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Tangent Bundles of P-Sasakian Manifolds Endowed with a Quarter-Symmetric Metric Connection

Mohammad Nazrul Islam Khan [1], Fatemah Mofarreh [2] and Abdul Haseeb [3,*]

[1] Department of Computer Engineering, College of Computer, Qassim University, Buraydah 51452, Saudi Arabia
[2] Mathematical Science Department, Faculty of Science, Princess Nourah Bint Abdulrahman University, Riyadh 11546, Saudi Arabia
[3] Department of Mathematics, College of Science, Jazan University, P.O. Box 2097, Jazan 45142, Saudi Arabia
* Correspondence: malikhaseeb80@gmail.com or haseeb@jazanu.edu.sa

Abstract: The purpose of this study is to evaluate the curvature tensor and the Ricci tensor of a P-Sasakian manifold with respect to the quarter-symmetric metric connection on the tangent bundle TM. Certain results on a semisymmetric P-Sasakian manifold, generalized recurrent P-Sasakian manifolds, and pseudo-symmetric P-Sasakian manifolds on TM are proved.

Keywords: Sasakian manifolds; quarter-symmetric metric connection; mathematical operators; tangent bundles; pseudosymmetric manifolds; partial differential equations; generalized recurrent manifolds

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1. Introduction

Let M be a Riemannian manifold with a linear connection $\tilde{\nabla}$. If the torsion tensor T of $\tilde{\nabla}$

$$T(t_1, t_2) = \tilde{\nabla}_{t_1} t_2 - \tilde{\nabla}_{t_2} t_1 - [t_1, t_2] \tag{1}$$

satisfies

$$T(t_1, t_2) = h(t_2)\phi t_1 - h(t_1)\phi t_2, \tag{2}$$

where h is a 1-form and ϕ is a (1, 1) tensor field, then the connection $\tilde{\nabla}$ is called a quarter-symmetric connection [1,2]. In addition, if $\tilde{\nabla}$ holds the relation

$$(\tilde{\nabla}_{t_1} g)(t_2, t_3) = 0, \tag{3}$$

$\forall\ t_1, t_2, t_3 \in \Im(M)$, the set of all smooth vector fields on M, then $\tilde{\nabla}$ refers to the quarter-symmetric metric connection [3]. Many geometers such as [4–16] studied such connection on M and discussed some geometric properties of it. The quarter-symmetric connection generalizes the semi-symmetric connection that plays a key role in the geometry of Riemannian manifolds.

A Riemannian manifold M ($dimM = n \geq 3$) with respect to the Levi–Civita connection ∇ is said to be

- A generalized recurrent [17] if

$$(\nabla_{t_1} R)(t_2, t_3)t_4 = \alpha(t_1)R(t_2, t_3)t_4 + \beta(t_1)[g(t_3, t_4)t_2 - g(t_2, t_4)t_3], \tag{4}$$

  where α and β are 1-forms of which $\beta \neq 0$. If in Equation (4), α is non-zero and β is zero, then the manifold is named a recurrent manifold [18].

- A pseudosymmetric [19] if

$$(\nabla_{t_1}R)(t_2,t_3)t_4 = 2\alpha(t_1)R(t_2,t_3)t_4 + \alpha(t_2)R(t_1,t_3)t_4 + \alpha(t_3)R(t_2,t_1)t_4 + \alpha(t_4)R(t_2,t_3)t_1 + g(R(t_2,t_3)t_4,t_1)\rho, \quad (5)$$

for $\alpha \neq 0$. The 1-forms α and β associated with the vector fields ρ and σ are defined as follows:

$$g(t_1,\rho) = \alpha(t_1), \;\; g(t_1,\sigma) = \beta(t_1). \quad (6)$$

On the other hand, Yano and Ishihara [20] proposed the notion of the lifting of tensor fields and connections to its tangent bundle and established the basic properties of curvature tensors. In [21], Manev studied tangent bundles with a complete lift of the base metric and almost hypercomplex Hermitian–Norden structure and characterized it. The metallic structures on the tangent bundle of a Riemannian manifold by using complete and horizontal lifts were studied by Azami [22]. Bilen [23] introduced the deformed Sasaki metric, which is a Berger type, studied the metric connection to the tangent bundle, established some curvature properties of this metric, and characterized the projective vector field. The geometric structures and the connections from a manifold to its tangent bundle have been studied by many authors such as [24–27] and many others.

Our main findings in the paper are as follows:

- Some results on the curvature tensor of a P-Sasakian manifold with respect to $\tilde{\nabla}^C$ on TM are obtained.
- A theorem on a semisymmetric P-Sasakian manifold with respect to $\tilde{\nabla}^C$ on TM is proved.
- A relationship between one and the forms α^C and β^C on TM of a generalized recurrent P-Sasakian manifold is established.
- An expression of a pseudosymmetric P-Sasakian manifold with respect to $\tilde{\nabla}^C$ on TM is determined.

2. *P*-Sasakian Manifolds

Let M be a differentiable manifold ($dimM = n$) endowed with a tensor field ϕ of type (1, 1), a characteristic vector field κ, and a 1-form h such that

$$\phi^2 t_1 = t_1 - h(t_1)\kappa, \;\; \phi\kappa = 0, \;\; h(\kappa) = 1, \;\; h(\phi t_1) = 0 \quad (7)$$

and let g be a Riemannian metric satisfying

$$g(\kappa,t_1) = h(t_1), \;\; g(\phi t_1,\phi t_2) = g(t_1,t_2) - h(t_1)h(t_2); \quad (8)$$

then, the structure (M,ϕ,κ,h,g) is said to be an almost para-contact metric manifold [28,29] If M holds:

$$d\eta = 0, \;\; \nabla_{t_1}\kappa = \phi t_1, \quad (\nabla_{t_1}\phi)t_2 = -g(t_1,t_2)\kappa - h(t_2)t_1 + 2\eta(t_1)h(t_2)\kappa, \quad (9)$$

then M is called a para-Sasakian manifold or, briefly, a P-Sasakian manifold [30–32]. Moreover, if M satisfies

