) > \widetilde{I}_\lambda(y, l_{s_k}) \ \ or \ \ \widetilde{I}^c_\lambda(x, l_{s_k}) > \widetilde{I}^c_\lambda(y, l_{s_k})\}, \quad (29)$$

where $DIS_{FOEALG}(x,y)$ is referred to as the FOEALG discernibility linguistic concept set of $\mathcal{K}$, and $\widetilde{I}_\lambda$ denotes the fuzzy relationship under fuzzy credibility threshold λ.

We denote $\Lambda_{FOEALG} = (DIS_{FOEALG}(x,y))$ as the FOEALG discernibility matrix.

Definition 17. *Given a fuzzy linguistic concept formal context $\mathcal{K} = (G, L_{s_\alpha}, \widetilde{I})$, we define the FOEALG-discernibility function as follows:*

$$f_{FOEALG} = \bigwedge \bigvee_{x,y \in G} DIS_{FOEALG}(x,y). \quad (30)$$

Theorem 2. *Given a fuzzy linguistic concept formal context $\mathcal{K} = (G, L_{s_\alpha}, \widetilde{I})$, $D_{s_\alpha} \subseteq L_{s_\alpha}$. For any $x \in G$, we have $(x^{\widetilde{\diamond}\widetilde{\Box}} \cap x^{\overline{\widetilde{\diamond}}\overline{\widetilde{\Box}}}) \subseteq (x^{\widetilde{\diamond}D_{s_\alpha}\widetilde{\Box}D_{s_\alpha}} \cap x^{\overline{\widetilde{\diamond}}D_{s_\alpha}\overline{\widetilde{\Box}}D_{s_\alpha}})$.*

Proof. $\forall y \in G$. Suppose that $y \in (x^{\widetilde{\diamond}\widetilde{\Box}} \cap x^{\overline{\widetilde{\diamond}}\overline{\widetilde{\Box}}})$; then, $y^{\widetilde{\diamond}} \subseteq x^{\widetilde{\diamond}}$, which implies that $y^{\widetilde{\diamond}D_{s_\alpha}} = y^{\widetilde{\diamond}} \cap D_{s_\alpha} \subseteq x^{\widetilde{\diamond}} \cap D_{s_\alpha} = x^{\widetilde{\diamond}D_{s_\alpha}}$. Then, $y \in x^{\widetilde{\diamond}D_{s_\alpha}\Box D_{s_\alpha}}$; likewise, $y \in x^{\overline{\widetilde{\diamond}}D_{s_\alpha}\overline{\widetilde{\Box}}D_{s_\alpha}}$. Therefore, $y \in (x^{\widetilde{\diamond}D_{s_\alpha}\widetilde{\Box}D_{s_\alpha}} \cap x^{\overline{\widetilde{\diamond}}D_{s_\alpha}\overline{\widetilde{\Box}}D_{s_\alpha}})$. Thus, we conclude $(x^{\widetilde{\diamond}\widetilde{\Box}} \cap x^{\overline{\widetilde{\diamond}}\overline{\widetilde{\Box}}}) \subseteq (x^{\widetilde{\diamond}D_{s_\alpha}\widetilde{\Box}D_{s_\alpha}} \cap x^{\overline{\widetilde{\diamond}}D_{s_\alpha}\overline{\widetilde{\Box}}D_{s_\alpha}})$. $\square$

Example 2. *Continuing considering the fuzzy linguistic concept formal context in Table 2, we can calculate granular reduction based on the FOEAL lattice to simplify the formal context as follows. Firstly, the granular concepts of FOEAL-concepts can be obtained:* $(\{x_1\}, (\{(a_{s_2}, 0.7), (b_{s_2}, 0.6), (c_{s_0}, 0.7)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (c_{s_1}, 0.8), (c_{s_2}, 0.9)\}))$, $(\{x_2\}, (\{(a_{s_0}, 0.7), (b_{s_2}, 0.6), (c_{s_0}, 0.6)\}, \{(a_{s_1}, 0.9), (a_{s_2}, 0.8), (b_{s_0}, 0.8), (b_{s_1}, 0.8), (c_{s_0}, 0.6), (c_{s_1}, 1)\}))$, $(\{x_3\}, (\{(c_{s_1}, 0.6)\}, \{(a_{s_0}, 0.7), (a_{s_1}, 0.8), (b_{s_0}, 0.6), (b_{s_2}, 0.9), (c_{s_0}, 0.9), (c_{s_2}, 0.7)\}))$, $(\{x_4\}, (\{(a_{s_2}, 0.7), (c_{s_0}, 0.8)\}, \{(a_{s_0}, 0.9), (a_{s_1}, 0.8), (b_{s_0}, 0.7), (b_{s_1}, 0.8), (c_{s_1}, 1), (c_{s_2}, 0.8)\}))$

