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Orthomodular and Skew Orthomodular Posets

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Abstract: We present the smallest non-lattice orthomodular poset and show that it is unique up to isomorphism. Since not every Boolean poset is orthomodular, we consider the class of skew orthomodular posets previously introduced by the first and third author under the name "generalized orthomodular posets". We show that this class contains all Boolean posets and we study its subclass consisting of horizontal sums of Boolean posets. For this purpose, we introduce the concept of a compatibility relation and the so-called commutator of two elements. We show the relationship between these concepts and introduce a kind of ternary discriminator for horizontal sums of Boolean posets. Numerous examples illuminating these concepts and results are included in the paper.

Keywords: smallest non-lattice orthomodular poset; skew orthomodular poset; Boolean poset; horizontal sum; compatibility relation; commutator; ternary discriminator

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1. Introduction

It is well-known that the set of closed subspaces of a Hilbert space forms a complete orthomodular lattice with respect to set-inclusion. Because these subspaces correspond to self-adjoint bounded operators which correspond to observables in quantum measurements, this orthomodular lattice is often considered as an algebraic counterpart of the logic of quantum mechanics, see, e.g., [1] or [2]. Recall that an *ortholattice* is a bounded lattice $(L, \vee, \wedge, ', 0, 1)$ with an antitone involution $'$ which is a complementation, and an *orthomodular lattice* is an ortholattice $(L, \vee, \wedge, ', 0, 1)$ satisfying the so-called *orthomodular law*, i.e.,

(OM) if $x \leq y$ then $y = x \vee (y \wedge x')$

which in turn is equivalent to its dual

if $x \leq y$ then $x = y \wedge (x \vee y')$.

However, it was recognized later that if the elements x and y are not *orthogonal*, i.e., if not $x \leq y'$, then the join $x \vee y$ need not exist in accordance with quantum theory. Hence, so-called orthomodular posets were introduced (see, e.g., [3]) as follows:

An *orthomodular poset* is a bounded poset $(P, \leq, ', 0, 1)$ with an antitone involution $'$ that is a complementation satisfying the following conditions:

(i) if $x \perp y$ then $x \vee y$ is defined,
(ii) if $x \leq y$ then $y = x \vee (y \wedge x')$. (OM)

Hereinafter, $x \perp y$ means $x \leq y'$. Observe that the expression in (ii) is well-defined because $x \perp y'$ yields that $y' \vee x$ exists and, by De Morgan's laws, also $y \wedge x' = (y' \vee x)'$ is defined and, due to $y \wedge x' \perp x$ also $x \vee (y \wedge x')$ is defined. Of course, (ii) is equivalent to its dual

$$x \leq y \text{ implies } x = y \wedge (x \vee y').$$

It is evident that if the lattice $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ is Boolean, i.e., a distributive complemented lattice, then it is orthomodular. Unfortunately, a similar result does not hold for distributive posets. This is the reason why we introduced the concept of a generalized orthomodular poset (see, e.g., [4]) which in this paper we will call a skew orthomodular poset (since the name "generalized orthomodular poset" is also used with a different meaning) and which, as we will show, can also be a Boolean poset. Hence, we essentially extend the class of orthomodular posets in such a way that they share more natural properties with orthomodular lattices than orthomodular posets do. This represents one of our goals in this paper. The second problem connected with orthomodular posets is to find such a poset of minimal size not being a lattice. As far as we know, this problem has yet to be solved.

In fact, S. Pulmannová and P. Pták [5] applied the method of Greechie diagrams in order to construct an 18-element orthomodular poset. The considered Greechie diagram consists of four three-atomic blocks forming a square. However, it was not proved that this orthomodular poset is the minimal non-lattice one and that it is unique up to isomorphism. This motivated us to provide an exact proof of these statements. It is worth noting that orthomodular posets are closely related to the logic of quantum mechanics, which is a physical theory based on the idea of symmetry. Overall, the relationship between skew orthomodular posets and symmetry highlights the deep connection between different areas of mathematics and science, and underscores the importance of symmetry as a fundamental concept in understanding the structure and behaviour of physical systems.

2. Basic Concepts

In the following, we need several concepts and notations which we will present in this section.

Let $\mathbf{P} = (P, \leq)$ be a poset, $A, B \subseteq P$ and $a, b \in P$. We define $A \leq B$ if and only if $x \leq y$ for all $x \in A$ and all $y \in B$. Instead of $A \leq \{b\}$, $\{a\} \leq B$ and $\{a\} \leq \{b\}$ we simply write $A \leq b$, $a \leq B$ and $a \leq b$, respectively. The sets

$$L(A) := \{x \in P \mid x \leq A\},$$
$$U(A) := \{x \in P \mid A \leq x\}$$

are called the *lower cone* and *upper cone* of A, respectively. Instead of $L(A \cup B)$, $L(A \cup \{b\})$, $L(\{a, b\})$ and $L\big(U(A)\big)$ we write $L(A, B)$, $L(A, b)$, $L(a, b)$ and $LU(A)$, respectively. Analogously, we proceed in similar cases. Recall that $\mathbf{P}$ is called *distributive* (see, e.g., [6]) if it satisfies the identity

$$L\big(U(x, y), z\big) \approx LU\big(L(x, z), L(y, z)\big)$$

or, equivalently, one of the following identities:

$$UL\big(U(x, y), z\big) \approx U\big(L(x, z), L(y, z)\big),$$
$$U\big(L(x, y), z\big) \approx UL\big(U(x, z), U(y, z)\big),$$
$$LU\big(L(x, y), z\big) \approx L\big(U(x, z), U(y, z)\big).$$