$$(\nabla_{t_1}h)(t_2) = -g(t_1,t_2) + h(t_1)h(t_2), \quad (10)$$

then M is a called special para-Sasakian manifold or an SP-Sasakian manifold [33]. In a P-Sasakian manifold, we have [32]:

$$\begin{aligned} S(t_1,\kappa) &= -(n-1)h(t_1) \iff Q\kappa = -(n-1)\kappa, & (11)\\ h(R(t_1,t_2)t_3) &= g(t_1,t_3)h(t_2) - g(t_2,t_3)h(t_1), & (12)\\ R(t_1,\kappa)t_2 &= g(t_1,t_2)\kappa - h(t_2)t_1, & (13)\\ R(t_1,t_2)\kappa &= h(t_1)t_2 - h(t_2)t_1, & (14)\\ S(\phi t_1,\phi t_2) &= S(t_1,t_2) + (n-1)h(t_1)h(t_2), & (15)\\ h(R(t_1,t_2)\kappa) &= 0, & (16) \end{aligned}$$

$\forall\, t_1,t_2,t_3 \in \Im(M)$, where the curvature and the Ricci tensors are symbolized as R and S, respectively.

For further studies on P-Sasakian manifolds, we recommend the papers [31,32,34–37] and many others. An almost paracontact Riemannian manifold M is said to be an h-Einstein manifold if its Ricci tensor $S(\neq 0)$ satisfies

$$S(t_1,t_2) = ag(t_1,t_2) + bh(t_1)h(t_2),$$

where a and b are smooth functions on the manifold M. In particular, if $b = 0$, then M is named as an Einstein manifold.

Definition 1. *In an n-dimensional differentiable manifold M, $T_p(M)$ is the tangent space at a point p of M, i.e., the set of all tangent vectors of M at p. Then, the set $TM = \bigcup_{p\in M} T_p(M)$ is the tangent bundle over M.*

Definition 2. *Let us consider $(x^i), i = 1,\ldots,n$ as a local co-ordinate system on M and let $(x^i,y^i), i = 1,\ldots,n$ be an induced local co-ordinate system on TM. If $t_1 = X^i\frac{\partial}{\partial x^i}$ is a local vector field on M, then its vertical, complete, and horizontal lifts in terms of partial differential equations are provided by*

$$t_1^V = X^i\frac{\partial}{\partial y^i}, \quad (17)$$

$$t_1^C = X^i\frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^j}y^j\frac{\partial}{\partial y^i}. \quad (18)$$

Let f,h,t_1 and ϕ represent a function, the 1-form, the vector field, and the tensor field type (1,1), respectively, on M. The complete and vertical lifts of such quantities are $f^C, f^V, h^C, h^V, t_1^C, t_1^V, \phi^C, \phi^V$ on the tangent bundle TM.

Let the mathematical operators ∇ and ∇^C be the Levi–Civita connections on M and TM. Then, we have [38–40]:

$$(ft_1)^V = f^V t_1^V,\ (ft_1)^C = f^C t_1^V + f^V t_1^C, \quad (19)$$

$$t_1^V f^V = 0,\ t_1^V f^C = t_1^C f^V = (t_1 f)^V,\ t_1^C f^C = (t_1 f)^C, \quad (20)$$

$$h^V(f^V) = 0,\ h^V(t_1^C) = h^C(t_1^V) = h(t_1)^V,\ h^C(t_1^C) = h(t_1)^C, \quad (21)$$

$$\phi^V t_1^C = (\phi t_1)^V,\ \phi^C t_1^C = (\phi t_1)^C, \quad (22)$$

$$[t_1,t_2]^V = [t_1^C,t_2^V] = [t_1^V,t_2^C], [t_1,t_2]^C = [t_1^C,t_2^C], \quad (23)$$

$$\nabla^C_{t_1^C} t_2^C = (\nabla_{t_1} t_2)^C,\ \ \nabla^C_{t_1^C} t_2^V = (\nabla_{t_1} t_2)^V. \quad (24)$$

Employing the complete lift on (1)–(16), we acquire

$$\begin{aligned}
(\phi^2 t_1)^C &= t_1^C - h^C(t_1^C)\kappa^V - h^V(t_1^C)\kappa^C, & (25)\\
\phi^C\kappa^C &= \phi^V\kappa^V = \phi^C\kappa^V = \phi^V\kappa^C = 0, & (26)\\
h^C(\kappa^C) &= h^V(\kappa^V) = 0,\ \ h^C(\kappa^V) = h^V(\kappa^C) = 1, & (27)\\
h^C(\phi t_1)^C &= h^V(\phi t_1)^V = h^C(\phi t_1)^V = h^V(\phi t_1)^C = 0. & (28)
\end{aligned}$$

Let g^C on TM be the complete lift of g on M, then

$$\begin{aligned}
g^C(\kappa^C, t_1^C) &= h^C(t_1^C), & (29)\\
g^C((\phi t_1)^C, (\phi t_2)^C) &= g^C(t_1^C, t_2^C) - h^C(t_1^C)h^V(t_2^C)\\
&- h^V(t_1^C)h^C(t_2^C). & (30)
\end{aligned}$$