Then, we can get the FOEALG-discernibility matrix, as shown in Table 5, and calculate the FOEALG-discernibility function as follows:

Table 5. FOEALG-discernibility matrix of $(G, L_{s_\alpha}, \widetilde{I})$.

	x_1	x_2	x_3	x_4
x_1	$\emptyset$	$a_{s_0}a_{s_2}c_{s_0}c_{s_2}$	$a_{s_0}a_{s_2}b_{s_0}b_{s_1}b_{s_2}c_{s_0}c_{s_1}c_{s_2}$	$b_{s_0}b_{s_2}c_{s_2}$
x_2	$a_{s_0}a_{s_1}a_{s_2}c_{s_0}c_{s_1}c_{s_2}$	$\emptyset$	$a_{s_0}a_{s_1}a_{s_2}b_{s_0}b_{s_1}b_{s_2}c_{s_1}c_{s_2}$	$a_{s_0}a_{s_1}a_{s_2}b_{s_0}b_{s_2}c_{s_0}c_{s_2}$
x_3	$b_{s_2}c_{s_0}c_{s_1}$	$a_{s_0}b_{s_2}c_{s_0}c_{s_1}c_{s_2}$	$\emptyset$	$b_{s_2}c_{s_0}c_{s_1}$
x_4	$c_{s_0}c_{s_1}$	$a_{s_0}a_{s_2}c_{s_0}c_{s_2}$	$a_{s_0}a_{s_2}b_{s_0}b_{s_1}c_{s_0}c_{s_1}c_{s_2}$	$\emptyset$

$$\begin{aligned} & f_{FOEALG} = \wedge \bigvee_{x,y \in G} DIS_{FOEALG}(x,y) \\ & = (c_{s_0} \vee c_{s_1}) \wedge (a_{s_0} \vee a_{s_2} \vee c_{s_0} \vee c_{s_2}) \wedge (b_{s_0} \vee b_{s_2} \vee c_{s_2}) \\ & = (b_{s_0} \wedge c_{s_0}) \vee (b_{s_2} \wedge c_{s_0}) \vee (c_{s_0} \wedge c_{s_2}) \vee (c_{s_1} \wedge c_{s_2}) \vee (a_{s_0} \wedge b_{s_0} \wedge c_{s_1}) \\ & \vee (a_{s_2} \wedge b_{s_0} \wedge c_{s_1}) \vee (a_{s_0} \wedge b_{s_2} \wedge c_{s_1}) \vee (a_{s_2} \wedge b_{s_2} \wedge c_{s_1}) \end{aligned}$$

Therefore, we have eight FOEALG reductions from $(G, L_{s_\alpha}, \widetilde{I})$, which are $\{b_{s_0}, c_{s_0}\}$, $\{b_{s_2}, c_{s_0}\}$, $\{c_{s_0}, c_{s_2}\}$, $\{c_{s_1}, c_{s_2}\}$, $\{a_{s_0}, b_{s_0}, c_{s_1}\}$, $\{a_{s_2}, b_{s_0}, c_{s_1}\}$, $\{a_{s_0}, b_{s_2}, c_{s_1}\}$ and $\{a_{s_2}, b_{s_2}, c_{s_1}\}$, respectively. In addition to the fact that a_{s_1} and b_{s_1} are unnecessary linguistic concepts, the other linguistic concepts are all relatively necessary linguistic concepts.

4. The Relation between Granular Reduction and Entropy Reduction in the Fuzzy Linguistic Concept Formal Context

Information entropy is an essential method for measuring the importance of attributes in rough set theory. This section proposes entropy reduction in a fuzzy linguistic concept formal context by combining necessity and possibility operators and then studies its relationship with granular reduction.