Here and in the sequel, $L\big(U(x, y), z\big) \approx LU\big(L(x, z), L(y, z)\big)$ means that

$$L\big(U(x, y), z\big) = LU\big(L(x, z), L(y, z)\big) \text{ holds for all } x, y, z \in P.$$

It can be easily seen that if $\mathbf{P}$ is a lattice then it is a distributive poset if and only if it satisfies the distributive law

$$(x \vee y) \wedge z \approx (x \wedge z) \vee (y \wedge z).$$

Lemma 1. *Let $(P, \leq, 0, 1)$ be a bounded distributive poset and $a, b, c, a', b' \in P$. Then the following holds:*

(i) *If b and c are complements of a then $b = c$,*

(ii) *if* $a \leq b$, $U(a,a') = \{1\}$ *and* $L(b,b') = \{0\}$ *then* $b' \leq a'$.

Proof.

(i) We have

$$L(b) = L(1,b) = L\big(U(c,a),b\big) = LU\big(L(c,b),L(a,b)\big) = LU\big(L(c,b),0\big) = \\ = LU\big(L(b,c),0\big) = LU\big(L(b,c),L(a,c)\big) = L\big(U(b,a),c\big) = L(1,c) = L(c)$$

and hence $b = c$.

(ii) We have

$$\{0\} \subseteq L(a,b') \subseteq L(b,b') = \{0\}$$

and hence $L(a,b') = \{0\}$ which implies

$$b' \in L(b') = L(1,b') = L\big(U(a,a'),b'\big) = LU\big(L(a,b'),L(a',b')\big) = \\ = LU\big(0,L(a',b')\big) = LUL(a',b') = L(a',b') \subseteq L(a'),$$

i.e., $b' \leq a'$.

□

Boolean posets, i.e., distributive complemented posets, play an important role in the algebraic theory of posets since they share many important properties of Boolean algebras. As mentioned in the introduction, every Boolean algebra is an orthomodular lattice, but not every Boolean poset is an orthomodular one. In order to avoid this discrepancy, we define the following concept introduced in [4] under the name "generalized orthomodular poset" (cf. also the paper [7]):

Definition 1. *A* skew orthomodular poset *is a bounded poset* $\mathbf{P} = (P, \leq, ', 0, 1)$ *with an antitone involution which in turn is a complementation satisfying the condition*

(GOM) $x \leq y$ *implies* $U(y) = U\big(x, L(y,x')\big)$.

It is worth noting that (GOM) is equivalent to its dual

$$x \leq y \text{ implies } L(x) = L\big(y, U(x,y')\big).$$

If the poset is *orthogonal*, i.e., if for all $x, y \in P$ with $x \leq y'$ there exists $x \vee y$, then (GOM) is equivalent to (OM) and hence $\mathbf{P}$ is an orthomodular poset.

Recall that a *Boolean poset* is a distributive complemented poset.

By Lemma 1, the complementation in a Boolean poset is unique and antitone. We will use this fact when proving our results in Section 4.

It is easy to prove the following assertion.

Proposition 1. *Let* $\mathbf{B} = (B, \leq, ', 0, 1)$ *be a Boolean poset. Then* $\mathbf{B}$ *is a skew orthomodular poset.*

Proof. Since x and x'' are complements of x', we obtain $x'' \approx x$ by Lemma 1(i). According to Lemma 1(ii), $'$ is antitone. Finally, if $x \leq y$ then

$$U(y) = ULU(y) = UL\big(U(y),1\big) = UL\big(U(x,y),U(x,x')\big) = U\big(x,L(y,x')\big)$$

using distributivity of $\mathbf{B}$. □

Example 1. *The poset depicted in Figure 1 is a non-lattice Boolean poset and hence a skew orthomodular poset according to Proposition 1.*

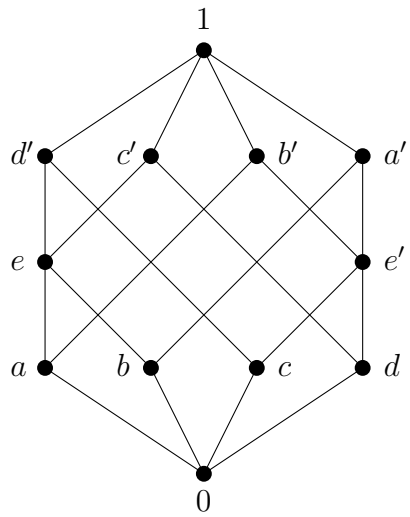

Figure 1. A non-orthomodular non-lattice Boolean poset.

