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Five Generalized Rough Approximation Spaces Produced by Maximal Rough Neighborhoods

A. A. Azzam [1,2,*] and Tareq M. Al-shami [3,4]

1 Department of Mathematics, Faculty of Science and Humanities, Prince Sattam bin Abdulaziz University, Alkharj 11942, Saudi Arabia
2 Department of Mathematics, Faculty of Science, New Valley University, Elkharga 72511, Egypt
3 Department of Mathematics, Sana'a University, Sana'a P.O. Box 1247, Yemen
4 Department of Engineering Mathematics & Physics, Faculty of Engineering & Technology, Future University, Cairo 11835, Egypt
* Correspondence: aa.azzam@psau.edu.sa

Abstract: In rough set theory, the multiplicity of methods of calculating neighborhood systems is very useful to calculate the measures of accuracy and roughness. In line with this research direction, in this article we present novel kinds of rough neighborhood systems inspired by the system of maximal neighborhood systems. We benefit from the symmetry between rough approximations (lower and upper) and topological operators (interior and closure) to structure the current generalized rough approximation spaces. First, we display two novel types of rough set models produced by maximal neighborhoods, namely, type 2 m_ξ-neighborhood and type 3 m_ξ-neighborhood rough models. We investigate their master properties and show the relationships between them as well as their relationship with some foregoing ones. Then, we apply the idea of adhesion neighborhoods to introduce three additional rough set models, namely, type 4 m_ξ-adhesion, type 5 m_ξ-adhesion and type 6 m_ξ-adhesion neighborhood rough models. We establish the fundamental characteristics of approximation operators inspired by these models and discuss how the properties of various relationships relate to one another. We prove that adhesion neighborhood rough models increase the value of the accuracy measure of subsets, which can improve decision making. Finally, we provide a comparison between Yao's technique and current types of adhesion neighborhood rough models.

Keywords: m_ξ-neighborhood and m_ξ-adhesion neighborhood systems; lower/upper approximation operators; accuracy value; rough set

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1. Introduction

In 1982, Pawlak introduced the rough set theory as a mathematical technique for dealing with incomplete data and knowledge [1,2] . The theory is based on describing each subset using two operators called upper and lower approximations. Following the development of rough set theory in 1982, the field has seen a resurgence of numerous applications in a variety of domains, including economics, engineering, medicine, chemistry, biology, finance, data mining, linguistics, networking, data analysis, and other fields; see [3–8]. Using the most common notion of binary relations as a foundation for formulation, interpretation, comparison of neighborhood systems, and approximations, Yao [9,10] gave a strong and engaging analysis of rough sets. Numerous researchers have studied various sorts of binary relations in addition to rough set theory, as in [11–15].

In this regard, several rough set models have been developed using minimal right neighborhoods and minimal left neighborhoods [16]. A finite number of any binary relations were used to generalize three sorts of lower and upper approximations of any set with respect to the right neighborhoods [17]. On the basis of a similarity relation, Abo-Tabl [18] provided a comparison of two types of definitions of rough approximations. Mareay [19], in 2016, explored four ways of approximating rough sets and provided their main features. Atef et al. [20] introduced $\aleph_\zeta$-neighborhood space and $\aleph_\zeta$-adhesion neighborhood space-based comparison of different forms of rough approximations. The intersection operation between $\aleph_\zeta$-neighborhoods has been employed to define E_ζ-neighborhoods by Al-shami et al. Additionally, using the relationship of inclusion between $\aleph_\zeta$-neighborhoods, new types of neighborhoods known as C_ζ-neighborhoods [21] and S_ζ-neighborhoods [22] have been established.

To improve the accuracy measures of subsets by enlarging lower approximations and shrinking upper approximations, Kandi et al. [23] integrated the ideal structures with rough neighborhoods to build new rough models, namely, "ideal approximation spaces". Following this, many authors adopted this approach of producing generalized rough approximations. Hosny [24] discussed generalized approximation spaces using ideal and $\aleph_\zeta$-neighborhoods. Two rough set models inspired by C_ζ-neighborhoods and S_ζ-neighborhoods systems were developed in terms of approximation operators using ideal structure by the authors of [25,26]. Recently, a type of rough set model based on maximal rough neighborhoods has been presented by Al-shami [27]. In [28–30], the authors exploited maximal rough neighborhoods and ideals to create new rough paradigms and show how they are applied to address some practical problems. Recent developments in research on rough set neighborhoods with ideal structures and their applications [23,31–33] have caught our attention. The investigation of generalized approximation space from topological views was an interesting and popular topic [34,35]. A generalization of topological rough sets was provided by Hosny [36], using two separate techniques. Moreover, extensions of topology have been applied to provide new rough paradigms with interesting medical applications, such as minimal structure [5], supra topology [37], infra topology [38], and fuzzy information systems [39] .

Despite all of this progress in rough set theory, there are significant contributions still to be added in the upper and lower approximations of the sets, and we are in the process of adding new approximation operators to improve the existing ones, as detailed in the remainder of this paper. The current paradigms produce higher measures of accuracy for subsets of data under research, which allows decision-makers to issue accurate decisions. Furthermore, the current paradigms offer different environments in which to deal with the data of information systems, which gives us a wide range of options to select the fit model that is convenient to describe the practical problems under study.

This article is organized as follows. We go over several fundamental ideas of rough neighborhood systems in Section 2. In Section 3, we generalize two different m_ζ-neighborhood space-based rough set types. An explanation of a few of the three relationships of rough sets' existing properties is provided. In Section 4, three brand-new relationships of rough sets depending on m_ζ-adhesion neighborhood spaces are defined and researched. In Section 5, a comparison of Yao's method [9,10] and our method is looked at. Finally, Section 6 presents conclusions and proposes some upcoming work.

2. Preliminaries

In this section, we review a number of fundamental ideas and the outcomes that will be incorporated into this paper.

2.1. Pawlak Approximation Space

Definition 1 ([1,2])**.** *Let ζ be an equivalence relation on a nonempty finite set Σ. Then, we call the ordered pair (Σ, ζ) Pawlak approximation space. If ζ is any non-equivalence relation, then we call (Σ, ζ) a generalized approximation space. The class of an element $\mu \in \Sigma$, denoted by $\zeta(\mu)$, is given by*

$$\zeta(\mu) = \{\nu \in \Sigma : \mu\zeta\nu\}.$$

Definition 2 ([1,2])**.** *Let* (Σ, ζ) *be Pawlak approximation space. The pair* $(\zeta^{-}(Z), \zeta^{+}(Z))$ *defined by*

$$\zeta^{-}(Z) = \{\mu \in \Sigma : \zeta(\mu) \sqsubseteq Z\} \text{ (named the first lower approximation of Z), and}$$

$$\zeta^{+}(Z) = \{\mu \in \Sigma : \zeta(\mu) \sqcap Z \neq \phi\} \text{ (named the first upper approximation of Z)}$$

are the initial specification for the lower and upper approximation operators of $Z \sqsubseteq \Sigma$ *with respect to* ζ.

A number of interesting qualities were proved, and we list these in the following.

Let $Z, Z_1, Z_2 \sqsubseteq \Sigma$:

(1) $\zeta^{-}(Z) = (\zeta^{+}((Z^{c}))^{c}$.
(2) $\zeta^{-}(\Sigma) \sqsubseteq \Sigma$.
(3) $\zeta^{-}(Z_1 \sqcap Z_2) = \zeta^{-}(Z_1) \sqcap \zeta^{-}(Z_2)$.
(4) $\zeta^{-}(Z_1) \sqcup \zeta^{-}(Z_2) \sqsubseteq \zeta^{-}(Z_1 \sqcup Z_2)$.
(5) $\zeta^{-}(Z_1) \sqsubseteq \zeta^{-}(Z_2)$ whenever $Z_1 \sqsubseteq Z_2$.
(6) $\zeta^{-}(\phi) = \phi$.
(7) $\zeta^{-}(Z) \sqsubseteq Z$.
(8) $Z \sqsubseteq \zeta^{-}(\zeta^{+}(Z))$.
(9) $\zeta^{-}(\zeta^{-}(Z)) = \zeta^{-}(Z)$.
(10)$\zeta^{+}(Z) = \zeta^{-}(\zeta^{+}(Z))$.
(11)$\zeta^{+}(Z) = (\zeta^{-}(Z^{c}))^{c}$.
(12) $\zeta^{+}(\phi) = \phi$.
(13) $\zeta^{+}(Z_1 \sqcap Z_2) \sqsubseteq \zeta^{+}(Z_1) \sqcap \zeta^{+}(Z_2)$.
(14) $\zeta^{+}(Z_1 \sqcup Z_2) = \zeta^{+}(Z_1) \sqcup \zeta^{-}(Z_2)$.
(15) $\zeta^{+}(Z_1) \sqsubseteq \zeta^{+}(Z_2)$ whenever $Z_1 \sqsubseteq Z_2$.
(16) $\zeta^{+}(\Sigma) = \Sigma$.
(17) $Z \sqsubseteq \zeta^{+}(Z)$.
(18) $\zeta^{+}(\zeta^{-}(Z)) \sqsubseteq Z$.
(19) $\zeta^{+}(\zeta^{+}(Z)) = \zeta^{+}(Z)$.
(20) $\zeta^{-}(Z) = \zeta^{+}(\zeta^{-}(Z))$.
(21) $\zeta^{-}(Z_1^{c} \sqcup Z_2) \sqsubseteq (\zeta^{-}(Z_1))^{c} \sqcup \zeta^{-}(Z_2)$.
(22) $\zeta^{-}(Z) \sqsubseteq \zeta^{+}(Z)$.

Definition 3 ([1,2])**.** *Let* (Σ, ζ) *be Pawlak approximation space. The pair* $(\zeta^{*}(Z), \zeta^{\star}(Z))$ *defined by*

$$\zeta^{*}(Z) = \sqcup\{\zeta(\mu) : \zeta(\mu) \sqsubseteq Z\} \text{ (named the second lower approximation of Z), and}$$

$$\zeta^{\star}(Z) = [\zeta^{*}(Z^{c})]^{c} \text{ (named the second upper approximation of Z)}$$

are the second specification for the lower and upper approximation operators of $Z \sqsubseteq \Sigma$ *with respect to* ζ.

Definition 4 ([1,2])**.** *Let* (Σ, ζ) *be Pawlak approximation space. The pair* $(\zeta^{\ominus}(Z), \zeta^{\oplus}(Z))$ *defined by*

$$\zeta^{\ominus}(Z) = [\zeta^{\oplus}(Z^{c})]^{c} \text{ (named the third lower approximation of Z), and}$$

$$\zeta^{\oplus}(Z) = \sqcup\{\zeta(\mu) : \zeta(\mu) \sqcap Z \neq \phi\} \text{ (named the third upper approximation of Z)}$$

are the third specification for the lower and upper approximations of $Z \sqsubseteq \Sigma$ *with respect to* ζ.

Yao and other researchers [9,10,16] expanded the idea of a rough set to any relation ζ as follows.