Definition 18. *Given a fuzzy linguistic concept formal context $\mathcal{K} = (G, L_{s_\alpha}, \widetilde{I})$, for $B_{s_\alpha}, C_{s_\alpha} \subseteq L_{s_\alpha}$, we define the information entropy of the FOEAL lattice as follows:*

$$H(L_{s_\alpha}) = -\frac{1}{|G|}\sum_{x\in G} log_2 \frac{|x^{\preceq L_{s_\alpha} \succeq L_{s_\alpha}}|}{|G|},$$

and the conditional information entropy of B_{s_α} with respect to C_{s_α} is as follows:

$$H(B_{s_\alpha}/C_{s_\alpha}) = -\frac{1}{|G|}\sum_{x\in G} log_2 \frac{|x^{\preceq B_{s_\alpha} \succeq B_{s_\alpha}} \bigcap x^{\preceq C_{s_\alpha} \succeq C_{s_\alpha}}|}{|x^{\preceq C_{s_\alpha} \succeq C_{s_\alpha}}|},$$

where $|\cdot|$ indicates the number of elements in the object set.

Theorem 3. *Given a fuzzy linguistic concept formal context $\mathcal{K} = (G, L_{s_\alpha}, \widetilde{I})$, for any $x \in G$, $C_{s_\alpha} \subseteq B_{s_\alpha} \subseteq L_{s_\alpha}$, the two information entropies $H(C_{s_\alpha})$ and $H(B_{s_\alpha})$, based on the FOEAL lattice, satisfy:*

1. $H(C_{s_\alpha}) \le H(B_{s_\alpha})$ *and* $x^{\preceq C_{s_\alpha} \succeq C_{s_\alpha}} \supseteq x^{\preceq B_{s_\alpha} \succeq B_{s_\alpha}}$;
2. *if* $H(C_{s_\alpha}) = H(B_{s_\alpha})$, *then* $x^{\preceq C_{s_\alpha} \succeq C_{s_\alpha}} = x^{\preceq B_{s_\alpha} \succeq B_{s_\alpha}}$.

Proof. This can be easily proved by Theorem 2. □

Definition 19. *Given a fuzzy linguistic concept formal context $\mathcal{K} = (G, L_{s_\alpha}, \widetilde{I})$, $\forall x \in G$, a linguistic concept subset $D_{s_\alpha} \subseteq L_{s_\alpha}$ is called an entropy consistent set of the FOEAL lattice if $H(D_{s_\alpha}) = H(L_{s_\alpha})$. If there is no proper linguistic concept subset $E_{s_\alpha} \subset L_{s_\alpha}$ such that E_{s_α} is an entropy consistent set of the FOEAL lattice, then D_{s_α} is called an entropy reduction of the FOEAL lattice.*

Given a fuzzy linguistic concept formal context $\mathcal{K} = (G, L_{s_\alpha}, \widetilde{I})$, denote all entropy reductions of the FOEAL lattice by $Red(\mathcal{K}_{AE})$. Similarly, the linguistic concept set can be divided as follows:

(1) core linguistic concept set C_t: $C_t = \cap Red(\mathcal{K}_{AE})$,
(2) relatively necessary linguistic concept set K_t: $K_t = \cup Red(\mathcal{K}_{AE}) - \cap Red(\mathcal{K}_{AE})$,
(3) unnecessary linguistic concept set I_t: $I_t = G - \cup Red(\mathcal{K}_{AE})$.

Theorem 4. *Given a fuzzy linguistic concept formal context $\mathcal{K} = (G, L_{s_\alpha}, \widetilde{I})$, $\forall l_{s_k} \in L_{s_\alpha}$, $l_{s_k} \in I_t$ if and only if $H(\{l_{s_k}\}|L_{s_\alpha} - \{l_{s_k}\}) = 0$.*

Proof. $(\Rightarrow)$ Since $l_{s_k} \in I_t$, we have $H(L_{s_\alpha} - \{l_{s_k}\}) \le H(L_{s_\alpha})$ according to the Theorem 3.