This poset is not an orthomodular poset since $a \leq c'$, but $a \vee c$ does not exist.

3. The Smallest Non-Lattice Orthomodular Poset

As mentioned in the introduction, as far as we know, the smallest non-lattice orthomodular poset is not known up to now. Sometimes, the following 20-element non-lattice orthomodular poset was considered (Figure 2). It is the poset of all subsets A of the set $\{1,\ldots,6\}$ having an even number of elements and satisfying $|A \cap \{1,2,3\}| = |A \cap \{4,5,6\}|$.

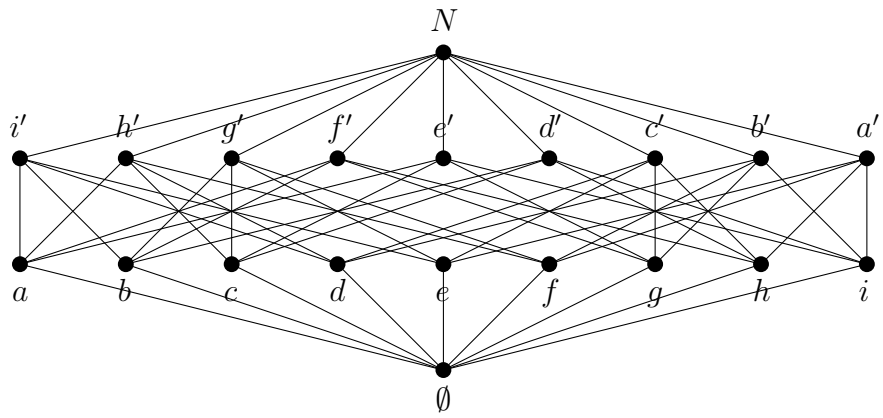

Figure 2. A 20-element non-lattice orthomdular poset.

Here $a = \{1,4\}$, $b = \{1,5\}$, $c = \{1,6\}$, $d = \{2,4\}$, $e = \{2,5\}$, $f = \{2,6\}$, $g = \{3,4\}$, $h = \{3,5\}$, $i = \{3,6\}$, $a' = \{2,3,5,6\}$, $b' = \{2,3,4,6\}$, $c' = \{2,3,4,5\}$, $d' = \{1,3,5,6\}$, $e' = \{1,3,4,6\}$, $f' = \{1,3,4,5\}$, $g' = \{1,2,5,6\}$, $h' = \{1,2,4,6\}$, $i' = \{1,2,4,5\}$ and $N = \{1,\ldots,6\}$. For

$$P := \{A \subseteq N \mid |A \cap \{1,2,3\}| = |A \cap \{4,5,6\}|\},$$

the poset $\mathbf{P} = (P, \subseteq, ', \emptyset, N)$ is not a lattice since, e.g., $a \vee b$ does not exist. Note that $\mathbf{P}$ is the smallest orthomodular subposet of the orthomodular poset $(Q, \subseteq, ', \emptyset, N)$ with $Q := \{A \in 2^N \mid A \text{ has an even number of elements}\}$ containing $a = \{1,4\}$ and $b = \{1,5\}$.

However, we will prove the following result.

Theorem 1. *The smallest non-lattice orthomodular poset is depicted in Figure 3 and is unique up to isomorphism.*

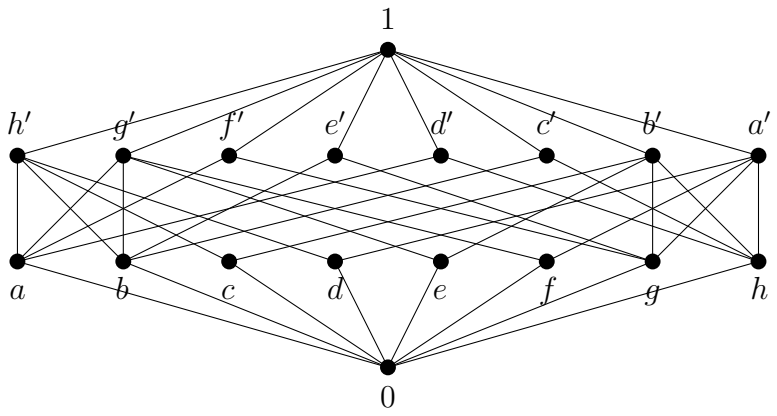

Figure 3. The smallest non-lattice orthomodular poset.

Proof. Let $\mathbf{P} = (P, \leq, ', 0, 1)$ be a minimal non-lattice orthomodular poset. Then there exist $a, b \in P$ having no supremum. Let g' and h' be two minimal upper bounds of a and b. If g' and h' had an infimum, they would not be minimal upper bounds of a and b. Hence, g' and h' have no infimum. Thus, the Hasse diagram of $\mathbf{P}$ must contain the configuration shown in Figure 4:

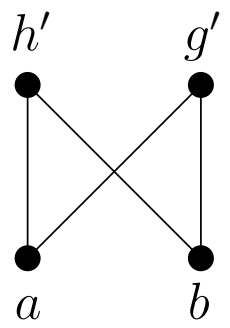

Figure 4. The conguration.

