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2-Absorbing Vague Weakly Complete Γ-Ideals in Γ-Rings

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Abstract: The aim of this study is to provide a generalization of prime vague Γ-ideals in Γ-rings by introducing non-symmetric 2-absorbing vague weakly complete Γ-ideals of commutative Γ-rings. A novel algebraic structure of a primary vague Γ-ideal of a commutative Γ-ring is presented by 2-absorbing weakly complete primary ideal theory. The approach of non-symmetric 2-absorbing K-vague Γ-ideals of Γ-rings are examined and the relation between a level subset of 2-absorbing vague weakly complete Γ-ideals and 2-absorbing Γ-ideals is given. The image and inverse image of a 2-absorbing vague weakly complete Γ-ideal of a Γ-ring and 2-absorbing K-vague Γ-ideal of a Γ-ring are studied and a 1-1 inclusion-preserving correspondence theorem is given. A vague quotient Γ-ring of R induced by a 2-absorbing vague weakly complete Γ-ideal of a 2-absorbing Γ-ring is characterized, and a diagram is obtained that shows the relationship between these concepts with a 2-absorbing Γ-ideal.

Keywords: 2-absorbing ideal; vague weakly complete Γ-ideal; 2-absorbing K-vague Γ-ideal; radical

MSC: 03E72; 08A72

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1. Introduction

As it is known, the study of symmetric rings and, in particular, symmetric ideals plays an important role in ring theory and module theory. Symmetric rings have many applications in all sciences with symmetries. However, sometimes, we cannot use symmetric notions in our theory. For instances of non-symmetric cases, Badawi suggested and developed the idea of a 2-absorbing ideal which corresponds to the universality of the prime ideal in [1] and presented it based on the existing structure in [2–6]. Many authors have extended this idea considerably (see, e.g., [7–11]).

Zadeh, in [12], studied the concept of a fuzzy set. Symmetry is presented in many works involving fuzzy sets and fuzzy systems. Whenever a human is involved in design of a fuzzy system, they naturally tend to opt for symmetric features. The most common examples are the fuzzy membership functions and linguistic terms that are often designed symmetrically and regularly distributed over the universe of discourse. Until now, the most efficient way to better describe symmetry is to use mathematical tools offered by the group and ring theory and their respective fuzzy structures. After the introduction of the notion of fuzzy sets, Rosenfeld [13] perused it to attach fuzzy ideas to algebraic forms. Then, it had been categorized by many authors, such as Liu [14] who turned to the perception of a fuzzy ideal. Nobusawa [15] expressed the idea of a Γ-ring that it tends to be more inclusive than with a ring. Barnes [16] relaxed the requirements in Nobusawa's theory about the term of Γ-rings. Kyuno [17] clarified the structure of Γ-rings and implemented divergent rationalizations interrelated to units in ring theory and also to other works written on this idea [18–20]. In fuzzy commutative algebra (where symmetry refers to the property of the operations of addition and multiplication satisfying the commutative law), prime ideals are the adequate weight structures. Dutta and Chanda [21] proposed fuzzy prime ideals in Γ-rings. Ersoy [22] specified fuzzy semiprime ideals in Γ-rings. Darani [23] indicated the notion of a 2-absorbing ideal based on L-fuzzy structures and constructed thought-provoking outcomes on these concepts. Then, Darani and Hashempoor [24] applied the idea of 2-absorbing ideals on a fuzzy semiring. Elkettani and Kasem [25] merged the ideas of 2-absorbing Γ-ideal and δ-primary ideal, and also suggested simulation issues concerning these notions. Sönmez et al. [26] extended 2-absorbing primary ideals of fuzzy set theory in the context of commutative rings and made some relations between these 2-absorbing ideals on fuzzy primary algebras and 2-absorbing ideals on classical primary algebras.

Hanoon et al. [27] investigated nearly 2-absorbing fuzzy submodules and weakly nearly 2-absorbing fuzzy submodules under the class of multiplication fuzzy modules. Yiarayong [28] introduced the notion of 2-absorbing bipolar fuzzy ideals and strongly 2-absorbing bipolar fuzzy ideals in LA-semigroups, and gave some characterizations of quasi-strongly 2-absorbing bipolar fuzzy ideals on an LA-semigroup by bipolar fuzzy points. Sharma et al. [29] studied a generalization of intuitionistic fuzzy primary ideals in Γ-rings by introducing intuitionistic fuzzy 2-absorbing primary ideals. Mandal [30] described the notion of 2-absorbing fuzzy ideals in commutative semirings and explored some of its fundamental results. Nimbhorkar et al. [31] suggested a fuzzy weakly 2-absorbing ideal of a lattice.

Some notable recent developments in fuzzy set theory can be found, for example, in artificial intelligence [32], supply chain [33], symmetry [34], computational intelligence [35], and some of them use fuzzy set theory in algebraic systems with different applications such as civil engineering [36–39], etc., where both symmetry and asymmetry have occurred.

To a certain extent, vague sets are more suitable than fuzzy rules for structuring fuzzy knowledge. In practice, it turned out that it was necessary to illustrate a vague term and its ambiguity as an intriguing worth-exploring subject. Gau and Buehrer [40] first identified the evaluability of vague sets as diversification of a fuzzy set approach, and vague sets are generally considered to classify perspective fuzzy sets. Ren et al. [41] defined a vague ring and a vague ideal based on the vague binary operation. Sezer [42] introduced the concepts of vague subring, vague ideal, vague prime ideal, and vague maximal ideal. Yin et al. [43] applied the notion of vague soft sets to hemiring theory. Davvaz et al. [44] reviewed a vague subsemigroup, a vague bi-ideal, and a vague $(1, 2)$-ideal in a Γ-semigroup. Bhaskar et al. [45] studied the notion of sum and direct sum of vague ideals of a near-ring. Baghernejad et al. [46] applied the notion of vague sets to multigraphs. Ragamayi et al. [47] were interested in lattice vague ideals of a Γ-near ring. Bhargavi [48] studied the notions of a translational invariant vague set of a Γ-semiring and units, associates, and prime elements concerning a vague set. Gahlot et al. [49] offered interval-valued vague ideals in Γ-near-rings. For more information, see [50–54].

The study of a 2-absorbing vague weakly complete Γ-ideal for a vague set has several motivations including:

(a) **The need to extend classical algebraic concepts to the domain of vague sets:**
As we have discussed with the other topics, the study of the 2-absorbing vague weakly complete Γ-ideal is motivated by the need to understand classical algebraic concepts. In this case, the focus is on 2-absorbing vague weakly complete Γ-ideals; i.e., a type

of ideal found in a certain class of groups and rings. The extension of this field is important for developing a more comprehensive theory of vague algebraic structures.

(b) **A desire to develop a broader theory of vague sets:**
As we mentioned with regard to other studies, 2-absorbing vague weakly complete Γ-ideals on vague sets are a part of a broader effort to develop a broader theory of vague sets. This theory can be used from the point of view of generalizing fuzzy set structures. Furthermore, for a vague set, a 2-absorbing vague weakly complete Γ-ideal is an important type of ideal given in the literature in the theory of Γ-rings. These ideals are also important for development of vague Γ-rings. This theory can be used to extend other structures in a wide range of applications.

(c) **The potential applications of vague sets in various fields:**
The study of a 2-absorbing vague weakly complete Γ-ideal has potential applications in some fields such as graph theory, lattice theory, and semigroups.

It is notable that there is no study on 2-absorbing vague weakly complete Γ-ideals, while in other literatures, we see vague weakly complete Γ-ideals of Γ-rings. Therefore, to fill this gap in the literature, in this study, we introduce the notion of a 2-absorbing vague weakly complete Γ-ideal. This work presents an interesting algebraic structure of a primary vague Γ-ideal of a commutative Γ-ring by using the 2-absorbing weakly complete primary ideal theory. The major contributions of this work are stated as follows:

(1) The notion of prime vague weakly complete Γ-ideals and 2-absorbing vague weakly complete Γ-ideals in a Γ-ring are presented and their algebraic properties are given.
(2) The notion of prime K-vague Γ-ideals and 2-absorbing K-vague Γ-ideals of a Γ-ring are defined and some theorems in relation to them are proposed. The relation between a level subset of a 2-absorbing vague weakly complete Γ-ideal and a 2-absorbing Γ-ideal is presented.
(3) The notion of prime K-vague Γ-ideals, primary K-vague Γ-ideals, 2-absorbing K-vague ideals, 2-absorbing primary vague weakly complete Γ-ideals, and 2-absorbing primary K-vague ideals of $\Re$ are suggested and various properties of them are investigated.
(4) A novel image and inverse image of 2-absorbing vague weakly complete Γ-ideals of a Γ-ring and 2-absorbing K-vague Γ-ideals of a Γ-ring is presented.
(5) A 1-1 inclusion-preserving correspondence theorem is obtained about these algebraic structures.
(6) A vague quotient Γ-ring of R induced by a 2-absorbing vague weakly complete Γ-ideal is characterized.
(7) A diagram that transitions the relationship between these concepts with the notion of the 2-absorbing Γ-ideal is given.

The organization of this paper is as follows: In Section 2, fundamental concepts of vague set, 2-absorbing, and Γ-rings are presented. In Section 3, we have defined 2-absorbing vague weakly complete Γ-ideals, and some properties of them are provided. Section 4 introduces the concept of a 2-absorbing K-vague Γ-ideal. Section 5 proposes the notion of 2-absorbing primary vague weakly complete Γ-ideals and an image and inverse image of them under homomorphisms. Section 6 presents 2-absorbing primary K-vague Γ-ideal. Section 7 offers a vague quotient Γ-ring of $\Re$ induced by a 2-absorbing vague weakly complete Γ-ideal. The conclusions are given in Section 8.

