symmetry

MDPI

Article

On Some Topological and Geometric Properties of Some q-Cesáro Sequence Spaces

Yılmaz Yılmaz [1] and Ahmet Ocak Akdemir [2,*]

[1] Faculty of Science and Letters, Department of Mathematics, İnönü University, Malatya 44280, Türkiye; yilmaz.yilmaz@inonu.edu.tr
[2] Faculty of Science and Letters, Department of Mathematics, Ağrı İbrahim Çeçen University, Ağrı 04100, Türkiye
* Correspondence: ahmetakdemir@agri.edu.tr

Abstract: Mathematical concepts are aesthetic tools that are useful to create methods or solutions to real-world problems in theory and practice, and that sometimes contain symmetrical and asymmetrical structures due to the nature of the problems. In this study, we investigate whether the sequence spaces $\mathcal{X}_p^q$, $0 \le p < \infty$, and $\mathcal{X}_\infty$, which are constructed by q-Cesáro matrix, satisfy some of the further properties described with respect to the bounded linear operators on them. More specifically, we answer to the question: "Which of these spaces have the Approximation, Dunford-Pettis, Radon–Riesz and Hahn–Banach extension properties?". Furthermore, we try to investigate some geometric properties such as rotundity and smoothness of these spaces.

Keywords: Cesáro matrix; sequence spaces; Hahn–Banach operator

MSC: primary 05C38, 15A15; secondary 05A15, 15A18

1. Introduction

In some fields of the quantum mechanics, especially in the field of combinatorics, fractals, dynamical systems, and quantum groups, the q-analogue of some mathematical topics is highly used. In almost all of these areas, the symmetrical structures in the immense nature of the universe are at the forefront. Mathematical analysis methods such as fractional analysis and q-analysis are well-known methods that help explain and solve these symmetric–asymmetric structures. The fact that uniform smooth spaces contain a set of points that spread symmetrically around the zero point and that Banach spaces, which are frequently used in our studies, show symmetrical behavior due to their metric structure reveals the interesting structure of mathematical concepts in the context of symmetry.

For $q \in (0,1)$, the q-analogue of some known scientific concepts is the generalization of that expression using a new parameter q and which returns back to the original expression for $q \to 1$. Additionally, these concepts have vast applications in engineering sciences. It is widely used by researchers in approximation theory, operator theory, and quantum algebras, as well (see [1–4]).

In [5], Yaying et al. have given several new findings via $(p,q)-$calculus and compact matrix operators. Some different type operators have been investigated on homogeneous Siegel domains by Calzi and Peloso in [6]. In [7], the authors have provided some new bounds for the operator norm on fractional sequence spaces. In [8], Çiçek et al. have given some generalizations on weighted spaces. In [9–11], some new findings have been given for sequence spaces.

Citation: Yılmaz, Y.; Akdemir, A.O. On Some Topological and Geometric Properties of Some q-Cesáro Sequence Spaces. *Symmetry* **2023**, *15*, 791. https://doi.org/10.3390/sym15040791

Academic Editor: Bipan Hazarika

Received: 21 February 2023
Revised: 16 March 2023
Accepted: 20 March 2023
Published: 24 March 2023

The q-analogue of a non-negative natural number n is defined by (see [5]):

$$[n]_q = \begin{cases} \sum\limits_{k=0}^{n-1} q^k, & n = 1,2,3,\dots \\ 0, & n = 0 \end{cases}$$

Its factorial, also known as the q-factorial, is defined as (see [5]):

$$[n]_q! = \begin{cases} [n]_q[n-1]_q \cdots [1]_q & n \geq 1 \\ 1 & n = 0 \end{cases}$$

Similarly, the q-binomial coefficient of integers n and m is given by (see [5]):

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[n-m]_q![m]_q!}.$$

q-analogs of Cesáro matrices of order α and its properties are studied in [12]. Further, in the same paper, Aktuglu and Bekar defined a density function and q-statistical convergence using the classical Cesáro matrix. In this study, we examine some properties of the sequence spaces $\mathcal{X}_p^q$, $0 \leq p < \infty$, and $\mathcal{X}_\infty$, which are defined by [13]. These spaces are constructed by q-Cesàro matrix $C(q)$, and it can be seen that $C(q)$ is the q-analogue of the classical Cesàro matrix. It is shown in [13] that these spaces are Banach spaces by their special norms and they have some interesting properties. In this study, we will first look at whether these spaces satisfy some of the further properties described with respect to the bounded linear operators on them. More specifically, we will investigate which of these spaces have such properties. Approximation and Dunford-Pettis properties are other important affliations of Banach spaces. We will see that $\mathcal{X}_1^q$ has Dunford-Pettis, Radon–Riesz, and Hahn–Banach extension properties. An interesting result in classical Banach spaces is related to ℓ_∞, discovered by Philips in [14]. This is the Hahn–Banach extension property of ℓ_∞-valued bounded linear operators. We will see that $\mathcal{X}_\infty^q$ has this distinguished property, as well. Secondly, in this work we try to investigate some geometric properties such as the rotundity and smoothness of some these spaces.