Definition 5 ([9,10,16]). *Let ζ be an arbitrary relation on Σ. Then, $\aleph_\zeta$-neighborhoods of an $\mu \in \Sigma$ (denoted by $\aleph_\zeta(\mu)$) are defined as follows.*
(1) $\aleph_{\zeta_1}(\mu) = \{\nu \in \Sigma : \mu\zeta\nu\}$.
(2) $\aleph_{\zeta_2}(\mu) = \{\nu \in \Sigma : \nu\zeta\mu\}$.
(3) $\aleph_{\zeta_3}(\mu) = \sqcap_{\mu \in \aleph_{\zeta_1}(\nu)} \aleph_{\zeta_1}(\nu)$.
(4) $\aleph_{\zeta_4}(\mu) = \sqcap_{\mu \in \aleph_{\zeta_2}(\nu)} \aleph_{\zeta_2}(\nu)$.
(5) $\aleph_{\zeta_5}(\mu) = \aleph_{\zeta_1}(\mu) \sqcup \aleph_{\zeta_2}(\mu)$.
(6) $\aleph_{\zeta_6}(\mu) = \aleph_{\zeta_1}(\mu) \sqcap \aleph_{\zeta_2}(\mu)$.
(7) $\aleph_{\zeta_7}(\mu) = \aleph_{\zeta_3}(\mu) \sqcup \aleph_{\zeta_4}(\mu)$.
(8) $\aleph_{\zeta_8}(\mu) = \aleph_{\zeta_3}(\mu) \sqcap \aleph_{\zeta_4}(\mu)$.

2.2. Rough Set Models Inspired m_ζ-Neighborhoods

Definition 6 ([12]). *Let ζ be an arbitrary relation on Σ. The maximal right neighborhood of an $\mu \in \Sigma$ (short for, $m_{\zeta_1}(\mu)$) is defined as follows.*

$$m_{\zeta_1}(\mu) = \sqcup_{\mu \in \aleph_{\zeta_1}(\nu)} \aleph_{\zeta_1}(\nu).$$

Definition 7 ([27]). *The other maximal neighborhoods of an element $\mu \in \Sigma$, designated by $m_\zeta(\mu)$, induced from any binary relation on Σ, are defined by:*
(1) $m_{\zeta_2}(\mu) = \sqcup_{\mu \in \aleph_{\zeta_2}(\nu)} \aleph_{\zeta_2}(\nu)$.
(2) $m_{\zeta_3}(\mu) = m_{\zeta_1}(\mu) \sqcap m_{\zeta_2}(\mu)$.
(3) $m_{\zeta_4}(\mu) = m_{\zeta_1}(\mu) \sqcup m_{\zeta_2}(\mu)$.
(4)

$$m_{\zeta_5}(\mu) = \begin{cases} \sqcap_{\mu \in m_{\zeta_1}(\nu)} m_{\zeta_1}(\nu) & : \quad \exists m_{\zeta_1}(\nu) \text{ containing } \mu \\ \phi & : \quad \text{otherwise.} \end{cases}$$

(5)

$$m_{\zeta_6}(\mu) = \begin{cases} \sqcap_{\mu \in m_{\zeta_2}(\nu)} m_{\zeta_2}(\nu) & : \quad \exists m_{\zeta_2}(\nu) \text{ containing } \mu \\ \phi & : \quad \text{otherwise.} \end{cases}$$

(6) $m_{\zeta_7}(\mu) = m_{\zeta_5}(\mu) \sqcap m_{\zeta_6}(\mu)$.
(7) $m_{\zeta_8}(\mu) = m_{\zeta_5}(\mu) \sqcup m_{\zeta_6}(\mu)$.

We illustrate the preceding definitions as follows.

Example 1. *Let $\zeta = \{(\nu,\nu),(\mu,\nu),(\nu,\lambda),(\lambda,\kappa)\}$ be any arbitrary relation on $\Sigma = \{\mu,\nu,\lambda,\kappa\}$. Next, we determine the maximal neighborhoods for each element $\mu \in \Sigma$ in Table 1.*

Table 1. $\aleph_\zeta$-neighborhoods and m_ζ-neighborhoods of all elements in Σ.

	μ	ν	λ	κ
$\aleph_{\zeta_1}$	$\{\nu\}$	$\{\nu,\lambda\}$	$\{\kappa\}$	ϕ
$\aleph_{\zeta_2}$	ϕ	$\{\mu,\nu\}$	$\{\nu\}$	$\{\lambda\}$
$\aleph_{\zeta_3}$	ϕ	$\{\nu\}$	$\{\nu,\lambda\}$	$\{\kappa\}$
$\aleph_{\zeta_4}$	$\{\mu,\nu\}$	$\{\nu\}$	$\{\lambda\}$	ϕ
$\aleph_{\zeta_5}$	ϕ	$\{\nu\}$	ϕ	ϕ
$\aleph_{\zeta_6}$	$\{\nu\}$	$\{\mu,\nu,\lambda\}$	$\{\nu,\kappa\}$	$\{\lambda\}$
$\aleph_{\zeta_7}$	ϕ	$\{\nu\}$	$\{\lambda\}$	ϕ
$\aleph_{\zeta_8}$	$\{\mu,\nu\}$	$\{\nu\}$	$\{\nu,\lambda\}$	$\{\kappa\}$
m_{ζ_1}	ϕ	$\{\nu,\lambda\}$	$\{\nu,\lambda\}$	$\{\kappa\}$
m_{ζ_2}	$\{\mu,\nu\}$	$\{\mu,\nu\}$	$\{\lambda\}$	ϕ
m_{ζ_3}	ϕ	$\{\nu\}$	$\{\lambda\}$	ϕ
m_{ζ_4}	$\{\mu,\nu\}$	$\{\mu,\nu,\lambda\}$	$\{\nu,\lambda\}$	$\{\kappa\}$
m_{ζ_5}	ϕ	$\{\nu,\lambda\}$	$\{\nu,\lambda\}$	$\{\kappa\}$
m_{ζ_6}	$\{\mu,\nu\}$	$\{\mu,\nu\}$	$\{\lambda\}$	ϕ
m_{ζ_7}	ϕ	$\{\nu\}$	$\{\lambda\}$	ϕ
m_{ζ_8}	$\{\mu,\nu\}$	$\{\mu,\nu,\lambda\}$	$\{\nu,\lambda\}$	$\{\kappa\}$

As shown below, we shall extend Definitions 3 and 4 to the m_ξ-neighborhood space in this section. This is an extension of the work presented by Al-shami [27] in which he introduced the first type of rough set models based on m_ξ-neighborhoods.

Definition 8 ([27])**.** *Let (Σ, ζ) be a generalized approximation space. We link two approximations $(1\aleph_{m_\xi}(Z), 1\aleph^{m_\xi}(Z))$ to a set Z using the definitions given below.*

$1\aleph_{m_\xi}(Z) = \{\mu \in \Sigma : m_\xi(\mu) \sqsubseteq Z\}$ *(named the type 1 lower approximation of Z), and*

$1\aleph^{m_\xi}(Z) = \{\mu \in \Sigma : m_\xi(\mu) \sqcap Z \neq \phi\}$ *(named the type 1 upper approximation of Z).*

Definition 9. *Let (Σ, ζ) be a generalized approximation space, and $Z \sqsubseteq \Sigma$. A type 1 mξ-neighborhood boundary region of Z, denoted by $1B_{m_\xi}$, and a type 1 m_ξ-neighborhood accuracy measure of Z, denoted by $1A_{m_\xi}$, are respectively defined as follows for each $m_\xi = \{m_{\xi_1}, m_{\xi_2}, m_{\xi_3}, m_{\xi_4}, m_{\xi_5}, m_{\xi_6}, m_{\xi_7}, m_{\xi_8}\}$:*
(1) $1B_{m_\xi}(Z) = 1\aleph^{m_\xi}(Z) - 1\aleph_{m_\xi}(Z)$.
(2) $1A_{m_\xi}(Z) = \frac{|1\aleph_{m_\xi}(Z) \sqcap Z|}{|1\aleph^{m_\xi}(Z) \sqcup Z|}$.

Example 2. *Consider $Z = \{\mu, \lambda\}$, in Example 1, as a subset of a m_ξ-NS $(\Sigma, \zeta, \eta_\xi)$. We have the calculations listed below.*

(1) *If $m_\xi \in \{m_{\xi_1}, m_{\xi_5}\}$,, then $1\aleph_{m_\xi}(Z) = \{\mu\}$, $1\aleph^{m_\xi}(Z) = \{\nu, \lambda\}$, $1B_{m_\xi}(Z) = \{\nu, \lambda\}$ and $1A_{m_\xi}(Z) = \frac{1}{3}$,*
(2) *If $m_\xi \in \{m_{\xi_2}, m_{\xi_6}\}$, then $1\aleph_{m_\xi}(Z) = \{\mu, \kappa\}$, $1\aleph^{m_\xi}(Z) = \{\mu, \nu, \lambda\}$, $1B_{m_\xi}(Z) = \{\mu, \nu\}$ and $1A_{m_\xi}(Z) = \frac{1}{3}$.*
(3) *If $m_\xi \in \{m_{\xi_3}, m_{\xi_7}\}$, then $1\aleph_{m_\xi}(Z) = \{\mu, \lambda, \kappa\}$, $1\aleph^{m_\xi}(Z) = \{\lambda\}$, $1B_{m_\xi}(Z) = \phi$ and $1A_{m_\xi}(Z) = 1$,*
(4) *If $m_\xi \in \{m_{\xi_4}, m_{\xi_8}\}$,, then $1\aleph_{m_\xi}(Z) = \phi$, $1\aleph^{m_\xi}(Z) = \{\mu, \nu, \lambda\}$, $1B_{m_\xi}(Z) = \{\mu, \nu, \lambda\}$ and $1A_{m_\xi}(Z) = 0$*

3. Two Generalized Types of Rough Set Model Based on m_ξ-Neighborhood Space

To address the boundary region issue, we shall extend Definitions 3 and 4 to m_ξ-neighborhood space in this section. The basic characterizations of novel types of rough set models presented herein are studied, and their relationships with each other and with previous models are illustrated. To point out the invalidity of some findings, we supply some elucidative examples.

Definition 10. *Let (Σ, ζ) be a generalized approximation space. A type 2 based on m_ξ-neighborhood (briefly, type 2 m_ξ-neighborhood rough set) of $Z \sqsubseteq \Sigma$ with respect to η_ξ is a pair $(2\aleph_{m_\xi}(Z), 2\aleph^{m_\xi}(Z))$, defined by*

$$2\aleph_{m_\xi}(Z) = \sqcup\{m_\xi(\mu) : m_\xi(\mu) \sqsubseteq Z\}$$
(named the type 2 lower approximation of Z), and

$$2\aleph^{m_\xi}(Z) = [2\aleph_{m_\xi}(Z^c)]^c$$
(named the type 2 upper approximation of Z).