2. Preliminaries

In the present part of the paper, we first give the required fundamental concepts and properties. $\Re$ is an Abelian Γ-ring with the identity element $1 \neq 0$, and $L = [0,1]$ denotes the complete lattice throughout the paper.

Now, we will define Γ-ring, where symmetry refers to the property of the operations of addition and Γ-multiplication satisfying the commutative law and distributive law.

Definition 1 ([55])**.** *Consider additive groups $\Re$ and Γ with the Abelian property. If this type of mapping exists,*

$$\begin{aligned} \Re \times \Gamma \times \Re &\to \Re, \\ (\theta, \gamma, \varphi) &\mapsto \theta\gamma\varphi, \end{aligned}$$

satisfying the following conditions,

1. $(\theta + \varphi)\gamma\phi = \theta\gamma\phi + \varphi\gamma\phi$,
2. $\theta\gamma(\varphi + \phi) = \theta\gamma\varphi + \theta\gamma\phi$,
3. $\theta(\gamma + \zeta)\varphi = \theta\gamma\varphi + \theta\zeta\varphi$,
4. $\theta\gamma(\varphi\zeta\phi) = (\theta\gamma\varphi)\zeta\phi$,

for all $\theta, \varphi, \phi \in \Re$, and $\gamma, \zeta \in \Gamma$, then $\Re$ is named as a Γ-ring. Furthermore, $\Re$ is called commutative if $\theta\gamma\varphi = \varphi\gamma\theta$ for any $\theta, \varphi \in \Re$, and $\gamma \in \Gamma$.

Based on [55], a left (right) Γ-ideal of $\Re$ is a subset $\mathcal{A}$ of $\Re$ that is an additive subgroup of $\Re$ and $\Re\Gamma\mathcal{A} \subseteq \mathcal{A}(\mathcal{A}\Gamma\Re \subseteq \mathcal{A})$ given as

$$\Re\Gamma\mathcal{A} = \{\theta\gamma\varphi \mid \theta \in \Re, \gamma \in \Gamma, \varphi \in \mathcal{A}\}.$$

When $\mathcal{A}$ is a left Γ- and a right Γ-ideal, in this case, $\mathcal{A}$ is named as a Γ-ideal of $\Re$. Let $\Re$ and $\mathcal{S}$ be Γ-rings, and ρ be a function of $\Re$ into $\mathcal{S}$. In this case, ψ is called a Γ-homomorphism whenever

$$\psi(\theta + \varphi) = \rho(\theta) + \rho(\varphi),$$

and

$$\psi(\theta\gamma\varphi) = \rho(\theta)\gamma\rho(\varphi),$$

for all $\theta, \varphi \in \Re$ and $\gamma \in \Gamma$.

Based on [16], let $\Re$ be a Γ-ring. A proper ideal $\mathcal{P}$ of $\Re$ is said to be a prime Γ-ideal whenever for all pairs of Γ-ideals $\mathcal{S}$ and $\mathcal{T}$ of $\Re$,

$$\mathcal{S}\Gamma\mathcal{T} \subseteq \mathcal{P} \quad \Rightarrow \quad \mathcal{S} \subseteq \mathcal{P} \text{ or } \mathcal{T} \subseteq \mathcal{P}.$$

Based on [19], we consider the Γ-ideal $\mathcal{P}$ of $\Re$. Then the inclusion criteria are identical:

(a) $\mathcal{P}$ is a prime Γ-ideal of $\Re$;

(b) If $\theta, \varphi \in \Re$ and $\theta\Gamma\Re\Gamma\varphi \subseteq \mathcal{P}$, then $\theta \in \mathcal{P}$ or $\varphi \in \mathcal{P}$.

Then, we present the notion of 2-absorbing for ideals.

Definition 2 ([1])**.** *A proper ideal $\mathcal{I}$ of the commutative ring $\Re$ is called a 2-absorbing ideal of $\Re$ so that if $\theta, \varphi, \phi \in \Re$ and $\theta\varphi\phi \in \mathcal{I}$, then $\theta\varphi \in \mathcal{I}$ or $\theta\phi \in \mathcal{I}$ or $\varphi\phi \in \mathcal{I}$.*

A proper Γ-ideal $\mathcal{I}$ of a Γ-ring $\Re$ is called a 2-absorbing Γ-ideal (2A-Γ-ideal) of $\Re$ with this condition that if $\theta, \varphi, \phi \in \Re$, $\gamma, \zeta \in \Gamma$ and $\theta\gamma\varphi\zeta\phi \in \mathcal{I}$, then $\theta\gamma\varphi \in \mathcal{I}$ or $\theta\zeta\phi \in \mathcal{I}$ or $\varphi\zeta\phi \in \mathcal{I}$ [25]. Note that each prime Γ-ideal of $\Re$ is a 2-absorbing Γ-ideal of $\Re$ [25].

Next, we recall the idea of a vague set.

Definition 3 ([40])**.** *A vague set ω in the universe X is a pair (t_ω, f_ω) in which*

$$t_\omega : X \to [0,1] \ \text{ and } f_\omega : X \to [0,1],$$

are mappings such that for each $x \in X$, $0 \leq t_\omega(x) + f_\omega(x) \leq 1$, where t_ω and f_ω are called true and false membership mappings, respectively.

For a vague set $\omega = (t_\omega, f_\omega)$, the interval $[t_\omega(x), 1 - f_\omega(x)]$ is known as the vague value of x in ω and is denoted by $\omega(x) = (t_\omega(x), 1 - f_\omega(x))$ [40].

Based on [40], let $\omega = (t_\omega, f_\omega)$ be a vague set and the vague cut of ω is given by

$$\omega_{(\varepsilon,\kappa)} = \{x \in X : t_\omega(x) \geq \varepsilon \text{ and } 1 - f_\omega(x) \geq \kappa\},$$

for $\varepsilon, \kappa \in [0, 1]$ with $\varepsilon \leq \kappa$.

A vague set ω of $\Re$ is called a vague Γ-ring of $\Re$ if for all $\theta, \varphi \in \Re$ and $\gamma \in \Gamma$:

1. $t_\omega(\theta - \varphi) \geq \min\{t_\omega(\theta), t_\omega(\varphi)\}$ and $1 - f_\omega(\theta - \varphi) \geq \min\{1 - f_\omega(\theta), 1 - f_\omega(\varphi)\}$;
2. $t_\omega(\theta\gamma\varphi) \geq \min\{t_\omega(\theta), t_\omega(\varphi)\}$ and $1 - f_\omega(\theta\gamma\varphi) \geq \min\{1 - f_\omega(\theta), 1 - f_\omega(\varphi)\}$.

A vague set ω of $\Re$ is called a vague Γ-ideal (V-Γ-ideal) of $\Re$ if:

(i) $t_\omega(\theta - \varphi) \geq \min\{t_\omega(\theta), t_\omega(\varphi)\}$ and $1 - f_\omega(\theta - \varphi) \geq \min\{1 - f_\omega(\theta), 1 - f_\omega(\varphi)\}$;
(ii) $t_\omega(\theta\gamma\varphi) \geq \max\{t_\omega(\theta), t_\omega(\varphi)\}$ and $1 - f_\omega(\theta\gamma\varphi) \geq \max\{1 - f_\omega(\theta), 1 - f_\omega(\varphi)\}$,

for all $\theta, \varphi \in \Re$ and $\gamma \in \Gamma$.

3. 2-Absorbing Vague Weakly Complete Γ-Ideals

In the present part, we will characterize the prime vague weakly complete Γ-ideals (PVWC-Γ-ideals) and 2-absorbing vague weakly complete Γ-ideals in a Γ-ring.

Definition 4. *A vague Γ-ideal ω of $\Re$ is called a prime vague weakly complete Γ-ideal of $\Re$ if ω is a non-constant function and for all $\theta, \varphi \in \Re$ and $\gamma \in \Gamma$,*

$$t_\omega(\theta\gamma\varphi) = \max\{t_\omega(\theta), t_\omega(\varphi)\} \text{ and } 1 - f_\omega(\theta\gamma\varphi) = \max\{1 - f_\omega(\theta), 1 - f_\omega(\varphi)\}.$$

Definition 5. *Let $\omega = \langle t_\omega, 1 - f_\omega\rangle$ be a vague Γ-ideal of $\Re$. ω is called a 2-absorbing vague weakly complete Γ-ideal of $\Re$ if*

$$\omega(\theta\gamma\varphi\zeta\phi) = \omega(\theta\gamma\varphi) \text{ or } \omega(\theta\gamma\varphi\zeta\phi) = \omega(\theta\zeta\phi) \text{ or } \omega(\theta\gamma\varphi\zeta\phi) = \omega(\varphi\zeta\phi),$$

i.e.,

$$t_\omega(\theta\gamma\varphi\zeta\phi) = t_\omega(\theta\gamma\varphi) \text{ or } t_\omega(\theta\gamma\varphi\zeta\phi) = t_\omega(\theta\zeta\phi) \text{ or } t_\omega(\theta\gamma\varphi\zeta\phi) = t_\omega(\varphi\zeta\phi),$$

and

$$1 - f_\omega(\theta\gamma\varphi\zeta\phi) = 1 - f_\omega(\theta\gamma\varphi) \text{ or } 1 - f_\omega(\theta\gamma\varphi\zeta\phi) = 1 - f_\omega(\theta\zeta\phi)$$

or $1 - f_\omega(\theta\gamma\varphi\zeta\phi) = 1 - f_\omega(\varphi\zeta\phi)$ for all $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$.