If (TM, g^C) satisfies

$$\begin{aligned}
(d\eta)^C &= 0,\ \nabla^C_{t_1^C}\kappa^C = (\phi t_1)^C,\\
(\nabla^C_{t_1^C}\phi^C)t_2^C &= -g^C(t_1^C, t_2^C)\kappa^V - g^C(t_1^V, t_2^C)\kappa^C\\
&- h^C(t_2^C)t_1^V - h^V(t_2^C)t_1^C + 2\{h^C(t_1^C)h^C(t_2^C)\kappa^V\\
&+ h^C(t_1^C)h^V(t_2^C)\kappa^C + h^V(t_1^C)h^C(t_2^C)\kappa^C\}, & (31)
\end{aligned}$$

$$(\nabla^C_{t_1^C}h^C)(t_2^C) = -g^C(t_1^C, t_2^C) + h^C(t_1^C)h^V(t_2^C) + h^V(t_1^C)h^C(t_2^C), \tag{32}$$

then the (TM, g^C) is called an SP-Sasakian manifold. Furthermore, we have

$$\begin{aligned}
S^C(t_1^C, \kappa^C) &= -(n-1)h^C(t_1^C),\ (Q\xi)^C = -(n-1)\kappa^C, & (33)\\
h^C(R^C(t_1^C, t_2^C)t_3^C) &= g^C(t_1^C, t_3^C)h^V(t_2^C) + g^C(t_1^V, t_3^C)h^C(t_2^C)\\
&- g^C(t_2^C, t_3^C)h^V(t_1^C) - g^C(t_2^V, t_3^C)h^C(t_1^C), & (34)\\
R^C(t_1^C, \kappa^C)t_2^C &= g^C(t_1^C, t_2^C)\kappa^V + g^C(t_1^V, t_2^C)\kappa^C\\
&- h^C(t_2^C)t_1^V - h^V(t_2^C)t_1^C, & (35)\\
R^C(t_1^C, t_2^C)\kappa^C &= h^C(t_1^C)t_2^V + h^V(t_1^C)t_2^C\\
&- h^C(t_2^C)t_1^V + h^V(t_2^C)t_1^C, & (36)\\
S^C((\phi t_1)^C, (\phi t_2)^C) &= S^C(t_1^C, t_2^C) + (n-1)\{h^C(t_1^C)h^V(t_2^C)\\
&+ h^V(t_1^C)h^C(t_2^C)\}, & (37)\\
h^C(R^C(t_1^C, t_2^C)\kappa^C) &= 0, & (38)
\end{aligned}$$

such that

$$g^C((Qt_1)^C, t_2^C) = S^C(t_1^C, t_2^C),$$
$$S^C(t_1^C, t_2^C) = ag^C(t_1^C, t_2^C) + b\{h^C(t_1^C)h^V(t_2^C) + h^V(t_1^C)h^C(t_2^C)\},$$

$\forall\ t_1^C, t_2^C, t_3^C \in \Im(TM)$.

3. Expression of the Curvature Tensor of a *P*-Sasakian Manifold with Respect to $\tilde{\nabla}^C$ on TM

Let $\tilde{\nabla}$ be a linear connection and ∇ be the Levi–Civita connection of a P-Sasakian manifold M such that

$$\tilde{\nabla}_{t_1}t_2 = \nabla_{t_1}t_2 + \mathcal{H}(t_1, t_2), \tag{39}$$

where $\mathcal{H}$ is a (1, 1)-type tensor and is provided by [1]

$$\mathcal{H}(t_1, t_2) = \frac{1}{2}[T(t_1, t_2) + T'(t_1, t_2) + T'(t_2, t_1)], \tag{40}$$

such that

$$g(T'(t_1, t_2), t_3) = g(T(t_3, t_1), t_2). \tag{41}$$

Applying the complete lift on (1), (2), (6), and using (39)–(41), we infer

$$T^C(t_1^C, t_2^C) = \tilde{\nabla}^C_{t_1^C} t_2^C - \tilde{\nabla}^V_{t_2^C} t_1^C - [t_1^C, t_2^C], \tag{42}$$

which satisfies

$$T^C(t_1^C, t_2^C) = h^C(t_2^C)(\phi t_1)^C - h^C(t_1^C)(\phi t_2)^C, \tag{43}$$

$$(\tilde{\nabla}^C_{t_1^C} g^C)(t_2^C, t_3^C) = 0, \tag{44}$$

$$g^C(t_1^C, \rho^C) = \alpha^C(t_1^C), \tag{45}$$

$$\tilde{\nabla}^C_{t_1^C} t_2^C = \nabla^C_{t_1^C} t_2^C + t_5^C(t_1^C, t_2^C), \tag{46}$$

where

$$\mathcal{H}^C(t_1^C, t_2^C) = \frac{1}{2}[T^C(t_1^C, t_2^C) + T'^C(t_1^C, t_2^C) + T'^C(t_2^C, t_1^C)], \tag{47}$$

and

$$g^C(T'^C(t_1^C, t_2^C), t_3^C) = g^C(T^C(t_3^C, t_1^C), t_2^C). \tag{48}$$

From (43) and (48), we lead to

$$\begin{aligned} T'^C(t_1^C, t_2^C) &= h^C(t_1^C)(\phi t_2)^C + h^V(t_1^C)(\phi t_2)^C \\ &- g^C((\phi t_1)^C, t_2^C)\kappa^V - g^C((\phi t_1)^V, t_2^C)\kappa^C. \end{aligned} \tag{49}$$

Using (43) and (49) in (47), we have

$$\begin{aligned} \mathcal{H}^C(t_1^C, t_2^C) &= h^C(t_2^C)(\phi t_1)^V + h^V(t_2^C)(\phi t_1)^C \\ &- g^C((\phi t_1)^C, t_2^C)\kappa^V + g^C((\phi t_1)^V, t_2^C)\kappa^C. \end{aligned} \tag{50}$$

Therefore, a quarter-symmetric metric connection $\tilde{\nabla}^C$ on TM is provided by

$$\begin{aligned} \tilde{\nabla}^C_{t_1^C} t_2^C &= \nabla^C_{t_1^C} t_2^C + h^C(t_2^C)(\phi t_1)^V + h^V(t_2^C)(\phi t_1)^C \\ &- g^C((\phi t_1)^C, t_2^C)\kappa^V + g^C((\phi t_1)^V, t_2^C)\kappa^C. \end{aligned} \tag{51}$$