Assume that $H(L_{s_\alpha} - \{l_{s_k}\}) < H(L_{s_\alpha})$; obviously, $L_{s_\alpha} - \{l_{s_k}\}$ is not a consistent set. It follows that $l_{s_k} \in C_t$, which contradicts $l_{s_k} \in I_t$. Therefore, $H(L_{s_\alpha} - \{l_{s_k}\}) = H(L_{s_\alpha})$, $\forall x \in G$, $x^{\preceq L_{s_\alpha} \succeq L_{s_\alpha}} = x^{\preceq L_{s_\alpha} - \{l_{s_k}\} \succeq L_{s_\alpha} - \{l_{s_k}\}}$ is held; i.e., $H(\{l_{s_k}\}|L_{s_\alpha} - \{l_{s_k}\}) = -\frac{1}{|G|}\sum_{x\in G} log_2 \frac{|x^{\preceq\{l_{s_k}\}\succeq\{l_{s_k}\}} \bigcap x^{\preceq L_{s_\alpha} - \{l_{s_k}\} \succeq L_{s_\alpha} - \{l_{s_k}\}}|}{|x^{\preceq L_{s_\alpha} - \{l_{s_k}\} \succeq L_{s_\alpha} - \{l_{s_k}\}}|} = -\frac{1}{|G|}\sum_{x\in G} log_2 \frac{|x^{\preceq\{l_{s_k}\}\succeq\{l_{s_k}\}} \bigcap x^{\preceq L_{s_\alpha} \succeq L_{s_\alpha}}|}{|x^{\preceq L_{s_\alpha} \succeq L_{s_\alpha}}|} = 0.$

$(\Leftarrow)$ Since $H(\{l_{s_k}\}|L_{s_\alpha} - \{l_{s_k}\}) = 0$, $\forall x \in G$, $log_2 \frac{|x^{\preceq\{l_{s_k}\}\succeq\{l_{s_k}\}} \bigcap x^{\preceq L_{s_\alpha} - \{l_{s_k}\} \succeq L_{s_\alpha} - \{l_{s_k}\}}|}{|x^{\preceq L_{s_\alpha} - \{l_{s_k}\} \succeq L_{s_\alpha} - \{l_{s_k}\}}|} \le 0$, we have $H(\{l_{s_k}\}|L_{s_\alpha} - \{l_{s_k}\}) \ge 0$. Suppose that $\exists x_0 \in G$, s.t. $x_0^{\preceq\{l_{s_k}\}\succeq\{l_{s_k}\}} \bigcap x_0^{\preceq L_{s_\alpha} - \{l_{s_k}\}}$

${}^{\succeq L_{s_\alpha}-\{l_{s_k}\}} \subset x_0^{\preceq L_{s_\alpha}-\{l_{s_k}\}\succeq L_{s_\alpha}-\{l_{s_k}\}}$; then $log_2 \frac{|x^{\preceq\{l_{s_k}\}\succeq\{l_{s_k}\}} \bigcap x^{\preceq L_{s_\alpha}-\{l_{s_k}\}\succeq L_{s_\alpha}-\{l_{s_k}\}}|}{|x^{\preceq L_{s_\alpha}-\{l_{s_k}\}\succeq L_{s_\alpha}-\{l_{s_k}\}}|} < 0$; i.e., $H(\{l_{s_k}\} |L_{s_\alpha} - \{l_{s_k}\}) > 0$, which contradicts $H(\{l_{s_k}\}|L_{s_\alpha} - \{l_{s_k}\}) = 0$. Therefore, $\forall x \in G$, $x^{\preceq\{l_{s_k}\}\succeq\{l_{s_k}\}} \bigcap x^{\preceq L_{s_\alpha}-\{l_{s_k}\}\succeq L_{s_\alpha}-\{l_{s_k}\}} = x^{\preceq L_{s_\alpha}-\{l_{s_k}\}\succeq L_{s_\alpha}-\{l_{s_k}\}}$, and then $x^{\preceq L_{s_\alpha}-\{l_{s_k}\}\succeq L_{s_\alpha}-\{l_{s_k}\}} = x^{\preceq L_{s_\alpha}\succeq L_{s_\alpha}}$, so we conclude that $l_{s_k} \in I_t$. □

Theorem 5. *Given a fuzzy linguistic concept formal context* $\mathcal{K} = (G, L_{s_\alpha}, \widetilde{I})$, $\forall l_{s_k} \in L_{s_\alpha}$, $l_{s_k} \in C_t$ *if and only if* $H(\{l_{s_k}\}|L_{s_\alpha} - \{l_{s_k}\}) > 0$.