Proposition 1. *Let ω be a non-constant vague Γ-ideal of $\Re$. ω is a 2-absorbing vague weakly complete Γ-ideal of $\Re$ if and only if*

$$\begin{aligned} t_\omega(\theta\gamma\varphi\zeta\phi) &= \max\{t_\omega(\theta\gamma\varphi), t_\omega(\theta\zeta\phi), t_\omega(\varphi\zeta\phi)\}, \\ 1 - f_\omega(\theta\gamma\varphi\zeta\phi) &= \max\{1 - f_\omega(\theta\gamma\varphi), 1 - f_\omega(\theta\zeta\phi), 1 - f_\omega(\varphi\zeta\phi)\}, \end{aligned}$$

for every $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$.

Theorem 1. *Every prime vague weakly complete Γ-ideal of $\Re$ is a 2-absorbing vague weakly complete Γ-ideal of $\Re$.*

Proof. Let ω be a prime vague weakly complete Γ-ideal of $\Re$. Then for every $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$,

$$t_\omega(\theta\gamma\varphi\zeta\phi) = t_\omega(\theta) \text{ or } t_\omega(\theta\gamma\varphi\zeta\phi) = t_\omega(\varphi) \text{ or } t_\omega(\theta\gamma\varphi\zeta\phi) = t_\omega(\phi),$$

and

$$1 - f_{\omega}(\theta\gamma\varphi\zeta\phi) = 1 - f_{\omega}(\theta) \text{ or } 1 - f_{\omega}(\theta\gamma\varphi\zeta\phi) = 1 - f_{\omega}(\varphi) \text{ or } 1 - f_{\omega}(\theta\gamma\varphi\zeta\phi) = 1 - f_{\omega}(\phi).$$

Suppose that

$$t_{\omega}(\theta\gamma\varphi\zeta\phi) = t_{\omega}(\theta) \text{ and } 1 - f_{\omega}(\theta\gamma\varphi\zeta\phi) = 1 - f_{\omega}(\theta).$$

By

$$t_{\omega}(\theta\gamma\varphi\zeta\phi) \geq t_{\omega}(\theta\gamma\varphi) \geq t_{\omega}(\theta) \text{ and } 1 - f_{\omega}(\theta\gamma\varphi\zeta\phi) \leq 1 - f_{\omega}(\theta\gamma\varphi) \leq 1 - f_{\omega}(\theta),$$

it follows that $t_{\omega}(\theta\gamma\varphi\zeta\phi) = t_{\omega}(\theta\gamma\varphi)$ and $1 - f_{\omega}(\theta\gamma\varphi\zeta\phi) = 1 - f_{\omega}(\theta\gamma\varphi)$. In a similar manner, we can write that if

$$t_{\omega}(\theta\gamma\varphi\zeta\phi) = t_{\omega}(\varphi) \text{ or } t_{\omega}(\theta\gamma\varphi\zeta\phi) = t_{\omega}(\phi),$$

and

$$1 - f_{\omega}(\theta\gamma\varphi\zeta\phi) = 1 - f_{\omega}(\varphi) \text{ or } 1 - f_{\omega}(\theta\gamma\varphi\zeta\phi) = 1 - f_{\omega}(\phi),$$

then

$$t_{\omega}(\theta\gamma\varphi\zeta\phi) = t_{\omega}(\varphi\zeta\phi) \text{ or } t_{\omega}(\theta\gamma\varphi\zeta\phi) = t_{\omega}(\theta\zeta\phi),$$

and

$$1 - f_{\omega}(\theta\gamma\varphi\zeta\phi) = 1 - f_{\omega}(\varphi\zeta\phi) \text{ or } 1 - f_{\omega}(\theta\gamma\varphi\zeta\phi) = 1 - f_{\omega}(\theta\zeta\phi).$$

In consequence, we find that ω is a 2-absorbing vague weakly complete Γ-ideal of $\Re$. □

Theorem 2. *Assume that ω is a vague Γ-ideal of $\Re$. Then:*

1. *ω is a 2-absorbing vague weakly complete Γ-ideal of $\Re$.*
2. *For every $\varepsilon, \kappa \in [0,1]$, the level subset $\omega_{(\varepsilon,\kappa)}$ of ω is a 2-absorbing Γ-ideal (2A-Γ-ideal) of $\Re$.*

Proof. $(1) \Rightarrow (2)$: Suppose that ω is a 2-absorbing vague weakly complete Γ-ideal of $\Re$ and let $\theta, \varphi, \phi \in \Re$, $\gamma, \zeta \in \Gamma$ and $\theta\gamma\varphi\zeta\phi \in \omega_{(\varepsilon,\kappa)}$ for some $\varepsilon, \kappa \in [0,1]$. Then

$$\begin{aligned} \max\{t_{\omega}(\theta\gamma\varphi), t_{\omega}(\theta\zeta\phi), t_{\omega}(\varphi\zeta\phi)\} &= t_{\omega}(\theta\gamma\varphi\zeta\phi) \geq \varepsilon, \\ \max\{1 - f_{\omega}(\theta\gamma\varphi), 1 - f_{\omega}(\theta\zeta\phi), 1 - f_{\omega}(\varphi\zeta\phi)\} &= 1 - f_{\omega}(\theta\gamma\varphi\zeta\phi) \geq \kappa. \end{aligned}$$

It follows that $t_{\omega}(\theta\gamma\varphi) \geq t$ or $t_{\omega}(\theta\zeta\phi) \geq \varepsilon$ or $t_{\omega}(\varphi\zeta\phi) \geq \varepsilon$, and $1 - f_{\omega}(\theta\gamma\varphi) \geq \kappa$ or $1 - f_{\omega}(\theta\zeta\phi) \geq \kappa$ or $1 - f_{\omega}(\varphi\zeta\phi) \geq \kappa$ which give that $\theta\gamma\varphi \in \omega_{(\varepsilon,\kappa)}$ or $\theta\zeta\phi \in \omega_{(\varepsilon,\kappa)}$ or $\varphi\zeta\phi \in \omega_{(\varepsilon,\kappa)}$. Thus, $\omega_{(\varepsilon,\kappa)}$ is a 2-absorbing Γ-ideal of $\Re$.

$(2) \Rightarrow (1)$: Assume that $\omega_{(t,\kappa)}$ is a 2-absorbing Γ-ideal of $\Re$ for every $\varepsilon, \kappa \in [0,1]$. For $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$, let $t_{\omega}(\theta\gamma\varphi\zeta\phi) = \varepsilon$ and $1 - f_{\omega}(\theta\gamma\varphi\zeta\phi) = \kappa$. Then $\theta\gamma\varphi\zeta\phi \in \omega_{(\varepsilon,\kappa)}$ and $\omega_{(\varepsilon,\kappa)}$ is a 2-absorbing Γ-ideal which gives $\theta\gamma\varphi \in \omega_{(\varepsilon,\kappa)}$ or $\theta\zeta\phi \in \omega_{(\varepsilon,\kappa)}$ or $\varphi\zeta\phi \in \omega_{(\varepsilon,\kappa)}$. Thus, $t_{\omega}(\theta\gamma\varphi) \geq \varepsilon$ or $t_{\omega}(\theta\zeta\phi) \geq \varepsilon$ or $t_{\omega}(\varphi\zeta\phi) \geq \varepsilon$, and $1 - f_{\omega}(\theta\gamma\varphi) \geq \kappa$ or $1 - f_{\omega}(\theta\zeta\phi) \geq \kappa$ or $1 - f_{\omega}(\varphi\zeta\phi) \geq \kappa$. It follows that $\max\{t_{\omega}(\theta\gamma\varphi), t_{\omega}(\theta\zeta\phi), t_{\omega}(\varphi\zeta\phi)\} \geq \varepsilon = t_{\omega}(\theta\gamma\varphi\zeta\phi)$ and

$$\max\{1 - f_{\omega}(\theta\gamma\varphi), 1 - f_{\omega}(\theta\zeta\phi), 1 - f_{\omega}(\varphi\zeta\phi)\} \geq \kappa = 1 - f_{\omega}(\theta\gamma\varphi\zeta\phi).$$

Moreover, since ω is a vague Γ-ideal of $\Re$, we have

$$\begin{aligned} t_{\omega}(\theta\gamma\varphi\zeta\phi) &\geq \max\{t_{\omega}(\theta\gamma\varphi), t_{\omega}(\theta\zeta\phi), t_{\omega}(\varphi\zeta\phi)\}, \\ 1 - f_{\omega}(\theta\gamma\varphi\zeta\phi) &\geq \max\{1 - f_{\omega}(\theta\gamma\varphi), 1 - f_{\omega}(\theta\zeta\phi), 1 - f_{\omega}(\varphi\zeta\phi)\}. \end{aligned}$$

Thus,

$$\begin{aligned} t_\omega(\theta\gamma\varphi\zeta\phi) &= \max\{t_\omega(\theta\gamma\varphi), t_\omega(\theta\zeta\phi), t_\omega(\varphi\zeta\phi)\} \text{ and} \\ 1 - f_\omega(\theta\gamma\varphi\zeta\phi) &= \max\{1 - f_\omega(\theta\gamma\varphi), 1 - f_\omega(\theta\zeta\phi), 1 - f_\omega(\varphi\zeta\phi)\}, \end{aligned}$$

and we find that ω is a 2-absorbing vague weakly complete Γ-ideal of $\Re$. □

Theorem 3. *Let $\psi : \Re \to \mathcal{S}$ be an onto Γ-ring homomorphism. If ω is a 2-absorbing vague weakly complete Γ-ideal of $\Re$ which is constant on $Ker\psi$, then $\psi(\omega)$ is a 2-absorbing vague weakly complete Γ-ideal of $\mathcal{S}$.*