2. Prerequisites

The classical Cesàro matrix C is defined by $C = (c_{nk})$ such that

$$c_{nk} = \begin{cases} \frac{1}{n} & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}.$$

Later, Ng and Lee [15] introduced the Cesàro sequence space $\mathcal{X}_p$, $0 \leq p < \infty$, and $\mathcal{X}_\infty$ as follows:

$$\mathcal{X}_p = \left\{ u = (u_n) \in w : \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} u_k \right|^p < \infty \right\}$$

and

$$\mathcal{X}_\infty = \left\{ u = (u_n) \in w : \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} u_k \right| < \infty \right\}.$$

For $0 < q < 1$, the q-Cesàro matrix $C(q) = \left(c_{nk}^q\right)$ is given in [13], and it is defined by

$$c_{nk}^q = \begin{cases} \frac{q^{k-1}}{[n]_q}, & \text{if } 0 \leq k \leq n \\ 0, & \text{if } k > n \end{cases}.$$

Moreover,

$$C(q) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{[2]_q} & \frac{q}{[2]_q} & 0 & 0 & \cdots \\ \frac{1}{[3]_q} & \frac{q}{[3]_q} & \frac{q^2}{[3]_q} & 0 & \cdots \\ \frac{1}{[4]_q} & \frac{q}{[4]_q} & \frac{q^2}{[4]_q} & \frac{q^3}{[4]_q} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

One can easily observe that the q-Cesàro matrix $C(q)$ reduces to the ordinary Cesàro matrix C as $q \to 1$. The inverse of $C(q)$ is the matrix $C^{-1}(q) = \left(c_{nk}^{-1}\right)$ such that

$$c_{nk}^{-1} = \begin{cases} (-1)^{n-k}\frac{[k]_q}{q^{n-1}}, & \text{if } n-1 \leq k \leq n \\ 0, & \textit{otherwise.} \end{cases}$$

$C(q)$-transform of the sequence $v = (v_k)$ is denoted by the sequence $u = (u_i)$ and so

$$u_n = (C(q)v)_n = \sum_{k=0}^{n} \frac{q^{k-1}}{[n]_q} v_k.$$

In [13], the sequence spaces $\mathcal{X}_p^q$, $0 \leq p < \infty$, and $\mathcal{X}_\infty^q$ are defined as the set of all sequences whose $C(q)$-transforms are in the spaces ℓ_p and ℓ_∞, respectively; that is,

$$\mathcal{X}_p^q = \{v = (v_n) \in w : C(q)v \in \ell_p\}$$

and

$$\mathcal{X}_\infty^q = \{v = (v_n) \in w : C(q)v \in \ell_\infty\}.$$

They proved that $\mathcal{X}_p^q$, $0 \leq p < \infty$, and $\mathcal{X}_\infty(q)$ are Banach spaces with the norms

$$\begin{aligned} \|v\|_{\mathcal{X}_p^q} &= \left(\sum_{n=1}^{\infty}\left|\sum_{k=1}^{n}\frac{q^{k-1}}{[n]_q}v_k\right|^p\right)^{1/p}, \\ \|v\|_{\mathcal{X}_\infty^q} &= \sup_n\left|\sum_{k=1}^{n}\frac{q^{k-1}}{[n]_q}v_k\right|, \end{aligned}$$

respectively. Further, they investigated some topological properties of these spaces and gave some characterization about matrix transformations between them.

We will highly benefited from the book [16] in this work. Suppose that X and Y are Banach spaces. A linear operator T from X into Y is compact if $T(B)$ is a relatively compact (means $\overline{T(B)}$ is compact) subset of Y whenever B is a bounded subset of X. The collection of all compact linear operators from X into Y is denoted by $K(X,Y)$, or by just $K(X)$ if $X = Y$. The range of a compact linear operator from a Banach space into a Banach space is closed if and only if the operator has finite rank; that is, the range of the operator is finite-dimensional [16].

Definition 1 ([17])**.** *A normed space X is* rotund *or strictly convex or strictly normed if*

$$\|tx_1 + (1-t)x_2\| < 1$$

whenever x_1 and x_2 are different points of unit sphere S_X and $0 < t < 1$.

An easier and more useful characterization of rotundity is the following theorem.

Theorem 1 ([16]). *Suppose that X is a normed space. Then, X is rotund if and only if*

$$\left\| \frac{1}{2}(x_1 + x_2) \right\| < 1$$

whenever x_1 and x_2 are different points of S_X.