Let $\tilde{R}^C$ and R^C be the curvature tensors in respect of the connections $\tilde{\nabla}^C$ and ∇^C on TM, respectively. Then, from (51), we have

$$\begin{aligned}
\tilde{R}^C(t_1^C,t_2^C)t_5^C &= R^C(t_1^C,t_2^C)t_5^C \\
&+ 3\{g^C((\phi t_1)^C,t_5^C)(\phi t_2)^V + g^C((\phi t_1)^V,t_5^C)(\phi t_2)^C\} \\
&- 3\{g^C((\phi t_2)^C,t_5^C)(\phi t_1)^V - g^C((\phi t_2)^V,t_5^C)(\phi t_1)^C\} \\
&+ h^C(t_5^C)h^C(t_1^C)t_2^V + h^C(t_5^C)h^V(t_1^C)t_2^C \\
&+ h^V(t_5^C)h^C(t_1^C)t_2^C - h^C(t_5^C)h^C(t_2^C)t_1^V \\
&- h^C(t_5^C)h^V(t_2^C)t_1^C - h^V(t_5^C)h^C(t_2^C)t_1^C \\
&- h^C(t_1^C)g^C(t_2^C,t_5^C)\kappa^V - h^C(t_1^C)g^C(t_2^V,t_5^C)\kappa^C \\
&- h^V(t_1^C)g^C(t_2^C,t_5^C)\kappa^C + h^C(t_2^C)g^C(t_1^C,t_5^C)\kappa^V \\
&+ h^C(t_2^C)g^C(t_1^V,t_5^C)\kappa^C \\
&+ h^V(t_2^C)g^C(t_1^C,t_5^C)\kappa^C,
\end{aligned} \tag{52}$$

where $\tilde{R}^C(t_1^C,t_2^C)t_5^C = \tilde{\nabla}^C_{t_1^C}\tilde{\nabla}^C_{t_2^C}t_5^C - \tilde{\nabla}^C_{t_2^C}\tilde{\nabla}^C_{t_1^C}t_5^C - \tilde{\nabla}^C_{[t_1^C,t_2^C]}t_5^C$, and $t_1^C,t_2^C,t_3^C \in \Im(TM)$. By using an appropriate contraction, from (52), we obtain that

$$\begin{aligned}
\tilde{S}^C(t_2^C,t_5^C) &= S^C(t_2^C,t_5^C) + 2g^C(t_2^C,t_5^C) \\
&- (n+1)\{h^C(t_2^C)h^V(t_5^C) + h^V(t_2^C)h^C(t_5^C)\} \\
&- 3trace\phi^C g^C((\phi t_2)^C,t_5^C),
\end{aligned} \tag{53}$$

where $\tilde{S}^C$ and S^C are the Ricci tensors of $\tilde{\nabla}^C$ and ∇^C on TM, respectively. This leads to the following theorem:

Theorem 1. *Let TM be the tangent bundle of the P-Sasakian manifold with $\tilde{\nabla}^C$. Then, we have*
(1) (52) *provides R^C;*
(2) *$\tilde{S}^C$ is symmetric;*
(3) $\tilde{R}^C(t_1^C,t_2^C,t_3^C,t_4^C) + \tilde{R}^C(t_1^C,t_2^C,t_4^C,t_3^C) = 0$;
(4) $\tilde{R}^C(t_1^C,t_2^C,t_3^C,t_4^C) + \tilde{R}^C(t_2^C,t_1^C,t_3^C,t_4^C) = 0$;
(5) $\tilde{R}^C(t_1^C,t_2^C,t_3^C,t_4^C) = \tilde{R}^C(t_3^C,t_4^C,t_1^C,t_2^C)$;
(6) $\tilde{S}^C(t_2^C,\kappa^C) = -2(n-1)h^C(t_2^C)$;
for all $t_1^C,t_2^C,t_3^C \in \Im(TM)$.

With the help of (25)–(28), (35) and (36) from (52) we obtain

$$\begin{aligned}
\tilde{R}^C(\kappa^C,t_2^C)t_5^C &= 2[h^C(t_5^C)t_2^V + h^V(t_5^C)t_2^C \\
&- g^C(t_5^C,t_2^C)\kappa^V - g^C(t_5^V,t_2^C)\kappa^C],
\end{aligned} \tag{54}$$

and

$$\begin{aligned}
\tilde{R}^C(t_1^C,t_2^C)\kappa^C &= 2[h^C(t_1^C)t_2^V + h^V(t_1^C)t_2^C \\
&- h^C(t_2^C)t_1^V - h^V(t_2^C)t_1^C],
\end{aligned} \tag{55}$$

where $t_1^C,t_2^C \in \Im(TM)$.

4. Expression of Semi-Symmetric *P*-Sasakian Manifolds with Respect to $\tilde{\nabla}^C$ on TM

In 2015, Mandal and De [41] characterized semisymmetric *P*-Sasakian manifolds with respect to the quarter-symmetric metric connection, that is, the curvature tensor satisfies the condition:

$$\tilde{R}(\kappa,t_2)\cdot\tilde{R}(t_5,t_6)t_4 = 0.$$

This implies

$$\begin{aligned}
\tilde{R}(\kappa,t_2)\tilde{R}(t_5,t_6)t_4 &- \tilde{R}(\tilde{R}(\kappa,t_2)t_5,t_6)t_4 - \tilde{R}(t_5,\tilde{R}(\kappa,t_2)t_6)t_4 \\
&- \tilde{R}(t_5,t_6)\tilde{R}(\kappa,t_2)t_4 = 0.
\end{aligned} \tag{56}$$

Applying the complete lift on (56), we infer

$$\begin{aligned}(\tilde{R}(\kappa,t_2)\tilde{R}(t_5,t_6)t_4)^C &- (\tilde{R}(\tilde{R}(\kappa,t_2)t_5,t_6)t_4)^C-(\tilde{R}(t_5,\tilde{R}(\kappa,t_2)t_6)t_4)^C\\ &- (\tilde{R}(t_5,t_6)\tilde{R}(\kappa,t_2)t_4)^C=0.\end{aligned} \tag{57}$$