Proof. Let us consider $\psi(\omega)(\theta\gamma\varphi\zeta\phi) \neq \psi(\omega)(\theta\gamma\varphi)$ and $\psi(1 - f_\omega)(\theta\gamma\varphi\zeta\phi) \neq \psi(\omega)(\theta\gamma\varphi)$ for any $\theta, \varphi, \phi \in \mathcal{S}$ and $\gamma, \zeta \in \Gamma$. Since ψ is an onto Γ-ring homomorphism, then

$$\psi(a) = \theta,\ \psi(b) = \varphi,\ \psi(c) = \phi \quad \exists\, a, b, c \in \Re.$$

Thus,

$$\begin{aligned} \psi(t_\omega)(\theta\gamma\varphi\zeta\phi) &= \psi(t_\omega)(\psi(a)\gamma\varphi(b)\zeta\varphi(c)) = \psi(t_\omega)(\psi(a\gamma b\zeta c)) \\ &\neq \psi(t_\omega)(\theta\gamma\varphi) = \psi(t_\omega)(\psi(a)\gamma\varphi(b)) = \psi(t_\omega)(\psi(a\gamma b)), \end{aligned}$$

and

$$\begin{aligned} \psi(1 - f_\omega)(\theta\gamma\varphi\zeta\phi) &= \psi(1 - f_\omega)(\psi(a)\gamma\varphi(b)\zeta\varphi(c)) = \psi(1 - f_\omega)(\psi(a\gamma b\zeta c)) \\ &\neq \psi(1 - f_\omega)(\theta\gamma\varphi) = \psi(1 - f_\omega)(\psi(a)\gamma\varphi(b)) = \psi(1 - f_\omega)(\psi(a\gamma b)). \end{aligned}$$

As ω is constant on *Ker* ψ, we obtain

$$\begin{aligned} \psi(t_\omega)(\psi(a\gamma b\zeta c)) &= t_\omega(a\gamma b\zeta c) \text{ and } \psi(t_\omega)(\psi(a\gamma b)) = t_\omega(a\gamma b), \\ \psi(1 - f_\omega)(\psi(a\gamma b\zeta c)) &= 1 - f_\omega(a\gamma b\zeta c) \text{ and } \psi(1 - f_\omega)(\psi(a\gamma b)) = 1 - f_\omega(a\gamma b). \end{aligned}$$

It follows that

$$\begin{aligned} \psi(t_\omega)(\psi(a\gamma b\zeta c)) &= t_\omega(a\gamma b\zeta c) \neq t_\omega(a\gamma b) = \psi(t_\omega)(\psi(a\gamma b)), \\ \psi(1 - f_\omega)(\psi(a\gamma b\zeta c)) &= 1 - f_\omega(a\gamma b\zeta c) \neq 1 - f_\omega(a\gamma b) = \psi(1 - f_\omega)(\psi(a\gamma b)). \end{aligned}$$

By the fact that ω is a 2-absorbing vague weakly complete Γ-ideal of $\Re$, then

$$\begin{aligned} t_\omega(a\gamma b\zeta c) &= \psi(t_\omega)(\psi(a)\gamma\varphi(b)\zeta\varphi(c)) = \psi(t_\omega)(\theta\gamma\varphi\zeta\phi) \\ &= t_\omega(a\zeta c) = \psi(t_\omega)(\psi(a\zeta c)) = \psi(t_\omega)(\psi(a)\zeta\varphi(c)) = \psi(t_\omega)(\theta\zeta\phi), \end{aligned}$$

and

$$\begin{aligned} 1 - f_\omega(a\gamma b\zeta c) &= \psi(1 - f_\omega)(\psi(a)\gamma\varphi(b)\zeta\varphi(c)) = \psi(1 - f_\omega)(\theta\gamma\varphi\zeta\phi) \\ &= 1 - f_\omega(a\zeta c) = \psi(1 - f_\omega)(\psi(a\zeta c)) = \psi(1 - f_\omega)(\psi(a)\zeta\varphi(c)) \\ &= \psi(1 - f_\omega)(\theta\zeta\phi). \end{aligned}$$

So, we obtain $\psi(t_\omega)(\theta\gamma\varphi\zeta\phi) = \psi(t_\omega)(\theta\zeta\phi)$ and $\psi(1 - f_\omega)(\theta\gamma\varphi\zeta\phi) = \psi(1 - f_\omega)(\theta\zeta\phi)$ or

$$\begin{aligned} t_\omega(a\gamma b\zeta c) &= \psi(t_\omega)(\psi(a)\gamma\varphi(b)\zeta\varphi(c)) = \psi(t_\omega)(\theta\gamma\varphi\zeta\phi) \\ &= t_\omega(b\zeta c) = \psi(t_\omega)(\psi(b\zeta c)) = \psi(t_\omega)(\psi(b)\zeta\varphi(c)) = \psi(t_\omega)(\varphi\zeta\phi), \end{aligned}$$

and

$$\begin{aligned} 1 - f_\omega(a\gamma b\zeta c) &= \psi(1 - f_\omega)(\psi(a)\gamma\varphi(b)\zeta\varphi(c)) = \psi(1 - f_\omega)(\theta\gamma\varphi\zeta\phi) \\ &= 1 - f_\omega(b\zeta c) = \psi(1 - f_\omega)(\psi(b\zeta c)) = \psi(1 - f_\omega)(\psi(b)\zeta\varphi(c)) \\ &= \psi(1 - f_\omega)(\varphi\zeta\phi). \end{aligned}$$

We obtain $\psi(t_\omega)(\theta\gamma\varphi\zeta\phi) = \psi(t_\omega)(\varphi\zeta\phi)$ and $\psi(1 - f_\omega)(\theta\gamma\varphi\zeta\phi) = \psi(1 - f_\omega)(\varphi\zeta\phi)$. Consequently, $\psi(\omega)$ is a 2-absorbing vague weakly complete Γ-ideal of $\mathcal{S}$. □

Theorem 4. *Let $\psi : \Re \to \mathcal{S}$ be a homomorphism of Γ-ring. If $\eta = \langle t_\eta, 1 - f_\eta \rangle$ is a 2-absorbing vague weakly complete Γ-ideal of $\mathcal{S}$, then $\psi^{-1}(\eta)$ is a 2-absorbing vague weakly complete Γ-ideal of $\Re$.*

Proof. Assume that $\psi^{-1}(t_\eta)(\theta\gamma\varphi\zeta\phi) \neq \psi^{-1}(t_\eta)(\theta\gamma\varphi)$ and

$$\psi^{-1}(1 - f_\eta)(\theta\gamma\varphi\zeta\phi) \neq \psi^{-1}(1 - f_\eta)(\theta\gamma\varphi),$$

for any $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$. Then,

$$\begin{aligned} \psi^{-1}(t_\eta)(\theta\gamma\varphi\zeta\phi) &= t_\eta(\psi(\theta\gamma\varphi\zeta\phi)) = t_\eta(\psi(\theta)\gamma\varphi(\varphi)\zeta\varphi(\phi)) \\ &\neq \psi^{-1}(t_\eta)(\theta\gamma\varphi) = t_\eta(\psi(\theta\gamma\varphi)) = t_\eta(\psi(\theta)\gamma\varphi(\varphi)), \end{aligned}$$

and

$$\begin{aligned} \psi^{-1}(1 - f_\eta)(\theta\gamma\varphi\zeta\phi) &= 1 - f_\eta(\psi(\theta\gamma\varphi\zeta\phi)) = 1 - f_\eta(\psi(\theta)\gamma\varphi(\varphi)\zeta\varphi(\phi)) \\ &\neq \psi^{-1}(1 - f_\eta)(\theta\gamma\varphi) = 1 - f_\eta(\psi(\theta\gamma\varphi)) = 1 - f_\eta(\psi(\theta)\gamma\varphi(\varphi)). \end{aligned}$$

We know that η is a 2-absorbing vague weakly complete Γ-ideal of $\mathcal{S}$. We obtain

$$\begin{aligned} t_\eta(\psi(\theta)\gamma\varphi(\varphi)\zeta\varphi(\phi)) &= \psi^{-1}(t_\eta)(\theta\gamma\varphi\zeta\phi) \\ &= t_\eta(\psi(\theta)\zeta\varphi(\phi)) = t_\eta(\psi(\theta\zeta\phi)) \\ &= \psi^{-1}(t_\eta)(\theta\zeta\phi), \end{aligned}$$

and

$$\begin{aligned} 1 - f_\eta(\psi(\theta)\gamma\varphi(\varphi)\zeta\varphi(\phi)) &= \psi^{-1}(1 - f_\eta)(\theta\gamma\varphi\zeta\phi) \\ &= 1 - f_\eta(\psi(\theta)\zeta\varphi(\phi)) = 1 - f_\eta(\psi(\theta\zeta\phi)) \\ &= \psi^{-1}(1 - f_\eta)(\theta\zeta\phi), \end{aligned}$$

or

$$\begin{aligned} t_\eta(\psi(\theta)\gamma\varphi(\varphi)\zeta\varphi(\phi)) &= \psi^{-1}(t_\eta)(\theta\gamma\varphi\zeta\phi) \\ &= t_\eta(\psi(\varphi)\zeta\varphi(\phi)) = t_\eta(\psi(\varphi\zeta\phi)) \\ &= \psi^{-1}(t_\eta)(\varphi\zeta\phi), \end{aligned}$$

and

$$\begin{aligned} 1 - f_\eta(\psi(\theta)\gamma\varphi(\varphi)\zeta\varphi(\phi)) &= \psi^{-1}(1 - f_\eta)(\theta\gamma\varphi\zeta\phi) \\ &= 1 - f_\eta(\psi(\varphi)\zeta\varphi(\phi)) = 1 - f_\eta(\psi(\varphi\zeta\phi)) \\ &= \psi^{-1}(1 - f_\eta)(\varphi\zeta\phi). \end{aligned}$$

Therefore, $\psi^{-1}(\eta)$ is a 2-absorbing vague weakly complete Γ-ideal of $\Re$. □

Corollary 1. *Let ψ be a Γ-ring homomorphism from $\Re$ onto $\mathcal{S}$. ψ induces a one to one inclusion preserving correspondence between the 2-absorbing vague weakly complete Γ-ideal of $\mathcal{S}$ in such a way that if ω is a 2-absorbing vague weakly complete Γ-ideal of $\Re$ and it is constant on Kerψ, then $\psi(\omega)$ is the corresponding 2-absorbing vague weakly complete Γ-ideal of $\mathcal{S}$, and if η is a 2-absorbing vague weakly complete Γ-ideal of $\mathcal{S}$, in that case, $\psi^{-1}(\eta)$ is the corresponding 2-absorbing vague weakly complete Γ-ideal of $\Re$.*

4. 2-Absorbing K-Vague Γ-Ideals

In this section, we will introduce the notion of prime K-vague Γ-ideal (PKV-Γ-ideal) and 2-absorbing K-vague Γ-ideal (2AKV-Γ-ideal) of a Γ-ring.