Since $\mathbf{P}$ is bounded and its unary operation $'$ is an antitone involution being a complementation, $\mathbf{P}$ must contain the configuration visualized in Figure 5 and P must have an even number of elements. We also conclude that a' and b' have no infimum and g and h have no supremum.

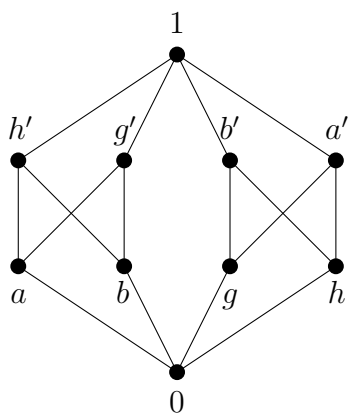

Figure 5. An orthogonal poset.

It is clear that these 10 elements are pairwise distinct. Let us mention that this poset is orthogonal. Put

$$\begin{aligned} c &:= h' \wedge b', \\ d &:= h' \wedge a', \\ e &:= g' \wedge b', \\ f &:= g' \wedge a'. \end{aligned}$$

Because of orthomodularity we have

$$\begin{aligned} h' &= b \vee c, \\ h' &= a \vee d, \\ g' &= b \vee e, \\ g' &= a \vee f. \end{aligned}$$

Using the facts $b \neq 0$, $h \neq 0$, $h \neq b'$ and that neither $a \vee b$ nor $g \vee h$ exists, we can prove $c \neq 0, a, b, g, h, a', b', g', h', 1$.

$c = 0$ would imply $h' = b \vee 0 = b$, a contradiction,
$c = a$ would imply $a = h' \wedge b' \leq b'$ and hence $a \vee b$ would exist, a contradiction,
$c = b$ would imply $h' = b \vee b = b$, a contradiction,
$c = g$ would imply $g = h' \wedge b' \leq h'$ and hence $g \vee h$ would exist, a contradiction,
$c = h$ would imply $h \leq b \vee h = h'$ and hence $h = h \wedge h' = 0$, a contradiction,
$c = a'$ would imply $a' = h' \wedge b' \leq b'$ and hence $a \vee b$ would exist, a contradiction,
$c = b'$ would imply $h' = b \vee b' = 1$, a contradiction,
$c = g'$ would imply $g' \leq b \vee g' = h'$ and hence $g \vee h$ would exist, a contradiction,
$c = h'$ would imply $b \leq b \vee h' = h' = h' \wedge b' \leq b'$ and hence $b = b \wedge b' = 0$, a contradiction,
$c = 1$ would imply $h' = b \vee 1 = 1$, a contradiction.

This shows $c \neq 0, a, b, g, h, a', b', g', h', 1$. Hence also c' is different from these 10 elements. Because of symmetry reasons, also d, e, f, d', e', f' are different from these 10 elements. Altogether, we see that any of the elements $c, d, e, f, c', d', e', f'$ is different from $0, a, b, g, h, a', b', g', h', 1$. Using the facts $c \neq 0$, $h \neq 0$ and that $g \vee h$ does not exist, we can prove $c \neq c', d', e', f'$.

$c = c'$ would imply $c = c \wedge c' = 0$, a contradiction,
$c = d'$ would imply $h \leq h \vee a = d' = h' \wedge b \leq h'$ and hence $h = h \wedge h' = 0$, a contradiction,
$c = e'$ would imply $g \leq g \vee b = e' = h' \wedge b' \leq h'$ and hence $g \vee h$ would exist, a contradiction,
$c = f'$ would imply $g \leq g \vee a = f' = h' \wedge b' \leq h'$ and hence $g \vee h$ would exist, a contradiction.

This shows $c \neq c', d', e', f'$. Because of symmetry reasons, also d, e, f are different from c', d', e', f'. Altogether, we see that any of the elements c, d, e, f is different from c', d', e', f'. Using the facts $a \neq b$ and $g \neq h$, we can prove that c, d, e, f are pairwise different.

$c = d$ would imply $h' \wedge b' = h' \wedge a'$ and hence $a' = h \vee (a' \wedge h') = h \vee (b' \wedge h') = b'$, a contradiction.
$c = e$ would imply $g' = b \vee e = b \vee c = h'$, a contradiction.
$c = f$ would imply $a \leq h'$, $f = c = h' \wedge b' \leq h'$, $b \leq g'$ and $c = f = g' \wedge a' \leq g'$ and hence $g' = a \vee f \leq h' = b \vee c \leq g'$ whence $g' = h'$, a contradiction.
$d = e$ would imply $b \leq h'$, $e = d = h' \wedge a' \leq h'$, $a \leq g'$ and $d = e = g' \wedge b' \leq g'$ and hence $g' = b \vee e \leq h' = a \vee d \leq g'$ whence $g' = h'$, a contradiction.
$d = f$ would imply $g' = a \vee f = a \vee d = h'$, a contradiction.