Proof. This is easy to prove by Theorem 4. □

Remark 1. *We can obtain the method to calculate the core linguistic concept set and unnecessary linguistic concept set according to the above theorems; therefore, we can also find the linguistic concept's reduction by calculating the relatively unnecessary linguistic concept set indirectly through these two theorems.*

Theorem 6. *Given a fuzzy linguistic concept formal context* $\mathcal{K} = (G, L_{s_\alpha}, \widetilde{I})$, $D_{s_\alpha} \subseteq L_{s_\alpha}$, *for any* $l_{s_k} \in D_{s_\alpha}$, D_{s_α} *is an entropy reduction of the FOEAL lattice if and only if* $H(D_{s_\alpha}) = H(L_{s_\alpha})$ *and* $H(\{l_{s_k}\}|L_{s_\alpha} - \{l_{s_k}\}) > 0$.

Proof. ($\Rightarrow$) Since D_{s_α} is an entropy reduction of the FOEAL lattice, we have $H(D_{s_\alpha}) = H(L_{s_\alpha})$. Obviously, for any $l_{s_k} \in D_{s_\alpha}$, it satisfies $log_2 \frac{|x^{\preceq\{l_{s_k}\}\succeq\{l_{s_k}\}} \bigcap x^{\preceq D_{s_\alpha}-\{l_{s_k}\}\succeq D_{s_\alpha}-\{l_{s_k}\}}|}{|x^{\preceq D_{s_\alpha}-\{l_{s_k}\}\succeq D_{s_\alpha}-\{l_{s_k}\}}|} \leq 0$; then, $H(\{l_{s_k}\}|D_{s_\alpha} - \{l_{s_k}\}) = -\frac{1}{|G|}\sum\limits_{x\in G} log_2 \frac{|x^{\preceq\{l_{s_k}\}\succeq\{l_{s_k}\}} \bigcap x^{\preceq D_{s_\alpha}-\{l_{s_k}\}\succeq D_{s_\alpha}-\{l_{s_k}\}}|}{|x^{\preceq D_{s_\alpha}-\{l_{s_k}\}\succeq D_{s_\alpha}-\{l_{s_k}\}}|} = -\frac{1}{|G|}\sum\limits_{x\in G} log_2 \frac{|x^{\preceq D_{s_\alpha}\succeq D_{s_\alpha}}|}{|x^{\preceq D_{s_\alpha}-\{l_{s_k}\}\succeq D_{s_\alpha}-\{l_{s_k}\}}|} \geq 0$. Suppose that $\exists h_{s_k} \in D_{s_\alpha}$, s.t. $H(\{h_{s_k}\}|D_{s_\alpha} - \{h_{s_k}\}) = 0$. Then, $h_{s_k} \in I_t$, $x^{\preceq D_{s_\alpha}\succeq D_{s_\alpha}} = x^{\preceq D_{s_\alpha}-\{h_{s_k}\}\succeq D_{s_\alpha}-\{h_{s_k}\}}$. It follows that $H(D_{s_\alpha} - \{h_{s_k}\}) = H(D_{s_\alpha})$, which contradicts that D_{s_α} is an entropy reduction of the FOEAL lattice. Therefore, $\forall l_{s_k} \in D_{s_\alpha}$, $H(\{l_{s_k}\}|L_{s_\alpha} - \{l_{s_k}\}) > 0$ holds.

($\Leftarrow$) Assume that $H(D_{s_\alpha}) = H(L_{s_\alpha})$ and $H(\{l_{s_k}\}|L_{s_\alpha} - \{l_{s_k}\}) > 0$ for any $l_{s_k} \in D_{s_\alpha}$. Then, $H(\{l_{s_k}\}|D_{s_\alpha} - \{l_{s_k}\}) = -\frac{1}{|G|}\sum\limits_{x\in G} log_2 \frac{|x^{\preceq D_{s_\alpha}\succeq D_{s_\alpha}}|}{|x^{\preceq D_{s_\alpha}-\{l_{s_k}\}\succeq D_{s_\alpha}-\{l_{s_k}\}}|} > 0$; i.e., $x^{\preceq D_{s_\alpha}\succeq D_{s_\alpha}} \subseteq x^{\preceq D_{s_\alpha}-\{l_{s_k}\}\succeq D_{s_\alpha}-\{l_{s_k}\}}$. According to Definition 18, $H(D_{s_\alpha}) = -\frac{1}{|G|}\sum\limits_{x\in G} log_2 \frac{|x^{\preceq D_{s_\alpha}\succeq D_{s_\alpha}}|}{|G|} > -\frac{1}{|G|}\sum\limits_{x\in G} log_2 \frac{|x^{\preceq D_{s_\alpha}-\{l_{s_k}\}\succeq D_{s_\alpha}-\{l_{s_k}\}}|}{|G|} = H(D_{s_\alpha} - \{l_{s_k}\})$. Therefore, $D_{s_\alpha} - \{l_{s_k}\}$ is not an entropy-consistent set of the FOEAL lattice. We can conclude that D_{s_α} is an entropy reduction of the FOEAL lattice. □