Definition 6. *Let ω be a vague Γ-ideal of $\Re$. Then ω is called a prime K-vague Γ-ideal of $\Re$ if*

$$\omega(\theta\gamma\varphi) = \omega(0) \textit{ implies that } \omega(\theta) = \omega(0) \textit{ or } \omega(\varphi) = \omega(0),$$

for $\theta, \varphi \in \Re$ and $\gamma, \zeta \in \Gamma$.

Definition 7. *Let ω be a vague Γ-ideal of $\Re$. Then, ω is called a 2-absorbing K-vague Γ-ideal of $\Re$ if for all $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$,*

$$\omega(\theta\gamma\varphi\zeta\phi) = \omega(0) \textit{ implies } \omega(\theta\gamma\varphi) = \omega(0) \textit{ or } \omega(\theta\zeta\phi) = \omega(0) \textit{ or } \omega(\varphi\zeta\phi) = \omega(0).$$

Theorem 5. *Every 2-absorbing vague weakly complete Γ-ideal of $\Re$ is a 2-absorbing K-vague Γ-ideal of $\Re$.*

Proof. Assume that ω is a 2-absorbing vague weakly complete Γ-ideal of $\Re$. If $\omega(\theta\gamma\varphi\zeta\phi) = \omega(0)$ for any $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$, then we write

$$\begin{aligned} \omega(0) &= \omega(\theta\gamma\varphi\zeta\phi) \leq \omega(\theta\gamma\varphi) \leq \omega(0) \text{ or} \\ \omega(0) &= \omega(\theta\gamma\varphi\zeta\phi) \leq \omega(\theta\zeta\phi) \leq \omega(0) \text{ or} \\ \omega(0) &= \omega(\theta\gamma\varphi\zeta\phi) \leq \omega(\varphi\zeta\phi) \leq \omega(0). \end{aligned}$$

Since ω is a 2-absorbing vague weakly complete Γ-ideal of $\Re$, the following equalities hold

$$\omega(\theta\gamma\varphi) = \omega(0) \text{ or } \omega(\theta\zeta\phi) = \omega(0) \text{ or } \omega(\varphi\zeta\phi) = \omega(0).$$

It follows that ω is a 2-absorbing K-vague Γ-ideal of $\Re$. □

The following example shows that the converse of the above theorem is not true.

Example 1. *Let $\Re = \mathbb{Z}$ and $\Gamma = 2\mathbb{Z}$. So, $\Re$ is a Γ-ring. We consider the vague Γ-ideal ω of $\Re$ as*

$$t_\omega(\theta) = \begin{cases} 1, & \textit{if } \theta = 0; \\ 1/4, & \textit{if } \theta \in 36\mathbb{Z} - \{0\}; \\ 1/5, & \textit{if } \theta \in \mathbb{Z} - 36\mathbb{Z}, \end{cases} \textit{ and } f_\omega(\theta) = \begin{cases} 0, & \textit{if } \theta = 0; \\ 1/4, & \textit{if } \theta \in 36\mathbb{Z} - \{0\}; \\ 1/5, & \textit{if } \theta \in \mathbb{Z} - 36\mathbb{Z}. \end{cases}$$

Then, ω is a 2-absorbing K-vague Γ-ideal. However, for $\gamma, \zeta \in 2\mathbb{Z}$, we have

$$1 - f_\omega(3\gamma3\zeta6) = 1 - 1/5 = 4/5 > 1/4 = \max\{f_\omega(3\gamma3), f_\omega(3\zeta6), f_\omega(3\zeta6)\}.$$

Thus, ω is not a 2-absorbing vague weakly complete Γ-ideal of $\Re$.

Theorem 6. *Every prime K-vague Γ-ideal of $\Re$ is a 2-absorbing K-vague Γ-ideal of $\Re$.*

Proof. Let ω be a prime K-vague Γ-ideal of $\Re$. Then for every $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$, the equality $\omega(\theta\gamma\varphi\zeta\phi) = \omega(0)$ implies that

$$\omega(\theta) = \omega(0) \text{ or } \omega(\varphi) = \omega(0) \text{ or } \omega(\phi) = \omega(0).$$

Suppose that $\omega(\theta) = \omega(0)$. Then by

$$\omega(0) = \omega(\theta) \leq \omega(\theta\gamma\varphi) \leq \omega(\theta\gamma\varphi\zeta\phi) = \omega(0),$$

we obtain $\omega(\theta\gamma\varphi) = \omega(0)$ or in a similar manner, we can deduce that $\omega(\theta\zeta\phi) = \omega(0)$ or $\omega(\varphi\zeta\phi) = \omega(0)$. As a result, ω is a 2-absorbing K-vague Γ-ideal of $\Re$. □

Theorem 7. *Let $\psi : \Re \to \mathcal{S}$ be an onto Γ-ring homomorphism. If ω is a 2-absorbing K-vague Γ-ideal of $\Re$ which is constant on $Ker\psi$, then $\psi(\omega)$ is a 2-absorbing K-vague Γ-ideal of $\mathcal{S}$.*

Proof. The proof is nearly identical to that of Theorem 3 and so the proof is omitted. □

Theorem 8. *Let $\psi : \Re \to \mathcal{S}$ be a Γ-ring homomorphism. If η is a 2-absorbing K-vague Γ-ideal of $\mathcal{S}$, then $\psi^{-1}(\eta)$ is a 2-absorbing K-vague Γ-ideal of $\Re$.*

Proof. We omit the proof, since it is similar to the proof of Theorem 4. □

Corollary 2. *Let ψ be a Γ-ring homomorphism from $\Re$ onto $\mathcal{S}$. ψ induces a 1-1 inclusion preserving correspondence between 2-absorbing K-vague Γ-ideal of $\mathcal{S}$ in such a way that if ω is a 2-absorbing K-vague Γ-ideal of $\Re$ which is constant on $Ker\psi$, then $\psi(\omega)$ is the corresponding 2-absorbing K-vague Γ-ideal of $\mathcal{S}$, and if η is a 2-absorbing K-vague Γ-ideal of $\mathcal{S}$, then $\psi^{-1}(\eta)$ is the corresponding 2-absorbing K-vague Γ-ideal of $\Re$.*

5. 2-Absorbing Primary Vague Weakly Complete Γ-Ideals

Let ω be a vague Γ-ideal of $\Re$. Then $\sqrt{\omega}$ is called the radical of ω and is characterized by

$$\sqrt{\omega} = \left(\sqrt{t_\omega}, \sqrt{f_\omega}\right),$$

where

$$\sqrt{t_\omega}(\theta) = \bigvee_{k\geq 1} t_\omega\left(\theta^k\right) \text{ and } \sqrt{f_\omega}(\theta) = \bigwedge_{k\geq 1} f_\omega\left(\theta^k\right).$$

In the following, we give the definition of primary vague weakly complete Γ-ideal (PRVWC-Γ-ideal) of $\Re$.

Definition 8. *Let ω be a vague Γ-ideal of $\Re$. ω is called a primary vague weakly complete Γ-ideal of $\Re$ if for $\forall\theta, \varphi \in \Re$ and $\gamma \in \Gamma$,*

$$\omega(\theta\gamma\varphi) \leq \omega(\theta) \text{ or } \omega(\theta\gamma\varphi) \leq \sqrt{\omega}(\varphi),$$

i.e.,

$$\begin{aligned} t_\omega(\theta\gamma\varphi) &\leq t_\omega(\theta) \text{ and } 1 - f_\omega(\theta\gamma\varphi) \leq f_\omega(\theta) \text{ , or} \\ t_\omega(\theta\gamma\varphi) &\leq \sqrt{t_\omega}(\varphi) \text{ and } 1 - f_\omega(\theta\gamma\varphi) \leq 1 - \sqrt{f_\omega}(\varphi). \end{aligned}$$

Proposition 2. *A vague Γ-ideal ω of $\Re$ is a primary vague weakly complete Γ-ideal of $\Re$ if ω is a non-constant function and for all $\theta, \varphi \in \Re$ and $\gamma \in \Gamma$,*

$$\begin{aligned} t_\omega(\theta\gamma\varphi) &= \max\left\{t_\omega(\theta), \sqrt{t_\omega}(\varphi)\right\} \text{ and} \\ 1 - f_\omega(\theta\gamma\varphi) &= \max\left\{1 - f_\omega(\theta), 1 - \sqrt{f_\omega}(\varphi)\right\}. \end{aligned}$$