$e = f$ would imply $g' \wedge b' = g' \wedge a'$ and hence $a' = g \vee (a' \wedge g') = g \vee (b' \wedge g') = b'$, a contradiction.

This shows that c, d, e, f are pairwise different. Thus, c', d', e', f' are also pairwise different. Altogether, we have proved that the 18 elements

$$0, a, b, c, d, e, f, g, h, a', b', c', d', e', f', g', h', 1$$

are pairwise different. Therefore, **P** must contain the poset depicted in Figure 3. However, this is already a non-lattice orthomodular poset and hence the smallest one with respect to the number of its elements. We need to show that it is unique up to isomorphism. If another 18-element orthomodular poset not isomorphic to **P** existed, then its Hasse diagram would have to contain an edge not included in the Hasse diagram of **P**. Let us check this. Consider that the orthomodular poset in question contains, e.g., the edge (a, b') and hence also (b, a'). Then (OM) is violated since $h' \neq a \vee (h' \wedge a')$. If it contained, e.g., the edges (c, g') and (g, c'), similarly $g' \neq c \vee (g' \wedge c')$. If it contained, e.g., (d, b) and (b', d'), (OM) would also be violated since

by adding the edge (d, b) such that $d \leq b$ we would get $h' \neq a \vee (h' \wedge a')$,
by adding the edge (d, b) such that $b \leq d$ we would get $g' \neq a \vee (g' \wedge a')$.

All the remaining cases can be checked in a similar way. All possible cases would lead to a contradiction, which proves that the poset **P** is unique up to isomorphism. □

4. Horizontal Sums

The aim of this section is to describe a construction of skew orthomodular posets by means of so-called horizontal sums. For the reader's convenience, let us recall this concept.

Let $\mathbf{P}_i = (P_i, \leq, ', 0, 1), i \in I$, be a non-empty family of bounded posets with an antitone involution. By the *horizontal sum* of the $\mathbf{P}_i$ we mean a poset **P** being the union of disjoint copies of the posets $\mathbf{P}_i$ where the bottom and top elements of the $\mathbf{P}_i$, respectively, are identified.

Proposition 2. *Let $\mathbf{P}_i = (P_i, \leq, ', 0, 1), i \in I$, be a non-empty family of skew orthomodular posets. Then the horizontal sum $(P, \leq, ', 0, 1)$ of the $\mathbf{P}_i, i \in I$, is a skew orthomodular poset.*

Proof. If $a, b \in P$ and $a \leq b$ then there exists some $i \in I$ with $a, b \in P_i$, thus (GOM) surely holds. If there does not exist some $i \in I$ with $a, b \in P_i$ then $a \parallel b$ and hence (GOM) is obviously satisfied for these elements a and b. □

Let us note that if $|I| > 1$ and each $\mathbf{P}_i$ is non-trivial (i.e., has more than two elements). Then the horizontal sum of the $\mathbf{P}_i$ is a non-distributive skew orthomodular poset. Namely, if $j, k \in I$, $j \neq k$, $a \in P_j \setminus \{0, 1\}$ and $b \in P_k \setminus \{0, 1\}$ then

$$L\big(U(a, a'), b\big) = L(1, b) = L(b) \neq L(0) = LU(0) = LU(0, 0) = LU\big(L(a, b), L(a', b)\big).$$

Corollary 1. *The horizontal sum of a family of Boolean posets is a skew orthomodular poset.*

Proof. By Proposition 1, every Boolean poset is a skew orthomodular poset. The rest follows from Proposition 2. □

Example 2. *The orthomodular poset **P** depicted in Figure 3 is not a horizontal sum of Boolean posets.*

The question is whether every skew orthomodular poset is the horizontal sum of Boolean posets. In the next example we show that this is not the case.

Example 3. *Consider the poset **P** depicted in Figure 6:*

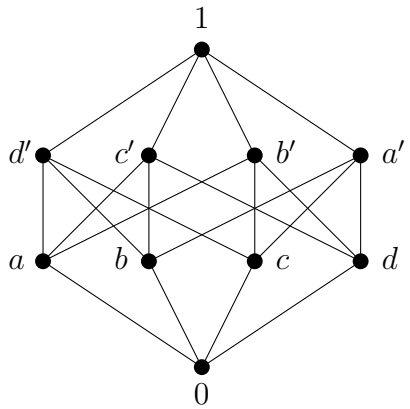

Figure 6. A non-orthomodular skew orthomodular poset not being a horizontal sum of Boolean posets.

P *is a non-lattice Boolean poset that is not orthomodular since* $a \leq d'$*, but* $a \vee (d' \wedge a')$ *is not defined. Considering the horizontal sum of* **P** *and some non-distributive skew orthomodular poset (e.g., the poset visualized in Figure 3), we obtain a non-lattice skew orthomodular poset being neither distributive, nor a horizontal sum of Boolean posets nor orthomodular.*

In what follows we will study horizontal sums of Boolean posets. For this purpose, we introduce the compatibility relation analogously as it was done for orthomodular lattices, see, e.g., [8].