Theorem 2 ([16]). *A normed space is rotund if and only if each of its two-dimensional subspaces is rotund.*

Definition 2 ([16]). *Suppose that x_0 is an element of the unit sphere S_X of a normed space X. Then, x_0 is a point of smoothness of the unit ball B_X if there is no more than one support hyperplane for B_X that supports B_X at x_0. The space X is smooth if each point of S_X is a point of smoothness of B_X.*

Suppose that X is a normed space, $x \in S_X$ and $y \in X$. Let

$$G_-(x,y) = \lim_{t \to 0^-} \frac{\|x + ty\| - \|x\|}{t}$$

and

$$G_+(x,y) = \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

Then, $G_-(x,y)$ and $G_+(x,y)$ are, respectively, the left-hand and right-hand *Gateaux derivative* of the norm at x in the direction y. The norm is *Gateaux differentiable* at x in the direction y if $G_-(x,y) = G_+(x,y)$, in which case the common value of $G_-(x,y)$ and $G_+(x,y)$ is denoted by $G(x,y)$ and is called the Gateaux derivative of the norm at x in the direction y. If the norm is Gateaux differentiable at x in every direction y, then the norm is Gateaux differentiable at x. Finally, if the norm is Gateaux differentiable at every point of the unit sphere S_X, then it is simply said that the norm is *Gateaux differentiable*.

Theorem 3. *(1) A normed space is smooth if and only if its norm is Gateaux differentiable [16].*
(2) A normed space is smooth if and only if each of its two-dimensional subspaces is smooth [16].

Definition 3 ([18]). *A Banach space X has the* approximation property *if, for every Banach space Y, the set of finite-rank members of $B(Y,X)$ is dense in $K(Y,X)$.*

Proposition 1. *The spaces c_0 and ℓ_p, $1 \le p < \infty$, have the approximation property [16].*

Suppose that X and Y are Banach spaces. A linear operator T from X into Y is weakly compact if $T(B)$ is a relatively weakly compact subset of Y whenever B is a bounded subset of X. The collection of all weakly compact linear operators from X into Y is denoted by $K^w(X,Y)$, or by just $K^w(X)$ if $X = Y$. Note that a subset U of X is relatively weakly compact, which means $\overline{U}$ is weakly compact subset of Y. $\overline{U}$ is weakly compact subset of Y if and only if $\overline{U}$ is compacy subset of Y in its weak topology. It is known by the Eberlein–Smulian theorem that [16]:

Proposition 2. *Suppose that T is a linear operator from a Banach space X into a Banach space Y. Then, T is weakly compact if and only if for any bounded sequence (x_n) in X has a subsequence $\left(x_{n_j}\right)_{j=0}^{\infty}$ such that $\left(Tx_{n_j}\right)$ converges weakly.*

Definition 4 ([19]). *Suppose that X and Y are Banach spaces. A linear operator T from X into Y is* completely continuous *or a Dunford-Pettis operotor if $T(K)$ is a compact subset of Y whenever K is a weakly compact subset of X.*

Proposition 3 ([16]). *ℓ_1 has the Dunford-Pettis property.*

Definition 5. *A normed space has the* Radon–Riesz property *or the* Kadets-Klee property *or property (H) and is called a Radon–Riesz space if it satisfies the following condition: Whenever* (x_n) *is a sequence in the space and* x *an element of the space such that* $x_n \overset{w}{\to} x$ *and* $\|x_n\| \to \|x\|$*, it follows that* $x_n \to x$*.*

An unusual property of the sequence space ℓ_∞, shown by Phillips, is its injectivity. More precisely, the property is given in the following theorem.

Theorem 4 ([14])**.** *Let* Y *be a linear subspace of the Banach space* X *and* $T : Y \to \ell_\infty$ *be a bounded linear operator. Then,* T *may be extended to a bounded linear operator* $S : X \to \ell_\infty$ *having the same norm as* T*.*

The bounded linear operator T is known as a Hahn–Banach operator, and then it is said that ℓ_∞ has the *Hahn–Banach extension property* in the literature.

3. Main Results

Now, we are in a position that to prove some new findings for sequence spaces with some further extension properties. Also, we will investigate the geometric properties of these spaces.

Theorem 5. *For* $1 \leq p < \infty$*, the Banach space* $\mathcal{X}_p^q$ *has the approximation property.*

Proof. Suppose that T is a compact linear operator from a Banach space Y into $\mathcal{X}_p^q$. We will find a sequence (T_n) of bounded linear operators of finite rank from Y into $\mathcal{X}_p^q$. For any $x \in Y$, $Tx \in \mathcal{X}_p^q$ and for any bounded sequnce (x_n) in Y, the sequence (Tx_n) has a convergent subsequence $\left(Tx_{n_j}\right)_{j=0}^{\infty}$ in $\mathcal{X}_p^q$. Hence,

$$\left\|Tx_{n_i} - Tx_{n_j}\right\|_{\mathcal{X}_p^q}^p = \left\|T\left(x_{n_i} - x_{n_j}\right)\right\|_{\mathcal{X}_p^q}^p \to 0 \text{ as } i, j \to \infty.$$

If we remember the definition of the space $\ell_p(C(q))$,

$$\left\|T\left(x_{n_i} - x_{n_j}\right)\right\|_{\mathcal{X}_p^q}^p = \left\|(C(q)T)\left(x_{n_i} - x_{n_j}\right)\right\|_{\ell_p}^p \to 0 \text{ as } i, j \to \infty.$$