Now, we give the definition of 2-absorbing primary vague weakly complete Γ-ideal (2APVWC-Γ-ideal) of $\Re$

Definition 9. *Let ω be a vague Γ-ideal of $\Re$. ω is called a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$ if*

$$\omega(\theta\gamma\varphi\zeta\phi) = \omega(\theta\gamma\varphi) \ \text{ or } \omega(\theta\gamma\varphi\zeta\phi) = \sqrt{\omega}(\theta\zeta\phi) \ \text{ or } \ \omega(\theta\gamma\varphi\zeta\phi) = \sqrt{\omega}(\varphi\zeta\phi),$$

i.e.,

$$\begin{aligned} t_\omega(\theta\gamma\varphi\zeta\phi) &= t_\omega(\theta\gamma\varphi),\ 1 - f_\omega(\theta\gamma\varphi\zeta\phi) = 1 - f_\omega(\theta\gamma\varphi), \text{ or} \\ t_\omega(\theta\gamma\varphi\zeta\phi) &= \sqrt{t_\omega}(\theta\phi),\ 1 - f_\omega(\theta\gamma\varphi\zeta\phi) = 1 - \sqrt{f_\omega}(\theta\zeta\phi), \text{ or} \\ t_\omega(\theta\gamma\varphi\zeta\phi) &= \sqrt{t_\omega}(\varphi\phi),\ 1 - f_\omega(\theta\gamma\varphi\zeta\phi) = 1 - \sqrt{f_\omega}(\varphi\zeta\phi), \end{aligned}$$

for all $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$.

Proposition 3. *Let ω be a non-constant vague Γ-ideal of $\Re$. ω is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$ if and only if*

$$\begin{aligned} t_\omega(\theta\gamma\varphi\zeta\phi) &= \max\{t_\omega(\theta\gamma\varphi), \sqrt{t_\omega}(\theta\zeta\phi), \sqrt{t_\omega}(\varphi\zeta\phi)\}, \\ 1 - f_\omega(\theta\gamma\varphi\zeta\phi) &= \max\Big\{1 - f_\omega(\theta\gamma\varphi), 1 - \sqrt{f_\omega}(\theta\zeta\phi), 1 - \sqrt{f_\omega}(\varphi\zeta\phi)\Big\}, \end{aligned}$$

for every $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$.

Theorem 9. *Every 2-absorbing vague weakly complete Γ-ideal of $\Re$ is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$.*

Proof. The proof is clear. □

The converse of Theorem 9 is not generally satisfied, as we can observe in the following example.

Example 2. *Let $\Re = \mathbb{Z}$ and $\Gamma = 2\mathbb{Z}$. We define the vague ideal ω of $\mathbb{Z}$ as*

$$\omega(\theta) = \left\{ \begin{array}{ll} (1,1) & ,\theta \in 27\mathbb{Z} \\ (0,0) & ,\theta \notin 27\mathbb{Z} \end{array} \right\}.$$

Assume that $\omega(\theta\gamma\varphi\zeta\phi) > \omega(\theta\gamma\varphi)$ for any $\theta, \varphi, \phi \in \mathbb{Z}$ and $\gamma, \zeta \in \Gamma$. Hence, $\omega(\theta\gamma\varphi\zeta\phi) = (1,1)$ and $\omega(\theta\gamma\varphi) = (0,0)$. It follows that $\theta\gamma\varphi\zeta\phi \in 27\mathbb{Z}$ and $\theta\gamma\varphi \notin 27\mathbb{Z}$. Since $27\mathbb{Z}$ is a primary ideal of $\mathbb{Z}$, we find that $\phi \in 3\mathbb{Z}$. From the definition of radical ω,

$$\sqrt{\omega}(\theta) = \left\{ \begin{array}{ll} (1,1) & ,\theta \in 3\mathbb{Z} \\ (0,0) & ,\theta \notin 3\mathbb{Z} \end{array} \right\},$$

we get $\sqrt{\omega}(\theta\gamma\varphi) = \sqrt{\omega}(\varphi\zeta\phi) = (1,1)$. Thus, $\sqrt{\omega}(\theta\gamma\varphi\zeta\phi) \geq \omega(\theta\gamma\varphi\zeta\phi)$ or $\sqrt{\omega}(\varphi\zeta\phi) \geq \omega(\theta\gamma\varphi\zeta\phi)$. Therefore, ω is a 2-absorbing primary vague weakly complete Γ-ideal. However, since $\omega(3.2.3.2.3) = (1,1) > (0,0) = \omega(3.2.3)$, it follows that ω is not a 2-absorbing vague weakly complete Γ-ideal.

Theorem 10. *Every primary vague weakly complete Γ-ideal of $\Re$ is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$.*

Proof. Let ω be a primary vague weakly complete Γ-ideal of $\Re$. Then for every $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$,

$$t_\omega(\theta\gamma\varphi\zeta\phi) = t_\omega(\theta) \text{ or } t_\omega(\theta\gamma\varphi\zeta\phi) = \sqrt{t_\omega}(\varphi) \text{ or } t_\omega(\theta\gamma\varphi\zeta\phi) = \sqrt{t_\omega}(\phi),$$
$$1 - f_\omega(\theta\gamma\varphi\zeta\phi) = f_\omega(\theta) \text{ or } 1 - f_\omega(\theta\gamma\varphi\zeta\phi) = 1 - \sqrt{f_\omega}(\varphi) \text{ or } 1 - f_\omega(\theta\gamma\varphi\zeta\phi) = 1 - \sqrt{f_\omega}(\phi).$$

Assume that $t_\omega(\theta\gamma\varphi\zeta\phi) = t_\omega(\theta)$ and $1 - f_\omega(\theta\gamma\varphi\zeta\phi) = 1 - f_\omega(\theta)$. By $t_\omega(\theta\gamma\varphi\zeta\phi) \geq t_\omega(\theta\gamma\varphi) \geq t_\omega(\theta)$ and $1 - f_\omega(\theta\gamma\varphi\zeta\phi) \geq 1 - f_\omega(\theta\gamma\varphi) \geq 1 - f_\omega(\theta)$, it follows that $t_\omega(\theta\gamma\varphi\zeta\phi) = t_\omega(\theta\gamma\varphi)$ and $1 - f_\omega(\theta\gamma\varphi\zeta\phi) = 1 - f_\omega(\theta\gamma\varphi)$. In a similar way, we can show that if $t_\omega(\theta\gamma\varphi\zeta\phi) = \sqrt{t_\omega}(\varphi)$ or $t_\omega(\theta\gamma\varphi\zeta\phi) = \sqrt{t_\omega}(\phi)$, and $1 - f_\omega(\theta\gamma\varphi\zeta\phi) = 1 - \sqrt{f_\omega}(\varphi)$ or $1 - f_\omega(\theta\gamma\varphi\zeta\phi) = 1 - \sqrt{f_\omega}(\phi)$, then $t_\omega(\theta\gamma\varphi\zeta\phi) = \sqrt{t_\omega}(\varphi\zeta\phi)$ or $t_\omega(\theta\gamma\varphi\zeta\phi) = \sqrt{t_\omega}(\theta\zeta\phi)$, and $1 - f_\omega(\theta\gamma\varphi\zeta\phi) = 1 - \sqrt{f_\omega}(\varphi\zeta\phi)$ or $1 - f_\omega(\theta\gamma\varphi\zeta\phi) = 1 - \sqrt{f_\omega}(\theta\zeta\phi)$. This implies that ω is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$. □

Theorem 11. *Let ω be a vague Γ-ideal of $\Re$. Then the following items are equivalent:*

1. *ω is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$;*
2. *For each $\varepsilon, \kappa \in [0, 1]$, the level subset $\omega_{(\varepsilon,\kappa)}$ of ω is a 2-absorbing primary Γ-ideal (2AP-Γ-ideal) of $\Re$.*

Proof. $(1) \Rightarrow (2)$: Take ω as a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$ and let $\theta, \varphi, \phi \in \Re$, $\gamma, \zeta \in \Gamma$ and $\theta\gamma\varphi\zeta\phi \in \omega_{(\varepsilon,\kappa)}$ for some $\varepsilon, \kappa \in [0, 1]$. Then

$$\begin{aligned}
\max\{t_\omega(\theta\gamma\varphi), \sqrt{t_\omega}(\theta\zeta\phi), \sqrt{t_\omega}(\varphi\zeta\phi)\} &= t_\omega(\theta\gamma\varphi\zeta\phi) \geq \varepsilon, \\
\max\Big\{1 - f_\omega(\theta\gamma\varphi), 1 - \sqrt{f_\omega}(\theta\zeta\phi), 1 - \sqrt{f_\omega}(\varphi\zeta\phi)\Big\} &= 1 - f_\omega(\theta\gamma\varphi\zeta\phi) \geq \kappa.
\end{aligned}$$

It follows that

$$\begin{aligned}
t_\omega(\theta\gamma\varphi) &\geq \varepsilon \text{ or } \sqrt{t_\omega}(\theta\zeta\phi) \geq \varepsilon \text{ or } \sqrt{t_\omega}(\varphi\zeta\phi) \geq \varepsilon \text{ and} \\
1 - f_\omega(\theta\gamma\varphi) &\geq \kappa \text{ or } 1 - \sqrt{f_\omega}(\theta\zeta\phi) \geq \kappa \text{ or } 1 - \sqrt{f_\omega}(\varphi\zeta\phi) \geq \kappa,
\end{aligned}$$

which give $\theta\gamma\varphi \in \omega_{(\varepsilon,\kappa)}$ or $\theta\zeta\phi \in \sqrt{\omega}_{(\varepsilon,\kappa)}$ or $\varphi\zeta\phi \in \sqrt{\omega}_{(\varepsilon,\kappa)}$. Thus, $\omega_{(\varepsilon,\kappa)}$ is a 2-absorbing primary Γ-ideal of $\Re$.