Definition 11. *Let (Σ, ζ) be a generalized approximation space, and $Z \sqsubseteq \Sigma$. A type 2 m_ξ-neighborhood boundary region of Z, denoted by $2B_{m_\xi}$, and a type 2 m_ξ-neighborhood accuracy measure, denoted by $2A_{m_\xi}$, are respectively given by*

$$2B_{m_\xi}(Z) = 2\aleph^{m_\xi}(Z) - 2\aleph_{m_\xi}(Z).$$

$$2A_{m_\xi}(Z) = \frac{|2\aleph_{m_\xi}(Z) \sqcap Z|}{|2\aleph^{m_\xi}(Z) \sqcup Z|}.$$

Example 3. *Consider $Z = \{\mu, \lambda\}$, in Example 1, as a subset of a generalized approximation space (Σ, ζ). We have the calculations listed below.*

(1) *If $m_\xi \in \{m_{\xi_1}, m_{\xi_5}\}$, then $2\aleph_{m_\xi}(Z) = \phi$, $2\aleph^{m_\xi}(Z) = \{\mu, \nu, \lambda\}$, $2B_{m_\xi}(Z) = \{\mu, \nu, \lambda\}$ and $2A_{m_\xi}(Z) = 0$,*

(2) *If $m_\xi \in \{m_{\xi_2}, m_{\xi_6}\}$, then $2\aleph_{m_\xi}(Z) = \{\lambda\}$, $2\aleph^{m_\xi}(Z) = \{\mu, \nu, \lambda, \kappa\}$, $2B_{m_\xi}(Z) = \{\mu, \nu, \kappa\}$ and $2A_{m_\xi}(Z) = \frac{1}{4}$.*

(3) *If $m_\xi \in \{m_{\xi_3}, m_{\xi_7}\}$, then $2\aleph_{m_\xi}(Z) = \{\lambda\}$, $2\aleph^{m_\xi}(Z) = \{\mu, \lambda, \kappa\}$, $2B_{m_\xi}(Z) = \{\mu, \kappa\}$ and $2A_{m_\xi}(Z) = \frac{1}{3}$,*

(4) *If $m_\xi \in \{m_{\xi_4}, m_{\xi_8}\}$, then $2\aleph_{m_\xi}(Z) = \phi$, $1\aleph^{m_\xi}(Z) = \{\mu, \nu, \lambda\}$, $2B_{m_\xi}(Z) = \{\mu, \nu, \lambda\}$ and $1A_{m_\xi}(Z) = 0$*

Proposition 1. *Let (Σ, ζ) be a generalized approximation space, and $Z_1, Z_2 \sqsubseteq \Sigma$. The following claims hold true.*
(1) $2\aleph_{m_\xi}(Z_1) \sqsubseteq 2\aleph_{m_\xi}(Z_2)$ whenever $Z_1 \sqsubseteq Z_2$.
(2) $2\aleph_{m_\xi}(Z_1 \sqcap Z_2) = 2\aleph_{m_\xi}(Z_1) \sqcap 2\aleph_{m_\xi}(Z_2)$.
(3) $2\aleph_{m_\xi}(Z_1^c) = (2\aleph^{m\xi}(Z_1))^c$.
(4) $2\aleph_{m_\xi}(\Sigma) \sqsubseteq \Sigma$.
(5) $2\aleph_{m_\xi}(\phi) = \phi$.
(6) $2\aleph_{m_\xi}(2\aleph_{m_\xi}(\Sigma)) \sqsubseteq 2\aleph_{m_\xi}(\Sigma)$.

Proof.

(1) Let $Z_1 \sqsubseteq Z_2$, hence $2\aleph_{m_\xi}(Z_1) = \sqcup\{m_\xi(\mu) : m_\xi(\mu) \sqsubseteq Z_1\} \sqsubseteq \sqcup\{m_\xi(\mu) : m_\xi(\mu) \sqsubseteq Z_2\} = 2\aleph_{m_\xi}(Z_2)$.

(2) From (1) it follows that: $2\aleph_{m_\xi}(Z_1 \sqcap Z_2) \sqsubseteq 2\aleph_{m_\xi}(Z_1) \sqcap 2\aleph_{m_\xi}(Z_2)$. As for the other direction, let $\mu \in 2\aleph_{m_\xi}(Z_1) \sqcap 2\aleph_{m_\xi}(Z_2)$, hence $\mu \in 2\aleph_{m_\xi}(Z_1)$ and $\mu \in 2\aleph_{m_\xi}(Z_2)$; this indicates $m_\xi(\mu) \sqsubseteq Z_1$ and $m_\xi(\mu) \sqsubseteq Z_2$. So, $m_\xi(\mu) \sqsubseteq Z_1 \sqcap Z_2$. Thus, $\mu \in 2\aleph_{m_\xi}(Z_1 \sqcap Z_2)$ and $2\aleph_{m_\xi}(Z_1) \sqcap 2\aleph_{m_\xi}(Z_2) \sqsubseteq 2\aleph_{m_\xi}(Z_1 \sqcap Z_2)$.

(3) Let $\mu \in 2\aleph_{m_\xi}(Z_1^c)$, then $m_\xi(\mu) \sqsubseteq Z_1^c$. i.e., $m_\xi(\mu) \sqcap Z_1^c = \phi$, $\mu \notin 2\aleph^{m\xi}(Z_1))^c$, and hence, $\mu \in (2\aleph^{m\xi}(Z_1))^c$.

(4) The fact that $m_\xi(\mu) \sqsubseteq \Sigma$ for each $\mu \in \Sigma$ proves it.

(5) Evident.

(6) Clear from (4).

□

Corollary 1. *Let (Σ, ζ) be a generalized approximation space. Then, $2\aleph_{m_\xi}(Z_1) \sqcup 2\aleph_{m_\xi}(Z_2) \sqsubseteq 2\aleph_{m_\xi}(Z_1 \sqcup Z_2)$ $\forall Z_1, Z_2 \sqsubseteq \Sigma$.*

Proposition 2. *Let (Σ, ζ) be a generalized approximation space, and $Z_1, Z_2 \sqsubseteq \Sigma$. The following claims are accurate.*
(1) If $Z_1 \sqsubseteq Z_2$, then $2\aleph^{m_\xi}(Z_1) \sqsubseteq 2\aleph^{m_\xi}(Z_2)$.
(2) $2\aleph^{m_\xi}(Z_1 \sqcup Z_2) = 2\aleph^{m_\xi}(Z_1) \sqcup 2\aleph^{m_\xi}(Z_2)$.
(3) $2\aleph^{m_\xi}(Z_1^c) = (2\aleph_{m\xi}(Z_1))^c$.
(4) $2\aleph^{m_\xi}(\Sigma) \sqsubseteq \Sigma$.
(5) $\phi \sqsubseteq 2\aleph^{m_\xi}(\phi)$.
(6) $2\aleph^{m_\xi}(2\aleph^{m_\xi}(\Sigma)) = 2\aleph^{m_\xi}(\Sigma)$.

Proof. The evidence is comparable to that in Proposition 1. □

Corollary 2. *Let (Σ, ζ) be a generalized approximation space. Then, $2\aleph^{m_\xi}(Z_1 \sqcap Z_2) \sqsubseteq 2\aleph^{m_\xi}(Z_1) \sqcap 2\aleph^{m^\xi}(Z_2)$ $\forall Z_1, Z_2 \sqsubseteq \Sigma$.*

Example 4. *Expansion of Example 1. Without loss of generality, take* $m_\xi = m_{\xi_4}$. *If* $Z_1 = \{\nu\}$, $Z_2 = \{\lambda\}$, *then* $2\aleph_{m_{\xi_4}}(Z_1 \sqcup Z_2) = \{\nu, \lambda\}$, $2\aleph^{m_{\xi_4}}(Z_1) = \phi$, *and* $2\aleph^{m_{\xi_4}}(Z_2) = \phi$ *i.e.,* $2\aleph_{m_{\xi_4}}(Z_1 \sqcup Z_2) \neq 2\aleph_{m_{\xi_4}}(Z_1) \sqcup 2\aleph_{m_{\xi_4}}(Z_2)$. *Additionally,* $2\aleph^{m_{\xi_4}}(Z_1 \sqcap Z_2) = \phi$, *and* $2\aleph^{m_{\xi_4}}(Z_1) \sqcap 2\aleph^{m_{\xi_4}}(Z_2) = \{\lambda\}$, *i.e.,* $2\aleph^{m_{\xi_4}}(Z_1 \sqcap Z_2) \neq 2\aleph^{m_{\xi_4}}(Z_1) \sqcap 2\aleph^{m_{\xi_4}}(Z_2)$.

Definition 12. *Let* (Σ, ζ) *be a generalized approximation space. A type 3 based on* m_ξ*-neighborhood (briefly, type 3* m_ξ*-neighborhood rough set) of* $Z \sqsubseteq \Sigma$ *with respect to* η_ξ *is a pair* $(3\aleph_{m_\xi}(Z), 3\aleph^{m_\xi}(Z))$ *given by*

$$3\aleph_{m_\xi}(Z) = [3\aleph^{m_\xi}(Z^c)]^c$$

(named the type 3 lower approximation of Z), and

$$3\aleph^{m_\xi}(Z) = \sqcup\{m_\xi(\mu) : m_\xi(\mu) \sqcap Z \neq \phi\}$$

(named the type 3 upper approximation of Z).

Definition 13. *Let* (Σ, ζ) *be a generalized approximation space, and* $Z \sqsubseteq \Sigma$. *A type 3* m_ξ*-neighborhood boundary region, denoted by* $3B_{m_\xi}$, *and a type 3* m_ξ*-neighborhood accuracy measure, denoted by* $3A_{m_\xi}$, *are respectively defined as follows.*

$$3B_{m_\xi}(Z) = 3\aleph^{m_\xi}(Z) - 3\aleph_{m_\xi}(Z).$$

$$3A_{m_\xi}(Z) = \frac{|3\aleph_{m_\xi}(Z) \sqcap Z|}{|3\aleph^{m_\xi}(Z) \sqcup Z|}.$$

Example 5. *Consider* $Z = \{\mu, \lambda\}$, *in Example 1, as a subset of a* m_ξ*-NS* $(\Sigma, \zeta, \eta_\xi)$. *We have the calculations listed below.*

(1) *If* $m_\xi \in \{m_{\xi_1}, m_{\xi_5}\}$, *then* $3\aleph_{m_\xi}(Z) = \{\mu\}$, $3\aleph^{m_\xi}(Z) = \{\nu, \lambda\}$, $3B_{m_\xi}(Z) = \{\nu, \lambda\}$ *and* $3A_{m_\xi}(Z) = \frac{1}{3}$.
(2) *If* $m_\xi \in \{m_{\xi_2}, m_{\xi_6}\}$, *then* $3\aleph_{m_\xi}(Z) = \{\lambda, \kappa\}$, $3\aleph^{m_\xi}(Z) = \{\mu, \nu, \lambda\}$, $3B_{m_\xi}(Z) = \{\mu, \nu\}$ *and* $3A_{m_\xi}(Z) = \frac{1}{3}$.
(3) *If* $m_\xi \in \{m_{\xi_3}, m_{\xi_7}\}$, *then* $3\aleph_{m_\xi}(Z) = \{\mu, \lambda, \kappa\}$, $3\aleph^{m_\xi}(Z) = \{\lambda\}$, $3B_{m_\xi}(Z) = \phi$ *and* $3A_{m_\xi}(Z) = 1$.
(4) *If* $m_\xi \in \{m_{\xi_4}, m_{\xi_8}\}$, *then* $3\aleph_{m_\xi}(Z) = \phi$, $3\aleph^{m_\xi}(Z) = \{\mu, \nu, \lambda\}$, $3B_{m_\xi}(Z) = \{\mu, \nu, \lambda\}$ *and* $3A_{m_\xi}(Z) = 0$.