This means the operator $C(q)T : Y \to \ell_p$ is well-defined and compact. The matrix transformation $C(q)$ is clearly bounded linear, so it can be denoted as $(C(q)T)$. Since ℓ_p has the approximation property, a sequence $(A_m)_{m=0}^{\infty}$ of bounded linear operators of finite rank from Y to ℓ_p exists such that $\|C(q)T - A_m\| \to 0$ as $m \to \infty$. Now, the sequence $\left(C(q)^{-1}A_m\right)_{m=0}^{\infty}$ is the desired sequece of finite rank from Y to $\mathcal{X}_p^q$. Easily, we can see that each $C(q)^{-1}A_m$ is bounded linear and has finite rank. Further ,

$$\begin{aligned}
\left\|T - C(q)^{-1}A_m\right\| &= \sup_{\|x\|=1} \left\|\left(T - C(q)^{-1}A_m\right)x\right\|_{\mathcal{X}_p^q} \\
&= \sup_{\|x\|=1} \left\|Tx - \left(C(q)^{-1}A_m\right)x\right\|_{\mathcal{X}_p^q}^p \\
&= \sup_{\|x\|=1} \left\|C(q)Tx - C(q)\left(C(q)^{-1}A_m\right)x\right\|_{\ell_p}^p \\
&= \sup_{\|x\|=1} \|(C(q)T - A_m)x\|_{\ell_p}^p \\
&\to 0 \text{ as } m \to \infty.
\end{aligned}$$

This completes the proof. □

Theorem 6. *$\mathcal{X}_1^q$ has the Dunford-Pettis property.*

Proof. Let $T : \mathcal{X}_1^q \to Y$ be a weakly compact linear operator and compose T with $C(q)^{-1}$. Then, $TC(q)^{-1}$ is obviously a bounded linear operator from ℓ_1 into Y. Further, it is weakly compact. Let us prove this: suppose U is a bounded in ℓ_1. By the boundedness of the matrix operator $C(q)^{-1}$, we conclude that $C(q)^{-1}(U)$ is a bounded subset of $\mathcal{X}_1^q$. Therefore,

$$T\Big(C(q)^{-1}(U)\Big) = \Big(TC(q)^{-1}\Big)(U)$$

is a relatively weakly compact set in Y. As a result, $TC(q)^{-1} : \ell_1 \to Y$ is a weakly compact operator. Now, since ℓ_1 has the Dunford-Pettis property, we obtain that $TC(q)^{-1}$ is completely continuous. Let W be a weakly compact subset of $\mathcal{X}_1^q$. Then, $C(q)(W)$ is a weakly compact subset of ℓ_1 [16] Exercise 3.50., and so

$$\Big(TC(q)^{-1}\Big)C(q)(W) = T(W)$$

is a compact subset in Y. $\square$

Let us present that $\mathcal{X}_\infty^q$ has the Hahn–Banach extension property.

Theorem 7. *Let Y be a linear subspace of the Banach space X and $T : Y \to \mathcal{X}_\infty^q$ be a bounded linear operator. Then, T may be extended to a bounded linear operator $S : X \to \mathcal{X}_\infty^q$ having the same norm as T.*

Proof. For any bounded linear operator $T : Y \to \mathcal{X}_\infty^q$, $C(q)T \in B(Y, \ell_\infty)$ and from the Theorem 4, ℓ_∞ has the Hahn–Banach extension property. Thus, $C(q)T$ may be extended to a bounded linear operator $U : X \to \ell_\infty$ having the same norm as $C(q)T$. Now, let us consider the operator $C(q)^{-1}U$. Classical operator algebra says that $C(q)^{-1}U = S$ is a bounded linear operator from X to $\mathcal{X}_\infty^q$. Only we will show that S is an extension of T and $\|T\| = \|S\|$. For any $y \in Y$,

$$\begin{aligned} Sy &= \Big(C(q)^{-1}U\Big)y = C(q)^{-1}(Uy) \\ &= C(q)^{-1}(C(q)T)y = Ty. \end{aligned}$$

Now,

$$\begin{aligned} \|S\| &= \Big\|C(q)^{-1}U\Big\| = \Big\|C(q)^{-1}(C(q)T)\Big\| \\ &= \Big\|\Big(C(q)^{-1}C(q)\Big)T\Big\| = \Big\|I_{\ell_\infty(\widetilde{C})}T\Big\| = \|T\|, \end{aligned}$$

where $I_{\mathcal{X}_\infty^q}$ is the identity operator on $\mathcal{X}_\infty^q$. $\square$