In orthomodular lattices $(L, \vee, \wedge)$ the compatibility relation C is defined as follows:

$$a \mathrm{C}\, b \text{ if } a = (a \wedge b) \vee (a \wedge b')$$

$(a, b \in L)$. Of course, in a Boolean algebra every two elements are compatible. For our reasons, we define the relation C in a skew orthomodular poset $(P, \leq, ', 0, 1)$ as follows:

$$a \mathrm{C}\, b \text{ if } U(a) = U\big(L(a,b), L(a,b')\big)$$

$(a, b \in P)$. Then a, b are called *compatible* and C is called the *compatibility relation*.

Lemma 2. *Let* $(P, \leq, ', 0, 1)$ *be a skew orthomodular poset and* $a, b \in P$*. Then the following holds:*

(i) $a \mathrm{C}\, b$ *if and only if* $a \mathrm{C}\, b'$*,*
(ii) $a \leq b$ *implies* $a \mathrm{C}\, b$*,*
(iii) *if* $\{a, b\} \cap \{0, 1\} \neq \emptyset$ *then* $a \mathrm{C}\, b$*.*

Proof.

(i) This is clear.
(ii) $a \leq b$ implies $U(a) = UL(a) = U\big(L(a), L(a,b')\big) = U\big(L(a,b), L(a,b')\big)$.
(iii) If $a = 0$ or $b = 1$ then $a \mathrm{C}\, b$ follows from (ii).
If $a = 1$ then $a \mathrm{C}\, b$ follows from

$$\begin{aligned} U(a) = U(1) = \{1\} = U(b, b') = U\big(L(b), L(b')\big) = U\big(L(1,b), L(1,b')\big) = \\ = U\big(L(a,b), L(a,b')\big). \end{aligned}$$

If $b = 0$ then $a \mathrm{C}\, b$ follows from (i) and (ii).

□

The next result is almost evident.

Lemma 3. *Let* $(B, \leq, ', 0, 1)$ *be a Boolean poset and* $a, b \in B$*. Then* $a \mathrm{C}\, b$*.*

Proof. We have $U(a) = UL(a) = UL(a,1) = UL\big(a, U(b,b')\big) = U\big(L(a,b), L(a,b')\big)$. □

However, we can prove a more interesting and important result.

Theorem 2. *Let $(P, \leq, ', 0, 1)$ be the horizontal sum of the Boolean posets $(B_i, \leq, ', 0, 1), i \in I$, and $a, b \in P$. Then the following are equivalent:*

(i) $a \mathrel{\mathrm{C}} b$,
(ii) *there exists some $i \in I$ with $a, b \in B_i$.*

Proof.
(i) ⇒ (ii):
If no $i \in I$ with $a, b \in B_i$ existed, then there would exist $j, k \in I$ with $j \neq k$, $a \in B_j \setminus \{0,1\}$ and $b \in B_k \setminus \{0,1\}$, which would imply

$$U(a) \neq U(0) = U(0,0) = U\big(L(a,b), L(a,b')\big),$$

a contradiction.
(ii) ⇒ (i):
If $\{a,b\} \cap \{0,1\} \neq \emptyset$ then $a \mathrel{\mathrm{C}} b$ according to Lemma 2. Now assume $a, b \in B_i \setminus \{0,1\}$. Because of Lemma 3 we have

$$U\big(L(a,b), L(a,b')\big) \cap B_i = U\big(L(a,b) \cap B_i, L(a,b') \cap B_i\big) \cap B_i = U(a) \cap B_i = U(a)$$

which implies $L(a,b) \cup L(a,b') \neq \{0\}$ and hence

$$U(a) = U\big(L(a,b), L(a,b')\big) \cap B_i = U\big(L(a,b), L(a,b')\big),$$

i.e., $a \mathrel{\mathrm{C}} b$. □

For orthomodular lattices $(L, \vee, \wedge)$, the commutator $c(x,y)$ was introduced as follows:

$$c(x,y) := (x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$$

for all $x, y \in L$ (cf. e.g., [8]). In skew orthomodular posets $(P, \leq, ', 0, 1)$ we analogously define

$$c(x,y) := \operatorname{Min} U\big(L(x,y), L(x,y'), L(x',y), L(x',y')\big)$$

for all $x, y \in P$. Here and in the following $\operatorname{Min} A$ for a subset A of a poset means the set of all minimal elements of A, and $\operatorname{Min} U(A)$ means $\operatorname{Min}\big(U(A)\big)$. Let us note that $\operatorname{Min} A$ if A may be empty.

In the following, we often identify singletons with their unique element.

Lemma 4. *Let $(P, \leq, ', 0, 1)$ be a skew orthomodular poset and $a, b \in P$. Then, the following holds:*

(i) $c(a,b) = c(b,a)$,
(ii) $c(a,b) = c(a,b') = c(a',b) = c(a',b')$,
(iii) $c(0,b) = c(1,b) = 1$,
(iv) *if $a \mathrel{\mathrm{C}} b$ and $a' \mathrel{\mathrm{C}} b$ then $c(a,b) = 1$.*

Proof.