Proposition 3. *Let* (Σ, ζ) *be a generalized approximation space, and* $Z_1, Z_2 \sqsubseteq \Sigma$. *The following claims are accurate.*
(1) If $Z_1 \sqsubseteq Z_2$, *then* $3\aleph_{m_\xi}(Z_1) \sqsubseteq 3\aleph_{m_\xi}(Z_2)$.
(2) $3\aleph_{m_\xi}(Z_1 \sqcap Z_2) = 3\aleph_{m_\xi}(Z_1) \sqcap 2\aleph_{m_\xi}(Z_2)$.
(3) $3\aleph_{m_\xi}(Z_1^c) = (3\aleph^{m\xi}(Z_1))^c$.
(4) $3\aleph_{m_\xi}(\Sigma) \sqsubseteq \Sigma$.
(5) $3\aleph_{m_\xi}(\phi) = \phi$.
(6) $3\aleph_{m_\xi}(3\aleph_{m_\xi}(\Sigma)) = 3\aleph_{m_\xi}(\Sigma)$.

Proof. The evidence is comparable to that in Proposition 1. □

Corollary 3. *Let* (Σ, ζ) *be a generalized approximation space. Then,* $3\aleph_{m_\xi}(Z_1) \sqcup 3\aleph_{m_\xi}(Z_2) \sqsubseteq 3\aleph_{m_\xi}(Z_1 \sqcup Z_2)\ \forall Z_1, Z_2 \sqsubseteq \Sigma$.

Proposition 4. *Let* (Σ, ζ) *be a generalized approximation space, and* $Z_1, Z_2 \sqsubseteq \Sigma$. *The following claims are accurate.*
(1) If $Z_1 \sqsubseteq Z_2$, *then* $3\aleph^{m_\xi}(Z_1) \sqsubseteq 3\aleph^{m_\xi}(Z_2)$.
(2) $3\aleph^{m_\xi}(Z_1 \sqcup Z_2) = 3\aleph^{m_\xi}(Z_1) \sqcup 3\aleph^{m_\xi}(Z_2)$.
(3) $3\aleph^{m_\xi}(Z_1^c) = (3\aleph_{m\xi}(Z_1))^c$.
(4) $3\aleph^{m_\xi}(\Sigma) \sqsubseteq \Sigma$.

(5) $\phi \sqsubseteq 3\aleph^{m_\xi}(\phi)$.
(6) $3\aleph^{m_\xi}(3\aleph^{m_\xi}(\Sigma)) = 3\aleph^{m_\xi}(\Sigma)$.
(7) $3\aleph^{m_\xi}(Z_1^c \sqcup Z_2) \sqsubseteq (3\aleph^{m_\xi}(Z_1))^c \sqcup 3\aleph^{m_\xi}(Z_2)$.

Proof. We only demonstrate (7).

Let $z \notin (3\aleph^{m_\xi}(Z_1))^c \sqcup 3\aleph^{m_\xi}(Z_2)$. Then, $z \notin (3\aleph^{m_\xi}(Z_1))^c$ and $z \notin 3\aleph^{m_\xi}(Z_2)$. Hence, $z \in 3\aleph^{m_\xi}Z_1$ and $z \notin 3\aleph^{m_\xi}(Z_2)$. So, $m_\xi(z) \sqsubseteq Z_1$ and $m_\xi(z) \not\sqsubseteq Z_2$. Then, $m_\xi(z) \not\sqsubseteq Z_1^c$ and $m_\xi(z) \not\sqsubseteq Z_2$, i.e., $m_\xi(z) \not\sqsubseteq (Z_1^c \sqcup Z_2)$, $z \notin m_\xi(Z_1^c \sqcup Z_2)$, i.e., $3\aleph^{m_\xi}(Z_1^c \sqcup Z_2) \sqsubseteq (3\aleph^{m_\xi}(Z_1))^c \sqcup 3\aleph^{m_\xi}(Z_2)$. □

Corollary 4. *Let (Σ, ζ) be a generalized approximation space. Then, $3\aleph^{m_\xi}(Z_1 \sqcap Z_2) = 3\aleph^{m_\xi}(Z_1) \sqcap 3\aleph^{m^\xi}(Z_2)$ $\forall Z_1, Z_2 \sqsubseteq \Sigma$.*

Proposition 5. *Let (Σ, ζ) be a generalized approximation space, and $Z_1, Z_2 \sqsubseteq \Sigma$. The following claims are accurate.*
(1) If $Z_1 \sqsubseteq Z_2$, then $3\aleph_{m_\xi}(Z_1) \sqsubseteq 3\aleph_{m_\xi}(Z_2)$.
(2) $3\aleph_{m_\xi}(Z_1 \sqcap Z_2) = 3\aleph_{m_\xi}(Z_1) \sqcap 2\aleph_{m_\xi}(Z_2)$.
(3) $3\aleph_{m_\xi}(Z_1^c) = (3\aleph^{m\xi}(Z_1))^c$.
(4) $3\aleph_{m_\xi}(\Sigma) = \Sigma$.
(5) $3\aleph_{m_\xi}(\phi) = \phi$.
(6) $3\aleph_{m_\xi}(3\aleph_{m_\xi}(\Sigma)) = 3\aleph_{m_\xi}(\Sigma)$.

Proof. The evidence is comparable to that in Proposition 4. □

Corollary 5. *Let (Σ, ζ) be a generalized approximation space. Then, $3\aleph_{m_\xi}(Z_1) \sqcup 3\aleph_{m_\xi}(Z_2) = 3\aleph_{m_\xi}(Z_1 \sqcup Z_2)$ $\forall Z_1, Z_2 \sqsubseteq \Sigma$.*

Example 6. *Expansion of Example 1. Without loss of generality, take $m_\xi = m_{\xi_4}$. If $Z = \{\mu, \lambda\}$, then $3\aleph_{m_{\xi_4}}(Z) = \phi$, $2\aleph^{m_{\xi_4}}(Z) = \{\mu, \nu, \lambda\}$, $3B_{m_{\xi_4}}(Z) = \{\mu, \nu, \lambda\}$, and $3A_{m_{\xi_4}}(Z) = 0$. Additionally, $3\aleph_{m_{\xi_7}}(Z) = Z$, $2\aleph^{m_{\xi_4}}(Z) = Z$, $3B_{m_{\xi_7}}(Z) = \phi$, and $3A_{m_{\xi_7}}(Z) = 1$.*

4. Novel Kinds of Adhesion Rough Set Model Generated by Maximal Neighborhood Systems

We shall extend Definition 7 to the m_ξ-adhesion neighborhood in this part and also provide three additional types of rough set models generated by maximal neighborhoods, as shown below.

Definition 14. *Let (Σ, ζ) be a generalized approximation space. The m_ξ-adhesion neighborhoods of element $z \in \Sigma$ are defined for each $m_\xi \in \{m_{\xi_1}, m_{\xi_2}, m_{\xi_3}, m_{\xi_4}, m_{\xi_5}, m_{\xi_6}, m_{\xi_7}, m_{\xi_8}\}$ as follows.*
(1) $\mathcal{M}_{m_{\xi_1}}(\mu) = \{\nu \in z : m_{\xi_1}(\mu) = m_{\xi_1}(\nu)\}$.
(2) $\mathcal{M}_{m_{\xi_2}}(\mu) = \{\nu \in z : m_{\xi_2}(\mu) = m_{\xi_2}(\nu)\}$.
(3) $\mathcal{M}m_{\xi_2}(\mu) = \{\nu \in z : \sqcap_{\mu \in m_{\xi_1}(\nu)} m_{\xi_1}(\nu) = \sqcap_{\nu \in m_{\xi_1}(\mu)} m_{\xi_1}(\mu)\}$.
(4) $\mathcal{M}_{m_{\xi_4}}(\mu) = \{\nu \in z : \sqcap_{\mu \in m_{\xi_2}(\nu)} m_{\xi_2}(\nu) = \sqcap_{\nu \in m_{\xi_2}(\mu)} m_{\xi_2}(\mu)\}$.
(5) $\mathcal{M}_{m_{\xi_5}}(\mu) = \mathcal{M}_{m\xi_1}(\mu) \sqcup \mathcal{M}_{m\xi_2}(\mu)$.
(6) $\mathcal{M}_{m_{\xi_6}}(\mu) = \mathcal{M}_{m\xi_1}(\mu) \sqcap \mathcal{M}_{m\xi_2}(\mu)$.
(7) $\mathcal{M}_{m_{\xi_7}}(\mu) = \mathcal{M}_{m\xi_3}(\mu) \sqcup \mathcal{M}_{m\xi_4}(\mu)$.
(8) $\mathcal{M}_{m_{\xi_8}}(\mu) = \mathcal{M}_{m\xi_3}(\mu) \sqcap \mathcal{M}_{m\xi_4}(\mu)$.

We provide an explanation of the aforementioned definition as follows.

Example 7. *Following Example 1, the neighborhood of m_ξ-adhesion is calculated in Table 2.*

Table 2. m_ξ-adhesion neighborhoods of all elements in Σ.

	μ	ν	λ	κ
$\mathcal{M}_{m_{\xi_1}}$	$\{\mu\}$	$\{\nu,\lambda\}$	$\{\nu,\lambda\}$	$\{\kappa\}$
$\mathcal{M}_{m_{\xi_2}}$	$\{\mu,\nu\}$	$\{\mu,\nu\}$	$\{\lambda\}$	$\{\kappa\}$
$\mathcal{M}_{m_{\xi_3}}$	$\{\mu\}$	$\{\nu,\lambda\}$	$\{\nu,\lambda\}$	$\{\kappa\}$
$\mathcal{M}_{m_{\xi_4}}$	$\{\mu,\nu\}$	$\{\mu,\nu\}$	$\{\lambda\}$	$\{\kappa\}$
$\mathcal{M}_{m_{\xi_5}}$	$\{\mu,\nu\}$	$\{\mu,\nu,\lambda\}$	$\{\nu,\lambda\}$	$\{\kappa\}$
$\mathcal{M}_{m_{\xi_6}}$	$\{\mu\}$	$\{\nu\}$	$\{\lambda\}$	$\{\kappa\}$
$\mathcal{M}_{m_{\xi_7}}$	$\{\mu,\nu\}$	$\{\mu,\nu,\lambda\}$	$\{\nu,\lambda\}$	$\{\kappa\}$
$\mathcal{M}_{m_{\xi_8}}$	$\{\mu\}$	$\{\nu\}$	$\{\lambda\}$	$\{\kappa\}$

Definition 15. *Let (Σ,ζ) be a generalized approximation space. We link two approximations $(4\aleph_{m_\xi}(Z), 4\aleph^{m_\xi}(Z))$ to a set Z using Definition 14, as follows.*

$4\aleph_{m_\xi}(Z) = \{\mu \in \Sigma : \mathcal{M}_{m_\xi}(\mu) \sqsubseteq Z\}$*(named the type 4 lower approximation of Z), and*

$4\aleph^{m_\xi}(Z) = \{\mu \in \Sigma : \mathcal{M}_{m_\xi}(\mu) \sqcap Z \neq \phi\}$ *(named the type 4 upper approximation of Z).*

Definition 16. *Let (Σ,ζ) be a generalized approximation space, and $Z \sqsubseteq \Sigma$. A type 4 m_ξ-adhesion neighborhood boundary region of Z, denoted by $4B_{m_\xi}$, and a type $4m_\xi$-adhesion neighborhood accuracy measure, denoted by $4A_{m_\xi}$, are respectively defined as follows*

$$4B_{m_\xi}(Z) = 4\aleph^{m_\xi}(Z) - 4\aleph_{m_\xi}(Z).$$

$$4A_{m_\xi}(Z) = \frac{|4\aleph_{m_\xi}(Z) \sqcap Z|}{|4\aleph^{m_\xi}(Z) \sqcup Z|}.$$

Example 8. *Consider $Z = \{\mu,\lambda\}$, in Example 1, as a subset of a generalized approximation space (Σ,ζ). We have the calculations listed below.*

(1) If $m_\xi \in \{m_{\xi_1}, m_{\xi_3}\}$, then $4\aleph_{m_\xi}(Z) = \{\mu\}$, $4\aleph^{m_\xi}(Z) = \{\mu,\nu,\lambda\}$, $4B_{m_\xi}(Z) = \{\nu,\lambda\}$, and $4A_{m_\xi}(Z) = \frac{1}{3}$.