Using (54) and (57) yields

$$\begin{aligned}h^C(\tilde{R}(t_5,t_6)t_4)^C t_2^C &- 2\{g^C(t_2^C,(\tilde{R}(t_5,t_6)t_4)^C)\kappa^V+g^C(t_2^V,(\tilde{R}(t_5,t_6)t_4)^C)\kappa^C\}\\ &- 2\{h^C(t_5^C)(\tilde{R}(t_2,t_6)t_4)^V+h^V(t_5^C)(\tilde{R}(t_2,t_6)t_4)^C\}\\ &+ 2\{g^C(t_2^C,t_5^C)(\tilde{R}(\kappa,t_6)t_4)^V+g^C(t_2^V,t_5^C)(\tilde{R}(\kappa,t_6)t_4)^C\}\\ &- 2\{h^C(t_6^C)(\tilde{R}(t_5,t_2)t_4)^V+h^V(t_6^C)(\tilde{R}(t_5,t_2)t_4)^C\}\\ &+ 2\{g^C(V^C,t_2^C)(\tilde{R}(t_5,\kappa)t_4)^V+g^C(t_6^V,t_2^C)(\tilde{R}(t_5,\kappa)t_4)^C\}\\ &- 2\{h^C(t_4^C)(\tilde{R}(t_5,t_6)t_2)^V+h^V(t_4^C)(\tilde{R}(t_5,t_6)t_2)^C\}\\ &+ 2\{g^C(t_2^C,t_4^C)(\tilde{R}(t_5,t_6)\kappa)^V\\ &+ g^C(t_2^V,t_4^C)(\tilde{R}(t_5,t_6)\kappa)^C\}=0.\end{aligned} \tag{58}$$

Using the inner product of (58) with κ and then using (52), (54) and (55), we obtain from (58) that

$$\begin{aligned}g^C((R(t_5,t_6)t_4)^C,t_2^C) &+ 3\{g^C((\phi t_5)^C,t_4^C)g^C((\phi t_6)^V,t_2^C)\\ &+ g^C((\phi t_6)^V,t_4^C)g^C((\phi t_6)^C,t_2^C)\}\\ &- 3\{g^C((\phi t_6)^V,t_4^C)g^C((\phi t_6)^C,t_2^C)\\ &+ g^C((\phi t_6)^C,t_4^C)g^C((\phi t_5)^V,t_2^C)\}\\ &+ g^C(t_6^C,t_2^C)h^C(t_5^C)h^V(t_4^C)+g^C(t_6^C,t_2^C)h^V(t_5^C)h^C(t_4^C)\\ &+ g^C(t_6^V,t_2^C)h^C(t_5^C)h^C(t_4^C)-g^C(t_5^C,t_2^C)h^C(t_6^C)h^V(t_4^C)\\ &- g^C(t_5^C,t_2^C)h^V(t_6^C)h^C(t_4^C)-g^C(t_5^V,t_2^C)h^C(t_6^C)h^C(t_4^C)\\ &- g^C(t_6^C,t_4^C)h^C(t_5^C)h^V(t_2^C)-g^C(t_6^C,t_4^C)h^V(t_5^C)h^C(t_2^C)\\ &- g^C(t_6^V,t_4^C)h^C(t_5^C)h^C(t_2^C)+g^C(t_5^C,t_4^C)h^C(t_6^C)h^V(t_2^C)\\ &+ g^C(t_5^C,t_4^C)h^V(t_6^C)h^C(t_2^C)\\ &+ g^C(t_5^V,t_4^C)h^C(t_6^C)h^C(t_2^C)+2\{g^C(t_2^C,t_5^C)g^C(t_6^V,t_4^C)\\ &+ g^C(t_2^V,t_5^C)g^C(t_6^C,t_4^C)-g^C(t_5^C,t_4^C)g^C(t_6^V,t_2^C)\\ &- g^C(t_5^V,t_4^C)g^C(t_6^C,t_2^C)=0.\end{aligned} \tag{59}$$

By contracting the above equation over t_4 and t_6, we infer

$$\begin{aligned}S^C(t_5^C,t_2^C) &= -2ng^C(t_5^C,t_2^C)+(n+1)\{h^V(t_5^C)h^C(t_2^C)\\ &+ h^C(t_5^C)h^V(t_2^C)\}+3trace\phi^C g^C((\phi t_5)^C,t_2^C).\end{aligned} \tag{60}$$

In view of (53) and (60), we obtain

$$\tilde{S}^C(t_5^C,t_2^C)=-2(n-1)g^C(t_5^C,t_2^C). \tag{61}$$

By contracting (61), we obtain

$$\tilde{r}^C=-2n(n-1). \tag{62}$$

This leads to the following theorem:

Theorem 2. *The tangent bundle TM of a quarter-symmetric P-Sasakian manifold M is an Eienstein manifold with0 respect to $\tilde{\nabla}^C$ and $\tilde{r}^C=-2n(n-1)$.*

5. Expression of Generalized Recurrent P-Sasakian Manifolds in Respect of $\tilde{\nabla}^C$ on TM

In this section, we consider generalized recurrent P-Sasakian manifolds with respect to the quarter-symmetric metric connection $\tilde{\nabla}$. Equation (4) with respect to $\tilde{\nabla}$ can be expressed as

$$(\tilde{\nabla}_{t_1}\tilde{R})(t_2,t_3)t_4 = \alpha(t_1)\tilde{R}(t_2,t_3)t_4 + \beta(t_1)[g(t_3,t_4)t_2 - g(t_2,t_4)t_3]. \tag{63}$$

Applying the complete lift on (63), we infer

$$\begin{aligned} ((\tilde{\nabla}_{t_1}\tilde{R})(t_2,t_3)t_4)^C &= (\alpha(t_1)(\tilde{R}(t_2,t_3)t_4)^C + \beta^C(t_1^C)g^C(t_3^C,t_4^C)t_2^V \\ &+ \beta^C(t_1^C)g^C(t_3^V,t_4^C)t_2^C + \beta^V(t_1^C)g^C(t_3^C,t_4^C)t_2^C \\ &- \beta^C(t_1^C)g^C(t_2^C,t_4^C)t_3^V - \beta^C(t_1^C)g^C(t_2^V,t_4^C)t_3^C \\ &- \beta^V(t_1^C)g^C(t_2^C,t_4^C)t_3^C \end{aligned} \tag{64}$$

for $t_1,t_2,t_3,t_4 \in \Im(M)$. Substituting $t_2 = t_4 = \kappa$ in (64),

$$\begin{aligned} ((\tilde{\nabla}_{t_1}\tilde{R})(\kappa,t_3)\kappa)^C &= \alpha^C(t_1^C)(\tilde{R}(\kappa,t_3)\kappa)^V + \alpha^V(t_1^C)(\tilde{R}(\kappa,t_3)\kappa)^C \\ &+ \beta^C(t_1^C)h^C(t_3^C)\kappa^V + \beta^C(t_1^C)h^V(t_3^C)\kappa^C \\ &+ \beta^V(t_1^C)h^C(t_3^C)\kappa^C - \beta^C(t_1^C)t_3^V - \beta^V(t_1^C)t_3^C. \end{aligned} \tag{65}$$