(i) and (ii) are clear.
(iii) According to (ii) we have

$$c(0,b) = c(1,b) = \operatorname{Min} U\big(L(1,b), L(1,b'), L(0,b), L(0,b')\big) = \operatorname{Min} U(b,b',0) = 1.$$

(iv) If $a \mathrel{\mathsf{C}} b$ and $a' \mathrel{\mathsf{C}} b$ then

$$\begin{aligned} c(a,b) &= \operatorname{Min} U\big(L(a,b), L(a,b'), L(a',b), L(a',b')\big) = \\ &= \operatorname{Min}\Big(U\big(L(a,b), L(a,b')\big) \cap U\big(L(a',b), L(a',b')\big)\Big) = \operatorname{Min}\big(U(a) \cap U(a')\big) = \\ &= \operatorname{Min} U(a,a') = 1. \end{aligned}$$

□

Corollary 2. *Let $(B, \leq, ', 0, 1)$ be a Boolean poset and $a, b \in B$. Then $c(a,b) = 1$.*

Proof. We have $a, a', b \in B$ and hence $a \mathrel{\mathsf{C}} b$ and $a' \mathrel{\mathsf{C}} b$ according to Lemma 3, which implies $c(a,b) = 1$ by Lemma 4. □

Now we prove a result similar to Theorem 2 for the commutator instead of compatibility.

Theorem 3. *Let $(P, \leq, ', 0, 1)$ be the horizontal sum of the Boolean posets $(B_i, \leq, ', 0, 1)$, $i \in I$, and $a, b \in P$. Then*

$$c(a,b) = \begin{cases} 1 & \text{if there exists some } i \in I \text{ with } a, b \in B_i \\ 0 & \text{otherwise.} \end{cases}$$

Hence $c(a,b) = 1$ if and only if there exists some $i \in I$ with $a, b \in B_i$.

Proof. First assume there exists some $i \in I$ with $a, b \in B_i$. If $\{a,b\} \cap \{0,1\} \neq \emptyset$ then $c(a,b) = 1$ according to Lemma 4. Now assume $a, b \in B_i \setminus \{0,1\}$. Because of Corollary 2 we have

$$\begin{aligned} &U\big(L(a,b) \cup L(a,b') \cup L(a',b) \cup L(a',b')\big) \cap B_i = \\ &= U\big(L(a,b) \cap B_i, L(a,b') \cap B_i, L(a',b) \cap B_i, L(a',b') \cap B_i\big) \cap B_i = 1 \end{aligned}$$

which implies $L(a,b) \cup L(a,b') \cup L(a',b) \cup L(a',b') \neq 0$ and therefore

$$\begin{aligned} c(a,b) &= \operatorname{Min} U\big(L(a,b), L(a,b'), L(a',b), L(a',b')\big) = \\ &= \operatorname{Min}\Big(U\big(L(a,b) \cup L(a,b') \cup L(a',b) \cup L(a',b')\big) \cap B_i\Big) = 1. \end{aligned}$$

Conversely, assume there exists no $i \in I$ with $a, b \in B_i$. Then there exist $j, k \in I$ with $j \neq k$, $a \in B_j \setminus \{0,1\}$ and $b \in B_k \setminus \{0,1\}$ and hence

$$c(a,b) = \operatorname{Min} U\big(L(a,b), L(a,b'), L(a',b), L(a',b')\big) = \operatorname{Min} U(0) = 0.$$

□

It is worth noting that the assumptions of Theorem 3 are essential. Namely, if the skew orthomodular poset $(P, \leq, ', 0, 1)$ is neither Boolean nor a horizontal sum of such posets then for $x, y \in P$ it may happen that $c(x,y)$ differs from both 0 and 1, see the following example.

Example 4.

(i) *Consider the orthomodular poset $(P, \leq, ', 0, 1)$ depicted in Figure 3. Then we compute*

$$\begin{aligned} c(a,b) &= \operatorname{Min} U\big(L(a,b), L(a,b'), L(a',b), L(a',b')\big) = \operatorname{Min} U(0,0,0,0,g,h\}) = \\ &= \operatorname{Min} U(g,h) = \{a', b'\} \end{aligned}$$

which differs from both 0 and 1.

(ii) *However, the condition from Theorem 3 does not characterize the class of horizontal sums of Boolean posets. For example, consider the ortholattice* $\mathbf{O}_6 = (O_6, \leq, ', 0, 1)$ *visualized in Figure 7:*

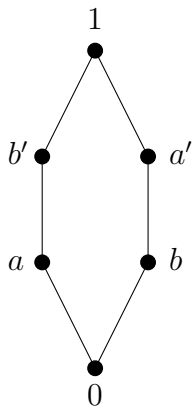

Figure 7. An ortholattice not being a horizontal sum of Boolean posets.