(2) If $m_\xi \in \{m_{\xi_2}, m_{\xi_4}\}$, then $4\aleph_{m_\xi}(Z) = \{\lambda\}$, $4\aleph^{m_\xi}(Z) = \{\mu,\nu,\lambda\}$, $4B_{m_\xi}(Z) = \{\mu,\lambda\}$, and $4A_{m_\xi}(Z) = \frac{1}{3}$.

(3) If $m_\xi \in \{m_{\xi_5}, m_{\xi_7}\}$, then $4\aleph_{m_\xi}(Z) = \phi$, $4\aleph^{m_\xi}(Z) = \{\mu,\nu,\lambda\}$, $4B_{m_\xi}(Z) = \{\mu,\nu,\lambda\}$, and $4A_{m_\xi}(Z) = 0$.

(4) If $m_\xi \in \{m_{\xi_6}, m_{\xi_8}\}$, then $4\aleph_{m_\xi}(Z) = \{\mu,\nu\}$, $4\aleph^{m_\xi}(Z) = \{\mu,\nu\}$, $4B_{m_\xi}(Z) = \phi$, and $4A_{m_\xi}(Z) = 1$.

Proposition 6. *Let (Σ,ζ) be a generalized approximation space, and $Z, Z_1, Z_2 \sqsubseteq \Sigma$. The following claims are accurate.*

(1) $4\aleph_{m_\xi}(Z) = 4\aleph^{m_\xi}(Z^c))^c$.

(2) $4\aleph_{m_\xi}(\phi) = \phi$.

(3) If $Z_1 \sqsubseteq Z_2$, then $4\aleph_{m_\xi}(Z_1) \sqsubseteq 4\aleph_{m_\xi}(Z_2)$.

(4) $4\aleph_{m_\xi}(Z_1 \sqcap Z_2) = 4\aleph_{m_\xi}(Z_1) \sqcap 4\aleph_{m_\xi}(Z_2)$.

(5) $4\aleph_{m_\xi}(Z_1) \sqcup 4\aleph_{m_\xi}(Z_2) \sqsubseteq 4\aleph_{m_\xi}(Z_1 \sqcup Z_2)$.

(6) $4\aleph_{m_\xi}(\Sigma) = \Sigma$.

(7) $4\aleph_{m_\xi}(Z) \sqsubseteq Z$.

(8) $Z \sqsubseteq 4\aleph_{m_\xi}(4\aleph^{m_\xi}(Z))$.

(9) $4\aleph_{m_\xi}(4\aleph_{m_\xi}(Z)) = 4\aleph_{m_\xi}(Z)$.

(10) $4\aleph^{m_\xi}(Z) = 4\aleph_{m_\xi}(4\aleph^{m_\xi}(Z))$.

Proof.

(1) There is $\mathcal{M}_{m\xi}(\mu) \sqsubseteq Z$ for each $\mu \in Z$ if $\mu \in 4\aleph_{m_\xi}(Z)$. Then, for every $\mu \in \Sigma - [\Sigma - Z]$, there exists $\mathcal{M}_{m_\xi}(\mu)$ such that $\mathcal{M}_{m_\xi}(\mu) \sqcap [\Sigma - Z] = \phi$. So, $\mu \notin 4\aleph^{m_\xi}(\Sigma - Z)$, $\mu \in \Sigma - [4\aleph^{m_\xi}(\Sigma - Z)]$. Therefore, $4\aleph_{m_\xi}(Z) = \Sigma - [4\aleph^{m_\xi}(\Sigma - Z)] = 4\aleph^{m_\xi}(Z^c))^c$/

(2) $4\aleph m_{\xi}(\phi) = \{\mu \in \Sigma : \mathcal{M}_{m_{\xi}}(\mu) \sqsubseteq \phi\} = \phi$.

(3) If $Z_1 \sqsubseteq Z_2$, then $4\aleph_{m_{\xi}}(Z_1) = \{\mu \in \Sigma : \mathcal{M}_{m_{\xi}}(\mu) \sqsubseteq Z_1\} \sqsubseteq \{\mu \in \Sigma : \mathcal{M}_{m_{\xi}}(\mu) \sqsubseteq Z_2\} = 4\aleph_{m_{\xi}}(Z_2)$.

(4) From Definition 15, we have $4\aleph_{m_{\xi}}(Z_1 \sqcap Z_2) = \{\mu \in \Sigma : \mathcal{M}_{m_{\xi}}(\mu) \sqsubseteq (Z_1 \sqcap Z_2)\}$. Then, $\mathcal{M}_{m_{\xi}}(\mu) \sqsubseteq Z_1$ and $\mathcal{M}_{m_{\xi}}(\mu) \sqsubseteq Z_1$ because of $(Z_1 \sqcap Z_2) \sqsubseteq Z_1$, and $(Z_1 \sqcap Z_2) \sqsubseteq Z_2$. Hence, $4\aleph_{m_{\xi}}(Z_1 \sqcap Z_2) \sqsubseteq 4\aleph_{m_{\xi}}(Z_1)$, and $4\aleph_{m_{\xi}}(Z_1 \sqcap Z_2) \sqsubseteq 4\aleph_{m_{\xi}}(Z_2)$. On the other hand, $4\aleph_{m_{\xi}}(Z_1) \sqcap 4\aleph_{m_{\xi}}(Z_2) = \{\mu \in \Sigma : \mathcal{M}_{m_{\xi}}(\mu) \sqsubseteq Z_1\}$, and $4\aleph_{m_{\xi}}(Z_1) \sqcap 4\aleph_{m_{\xi}}(Z_2) = \{\mu \in \Sigma : \mathcal{M}_{m_{\xi}}(\mu) \sqsubseteq Z_2\}$. i.e., $4\aleph_{m_{\xi}}(Z_1) \sqcap 4\aleph_{m_{\xi}}(Z_2) = \{\mu \in \Sigma : \mathcal{M}_{m_{\xi}}(\mu) \sqsubseteq (Z_1 \sqcap Z_2)\} = 4\aleph_{m_{\xi}}(Z_1 \sqcap Z_2) = 4\aleph_{m_{\xi}}(Z_1) \sqcap 4\aleph_{m_{\xi}}(Z_2)$.

(5) One can prove following similar lines to those in (4).

(6) By (2) and Definition 15, we obtain the proof.

By Definition 15, the proofs (7), (8), (9), and (10) are obtained. □

As illustrated in the following example, the equality of (5) in Proposition 6 does not hold.

Example 9. *Following Example 7, consider $m_{\xi} = m_{\xi_1}$. If $Z_1 = \{\nu\}$ and $Z_2 = \{\lambda\}$, then $4\aleph_{m_{\xi}}(Z_1) = \phi$ and $4\aleph_{m_{\xi}}(Z_2) = \phi$. So, $4\aleph_{m_{\xi}}(Z_1) \sqcup 4\aleph_{m_{\xi}}(Z_2) = \phi$. However, $4\aleph_{m_{\xi}}(Z_1 \sqcup Z_2) = \{\nu, \lambda\}$. Therefore, $4\aleph_{m_{\xi}}(Z_1) \sqcup 4\aleph_{m_{\xi}}(Z_2) \neq 4\aleph_{m_{\xi}}(Z_1 \sqcup Z_2)$.*

Proposition 7. *Let (Σ, ζ) be a generalized approximation space, and $Z, Z_1, Z_2 \sqsubseteq \Sigma$. The following claims are accurate.*
(1) $4\aleph^{m_{\xi}}(Z) = (4\aleph_{m_{\xi}}(Z^c))^c$.
(2) $4\aleph^{m_{\xi}}(\phi) = \phi$.
(3) If $Z_1 \sqsubseteq Z_2$, then $4\aleph^{m_{\xi}}(Z_1) \sqsubseteq 4\aleph^{m_{\xi}}(Z_2)$.
(4) $4\aleph^{m_{\xi}}(Z_1 \sqcap Z_2) \sqsubseteq 4\aleph^{m_{\xi}}(Z_1) \sqcap 4\aleph^{m_{\xi}}(Z_2)$.
(5) $4\aleph^{m_{\xi}}(Z_1) \sqcup 4\aleph^{m_{\xi}}(Z_2) = 4\aleph^{m_{\xi}}(Z_1 \sqcup Z_2)$.
(6) $4\aleph^{m_{\xi}}(\Sigma) = \Sigma$.
(7) $Z \sqsubseteq 4\aleph^{m_{\xi}}(Z)$.
(8) $4\aleph^{m_{\xi}}(4\aleph_{m_{\xi}}(Z)) \sqsubseteq Z$.
(9) $4\aleph^{m_{\xi}}(4\aleph^{m_{\xi}}(Z)) = 4\aleph^{m_{\xi}}(Z)$.
(10) $4\aleph_{m_{\xi}}(Z) = 4\aleph^{m_{\xi}}(4\aleph_{m_{\xi}}(Z))$.

Proof. It is analogous to the proof of Proposition 6. □

The equality relation of (4) in Proposition 7 need not be satisfied, as the next example elucidates.

Example 10. *Following Example 7, consider $m_{\xi} = m_{\xi_1}$. If $Z_1 = \{\mu, \kappa\}$ and $Z_2 = \{\lambda, \kappa\}$, then $4\aleph^{m_{\xi}}(Z_1) = \{\mu, \kappa\}$ and $4\aleph^{m_{\xi}}(Z_2) = \{\mu, \lambda, \kappa\}$, and $4\aleph^{m_{\xi}}(Z_1 \sqcap Z_2) = \{\kappa\}$. So, $4\aleph^{m_{\xi}}(Z_1 \sqcap Z_2) \neq 4\aleph^{m_{\xi}}(Z_1) \sqcap 4\aleph^{m_{\xi}}(Z_2)$.*

Definition 17. *Let (Σ, ζ) be a generalized approximation space. We link two approximations $(5\aleph_{m_{\xi}}(Z), 5\aleph^{m_{\xi}}(Z))$ to a set Z using Definition 14, as follows.*

$$5\aleph_{m_{\xi}}(Z) = \sqcup\{\mathcal{M}_{m\xi}(\mu) : \mathcal{M}_{m\xi}(\mu) \sqsubseteq Z\} \text{(named the type 5 lower approximation of } Z\text{), and}$$

$$5\aleph^{m_{\xi}}(Z) = [5\aleph_{m_{\xi}}(Z^c)]^c \text{ (named the type 5 upper approximation of } Z\text{).}$$

As can be seen from Definition 17 above, it is more precise than Definition 10 and addresses some of the issues raised by Propositions 8 and 9.