Theorem 8. *$\mathcal{X}_2^q$ has the Radon–Riesz property.*

Proof. Let (x_n) be a sequence in $\mathcal{X}_2^q$ and x an element of $\mathcal{X}_2^q$. Assume that $x_n \xrightarrow{w} x$ and $\|x_n\|_{\mathcal{X}_2^q} \to \|x\|_{\mathcal{X}_2^q}$. We will prove that $x_n \to x$. Now, the assumption $x_n \xrightarrow{w} x$ implies that $yx_n \to yx$ for each $y \in \Big(\mathcal{X}_2^q\Big)^*$. Let us show that $\|x_n - x\|_{\mathcal{X}_2^q} \to 0$ to complete the proof:

$$
\begin{aligned}
\|x_n - x\|^2_{\mathcal{X}_2^q} &= \|C(q)x_n - C(q)x\|^2_{\ell_2} \\
&= \langle C(q)x_n - C(q)x, C(q)x_n - C(q)x\rangle_{\ell_2} \\
&= \langle C(q)x_n, C(q)x_n\rangle_{\ell_2} - \langle C(q)x_n, C(q)x\rangle_{\ell_2} \\
&\quad - \langle C(q)x, C(q)x_n\rangle_{\ell_2} + \langle C(q)x, C(q)x\rangle_{\ell_2} \\
&= \|C(q)x_n\|^2_{\ell_2} + \|C(q)x\|^2_{\ell_2} - \langle C(q)x_n, C(q)x\rangle_{\ell_2} - \langle C(q)x, C(q)x_n\rangle_{\ell_2}
\end{aligned}
$$

Take $C(q)x = z \in \ell_2 = \ell_2^*$. Now, $z \circ C(q)$ is a continuous linear functional on $\mathcal{X}_2^q$ from the properties of the matrix $C(q)$. Further,

$$
(z \circ C(q))x_n = z(C(q)x_n) = \langle C(q)x_n, C(q)x\rangle_{\ell_2}
$$

from the Riesz's Theorem. By the assumption $x_n \overset{w}{\to} x$, we have

$$
\begin{aligned}
(z \circ C(q))(x_n) &= z(C(q)x_n) \\
&= \langle C(q)x_n, C(q)x\rangle_{\ell_2} \\
&\to (z \circ C(q))(x), \text{ as } n \to \infty, \\
&= \langle C(q)x, C(q)x\rangle_{\ell_2} \\
&= \|C(q)x\|^2_{\ell_2}
\end{aligned}
$$

Dually, let us now take $C(q)x_n = z_n \in \ell_2^* = \ell_2$ for each n. Then,

$$
(z_n \circ C(q))x = z_n(C(q)x) = \langle C(q)x, C(q)x_n\rangle_{\ell_2}.
$$

Again, each $z_n \circ C(q)$ is a continuous linear functional on $\mathcal{X}_2^q$, and by the assumption $x_n \overset{w}{\to} x$, we have

$$
\begin{aligned}
(z_n \circ C(q))(x) &= z_n(C(q)x) \\
&= \langle C(q)x, C(q)x_n\rangle_{\ell_2} \\
&= \overline{\langle C(q)x_n, C(q)x\rangle_{\ell_2}} \\
&\to \overline{(f \circ C(q))(x)}, \text{ as } n \to \infty, \\
&= \overline{\langle C(q)x, C(q)x\rangle_{\ell_2}} \\
&= \|C(q)x\|^2_{\ell_2}.
\end{aligned}
$$

Eventually, by the assumption $\|x_n\|_{\mathcal{X}_2^q} \to \|x\|_{\mathcal{X}_2^q}$, we have

$$
\begin{aligned}
\|x_n - x\|^2_{\mathcal{X}_2^q} &= \|C(q)x_n\|^2_{\ell_2} + \|C(q)x\|^2_{\ell_2} - \langle C(q)x_n, C(q)x\rangle_{\ell_2} - \langle C(q)x, C(q)x_n\rangle_{\ell_2} \\
&\to \|C(q)x\|^2_{\ell_2} + \|C(q)x\|^2_{\ell_2} - \|C(q)x\|^2_{\ell_2} - \|C(q)x\|^2_{\ell_2} \\
&= 0, \text{ as } n \to \infty.
\end{aligned}
$$

□

4. Uniform Smoothness and Rotundity

We know that the unit sphere of n-dimensional Euclidean space is rotund. Additionally, the geometry of all ℓ_p spaces, $1 < p < \infty$, tells us that the unit spheres of them are rotund. Is it true for the sequence space $\mathcal{X}_p^q$?

Theorem 9. *For $1 < p < \infty$, the space $\mathcal{X}_p^q$ is rotund.*

Proof. By Proposition 2, it is sufficient to prove rotundity of the space $span\{e_1, e_2\} = Z$ in $\mathcal{X}_p^q$ where e_1, e_2 are elements of the unit vector basis of ℓ_p. That is, we will consider two-dimensional subspace

$$Z = \Big\{(x_0, x_1, 0, 0\dots) : (x_0, x_1, 0, 0\dots) \in \mathcal{X}_p^q\Big\}.$$