Theorem 7. *Given a fuzzy linguistic concept formal context* $\mathcal{K} = (G, L_{s_\alpha}, \widetilde{I})$, $D_{s_\alpha} \subseteq L_{s_\alpha}$, D_{s_α} *is an entropy consistent set of the FOEAL lattice if and only if* D_{s_α} *is a FOEALG consistent set.*

Proof. ($\Rightarrow$) Assume that D_{s_α} is an entropy consistent set. We obtain $H(D_{s_\alpha}) = H(L_{s_\alpha})$. $\forall x \in G$, $x^{\preceq D_{s_\alpha}\succeq D_{s_\alpha}} = x^{\preceq L_{s_\alpha}\succeq L_{s_\alpha}}$; therefore, D_{s_α} is a FOEALG-consistent set.

($\Leftarrow$) Analogously to the necessity, it follows easily. □

Theorem 8. *Given a fuzzy linguistic concept formal context* $\mathcal{K} = (G, L_{s_\alpha}, \widetilde{I})$, $D_{s_\alpha} \subseteq L_{s_\alpha}$, D_{s_α} *is an entropy reduction of the FOEAL lattice if and only if* D_{s_α} *is a FOEALG reduction.*

Proof. Analogously to Theorem 7, it follows easily. □

Example 3. *Continuing considering the fuzzy linguistic concept formal context in Table 2. From Example 2, it is clear that* $x_1^{\preceq L_{s_\alpha}\succeq L_{s_\alpha}} = x_1$, $x_2^{\preceq L_{s_\alpha}\succeq L_{s_\alpha}} = x_2$, $x_3^{\preceq L_{s_\alpha}\succeq L_{s_\alpha}} = x_3$ *and* $x_4^{\preceq L_{s_\alpha}\succeq L_{s_\alpha}} = x_4$. *We can get* $H(L_{s_\alpha}) = -\frac{1}{|G|}\sum\limits_{x\in G} log_2 \frac{|x^{\preceq L_{s_\alpha}\succeq L_{s_\alpha}}|}{|G|} = 2$ *according to the Definition 18. By taking the* *linguistic concept* $D_{s_\alpha} = \{a_{s_0}, b_{s_0}, c_{s_1}\}$ *as an example, the calculation yields* $H(D_{s_\alpha}) = 2$*; however, if* $D^1_{s_\alpha} = \{a_{s_0}, b_{s_0}\}$*, then* $H(D^1_{s_\alpha}) = \frac{7}{4}$*; if* $D^2_{s_\alpha} = \{a_{s_0}, c_{s_1}\}$*, then* $H(D^2_{s_\alpha}) = \frac{3}{2}$*; if* $D^3_{s_\alpha} = \{b_{s_0}, c_{s_1}\}$*, then* $H(D^3_{s_\alpha}) = 2 - \frac{1}{4}log_2 3$*. Therefore, we can conclude that* $H(L_{s_\alpha}) = H(D_{s_\alpha})$*,* $\{a_{s_0}, b_{s_0}, c_{s_1}\}$ *is an entropy reduction of the FOEAL lattice. Similarly,* $\{b_{s_0}, c_{s_0}\}$*,* $\{b_{s_2}, c_{s_0}\}$*,* $\{c_{s_0}, c_{s_2}\}$*,* $\{c_{s_1}, c_{s_2}\}$*,* $\{a_{s_2}, b_{s_0}, c_{s_1}\}$*,* $\{a_{s_0}, b_{s_2}, c_{s_1}\}$ *and* $\{a_{s_2}, b_{s_2}, c_{s_1}\}$ *are also entropy reductions of the FOEAL lattice.*

From the above analysis, we confirm that the entropy reduction is equal to the granular reduction based on the FOEAL lattice, and we can obtain the granular concept information through the information entropy. In practical applications, the desired FOEAL-concept information can also be selected by setting the parameter λ.

5. Comparative Analysis

We compare our methods with other knowledge reduction methods in six aspects: the type of concept lattice that the paper can handle, the expression form of the concept, how many reduction methods the paper actually proposes, which conditions are preserved during reduction and whether the reduction method can handle linguistic information and fuzzy information. Table 6 displays the comparison results for the corresponding methods.