Using (55) in (65), we obtain

$$((\tilde{\nabla}_{t_1}\tilde{R})(t_2,t_3)\kappa)^C = 2[((\tilde{\nabla}^C_{t_1^C}h^C)t_2^C)t_3^C - (\tilde{\nabla}^C_{t_1^C}h^C)t_3^C)t_2^C]. \tag{66}$$

On the other hand, using (9), (44) and (51) we obtain

$$(\tilde{\nabla}^C_{t_1^C}h^C)t_2^C = 2g^C(t_2^C,(\phi t_1)^C). \tag{67}$$

Thus, from the differential Equations (66) and (67), we have

$$\begin{aligned} ((\tilde{\nabla}_{t_1}\tilde{R})(t_2,t_3)\kappa)^C &= 4[g^C(t_2^C,(\phi t_1)^C t_3^V + g^C(t_2^V,(\phi t_1)^C t_3^C \\ &- g^C(t_3^C,(\phi t_1)^C t_2^V - g^C(t_3^V,(\phi t_1)^C t_2^C], \end{aligned}$$

which, by putting $t_2 = \kappa$, yields

$$((\tilde{\nabla}_{t_1}\tilde{R})(\kappa,t_3)\kappa)^C = -4g^C(t_3^C,(\phi t_1)^C\xi^V - g^C(t_3^V,(\phi t_1)^C\kappa^C. \tag{68}$$

Again, from (55), we have

$$(\tilde{R}(\kappa,t_3)\kappa)^C = 2[t_3^C - h^C(t_3^C)\kappa^V - h^V(t_3^C)\kappa^C]. \tag{69}$$

Thus, from (65) and (69), we obtain

$$\begin{aligned} ((\tilde{\nabla}_{t_1}\tilde{R})(\kappa,t_3)\kappa)^C &= \alpha^C(t_1^C)[t_3^C - h^C(t_3^C)\kappa^V - h^V(t_3^C)\kappa^C] \\ &+ \beta^C(t_1^C)[h^C(t_3^C)\kappa^V + h^V(t_3^C)\kappa^C - t_3^C]. \end{aligned} \tag{70}$$

In view of (68) and (70), we obtain

$$\begin{aligned} -4\{g^C(t_3^C,(\phi t_1)^C)\kappa^V &+ g^C(t_3^V,(\phi t_1)^C)\kappa^C\} \\ &= 2\alpha^C(t_1^C)[t_3^C - h^C(t_3^C)\kappa^V - h^V(t_3^C)\kappa^C] \\ &- \beta^C(t_1^C)[t_3^C - h^C(t_3^C)\kappa^V - h^V(t_3^C)\kappa^C]. \end{aligned} \tag{71}$$

By applying ϕ on (71) and using (25)–(28), we infer

$$\beta^C(t_1^C) = 2\alpha^C(t_1^C). \tag{72}$$

This leads to the following theorem:

Theorem 3. *The 1-forms α^C and β^C on TM of a generalized recurrent P-Sasakian manifold are related by $\beta^C = 2\alpha^C$.*

Next, applying the complete lift on (4), we infer

$$\begin{aligned} ((\tilde{\nabla}_{t_1}\tilde{R})(t_2,t_3)t_4)^C &= \alpha^C(t_1^C)(\tilde{R}(t_2,t_3)t_4)^V + \alpha^V(t_1^C)(\tilde{R}(t_2,t_3)t_4)^C \\ &+ \beta^C(t_1^C)g^C(t_3^C,t_4^C)t_2^V + \beta^C(t_1^C)g^C(t_3^V,t_4^C)t_2^C \\ &+ \beta^V(t_1^C)g^C(t_3^C,t_4^C)t_2^C - \beta^C(t_1^C)g^C(t_2^C,t_4^C)t_3^V \\ &- \beta^C(t_1^C)g^C(t_2^V,t_4^C)t_3^C \\ &- \beta^V(t_1^C)g^C(t_3^C,t_4^C)t_2^C, \end{aligned} \tag{73}$$

where $\tilde{\nabla}^C$ is the complete lift of $\tilde{\nabla}$. From the above equation, it follows that

$$((\tilde{\nabla}_{t_1}\tilde{R})(t_2,t_3)t_4)^C = \alpha^C(t_1^C)(\tilde{R}(t_2,t_3)t_4)^V + \alpha^V(t_1^C)(\tilde{R}(t_2,t_3)t_4)^C, \tag{74}$$

$$\forall\, t_1^C, t_2^C, t_3^C, t_4^C \in \Im(TM).$$

Thus, in view of Theorem 3, we obtain $\alpha^C(t_1^C) = 0$. Hence, we have the following corollary:

Corollary 1. *The 1-form α^C on TM of a generalized recurrent P-Sasakian manifold vanishes.*

6. Expression of Pseudosymmetric P-Sasakian Manifolds with Respect to $\tilde{\nabla}^C$ on TM

In this section, we prove the following theorem:

Theorem 4. *There is no pseudosymmetric P-Sasakian manifold with respect to $\tilde{\nabla}^C$ on TM.*