Let x and y be arbitrary elements of S_Z and $x + y = (x_0 + y_0, x_1 + y_1, 0, 0, \cdots)$. Then,

$$\begin{aligned} & \left\|\frac{1}{2}(x+y)\right\|_{\mathcal{X}_p^q}^p \\ = & \left\|\frac{1}{2}\left((x_0+y_0), \frac{1}{[2]_q}(x_0+y_0) + \frac{q}{[2]_q}(x_1+y_1), \frac{1}{[3]_q}(x_0+y_0) + \frac{q}{[3]_q}(x_1+y_1), \dots\right)\right\|_{\ell_p}^p \\ = & \frac{1}{2^p}\left[|x_0+y_0|^p + \left|\frac{1}{[2]_q}(x_0+y_0) + \frac{q}{[2]_q}(x_1+y_1)\right|^p + \dots\right]. \end{aligned}$$

Remember,

$$\begin{aligned} \|x\|_{\mathcal{X}_p^q}^p &= |x_0|^p + \left|\frac{1}{[2]_q}x_0 + \frac{q}{[2]_q}x_1\right|^p + \dots = 1, \\ \|y\|_{\mathcal{X}_p^q}^p &= |y_0|^p + \left|\frac{1}{[2]_q}y_0 + \frac{q}{[2]_q}y_1\right|^p + \dots = 1 \end{aligned}$$

and write

$$u_1 = \frac{1}{[2]_q}x_0 + \frac{q}{[2]_q}x_1 + \dots$$

and

$$v_1 = \frac{1}{[2]_q}y_0 + \frac{q}{[2]_q}y_1 + \dots.$$

Hence,

$$\|x\|_{\mathcal{X}_p^q}^p = |x_0|^p + |u_1|^p = 1 \text{ and } \|y\|_{\mathcal{X}_p^q}^p = |v_0|^p + |v_1|^p = 1.$$

By the rotundity of two-dimensional Banach space ℓ_p^2 where (x_0, u_1) and (y_0, v_1) are elements of ℓ_p^2, we obtain

$$\left|\frac{x_0+y_0}{2}\right|^p + \left|\frac{u_1+v_1}{2}\right|^p < 1.$$

Again, remember that

$$\left|\frac{u_1+v_1}{2}\right|^p = \frac{1}{2^p}\left|\frac{x_0+y_0}{2} + \frac{x_1+y_1}{2}\right|^p.$$

Thus, we obtain $\left\|\frac{1}{2}(x+y)\right\|_{\mathcal{X}_p^q}^p < 1$. □

Theorem 10. *$\mathcal{X}_\infty^q$ and $\mathcal{X}_1^q$ are not rotund.*

Proof. Consider two special elements

$$x = e_1 + e_2 = (1, 1, 0, 0, \dots)$$

and

$$y = e_1 - e_2 = (1, -1, 0, 0, \dots)$$

in $\mathcal{X}_\infty^q$, and let us see that x and $y \in S_{\mathcal{X}_\infty^q}$. Indeed,

$$\begin{aligned}
\|y\|_{\mathcal{X}_\infty^q} &= \|(1,-1,0,0,\dots)\|_{\mathcal{X}_\infty^q} \\
&= \left\|\left(1,\frac{1-q}{[2]_q},\frac{1-q}{[3]_q},\dots\right)\right\|_{\ell_\infty} \\
&= \left\|\left(1,\frac{1-q}{1+q},\frac{1-q}{1+q+q^2},\dots\right)\right\|_{\ell_\infty} \\
&= 1
\end{aligned}$$

and similarly $\|x\|_{\mathcal{X}_\infty^q} = 1$. Now,

$$\begin{aligned}
\left\|\frac{1}{2}(x+y)\right\| &= \left\|\frac{1}{2}(2,0,0,\dots)\right\|_{\mathcal{X}_\infty^q} \\
&= \|(1,0,0,\dots)\|_{\mathcal{X}_\infty^q} \\
&= \left\|\left(1,\frac{1}{[2]_q},\frac{1}{[3]_q},\dots\right)\right\|_{\ell_\infty} \\
&= 1.
\end{aligned}$$

This means $\mathcal{X}_\infty^q$ is not rotund. The assertion for $\mathcal{X}_1^q$ can be done similarly. □

The uniform smoothness of a Banach spaces is an indication that the geometry of the balls in the space does not contain sharp edges or cliffs. We see that sequence spaces $\mathcal{X}_2^q$ has this property as well.

Definition 6 ([16]). *Suppose that X is a normed- space. Define a function $\rho_X : (0,\infty) \to [0,\infty)$ by the formula*

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x+ty\| + \|x-ty\|) - 1 : x, y \in S_X\right\}$$

if $X \neq \{0\}$, and by the formula

$$\rho_X(t) = \begin{cases} 0 & if\ 0 < t < 1 \\ t-1 & if\ t \geq 1 \end{cases}$$

if $X = \{0\}$. Then, ρ_X is the modulus of smoothness of X. The space X is uniformly smooth if $\lim_{t\to 0^+} \rho_X(t)/t = 0$.