One can easily check that $c(x, y) = 1$ *for all* $x, y \in O_6$ (*if we define the commutator in ortholattices in the same way as it was done for orthomodular lattices*). *Of course, this lattice is neither a horizontal sum of Boolean posets nor a skew orthomodular poset.*

On the other hand, for arbitrary skew orthomodular posets we can prove the following result.

Proposition 3. *Let* $(P, \leq, ', 0, 1)$ *be a skew orthomodular poset. Then the following are equivalent:*

(i) $c(x, y) \in \{0, 1\}$ *for all* $x, y \in P$,

(ii) *If* $x, y \in P$ *then either* $L(x, y) = L(x, y') = L(x', y) = L(x', y') = 0$ *or* $U\big(L(x, y), L(x, y'), L(x', y), L(x', y')\big) = 1$.

Proof. Let $a, b \in P$. Then the following are equivalent:

$$\begin{aligned} c(a, b) &= 0, \\ \operatorname{Min} U\big(L(a, b), L(a, b'), L(a', b), L(a', b')\big) &= 0, \\ 0 &\in U\big(L(a, b), L(a, b'), L(a', b), L(a', b')\big), \\ L(a, b) \cup L(a, b') \cup L(a', b) \cup L(a', b') &= 0, \\ L(a, b) = L(a, b') = L(a', b) = L(a', b') &= 0. \end{aligned}$$

Moreover, the following are equivalent:

$$\begin{aligned} c(a, b) &= 1, \\ \operatorname{Min} U\big(L(a, b), L(a, b'), L(a', b), L(a', b')\big) &= 1, \\ U\big(L(a, b), L(a, b'), L(a', b), L(a', b')\big) &= 1. \end{aligned}$$

□

Next, we describe the mutual relationship between the compatibility relation and the commutator.

Corollary 3. *If* $(P, \leq, ', 0, 1)$ *is a horizontal sum of Boolean posets and* $a, b \in P$ *then* $a \mathrel{\mathsf{C}} b$ *if and only if* $c(a, b) = 1$.

Proof. This follows from Theorems 2 and 3. □

Now we extend the notion of the commutator from elements to subsets. For a skew orthomodular poset $(P, \leq, ', 0, 1)$ and subsets A and B of P we define

$$c(A, B) := \bigcup_{a \in A, b \in B} c(a, b).$$

Corollary 4. *The class of horizontal sums of Boolean posets satisfies the identity* $c(c(x, y), z) \approx 1$.

Proof. We have $c(x, y) \in \{0, 1\}$ according to Theorem 3 and $c(0, z) \approx c(1, z) \approx 1$ according to Lemma 4. □

Let $(P, \leq, ', 0, 1)$ be a skew orthomodular poset and $A \subseteq P$. Then we put $A' := \{x' \mid x \in A\}$ and define

$$t(x, y, z) := \operatorname{Min} U\Big(L\big((c(x, y))', x\big), L\big(c(x, y), z\big)\Big)$$

for all $x, y, z \in P$.

The next theorem shows that t behaves on horizontal sums of Boolean posets similarly to the ternary discriminator.

Theorem 4. *Let* $(P, \leq, ', 0, 1)$ *be a horizontal sum of Boolean posets and* $a, b, c \in P$. *Then*

$$t(a, b, c) = \begin{cases} c & \text{if } a \mathrel{C} b \\ a & \text{otherwise.} \end{cases}$$

Proof. If $a \mathrel{C} b$ then according to Theorem 3 we have $c(a, b) = 1$ and hence

$$\begin{aligned} t(a, b, c) &= \operatorname{Min} U\Big(L\big((c(a, b))', a\big), L\big(c(a, b), c\big)\Big) = \operatorname{Min} U\big(L(0, a), L(1, c)\big) = \\ &= \operatorname{Min} U(0, c) = c. \end{aligned}$$

Otherwise, according to Theorem 3 we have $c(a, b) = 0$ and therefore

$$\begin{aligned} t(a, b, c) &= \operatorname{Min} U\Big(L\big((c(a, b))', a\big), L\big(c(a, b), c\big)\Big) = \operatorname{Min} U\big(L(1, a), L(0, c)\big) = \\ &= \operatorname{Min} U(a, 0) = a. \end{aligned}$$

□

5. Conclusions

We have proved that up to isomorphism, there exists exactly one 18-element non-lattice orthomodular poset and that it is the minimal one. Since orthomodular posets form an algebraic counterpart to the logic of quantum mechanics, this result is of some importance for the properties of this logical calculus. Concerning quantum mechanics and related structures, we refer the reader to [9,10]. Further, we have shown that contrary to the case of Boolean algebras, Boolean posets need not be orthomodular. Hence, we introduced the class of so-called skew orthomodular posets including the class of Boolean posets. In addition to the other properties of skew orthomodular posets investigated herein, we have introduced the compatibility relation and the commutator, which allowed us to describe horizontal sums of Boolean posets (which may be considered as particular skew orthomodular posets). Moreover, we have used the compatibility relation for introducing a kind of ternary discriminator for horizontal sums of Boolean posets.

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