Definition 18. *Let (Σ, ζ) be a generalized approximation space, and $Z \sqsubseteq \Sigma$. A type 5 m_{ξ}-adhesion neighborhood boundary region of Z, denoted by $5B_{m_{\xi}}$, and a type 5m_{ξ}-adhesion neighborhood accuracy measure, denoted by $5A_{m_{\xi}}$, are respectively defined as follows.*

$$5B_{m_\xi}(Z) = 5\aleph^{m_\xi}(Z) - 5\aleph_{m_\xi}(Z).$$

$$5A_{m_\xi}(Z) = \frac{|5\aleph_{m_\xi}(Z) \sqcap Z|}{|5\aleph^{m_\xi}(Z) \sqcup Z|}.$$

Example 11. *Consider $Z = \{\nu, \lambda\}$, in Example 7, as a subset of a generalized approximation space (Σ, ζ). We have the calculations listed below.*

(1) *If $m_\xi \in \{m_{\xi_1}, m_{\xi_3}, m_{\xi_6}, m_{\xi_8}\}$,, then $5\aleph_{m_\xi}(Z) = \{\nu, \lambda\}$, $5\aleph^{m_\xi}(Z) = \{\nu, \lambda\}$, $5B_\xi(Z) = \phi$ and $5A_{m\xi}(Z) = 1$,*

(2) *If $m_\xi \in \{m_{\xi_2}, m_{\xi_4}\}$, then $5\aleph_{m_\xi}(Z) = \{\lambda\}$, $5\aleph^{m_\xi}(Z) = \{\mu, \nu, \lambda\}$, $5B_\xi(Z) = \{\mu, \nu\}$ and $5A_{m\xi}(Z) = \frac{1}{3}$.*

(3) *If $m_\xi \in \{m_{\xi_5}, m_{\xi_7}\}$, then $5\aleph_{m_\xi}(Z) = \{\nu, \lambda\}$, $4\aleph^{m_\xi}(Z) = \{\mu, \nu, \lambda\}$, $5B_\xi(Z) = \{\mu\}$ and $5A_{m\xi}(Z) = \frac{2}{3}$.*

Proposition 8. *Let (Σ, ζ) be a generalized approximation space, and $Z, Z_1, Z_2 \sqsubseteq \Sigma$. The following claims are accurate.*

(1) $5\aleph_{m_\xi}(Z) = 5\aleph^{m_\xi}(Z^c))^c$.

(2) $5\aleph_{m_\xi}(\phi) = \phi$.

(3) If $Z_1 \sqsubseteq Z_2$, then $5\aleph_{m_\xi}(Z_1) \sqsubseteq 5\aleph_{m_\xi}(Z_2)$.

(4) $5\aleph_{m_\xi}(Z_1 \sqcap Z_2) = 5\aleph_{m_\xi}(Z_1) \sqcap 5\aleph_{m_\xi}(Z_2)$.

(5) $5\aleph_{m_\xi}(Z_1) \sqcup 5\aleph_{m_\xi}(Z_2) \sqsubseteq 5\aleph_{m_\xi}(Z_1 \sqcup Z_2)$.

(6) $5\aleph_{m_\xi}(\Sigma) = \Sigma$.

(7) $5\aleph_{m_\xi}(Z) \sqsubseteq Z$.

(8) $Z \sqsubseteq 5\aleph_{m_\xi}(5\aleph^{m_\xi}(Z))$.

(9) $5\aleph_{m_\xi}(5\aleph_{m_\xi}(Z)) = 5\aleph_{m_\xi}(Z)$.

(10) $5\aleph^{m_\xi}(Z) = 5\aleph_{m_\xi}(5\aleph^{m_\xi}(Z))$.

Proof.

(1) There is $\mathcal{M}_{m_\xi}(\mu) \sqsubseteq Z$ for each $\mu \in Z$ if $\mu \in 5\aleph_{m_\xi}(Z)$. Then, for every $\mu \in \Sigma - [\Sigma - Z]$, there exists $\mathcal{M}_{m_\xi}(\mu)$ such that $\mathcal{M}_{m_\xi}(\mu) \sqcap [\Sigma - Z] = \phi$. So, $\mu \notin 5\aleph^{m_\xi}(\Sigma - Z)$, $\mu \in \Sigma - [5\aleph^{m_\xi}(\Sigma - Z)]$. Therefore, $5\aleph m_\xi(Z) = \Sigma - [5\aleph^{m_\xi}(\Sigma - Z)] = 5\aleph^{m_\xi}(Z^c))^c$.

(2) $5\aleph m_\xi(\phi) = \sqcup\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqsubseteq \phi\} = \phi$.

(3) If $Z_1 \sqsubseteq Z_2$, then $5\aleph_{m_\xi}(Z_1) = \sqcup\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqsubseteq Z_1\} \sqsubseteq \sqcup\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqsubseteq Z_2\} = 5\aleph_{m_\xi}(Z_2)$.

(4) From Definition 17 , we have $5\aleph_{m_\xi}(Z_1 \sqcap Z_2) = \sqcup\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqsubseteq (Z_1 \sqcap Z_2)\}$. Then, $\mathcal{M}_{m_\xi}(\mu) \sqsubseteq Z_1$ and $\mathcal{M}_{m_\xi}(\mu) \sqsubseteq Z_2$ because of $(Z_1 \sqcap Z_2) \sqsubseteq Z_1$, and $(Z_1 \sqcap Z_2) \sqsubseteq Z_2$. Hence, $5\aleph_{m_\xi}(Z_1 \sqcap Z_2) \sqsubseteq 5\aleph_{m_\xi}(Z_1)$, and $5\aleph_{m_\xi}(Z_1 \sqcap Z_2) \sqsubseteq 5\aleph_{m_\xi}(Z_2)$. On the other hand, $5\aleph_{m_\xi}(Z_1) \sqcap 5\aleph_{m_\xi}(Z_2) = \sqcup\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqsubseteq Z_1\}$, and $5\aleph_{m_\xi}(Z_1) \sqcap 5\aleph_{m_\xi}(Z_2) = \sqcup\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqsubseteq Z_2\}$. i.e., $5\aleph_{m_\xi}(Z_1) \sqcap 5\aleph_{m_\xi}(Z_2) = \sqcup\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqsubseteq (Z_1 \sqcap Z_2)\} = 5\aleph_{m_\xi}(Z_1 \sqcap Z_2) = 5\aleph_{m_\xi}(Z_1) \sqcap 5\aleph_{m_\xi}(Z_2)$.

(5) One can prove following similar lines to those in (4).

(6) By (2) and Definition 17, we obtain the proof.

By Definition 17, the proofs (7), (8), (9), and (10) are obtained. □

Example 12. *Following Example 7, let $m_\xi = m_{\xi_1}$. If $Z_1 = \{\nu\}$ and $Z_2 = \{\lambda\}$, then $5\aleph_{m_\xi}(Z_1) = \phi$ and $5\aleph_{m_\xi}(Z_2) = \phi$. So, $5\aleph_{m_\xi}(Z_1) \sqcup 5\aleph_{m_\xi}(Z_2) = \phi$. But, $5\aleph_{m_\xi}(Z_1 \sqcup Z_2) = \{\nu, \lambda\}$. Therefore, $5\aleph_{m_\xi}(Z_1) \sqcup 5\aleph_{m_\xi}(Z_2) \neq 5\aleph_{m_\xi}(Z_1 \sqcup Z_2)$.*

Proposition 9. *Let (Σ, ζ) be a generalized approximation space, and $Z, Z_1, Z_2 \sqsubseteq \Sigma$. The following claims are accurate.*
(1) $5\aleph^{m_\xi}(Z) = (5\aleph_{m_\xi}(Z^c))^c$.
(2) $5\aleph^{m_\xi}(\phi) = \phi$.
(3) If $Z_1 \sqsubseteq Z_2$, *then* $5\aleph^{m_\xi}(Z_1) \sqsubseteq 5\aleph^{m_\xi}(Z_2)$.
(4) $5\aleph^{m_\xi}(Z_1 \sqcap Z_2) \sqsubseteq 5\aleph^{m_\xi}(Z_1) \sqcap 5\aleph^{m_\xi}(Z_2)$.
(5) $5\aleph^{m_\xi}(Z_1) \sqcup 5\aleph^{m_\xi}(Z_2) = 5\aleph^{m_\xi}(Z_1 \sqcup Z_2)$.
(6) $5\aleph^{m_\xi}(\Sigma) = \Sigma$.
(7) $Z \sqsubseteq 5\aleph^{m_\xi}(Z)$.
(8) $5\aleph^{m_\xi}(5\aleph_{m_\xi}(Z)) \sqsubseteq Z$.
(9) $5\aleph^{m_\xi}(5\aleph^{m_\xi}(Z)) = 5\aleph^{m_\xi}(Z)$.
(10) $5\aleph_{m_\xi}(Z) = 5\aleph^{m_\xi}(5\aleph_{m_\xi}(Z))$.

Proof. It is analogous to the proof of Proposition 8. □

The equality relation of (4) in Proposition 9 need not be satisfied, as the next example elucidates.

Example 13. *Following Example 7, let $m_\xi = m_{\xi_1}$. If $Z_1 = \{\mu, \kappa\}$ and $Z_2 = \{\lambda, \kappa\}$, then $5\aleph^{m_\xi}(Z_1) = \{\mu, \nu\}$ and $5\aleph^{m_\xi}(Z_2) = \{\lambda, \kappa\}$, and $5\aleph^{m_\xi}(Z_1 \sqcap Z_2) = \{\kappa\}$. So, $5\aleph^{m_\xi}(Z_1 \sqcap Z_2) \neq 5\aleph^{m_\xi}(Z_1) \sqcap 5\aleph^{m_\xi}(Z_2)$.*

Definition 19. *Let (Σ, ζ) be a generalized approximation space. We link two approximations $(6\aleph_{m_\xi}(Z), 6\aleph^{m_\xi}(Z))$ to a set Z using Definition 14 as follows.*

$$6\aleph_{m_\xi}(Z) = [6\aleph^{m_\xi}(Z^c)]^c \text{(named the type 6 lower approximation of Z), and}$$

$$6\aleph^{m_\xi}(Z) = \sqcup\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqcap Z \neq \phi\} \text{ (named the type 6 upper approximation of Z).}$$

As can be seen from Definition 19 above, it is more precise than Definition 12 and addresses some of the issues raised by Propositions 10 and 11.