Remark 1. *Uniformly smooth Banach spaces are important because they allow to establish, to some extent, concepts close to geometric structures that can be constructed in Hilbert spaces. Of course, every Hilbert space is uniformly smooth. However, the reverse is not true. The condition $\lim_{t\to 0^+} \rho_X(t)/t = 0$ also includes that the norm of the space is uniformly Gateaux differentiable, that is, Frechet differentiable at every point in every direction. Therefore, uniformly smooth spaces are smooth, but the reverse is not true.*

Theorem 11. *For $1 < p < \infty$, the space $\mathcal{X}_p^q$ is uniformly smooth.*

Proof. First of all, let us calculate $\|x+ty\|_{\mathcal{X}_p^q}$ and $\|x-ty\|_{\mathcal{X}_p^q}$.

$$\|x+ty\|_{\mathcal{X}_p^q}^p = \sum_{n=1}^{\infty}\left|\sum_{k=1}^{n}\frac{q^{k-1}}{[n]_q}(x_k+ty_k)\right|^p$$

and

$$\|x-ty\|^p_{\mathcal{X}^q_p} = \sum_{n=1}^{\infty}\left|\sum_{k=1}^{n}\frac{q^{k-1}}{[n]_q}(x_k-ty_k)\right|^p$$

$\lim_{t\to 0^+}\rho_X(t)/t$ gives the $0/0$ uncertainty in the primary stage, and then we can solve this limit with the help of L'Hospital rule. Then,

$$\lim_{t\to 0^+}\rho_X(t)/t = \lim_{t\to 0^+}\frac{d}{dt}(\rho_X(t)).$$

Now, let us try to determine $\frac{d}{dt}(\rho_X(t))$. By the properties of the supremum and by the linearity of the derivative, we can write

$$\begin{aligned}
&\frac{d}{dt}(\rho_X(t))\\
=\;& \sup\left\{\frac{1}{2}\left(\frac{d}{dt}\|x+ty\|+\frac{d}{dt}\|x-ty\|\right): x,y\in S_X\right\}.
\end{aligned}$$

Let us consider

$$\begin{aligned}
&\frac{d}{dt}\left(\|x+ty\|^p\right)\\
=\;& \frac{d}{dt}\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k+ty_k)\right|^p\\
=\;& \sum_{n=0}^{\infty}\frac{d}{dt}\left|\sum_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k+ty_k)\right|^p\\
=\;& \sum_{n=0}^{\infty}\left(p\left|\sum_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k+ty_k)\right|^{p-1}\frac{d}{dt}\left|\sum_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k+ty_k)\right|\right)
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\frac{d}{dt}\left(\|x-ty\|^p\right)\\
=\;& \sum_{n=0}^{\infty}\left(p\left|\sum_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k-ty_k)\right|^{p-1}\frac{d}{dt}\left|\sum_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k-ty_k)\right|\right).
\end{aligned}$$

Now, we should focuse on derivatives. Then, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left|\sum_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k+ty_k)\right|\\
=\;& \begin{cases}\frac{d}{dt}\left(\sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k+ty_k)\right), & \text{if } \sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k+ty_k)\geq 0\\ -\frac{d}{dt}\left(\sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k+ty_k)\right), & \text{if } \sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k+ty_k)< 0\end{cases}\\
=\;& \begin{cases}\sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}y_k, & \text{if } \sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k+ty_k)\geq 0\\ -\sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}y_k, & \text{if } \sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k+ty_k)< 0\end{cases}
\end{aligned}$$

and similarly,

$$\frac{d}{dt}\left|\sum_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k - ty_k)\right| = \begin{cases} -\sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}y_k, & \text{if } \sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k - ty_k) \geq 0 \\ \sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}y_k, & \text{if } \sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}(x_k - ty_k) < 0 \end{cases}.$$

Now, if we apply $t \to 0^+$, then

$$\lim_{t\to 0^+}\frac{d}{dt}\|x+ty\|^p = \begin{cases} p\sum\limits_{n=0}^{\infty}\left|\sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}x_k\right|^{p-1}\sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}y_k, & \text{if } \sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}x_k \geq 0 \\ -p\sum\limits_{n=0}^{\infty}\left|\sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}x_k\right|^{p-1}\sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}y_k, & \text{if } \sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}x_k < 0 \end{cases}$$

and

$$\lim_{t\to 0^+}\frac{d}{dt}\|x-ty\|^p = \begin{cases} -p\sum\limits_{n=0}^{\infty}\left|\sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}x_k\right|^{p-1}\sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}y_k, & \text{if } \sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}x_k \geq 0 \\ p\sum\limits_{n=0}^{\infty}\left|\sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}x_k\right|^{p-1}\sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}y_k, & \text{if } \sum\limits_{k=0}^{n}\frac{q^{k-1}}{[n]_q}x_k < 0 \end{cases}.$$

We just see that

$$\lim_{t\to 0^+}\frac{d}{dt}\|x+ty\|^p + \lim_{t\to 0^+}\frac{d}{dt}\|x-ty\|^p = 0.$$