$(2) \Rightarrow (1)$: Take $\omega_{(\varepsilon,\kappa)}$ as a 2-absorbing primary Γ-ideal of $\Re$ for each $\varepsilon, \kappa \in [0, 1]$. For $\theta, \varphi, \phi \in \Re$, $\gamma, \zeta \in \Gamma$, let $t_\omega(\theta\gamma\varphi\zeta\phi) = \varepsilon$ and $1 - f_\omega(\theta\gamma\varphi\zeta\phi) = \kappa$. Then $\theta\gamma\varphi\zeta\phi \in \omega_{(\varepsilon,\kappa)}$ and $\omega_{(\varepsilon,\kappa)}$ is 2-absorbing primary Γ-ideal. This gives $\theta\gamma\varphi \in \omega_{(\varepsilon,\kappa)}$ or $\theta\zeta\phi \in \sqrt{\omega}_{(\varepsilon,\kappa)}$ or $\varphi\zeta\phi \in \sqrt{\omega}_{(\varepsilon,\kappa)}$. Thus, $t_\omega(\theta\gamma\varphi) \geq \varepsilon$ or $\sqrt{t_\omega}(\theta\zeta\phi) \geq \varepsilon$ or $\sqrt{t_\omega}(\varphi\zeta\phi) \geq \varepsilon$, and $1 - f_\omega(\theta\gamma\varphi) \geq \kappa$ or $1 - \sqrt{f_\omega}(\theta\zeta\phi) \geq \kappa$ or $1 - \sqrt{f_\omega}(\varphi\zeta\phi) \geq \kappa$. These results follow that $\max\{t_\omega(\theta\gamma\varphi), \sqrt{t_\omega}(\theta\zeta\phi), \sqrt{t_\omega}(\varphi\zeta\phi)\} \geq \varepsilon = t_\omega(\theta\gamma\varphi\zeta\phi)$, and

$$\max\Big\{1 - f_\omega(\theta\gamma\varphi), 1 - \sqrt{f_\omega}(\theta\zeta\phi), 1 - \sqrt{f_\omega}(\varphi\zeta\phi)\Big\} \geq \kappa = 1 - f_\omega(\theta\gamma\varphi\zeta\phi).$$

Furthermore, since ω is a primary vague Γ-ideal of $\Re$, we have

$$\begin{aligned}
t_\omega(\theta\gamma\varphi\zeta\phi) &\geq \max\{t_\omega(\theta\gamma\varphi), \sqrt{t_\omega}(\theta\zeta\phi), \sqrt{t_\omega}(\varphi\zeta\phi)\}, \\
1 - f_\omega(\theta\gamma\varphi\zeta\phi) &\geq \max\Big\{1 - f_\omega(\theta\gamma\varphi), 1 - \sqrt{f_\omega}(\theta\zeta\phi), 1 - \sqrt{f_\omega}(\varphi\zeta\phi)\Big\}.
\end{aligned}$$

Hence

$$\begin{aligned}
t_\omega(\theta\gamma\varphi\zeta\phi) &= \max\{t_\omega(\theta\gamma\varphi), \sqrt{t_\omega}(\theta\zeta\phi), \sqrt{t_\omega}(\varphi\zeta\phi)\} \text{ and} \\
1 - f_\omega(\theta\gamma\varphi\zeta\phi) &= \max\Big\{1 - f_\omega(\theta\gamma\varphi), 1 - \sqrt{f_\omega}(\theta\zeta\phi), 1 - \sqrt{f_\omega}(\varphi\zeta\phi)\Big\}.
\end{aligned}$$

Therefore, ω is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$. □

Theorem 12. *If ω is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$, then $\sqrt{\omega}$ is a 2-absorbing vague weakly complete Γ-ideal of $\Re$.*

Proof. If ω is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$, then according to the previous theorem, we find that $\omega_{(\varepsilon,\kappa)}$ is a 2-absorbing primary Γ-ideal of $\Re$ for each $\varepsilon, \kappa \in [0,1]$. Since $\omega_{(\varepsilon,\kappa)}$ is a 2-absorbing primary Γ-ideal of $\Re$, then $\sqrt{\omega_{(\varepsilon,\kappa)}} = \sqrt{\omega}_{(\varepsilon,\kappa)}$ is a 2-absorbing Γ-ideal of $\Re$. Since $\sqrt{\omega}_{(\varepsilon,\kappa)}$ is a 2-absorbing Γ-ideal of $\Re$, we find that $\sqrt{\omega}$ is a 2-absorbing vague weakly complete Γ-ideal of $\Re$. Thus, we reach to this fact that $\sqrt{\omega}$ is a 2-absorbing vague weakly complete Γ-ideal of $\Re$. □

Let $\psi : \Re \to \mathcal{S}$ be a Γ-ring homomorphism, ω be a vague Γ-ideal of $\Re$ such that ω is constant on $Ker\psi$ and η be a vague Γ-ideal of $\mathcal{S}$. Then,

$$\sqrt{\psi(\omega)} = \psi(\sqrt{\omega}) \text{ and } \sqrt{\psi^{-1}(\eta)} = \psi^{-1}(\sqrt{\eta}).$$

Theorem 13. *Let $\psi : \Re \to \mathcal{S}$ be an onto Γ-ring homomorphism. If ω is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$ which is constant on $Ker\psi$, then $\psi(\omega)$ is a 2-absorbing primary vague weakly complete Γ-ideal of $\mathcal{S}$.*

Proof. Consider $\psi(\omega)(\theta\gamma\varphi\zeta\phi) \neq \psi(\omega)(\theta\gamma\varphi)$ for each $\theta, \varphi, \phi \in \mathcal{S}$ and $\gamma, \zeta \in \Gamma$. Since ψ is an onto Γ-ring homomorphism, then

$$\psi(a) = \theta,\ \psi(b) = \varphi,\ \psi(c) = \phi \quad \exists\, a, b, c \in \Re.$$

Thus

$$\begin{aligned}\psi(t_\omega)(\theta\gamma\varphi\zeta\phi) &= \psi(t_\omega)(\psi(a)\gamma\psi(b)\zeta\psi(c)) = \psi(t_\omega)(\psi(a\gamma b\zeta c)) \\ &\neq \psi(t_\omega)(\theta\gamma\varphi) = \psi(t_\omega)(\psi(a)\gamma\psi(b)) = \psi(t_\omega)(\psi(a\gamma b)),\end{aligned}$$

and

$$\begin{aligned}\psi(1-f_\omega)(\theta\gamma\varphi\zeta\phi) &= \psi(1-f_\omega)(\psi(a)\gamma\psi(b)\zeta\psi(c)) = \psi(1-f_\omega)(\psi(a\gamma b\zeta c)) \\ &\neq \psi(1-f_\omega)(\theta\gamma\varphi) = \psi(1-f_\omega)(\psi(a)\gamma\psi(b)) = \psi(1-f_\omega)(\psi(a\gamma b)).\end{aligned}$$

We know that ω is constant on $Ker\psi$, so

$$\begin{aligned}\psi(t_\omega)(\psi(a\gamma b\zeta c)) &= t_\omega(a\gamma b\zeta c) \text{ and } \psi(t_\omega)(\psi(a\gamma b)) = t_\omega(a\gamma b), \\ \psi(1-f_\omega)(\psi(a\gamma b\zeta c)) &= 1 - f_\omega(a\gamma b\zeta c) \text{ and } \psi(1-f_\omega)(\psi(a\gamma b)) = 1 - f_\omega(a\gamma b).\end{aligned}$$

It follows that

$$\begin{aligned}\psi(t_\omega)(\psi(a\gamma b\zeta c)) &= t_\omega(a\gamma b\zeta c) \neq t_\omega(a\gamma b) = \psi(t_\omega)(\psi(a\gamma b)), \\ \psi(1-f_\omega)(\psi(a\gamma b\zeta c)) &= 1 - f_\omega(a\gamma b\zeta c) \neq 1 - f_\omega(a\gamma b) = \psi(1-f_\omega)(\psi(a\gamma b)).\end{aligned}$$

Since ω is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$, we have

$$\begin{aligned}t_\omega(a\gamma b\zeta c) &= \psi(t_\omega)(\psi(a)\gamma\psi(b)\zeta\psi(c)) = \psi(t_\omega)(\theta\gamma\varphi\zeta\phi) \\ &= \sqrt{t_\omega}(a\zeta c) = \psi(\sqrt{t_\omega})(\psi(a\zeta c)) \\ &= \psi(\sqrt{t_\omega})(\psi(a)\zeta\psi(c)) = \psi(\sqrt{t_\omega})(\theta\zeta\phi),\end{aligned}$$

and

$$\begin{aligned} 1-f_\omega(a\gamma b\zeta c) &= \psi(1-f_\omega)(\psi(a)\gamma\psi(b)\zeta\psi(c)) = \psi(1-f_\omega)(\theta\gamma\varphi\zeta\phi) \\ &= 1-\sqrt{f_\omega}(a\zeta c) = \psi\left(1-\sqrt{f_\omega}\right)(\psi(a\zeta c)) \\ &= \psi\left(1-\sqrt{f_\omega}\right)(\psi(a)\zeta\psi(c)) = \psi\left(1-\sqrt{f_\omega}\right)(\theta\zeta\phi). \end{aligned}$$