Definition 20. *Let (Σ, ζ) be a generalized approximation space, and $Z \sqsubseteq \Sigma$. A type 6 m_ξ-adhesion neighborhood boundary region of Z, denoted by $6B_{m_\xi}$, and a type 5$m\xi$-adhesion neighborhood accuracy measure, denoted by $6A_{m_\xi}$, are given, respectively, by*

$$6B_{m\xi}(Z) = 6\aleph^{m_\xi}(Z) - 6\aleph_{m_\xi}(Z).$$

$$6A_{m\xi}(Z) = \frac{|6\aleph_{m_\xi}(Z) \sqcap Z|}{|6\aleph^{m_\xi}(Z) \sqcup Z|}.$$

Example 14. *Consider $Z = \{\nu, \lambda\}$, in Example 7, as a subset of a generalized approximation space (Σ, ζ). We have the calculations listed below.*

(1) *If $m_\xi \in \{m_{\xi_1}, m_{\xi_3}, m_{\xi_6}, m_{\xi_8}\}$, then $6\aleph_{m_\xi}(Z) = \{\nu, \lambda\}$, $6\aleph^{m_\xi}(Z) = \{\nu, \lambda\}$, $6B_{m_\xi}(Z) = \phi$ and $6A_{m_\xi}(Z) = 1$.*
(2) *If $m_\xi \in \{m_{\xi_2}, m_{\xi_4}\}$, then $6\aleph_{m_\xi}(Z) = \{\lambda\}$, $5\aleph^{m_\xi}(Z) = \{\mu, \nu, \lambda\}$, $6B_{m_\xi}(Z) = \{\mu, \nu\}$ and $6A_{m_\xi}(Z) = \frac{1}{3}$.*
(3) *If $m_\xi \in \{m_{\xi_5}, m_{\xi_7}\}$, then $6\aleph_{m_\xi}(Z) = \phi$, $4\aleph^{m_\xi}(Z) = \{\mu, \nu, \lambda\}$, $6B_{m_\xi}(Z) = \{\mu, \nu, \lambda\}$ and $6A_{m_\xi}(Z) = 0$.*

Proposition 10. *Let (Σ, ζ) be a generalized approximation space, and $Z, Z_1, Z_2 \sqsubseteq \Sigma$. The following claims are accurate.*
(1) $6\aleph_{m_\xi}(Z) = 6\aleph^{m_\xi}(Z^c))^c$.
(2) $6\aleph_{m_\xi}(\phi) = \phi$.
(3) If $Z_1 \sqsubseteq Z_2$, *then* $6\aleph_{m_\xi}(Z_1) \sqsubseteq 6\aleph_{m_\xi}(Z_2)$.
(4) $6\aleph_{m_\xi}(Z_1 \sqcap Z_2) = 6\aleph_{m_\xi}(Z_1) \sqcap 6\aleph_{m_\xi}(Z_2)$.

(5) $6\aleph_{m_\xi}(Z_1) \sqcup 6\aleph_{m_\xi}(Z_2) \sqsubseteq 6\aleph_{m_\xi}(Z_1 \sqcup Z_2)$.
(6) $6\aleph_{m_\xi}(\Sigma) = \Sigma$.
(7) $6\aleph_{m_\xi}(Z) \sqsubseteq Z$.
(8) $Z \sqsubseteq 6\aleph_{m_\xi}(6\aleph^{m_\xi}(Z))$.
(9) $6\aleph_{m_\xi}(6\aleph_{m_\xi}(Z)) = 6\aleph_{m_\xi}(Z)$.
(10) $6\aleph^{m_\xi}(Z) = 6\aleph_{m_\xi}(6\aleph^{m_\xi}(Z))$.

Proof.

(1) There is $\mathcal{M}_{m_\xi}(\mu) \sqsubseteq Z$ for each $\mu \in Z$ if $\mu \in 6\aleph_{m_\xi}(Z)$. Then, for every $\mu \in \Sigma - [\Sigma - Z]$, there exists $\mathcal{M}_{m_\xi}(\mu)$ such that $\mathcal{M}_{m_\xi}(\mu) \sqcap [\Sigma - Z] = \phi$. So, $\mu \notin 6\aleph^{m_\xi}(\Sigma - Z)$, $\mu \in \Sigma - [6\aleph^{m_\xi}(\Sigma - Z)]$. Therefore, $6\aleph m_\xi(Z) = \Sigma - [6\aleph^{m_\xi}(\Sigma - Z)] = 6\aleph^{m_\xi}(Z^c))^c$.

(2) $6\aleph m_\xi(\phi) = \sqcap\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqcap \Sigma \neq \phi\}^c = \phi$.

(3) If $Z_1 \sqsubseteq Z_2$, then $6\aleph_{m_\xi}(Z_1) = [\sqcup\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqcap Z_1^c \neq \phi\}]^c \sqsubseteq [\sqcup\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqcap Z_2^c \neq \phi\}]^c = 6\aleph_{m_\xi}(Z_2)$.

(4) From Definition 19, we have $6\aleph_{m_\xi}(Z_1 \sqcap Z_2) = \sqcap[\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqcap (Z_1 \sqcap Z_2)^c \neq \phi\}]^c$. Then, $\mathcal{M}_{m_\xi}(\mu) \sqsubseteq Z_1$ and $\mathcal{M}_{m_\xi}(\mu) \sqsubseteq Z_2$ because of $(Z_1 \sqcap Z_2) \sqsubseteq Z_1$, and $(Z_1 \sqcap Z_2) \sqsubseteq Z_2$. Hence, $6\aleph_{m_\xi}(Z_1 \sqcap Z_2) \sqsubseteq 6\aleph_{m_\xi}(Z_1)$, and $6\aleph_{m_\xi}(Z_1 \sqcap Z_2) \sqsubseteq 6\aleph_{m_\xi}(Z_2)$. On the other hand, $6\aleph_{m_\xi}(Z_1) \sqcap 6\aleph_{m_\xi}(Z_2) = \sqcap[\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqcap Z_1^c \neq \phi\}]^c$, and $6\aleph_{m_\xi}(Z_1) \sqcap 6\aleph_{m_\xi}(Z_2) = \sqcap[\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqsubseteq Z_2^c \neq \phi\}]^c$. i.e., $6\aleph_{m_\xi}(Z_1) \sqcap 6\aleph_{m_\xi}(Z_2) = \sqcap[\{\mathcal{M}_{m_\xi}(\mu) : \mathcal{M}_{m_\xi}(\mu) \sqsubseteq (Z_1 \sqcap Z_2)^c \neq \phi\}]^c = 6\aleph_{m_\xi}(Z_1 \sqcap Z_2) = 6\aleph_{m_\xi}(Z_1) \sqcap 6\aleph_{m_\xi}(Z_2)$.

(5) One can prove following similar lines to those in (4).

(6) By (2) and Definition 19, we obtain the proof.

By Definition 19, the proofs (7), (8), (9), and (10) are obtained. □

Example 15. *Following Example 7, take $m_\xi = m_{\xi_1}$. If $Z_1 = \{\nu\}$ and $Z_2 = \{\lambda\}$, then $6\aleph_{m_\xi}(Z_1) = \phi$ and $6\aleph_{m_\xi}(Z_2) = \phi$. So, $6\aleph_{m_\xi}(Z_1) \sqcup 6\aleph_{m_\xi}(Z_2) = \phi$. But, $6\aleph_{m_\xi}(Z_1 \sqcup Z_2) = \{\nu, \lambda\}$. Therefore, $6\aleph_{m_\xi}(Z_1) \sqcup 6\aleph_{m_\xi}(Z_2) \neq 6\aleph_{m_\xi}(Z_1 \sqcup Z_2)$.*

Proposition 11. *Let (Σ, ζ) be a generalized approximation space, and $Z, Z_1, Z_2 \sqsubseteq \Sigma$. The following claims are accurate.*
(1) $6\aleph^{m_\xi}(Z) = (6\aleph_{m_\xi}(Z^c))^c$.
(2) $6\aleph^{m_\xi}(\phi) = \phi$.
(3) If $Z_1 \sqsubseteq Z_2$, then $6\aleph^{m_\xi}(Z_1) \sqsubseteq 6\aleph^{m_\xi}(Z_2)$.
(4) $6\aleph^{m_\xi}(Z_1 \sqcap Z_2) \sqsubseteq 6\aleph^{m_\xi}(Z_1) \sqcap 6\aleph^{m_\xi}(Z_2)$.
(5) $6\aleph^{m_\xi}(Z_1) \sqcup 6\aleph^{m_\xi}(Z_2) = 6\aleph^{m_\xi}(Z_1 \sqcup Z_2)$.
(6) $6\aleph^{m_\xi}(\Sigma) = \Sigma$.
(7) $Z \sqsubseteq 6\aleph^{m_\xi}(Z)$.
(8) $6\aleph^{m_\xi}(6\aleph_{m_\xi}(Z)) \sqsubseteq Z$.
(9) $6\aleph^{m_\xi}(6\aleph^{m_\xi}(Z)) = 6\aleph^{m_\xi}(Z)$.
(10) $6\aleph_{m_\xi}(Z) = 6\aleph^{m_\xi}(6\aleph_{m_\xi}(Z))$.

Proof. It is analogous to the proof of Proposition 10. □

The equality relation of (4) in Proposition 11 need not be satisfied, as the next example elucidates.

Example 16. *Following Example 7, take $m_\xi = m_{\xi_1}$. If $Z_1 = \{\mu, \kappa\}$ and $Z_2 = \{\lambda, \kappa\}$, then $6\aleph^{m_\xi}(Z_1) = \{\mu, \nu, \lambda, \kappa\}$ and $6\aleph^{m_\xi}(Z_2) = \{\mu, \nu, \lambda, \kappa\}$, and $6\aleph^{m_\xi}(Z_1 \sqcap Z_2) = \{\kappa\}$. So, $6\aleph^{m_\xi}(Z_1 \sqcap Z_2) \neq 6\aleph^{m_\xi}(Z_1) \sqcap 6\aleph^{m_\xi}(Z_2)$.*

5. Comparison between Yao's Method [9,10] and the Present Method

The major purpose of this section is to compare our approximations to Yao's in order to demonstrate the good performance of the current models in reducing the boundary region by increasing lower approximations and decreasing upper approximations. Using Example 17, we will compare Yao's technique [9,10] (see Definitions 10 and 12) with our proposed method (see Definitions 17 and 19).

Example 17. *Let $\Sigma = \{\mu, \nu, \lambda, \kappa\}$ and $\zeta = \{(\mu, \nu), (\nu, \nu), (\nu, \lambda), (\lambda, \kappa)\}$.*

As in Definition 3 (i.e., Definition 10 when $m_\xi = m_{\xi_1}$), we shall utilize Yao's method [9,10] to calculate the type 2 lower approximations, kind 2 upper approximations, kind 2 m_ξ-neighborhood boundary region, and kind 2m_ξ-neighborhood accuracy measure of subsets Z of Σ. Then, we shall utilize our method displayed in Definition 17 to calculate the kind 5 lower approximations, kind 5 upper approximations, kind 5 m_ξ-neighborhood boundary region, and kind 5m_ξ-neighborhood accuracy measure of every subset of Σ (these calculations are provided in Table 3). As in Definition 4 (i.e., Definition 12 when $m\xi = m\xi_1$), we shall utilize Yao's method [9,10] to calculate the kind 3 lower approximations, kind 3 upper approximations, kind 3 m_ξ-neighborhood boundary region, and kind 3 m_ξ-neighborhood accuracy measure of every subset of Σ. Additionally, we shall utilize our method displayed in Definition 19 to calculate the kind 5 lower approximations, kind 5 upper approximations, kind 6 m_ξ-neighborhood boundary region, and kind 6m_ξ-neighborhood accuracy measure of every subset of Σ (these calculations are provided in Table 4).