Remember that $|a| \leq |a|^p$ for $1 < p < \infty$, and so

$$\lim_{t\to 0^+}\frac{d}{dt}\|x+ty\| + \lim_{t\to 0^+}\frac{d}{dt}\|x-ty\| = 0.$$

Eventually, we obtain

$$\lim_{t\to 0^+}\frac{d}{dt}(\rho_X(t)) = 0.$$

This completes the proof □

Theorem 12. *$\mathcal{X}_1^q$ and $\mathcal{X}_\infty^q$ are not uniformly smooth.*

5. Conclusions

The main motivation point of this study is to determine the various properties of some sequence spaces formed with the help of the q-Cesáro matrix and to examine their geometric structures in the context of rotundity and smoothness. It is thought that the findings of the research have the potential to be used in the fields of quantum mechanics, combinatorics, dynamical systems, functional analysis, topological spaces, and quantum groups. By using a similar methodology to researchers interested in the subject, the formation of new sequence spaces and the determination of their properties can be considered as problems for new studies.

Author Contributions: Conceptualization, Y.Y. and A.O.A; methodology, Y.Y.; formal analysis, Y.Y. and A.O.A.; investigation, Y.Y.; writing-original draft preparation, Y.Y.; writing-review and editing, Y.Y. and A.O.A.; supervision, Y.Y.; funding acquisition, A.O.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. İlkhan, M. A new conservative matrix derived by Catalan numbers and its matrix domain in the spaces c and c_0. *Linear Multilinear Algebra* **2020**, *68*, 417–434. [CrossRef]
2. Kara, M.İ.; Kara, E.E. Matrix transformations and compact operators on Catalan sequence spaces. *J. Math. Anal. Appl.* **2021**, *498*, 124925. [CrossRef]
3. Koshy, T. *Catalan Numbers with Applications*; Oxford University Press: New York, NY, USA, 2009.
4. Grimaldi, R.P. *Fibonacci and Catalan Numbers: An Introduction*; John Wiley & Sons: Hoboken, NJ, USA, 2012.
5. Yaying, T.; Hazarika, B.; Mursaleen, M. Cesáro sequence spaces via $(p,q)-$calculus and compact matrix operators. *J. Anal.* **2022**, *30*, 1535–1553. [CrossRef]
6. Calzi, M.; Peloso, M.M. Toeplitz and Cesáro-type operators on homogeneous Siegel domains. *Complex Var. Elliptic Equa.* **2023**, *68*, 167–199. [CrossRef]
7. Raj, K.; Choudhary, A.; Mursaleen, M. Bounds for the operator norm on weighted Cesáro fractional difference sequence spaces. *Sahand Commun. Math. Anal.* **2023**, *20*, 19–34.
8. Cicek, H.; Zainalabdin, S.J.; Izgi, A. New generalization of Szasz-Kantorovich operators on weighted space. *Turk. J. Sci.* **2022**, *7*, 85–106.
9. Bekar, Ş. q-Matrix Summability Methods. Ph.D. Dissertation, Applied Mathematics and Computer Science, Eastern Mediterranean University, Gazimağusa, North Cyprus, 2010.
10. Alotaibi, A.; Yaying, T.; Mohiuddine, S.A. Sequence spaces and spectrum of q-difference operator of second order. *Symmetry* **2022**, *14*, 1155. [CrossRef]
11. Atabey, K.İ.; Çınar, M.; Et, M. q-Fibonacci sequence spaces and related matrix transformations. *J. Appl. Math. Comput.* **2023**, *69*, 2135–2154. [CrossRef]
12. Aktuglu, H.; Bekar, Ş. q-Cesàro matrix and q- statistical convergence. *J. Comput. Appl. Math.* **2011**, *235*, 4717–4723. [CrossRef]
13. Yaying, T.; Hazarika, B.; Mursaleen, M. On sequence space derived by the domain of q-Cesàro matrix in ℓ_p space and the associated operator ideal. *J. Math. Anal. Appl.* **2021**, *493*, 124453. [CrossRef]
14. Phillips, R.S. On linear transformations. *Trans. Am. Math. Soc.* **1940**, *48*, 516–541. [CrossRef]
15. Ng, P.-N.; Lee, P.-Y. Cesáro sequence spaces of non-absolute type. *Comment. Math. Prace Mat.* **1978**, *20*, 429–433.
16. Megginson, R.E. *An Introduction to Banach Space Theory*; Springer: New York, NY, USA, 1998.
17. Akhiezer, N.I.; Krein, M.G. *Some Questions in the Theory of Moments*; Gosudarstv. Naucno-Tehn. Izdat. Ukrain: Kharkov, Ukraine, 1938.
18. Grothendieck, A. Sur les applications lineaires faiblement compactes d'espaces du type $C(K)$. *Canad. J. Math.* **1953**, *5*, 129–173. [CrossRef]
19. Hilbert, D. Grundzii.ge einer allgemeinen Theorie der linearen lntegralgleichungen, IV. *Nachr. Kgl. Gesells. Wiss. Gottingen Math. Phys. Kl.* **1906**, 1906, 157–227.