Proof. Let us suppose that TM is the tangent bundle of a pseudosymmetric P-Sasakian manifold with respect to $\tilde{\nabla}^C$. Using the complete lift on (5), we obtain

$$\begin{aligned} ((\tilde{\nabla}_{t_1}\tilde{R})(t_2,t_3)t_4)^C &= 2(\alpha(t_1)\tilde{R}(t_2,t_3)t_4)^C + (\alpha(t_2)\tilde{R}(t_1,t_3)t_4)^C \\ &+ (\alpha(t_3)\tilde{R}(t_2,t_1)t_4)^C + (\alpha(t_4)\tilde{R}(t_2,t_3)t_1)^C \\ &+ (g(\tilde{R}(t_2,t_3)t_4,t_1)\rho)^C. \end{aligned} \tag{75}$$

By contracting t_2 in (75) and substituting $t_4 = \kappa$, we have

$$\begin{aligned} ((\tilde{\nabla}_{t_1}\tilde{S})(t_3,\kappa))^C &= 2(\alpha(t_1)\tilde{S}(t_3,\kappa))^C + (\alpha(\tilde{R}(t_1,t_3)\kappa)^C \\ &+ (\alpha(t_3)\tilde{S}(t_1,\kappa))^C + (\alpha(\kappa)\tilde{S}(t_3,t_1))^C \\ &+ (g(\tilde{R}(\rho,t_3)\kappa,t_1))^C. \end{aligned} \tag{76}$$

In view of Theorem 1, we acquire

$$\tilde{S}^C(t_3^C,\kappa^C) = -2(n-1)h^C(t_3^C).$$

In consequence of (67), we infer

$$\tilde{\nabla}^C_{t_1^C}\tilde{S}^C(t_3^C,\kappa^C) = -4(n-1)g^C(t_3^C,(\phi t_1)^C). \tag{77}$$

Next, the consequences of (25)–(28) and Theorem 1, we infer

$$\begin{aligned}
\tilde{\nabla}^C_{t_1^C}\tilde{S}^C(t_3^C,\kappa^C) &= -4n\{\alpha^C(t_1^C)h^V(t_3^C)+\alpha^C(t_1^C)h^C(t_3^V)\} \\
&+ 2\{h^C(t_1^C)\alpha^V(t_3^C)+h^C(t_1^C)\alpha^C(t_3^V)\} \\
&- 2(n-1)\{\alpha^C(t_3^C)h^V(t_1^C)+\alpha^C(t_3^C)h^C(t_1^V)\} \\
&+ 2\{\alpha^C(\kappa^C)g^C(t_1^V,t_3^C)+\alpha^V(\kappa^C)g^C(t_1^C,t_3^C)\} \\
&+ \alpha^C(\kappa^C)\tilde{S}^C(t_1^V,t_3^C)+\alpha^V(\kappa^C)\tilde{S}^C(t_1^C,t_3^C). \qquad (78)
\end{aligned}$$

Equating the differential Equations (77) and (78) and then using $t_1=\kappa$, we obtain

$$\begin{aligned}
-4(n-1)g^C(t_3^C,(\phi\kappa)^C) &= -4n\{\alpha^C(\kappa^C)h^V(t_3^C)+\alpha^C(\kappa^C)h^C(t_3^V)\} \\
&+ 2\{h^C(\kappa^C)\alpha^V(t_3^C)+h^C(\kappa^C)\alpha^C(t_3^V)\} \qquad (79) \\
&- 2(n-1)\{\alpha^C(t_3^C)h^V(\kappa^C)+\alpha^C(t_3^C)h^C(\kappa^V)\} \\
&+ 2\{\alpha^C(\kappa^C)h^V(t_3^C)+\alpha^V(\kappa^C)h^C(t_3^C)\} \\
&+ \alpha^C(\kappa^C)\tilde{S}^C(\kappa^V,t_3^C)+\alpha^V(\kappa^C)\tilde{S}^C(\kappa^C,t_3^C).
\end{aligned}$$

By using (25)–(30), (45), and Theorem 1 in (79), we lead to

$$(2-3n)\{\alpha^C(\kappa^C)h^V(t_3^C)+\alpha^C(\kappa^C)h^C(t_3^V)\}+(2-n)\alpha^C(t_3^C)=0. \qquad (80)$$

By replacing t_3 by κ in (80), we obtain $\alpha^C\kappa^C=0$, which, used in (80), provides

$$\alpha^C t_3^C = 0 \Rightarrow \alpha^C = 0.$$

This goes against what we assumed. This completes the proof. □

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References

1. Golab, S. On semi-symmetric and quarter-symmetric linear connections. *Tensor NS* **1975**, *29*, 249–254.
2. Balcerzak, B. On symmetric brackets induced by linear connections. *Symmetry* **2021**, *13*, 1003. [CrossRef]
3. Schouten, J.A. *Ricci-Calculus—An Introduction to Tensor Analysis and Geometrical Applications*; Springer: Berlin/Heidelberg, Germany, 1954.
4. Biswas, S.C.; De, U.C. Quarter-symmetric metric connection in SP-Sasakian manifold. *Common. Fac. Sci. Univ. Ank. Al.* **1997**, *46*, 49–56.
5. Kumar, K.T.; Bagewadi, C.S. On ϕ-Recurrent para-Sasakian manifold admitting quarter-symmetric metric connection. *ISRN Geom.* **2012**, *2012*, 317253.
6. Prasad, R.; Haseeb, A. On a Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection. *Novi Sad J. Math.* **2016**, *46*, 103–116.