- In Ref. [30]: Utilizing the object-oriented concept lattice $L_O(G, M, I)$ and the attribute-oriented concept lattice $L_A(G, M, I)$, Qin et al. introduced a technique for attribute reduction that maintains decision rules.
- In Ref. [39]: Zhang et al. put forward a new fuzzy three-way concept lattice, denoted by $OFTL(G, M, \widetilde{I})$ and $AFTL(G, M, \widetilde{I})$, which takes into account the fuzziness of objects and attributes, respectively. Furthermore, they presented a granular matrix-based reduction method to handle fuzzy data in a fuzzy formal context.
- In Ref. [22]: Ren et al. developed four techniques for attribute reduction that preserve lattice structure, granular information and join (meet)-irreducible elements, utilizing three-way concept lattices $OEL(G, M, I)$ and $AEL(G, M, I)$.
- In Ref. [48]: Zou et al. presented a linguistic concept lattice $LL(G, L_{s_\alpha}, I)$, and further studied a multi-granularity linguistic-concept reduction algorithm based on the similarity relations in an incomplete linguistic concept formal context, which can deal with different types of linguistic information.

Remark 2. *Even though the concept lattices examined in references [22,39,48] pertain to knowledge reduction of the two concept lattices analyzed from the viewpoints of objects and attributes, their techniques essentially share the same nature. Therefore, only one of them is displayed in Table 6.*

Table 6. Comparison of other reduction methods.

Methods	The Type of Concept Lattice	Concept Extent	Concept Intent	Reduction Methods	Reduction Conditions for Preservation	Linguistic Information	Fuzzy Information
Ref. [30]	$L_O(G, M, I)$	X	B	1	decision rules	×	×
Ref. [39]	$OFTL(G, M, \widetilde{I})$	X	$(\widetilde{B}, \widetilde{C})$	1	granular matrix	×	✓
Ref. [22]	$OEL(G, M, I)$	X	(B, C)	4	lattice structure/ granular information/ join (meet)-irreducible elements	×	×
Ref. [48]	$LL(G, L_{s_\alpha}, I)$	X	B_{s_α}	2	multi-granularity similarity relations/ binary relation	✓	×
Our methods	$FOEAL(G, L_{s_\alpha}, \widetilde{I})$	X	$(\widetilde{B}_{s_\alpha}, \widetilde{C}_{s_\alpha})$	2	granular concept/ entropy information	✓	✓

Since people are always accustomed to evaluating things in natural language, the combination of linguistic-term sets and formal concept analysis plays a significant role. In light of the above analysis, we summarize some merits of our proposed approach:

(1) To accurately represent the uncertainty and complexity of real-world situations, we introduced a fuzzy linguistic concept formal context that establishes a fuzzy relation

between objects and linguistic concepts. This approach generates a FOEAL lattice that aligns more closely with human cognition.

(2) By combining the advantages of $OEL(G, M, I)$ and $L_A(G, M, I)$, we propose a FOEAL lattice in a fuzzy linguistic concept formal context, which can not only show the idea of three divisions in three-way decisions, but also express the complementary structure of a linguistic concept lattice compared with symmetric linguistic-evaluation information.

(3) In view of the validity and simplicity of granular reduction in formal concept analysis, two linguistic-concept-reduction methods preserving granular information and information entropy, granular reduction and entropy reduction based on the FOEAL lattice, are given to reduce the scale of linguistic concepts.

6. Conclusions

We have discussed the linguistic-concept-reduction methods combining granular computing with information entropy in a fuzzy linguistic concept formal context. A fuzzy-object-induced three-way attribute-oriented linguistic-concept lattice has been proposed, which describes the possible fuzzy relations between objects and linguistic concepts in terms of both positive and negative information. We have given a granular reduction method of the FOEAL lattice to preserve the fuzzy, granular concept information. In addition, we have also presented an entropy reduction method of the FOEAL lattice and discussed its relationship with the granular reduction mentioned above, which further enhances our understanding of the decision-making problem in artificial intelligence.

In future research, we intend to investigate different reduction methods in the fuzzy linguistic concept formal decision context and apply them to practical problems. Moreover, we will further employ fuzzy learning techniques to improve the effectiveness of the FOEAL concept lattice.

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