In Tables 3 and 4, one can notice the differences between Yao's methods [9,10] and the current methods in terms of approximation operators (lower and upper), the size of boundary regions, and values of accuracy. For instance, if we take $\{\mu, \kappa\}$ from Table 3, the kind 2 m_ξ-neighborhood boundary and kind 2 $m\xi$-neighborhood accuracy produced by the methods of Yao [9,10] according to Definition 3 (i.e., Definition 10 when $m_\xi = m_{\xi_1}$) are $\{\mu\}$ and $\frac{1}{2}$, respectively, while their counterparts obtained from adhesion neighborhoods given in Definition 17 are, respectively, ϕ and 1. Furthermore, if we take $\{\lambda\}$ from Table 4, the kind 6 $m\xi$-adhesion neighborhood boundary and kind 6 m_ξ-adhesion neighborhood accuracy according to Definition 19 are, respectively, ϕ and 1, whereas their counterparts produced by the methods of Yao [9,10] (as in Definition 4) (i.e., Definition 12 when $m\xi = m\xi_4$) are, respectively, $\{\mu, \nu, \lambda\}$ and 0.

The types 5 and 6 of lower and upper approximations based on m_ξ-adhesion neighborhood space are obtained to increase the accuracy measure and decrease the boundary region of set Z, as can be seen from Tables 3 and 4. As a result, in this work, Definitions 17 and 19 (i.e., Definitions 10 and 12 when $m_\xi = m_{\xi_4}$) produce results that are superior to Yao's approach [9,10] (as defined in Definitions 3 and 4).

Table 3. Comparison of our technique in Definitions 10 and 17 with Yao's approach [9,10].

	$2\aleph_{m_{\xi_1}}(Z)$	$2\aleph^{m_{\xi_1}}(Z)$	$2B_{m_{\xi_1}}(Z)$	$2A_{m_{\xi_1}}(Z)$	$5\aleph_{m_{\xi_1}}(Z)$	$5\aleph^{m_{\xi_1}}(Z)$	$5B_{m_{\xi_1}}(Z)$	$5A_{m_{\xi_1}}(Z)$
$\{\mu\}$	ϕ	$\{\mu\}$	$\{\mu\}$	0	$\{\mu\}$	$\{\mu\}$	ϕ	1
$\{\nu\}$	ϕ	$\{\mu,\nu,\lambda\}$	$\{\mu,\nu,\lambda\}$	0	ϕ	$\{\nu,\lambda\}$	$\{\nu,\lambda\}$	0
$\{\lambda\}$	ϕ	$\{\mu,\nu,\lambda\}$	$\{\mu,\nu,\lambda\}$	0	ϕ	$\{\nu,\lambda\}$	$\{\nu,\lambda\}$	0
$\{\kappa\}$	$\{\kappa\}$	$\{\mu,\kappa\}$	$\{\mu\}$	$\frac{1}{2}$	$\{\kappa\}$	$\{\kappa\}$	ϕ	1
$\{\mu,\nu\}$	ϕ	$\{\mu,\nu,\lambda\}$	$\{\mu,\nu,\lambda\}$	0	$\{\mu\}$	$\{\mu,\nu,\lambda\}$	$\{\nu,\lambda\}$	$\frac{1}{3}$
$\{\mu,\lambda\}$	ϕ	$\{\mu,\nu,\lambda\}$	$\{\mu,\nu,\lambda\}$	0	$\{\mu\}$	$\{\mu,\nu,\lambda\}$	$\{\nu,\lambda\}$	$\frac{1}{3}$
$\{\mu,\kappa\}$	$\{\kappa\}$	$\{\mu,\kappa\}$	$\{\mu\}$	$\frac{1}{2}$	$\{\mu,\kappa\}$	$\{\mu,\kappa\}$	ϕ	1
$\{\nu,\lambda\}$	$\{\nu,\lambda\}$	$\{\mu,\nu,\lambda\}$	$\{\mu\}$	$\frac{2}{3}$	$\{\nu,\lambda\}$	$\{\nu,\lambda\}$	ϕ	1
$\{\nu,\kappa\}$	$\{\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu,\nu,\lambda\}$	$\frac{1}{4}$	$\{\kappa\}$	$\{\nu,\lambda,\kappa\}$	$\{\nu,\lambda\}$	$\frac{1}{3}$
$\{\lambda,\kappa\}$	$\{\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu,\nu,\lambda\}$	$\frac{1}{4}$	$\{\kappa\}$	$\{\nu,\lambda,\kappa\}$	$\{\nu,\lambda\}$	$\frac{1}{3}$
$\{\mu,\nu,\lambda\}$	$\{\nu,\lambda\}$	$\{\mu,\nu,\lambda\}$	$\{\mu\}$	$\frac{2}{3}$	$\{\mu,\nu,\lambda\}$	$\{\mu,\nu,\lambda\}$	ϕ	1
$\{\mu,\nu,\kappa\}$	$\{\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu,\nu,\lambda\}$	$\frac{1}{4}$	$\{\mu,\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\nu,\lambda\}$	$\frac{1}{2}$
$\{\nu,\lambda,\kappa\}$	$\{\nu,\lambda,\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu\}$	$\frac{3}{4}$	$\{\nu,\lambda,\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu\}$	$\frac{3}{4}$
$\{\mu,\lambda,\kappa\}$	$\{\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu,\nu,\lambda\}$	$\frac{1}{4}$	$\{\mu,\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\nu,\lambda\}$	$\frac{1}{2}$
Σ	$\{\nu,\lambda,\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu\}$	$\frac{3}{4}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	ϕ	1

Table 4. Comparison of our technique in Definitions 12 and 19 with Yao's approach [9,10].

	$3\aleph_{m_{\xi_1}}(Z)$	$3\aleph^{m_{\xi_1}}(Z)$	$3B_{m_{\xi_1}}(Z)$	$3A_{m_{\xi_1}}(Z)$	$6\aleph_{m_{\xi_1}}(Z)$	$6\aleph^{m_{\xi_1}}(Z)$	$6B_{m_{\xi_1}}(Z)$	$6A_{m_{\xi_1}}(Z)$
$\{\mu\}$	ϕ	$\{\mu,\nu,\lambda\}$	$\{\mu,\nu,\lambda\}$	0	ϕ	$\{\mu,\nu\}$	$\{\mu,\nu\}$	0
$\{\nu\}$	ϕ	$\{\mu,\nu,\lambda\}$	$\{\mu,\nu,\lambda\}$	0	ϕ	$\{\mu,\nu\}$	$\{\mu,\nu\}$	0
$\{\lambda\}$	ϕ	$\{\mu,\nu,\lambda\}$	$\{\mu,\nu,\lambda\}$	0	$\{\lambda\}$	$\{\lambda\}$	ϕ	1
$\{\kappa\}$	$\{\kappa\}$	$\{\kappa\}$	ϕ	1	$\{\kappa\}$	$\{\kappa\}$	ϕ	1
$\{\mu,\nu\}$	ϕ	$\{\mu,\nu,\lambda\}$	$\{\mu,\nu,\lambda\}$	0	$\{\mu,\nu\}$	$\{\mu,\nu\}$	ϕ	1
$\{\mu,\lambda\}$	$\{\mu\}$	$\{\mu,\nu,\lambda\}$	$\{\nu,\lambda\}$	$\frac{1}{3}$	$\{\lambda\}$	$\{\mu,\nu,\lambda\}$	$\{\nu,\lambda\}$	$\frac{1}{3}$
$\{\mu,\kappa\}$	$\{\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu\}$	$\frac{1}{4}$	$\{\kappa\}$	$\{\mu,\nu,\kappa\}$	$\{\mu,\nu\}$	$\frac{1}{3}$
$\{\nu,\lambda\}$	ϕ	$\{\mu,\nu,\lambda\}$	$\{\mu,\nu,\lambda\}$	0	$\{\lambda\}$	$\{\mu,\nu,\lambda\}$	$\{\mu,\nu\}$	$\frac{1}{3}$
$\{\nu,\kappa\}$	$\{\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu,\nu,\lambda\}$	$\frac{1}{4}$	$\{\kappa\}$	$\{\mu,\nu,\kappa\}$	$\{\mu,\nu\}$	$\frac{1}{3}$
$\{\lambda,\kappa\}$	$\{\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu,\nu,\lambda\}$	$\frac{1}{4}$	$\{\kappa\}$	$\{\nu,\lambda,\kappa\}$	$\{\nu,\lambda\}$	$\frac{1}{3}$
$\{\mu,\nu,\lambda\}$	$\{\mu,\nu,\lambda\}$	$\{\mu,\nu,\lambda\}$	ϕ	1	$\{\lambda,\kappa\}$	$\{\lambda,\kappa\}$	ϕ	1
$\{\mu,\nu,\kappa\}$	$\{\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu,\nu,\lambda\}$	$\frac{1}{4}$	$\{\mu,\nu,\kappa\}$	$\{\mu,\nu,\kappa\}$	ϕ	1
$\{\nu,\lambda,\kappa\}$	$\{\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu,\nu,\lambda\}$	$\frac{1}{4}$	$\{\lambda,\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu,\nu\}$	$\frac{1}{2}$
$\{\mu,\lambda,\kappa\}$	$\{\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu,\nu,\lambda\}$	$\frac{1}{4}$	$\{\lambda,\kappa\}$	$\{\mu,\nu,\lambda,\kappa\}$	$\{\mu,\nu\}$	$\frac{1}{2}$
Σ	Σ	$\{\nu,\lambda,\kappa\}$	ϕ	1	Σ	Σ	ϕ	1

6. Conclusions

The rough set theory has recently been widely used in a variety of fields. As a result, improvements to its main concepts have been made by many researchers and scholars, particularly in the domain of decreasing the boundary region and increasing the accuracy of decision-making. Contributing to this area of research, we have presented a generalization for two kinds of rough set models based on m_ξ-neighborhood systems. We integrated the ideas of adhesion neighborhood and m_ξ-neighborhood systems to introduce three novel rough set models. These types of generalized rough approximation spaces are structured using the property of symmetry between interior and closure topological operators and their counterparts of lower and upper rough approximations.

The main characterizations and features of these models were scrutinized, and the advantages of the proposed models to improve the output of lower and upper approximations are given. Additionally, the relationships between them are provided, and their relationships with some foregoing models were researched with the assistance of some illustrative examples. The relationships and comparisons presented herein demonstrate the importance of the proposed models to address various situations with higher accuracy of measure.

This work can be continued in future by investigating the presented models from the topological standpoint. Additionally, these models can be hybridized with other uncertainty instruments, such as fuzzy sets, vague sets and rough sets.

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