7. Zhang, P.; Li, Y.; Roy, S.; Dey, S. Geometry of α-cosymplectic metric as $*$-conformal η-Ricci-Yamabe solitons admitting quarter symmetric metric connection. *Symmetry* **2021**, *13*, 2189. [CrossRef]
8. Li, J.; He, G.; Zhao, P. On submanifolds in a Riemannian manifold with a semi-symmetric non-metric connection. *Symmetry* **2017**, *9*, 112. [CrossRef]
9. Yano, K.; Imai, T. Quarter-symmetric metric connections and their curvature tensors. *Tensor NS* **1982**, *38*, 13–18.
10. Ahmad, M.; Jun, J.B.; Haseeb, A. Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a quarter symmetric metric connection. *Bull. Korean Math. Soc.* **2009**, *55*, 477–487. [CrossRef]
11. Li, Y.; Aldossary, M.T.; Abdel-Baky, R.A. Spacelike Circular Surfaces in Minkowski 3-Space. *Symmetry* **2023**, *15*, 173. [CrossRef]
12. Li, Y.; Chen, Z.; Nazra, S.H.; Abdel-Baky, R.A. Singularities for Timelike Developable Surfaces in Minkowski 3-Space. *Symmetry* **2023**, *15*, 277. [CrossRef]
13. Li, Y.; Erdoğdu, M.; Yavuz, A. Differential Geometric Approach of Betchow-Da Rios Soliton Equation. *Hacet. J. Math. Stat.* **2022**, 1–12 . [CrossRef]
14. Li, Y.; Abolarinwa, A.; Alkhaldi, A.H.; Ali, A. Some Inequalities of Hardy Type Related to Witten-Laplace Operator on Smooth Metric Measure Spaces. *Mathematics* **2022**, *10*, 4580. [CrossRef]
15. Li, Y.; Mofarreh, F.; Laurian-Ioan, P.; Ali, A. Geometric Inequalities on the Dirac Operator and the Frobenius Norms. *Adv. Math. Phys.* **2023**, *2023*, 1275374.
16. Li, Y.; Tuncer, O.O. On (contra)pedals and (anti)orthotomics of frontals in de Sitter 2-space. *Math. Meth. Appl. Sci.* **2023**, 1–15. [CrossRef]
17. De, U.C.; Guha, N. On generalized recurrent manifolds. *J. Natl. Acad. Math.* **1991**, *9*, 85–92.
18. Walker, A.G. On Ruse's spaces of recurrent curvature. *Proc. Lond. Math. Soc.* **1951**, *52*, 36–64. [CrossRef]
19. Chaki, M.C. On pseudosymmetric manifolds. *Ann. Alexandru Ioan Cuza Univ. Math.* **1987**, 33, 53–58.
20. Yano, K.; Ishihara, S. *Tangent and Cotangent Bundles*; Marcel Dekker, Inc.: New York, NY, USA, 1973.
21. Manev, M. Tangent bundles with complete lift of the base metric and almost hypercomplex Hermitian-Norden structure. *Comptes Rendus Acad. Bulg. Sci. Sci. Math. Nat.* **2014**, *67*, 313–322.
22. Azami, S. General natural metallic structure on tangent bundle. *Iran. J. Sci. Technol. Trans. A Sci.* **2018**, *42*, 81–88. [CrossRef]
23. Bilen, L. Metric connection on tangent bundle with Berger-type deformed Sasaki metric. *Turk. J Math.* **2022**, *46*, 1036–1047. [CrossRef]
24. Dida, H.M.; Ikemakhen, A. A class of metrics on tangent bundles of pseudo-Riemannian manifolds. *Arch. Math.* **2011**, *47*, 293–308.
25. Khan, M.N.I. Lifts of semi-symmetric non-metric connection on a Kähler manifold. *Afr. Mat.* **2016**, *27*, 345–352. [CrossRef]
26. Khan, M.N.I. Novel theorems for metallic structures on the frame bundle of the second order. *Filomat* **2022**, *36*, 4471–4482. [CrossRef]
27. Khan, M.N.I.; De, U.C.; Velimirovic, L.S. Lifts of a quarter-symmetric metric connection from a Sasakian manifold to its tangent bundle. *Mathematics* **2023**, *11*, 53. [CrossRef]
28. Sato, I. On a structure similar to the almost contact structure. *Tensor NS* **1976**, 30, 219–224.
29. Choudhary, M.A.; Khan, M.N.I.; Siddiqi, M.D. Some basic inequalities on (ϵ)-para Sasakian manifold. *Symmetry* **2022**, *14*, 2585. [CrossRef]
30. Adati, T.; Matsumoto, K. On conformally recurrent and conformally symmetric P-Sasakian manifolds. *TRU Math.* **1977**, *13*, 25–32.
31. Matsumoto, K.; Ianus, S.; Mihai, I. On P-Sasakian manifolds which admit certain tensor fields. *Publ. Math. Debr.* **1986**, *33*, 61–65. [CrossRef]
32. Özgür, C. On a class of para-Sasakian manifolds. *Turk. J. Math.* **2005**, *29*, 249–257.
33. Sato, I.; Matsumoto, K. On P-Sasakian manifolds satisfying certain conditions. *Tensor NS* **1979**, *33*, 173–178.
34. Haseeb, A. Some new results on para-Sasakian manifold with a quarter symmetric metric connection. *Facta Univ. (NIS) Ser. Math. Inform.* **2015**, *30*, 765–776.
35. Adati, T.; Miyazawa, T. Some properties of P-Sasakian manifolds. *TRU Math.* **1977**, *13*, 33–42.
36. De, U.C.; Özgür, C.; Arslan, K.; Murathan, C.; Yildiz, A. On a type of P-Sasakian manifolds. *Math. Balk.* **2008**, *22*, 25–36.
37. Deshmukh, S.; Ahmed, S. Para Sasakian manifolds isometrically immersed in spaces of constant curvature. *Kyungpook J. Math.* **1980**, *20*, 112–121.
38. Das, L.S.; Khan, M.N.I. Almost r-contact structures on the tangent bundle. *Differ. Geom.-Dyn. Syst.* **2005**, *7*, 34–41.
39. Khan, M.N.I.; De, U.C. Liftings of metallic structures to tangent bundles of order r. *AIMS Math.* **2022**, *7*, 7888–7897. [CrossRef]
40. Peyghan, E.; Firuzi, F.; De, U.C. Golden Riemannian structures on the tangent bundle with g-natural metrics. *Filomat* **2019**, *33*, 2543–2554. [CrossRef]
41. Mandal, K.; De, U.C. Quarter-symmetric metric connection in a P-Sasakian manifold. *Analele Univ. Vest Timis. Ser. Mat.-Inform.* **2015**, *LIII*, 137–150. [CrossRef]