So, we obtain

$$\begin{aligned} \psi(t_\omega)(\theta\gamma\varphi\zeta\phi) &= \psi(\sqrt{t_\omega})(\theta\zeta\phi) = \sqrt{\psi(t_\omega)}(\theta\zeta\phi), \text{ and} \\ \psi(1-f_\omega)(\theta\gamma\varphi\zeta\phi) &= \psi\left(1-\sqrt{f_\omega}\right)(\theta\zeta\phi) = \sqrt{\psi(1-f_\omega)}(\theta\zeta\phi), \end{aligned}$$

or

$$\begin{aligned} t_\omega(a\gamma b\zeta c) &= \psi(t_\omega)(\psi(a)\gamma\psi(b)\zeta\psi(c)) = \psi(t_\omega)(\theta\gamma\varphi\zeta\phi) \\ &= \sqrt{t_\omega}(b\zeta c) = \psi(\sqrt{t_\omega})(\psi(b\zeta c)) \\ &= \psi(\sqrt{t_\omega})(\psi(b)\zeta\psi(c)) = \psi(\sqrt{t_\omega})(\varphi\zeta\phi), \end{aligned}$$

and

$$\begin{aligned} 1-f_\omega(a\gamma b\zeta c) &= \psi(1-f_\omega)(\psi(a)\gamma\psi(b)\zeta\psi(c)) = \psi(1-f_\omega)(\theta\gamma\varphi\zeta\phi) \\ &= 1-\sqrt{f_\omega}(b\zeta c) = \psi\left(1-\sqrt{f_\omega}\right)(\psi(b\zeta c)) \\ &= \psi\left(1-\sqrt{f_\omega}\right)(\psi(b)\zeta\psi(c)) = \psi\left(1-\sqrt{f_\omega}\right)(\varphi\zeta\phi). \end{aligned}$$

Thus, we obtain $\psi(t_\omega)(\theta\gamma\varphi\zeta\phi) = \psi(\sqrt{t_\omega})(\varphi\zeta\phi) = \sqrt{\psi(t_\omega)}(\varphi\zeta\phi)$ and $\psi(1-f_\omega)(\theta\gamma\varphi\zeta\phi)$ $= \psi(1-f_\omega)(\varphi\zeta\phi) = \sqrt{\psi(1-f_\omega)}(\varphi\zeta\phi)$. As a deduction, $\psi(\omega)$ is a 2-absorbing primary vague weakly complete Γ-ideal of $\mathcal{S}$. □

Theorem 14. *Consider a Γ-ring homomorphism $\psi : \Re \to \mathcal{S}$. If $\eta = \langle t_\eta, f_\eta \rangle$ is a 2-absorbing primary vague weakly complete Γ-ideal of $\mathcal{S}$, then $\psi^{-1}(\eta)$ is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$.*

Proof. Suppose that $\psi^{-1}(t_\eta)(\theta\gamma\varphi\zeta\phi) > \psi^{-1}(t_\eta)(\theta\gamma\varphi)$ and

$$\psi^{-1}(1-f_\eta)(\theta\gamma\varphi\zeta\phi) > \psi^{-1}(1-f_\eta)(\theta\gamma\varphi),$$

for each $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$. Then,

$$\begin{aligned} \psi^{-1}(t_\eta)(\theta\gamma\varphi\zeta\phi) &= t_\eta(\psi(\theta\gamma\varphi\zeta\phi)) = t_\eta(\psi(\theta)\gamma\psi(\varphi)\zeta\psi(\phi)) \\ &> \psi^{-1}(t_\eta)(\theta\gamma\varphi) = t_\eta(\psi(\theta\gamma\varphi)) = t_\eta(\psi(\theta)\gamma\psi(\varphi)), \end{aligned}$$

and

$$\begin{aligned} \psi^{-1}(1-f_\eta)(\theta\gamma\varphi\zeta\phi) &= 1-f_\eta(\psi(\theta\gamma\varphi\zeta\phi)) = 1-f_\eta(\psi(\theta)\gamma\psi(\varphi)\zeta\psi(\phi)) \\ &> \psi^{-1}(1-f_\eta)(\theta\gamma\varphi) = 1-f_\eta(\psi(\theta\gamma\varphi)) = 1-f_\eta(\psi(\theta)\gamma\psi(\varphi)). \end{aligned}$$

Since η is a 2-absorbing primary vague weakly complete Γ-ideal of $\mathcal{S}$, we have

$$\begin{aligned} \psi^{-1}(t_\eta)(\theta\gamma\varphi\zeta\phi) &= t_\eta(\psi(\theta)\gamma\psi(\varphi)\zeta\psi(\phi)) \\ &\le \sqrt{t_\eta}(\psi(\theta)\zeta\psi(\phi)) = \sqrt{t_\eta}(\psi(\theta\zeta\phi)) \\ &= \sqrt{\psi^{-1}(t_\eta)}(\theta\zeta\phi), \end{aligned}$$

and

$$\begin{aligned}\psi^{-1}(1-f_\eta)(\theta\gamma\varphi\zeta\phi) &= 1-f_\eta(\psi(\theta)\gamma\psi(\varphi)\zeta\psi(\phi))\\ &\leq 1-\sqrt{f_\eta}(\psi(\theta)\zeta\psi(\phi)) = 1-\sqrt{f_\eta}(\psi(\theta\zeta\phi))\\ &= \sqrt{\psi^{-1}(1-f_\eta)}(\theta\zeta\phi),\end{aligned}$$

or

$$\begin{aligned}\psi^{-1}(t_\eta)(\theta\gamma\varphi\zeta\phi) &= t_\eta(\psi(\theta)\gamma\psi(\varphi)\zeta\psi(\phi))\\ &\leq \sqrt{t_\eta}(\psi(\varphi)\zeta\psi(\phi)) = \sqrt{t_\eta}(\psi(\varphi\zeta\phi))\\ &= \sqrt{\psi^{-1}(t_\eta)}(\varphi\zeta\phi),\end{aligned}$$

and

$$\begin{aligned}\psi^{-1}(1-f_\eta)(\theta\gamma\varphi\zeta\phi) &= 1-f_\eta(\psi(\theta)\gamma\psi(\varphi)\zeta\psi(\phi))\\ &\leq 1-\sqrt{f_\eta}(\psi(\varphi)\zeta\psi(\phi)) = 1-\sqrt{f_\eta}(\psi(\varphi\zeta\phi))\\ &= \sqrt{\psi^{-1}(1-f_\eta)}(\varphi\zeta\phi).\end{aligned}$$

Therefore, $\psi^{-1}(\eta)$ is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$. □

Corollary 3. *Let ψ be a Γ-ring homomorphism from $\Re$ onto $\mathcal{S}$. ψ induces a 1-1 inclusion preserving correspondence between 2-absorbing primary vague weakly complete Γ-ideal of $\mathcal{S}$ in such a way that if ω is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$ which is constant on Kerψ, then $\psi(\omega)$ is the corresponding 2-absorbing primary vague weakly complete Γ-ideal of $\mathcal{S}$, and if η is a 2-absorbing primary vague weakly complete Γ-ideal of $\mathcal{S}$, then $\psi^{-1}(\eta)$ is the corresponding 2-absorbing primary vague weakly complete Γ-ideal of $\Re$.*

6. 2-Absorbing Primary *K*-Vague Γ-Ideals

Now, we investigate the notion of the prime K-vague Γ-ideal , primary K-vague Γ-ideal (PRKV-Γ-ideal), 2-absorbing K-vague Γ-ideal, and 2-absorbing primary K-vague Γ-ideal (2APKV-Γ-ideal) of $\Re$.

Let ω be a vague Γ-ideal of $\Re$. ω is called a prime K-vague Γ-ideal of $\Re$ if

$$\omega(\theta\gamma\varphi) = \omega(0) \text{ implies that } \omega(\theta) = \omega(0) \text{ or } \omega(\varphi) = \omega(0).$$

Additionally, ω is called a primary K-vague Γ-ideal of $\Re$ if

$$\omega(\theta\gamma\varphi) = \omega(0) \text{ implies that } \omega(\theta) = \omega(0) \text{ or } \sqrt{\omega}(\varphi) = \omega(0),$$

for each $\theta, \varphi \in \Re$ and $\gamma \in \Gamma$.

Definition 10. *Let ω be a vague Γ-ideal of $\Re$. ω is called a 2-absorbing K-vague Γ-ideal of $\Re$ if*

$$\omega(\theta\gamma\varphi\zeta\phi) = \omega(0) \textit{ implies } \omega(\theta\gamma\varphi) = \omega(0) \textit{ or } \omega(\theta\zeta\phi) = \omega(0) \textit{ or } \omega(\varphi\zeta\phi) = \omega(0).$$

Furthermore, ω is called a 2-absorbing primary K-vague Γ-ideal of $\Re$ if

$$\omega(\theta\gamma\varphi\zeta\phi) = \omega(0) \textit{ implies } \omega(\theta\gamma\varphi) = \omega(0) \textit{ or} \sqrt{\omega}(\theta\zeta\phi) = \omega(0) \textit{ or} \sqrt{\omega}(\varphi\zeta\phi) = \omega(0),$$

for all $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$.

Theorem 15. *Every primary K-vague Γ-ideal of $\Re$ is a 2-absorbing primary K-vague Γ-ideal of $\Re$.*

Proof. Let ω be a primary K-vague Γ-ideal of $\Re$. In this case, we obtain

$$\omega(\theta\varphi\phi) = \omega(0) \text{ implies } \omega(\theta) = \omega(0) \text{ or } \sqrt{\omega}(\varphi) = \omega(0) \text{ or } \sqrt{\omega}(\phi) = \omega(0),$$

for each $\theta, \varphi, \phi \in \Re$. Suppose that $\omega(\theta) = \omega(0)$. Then, for all $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$, by the inequilities

$$\omega(0) = \omega(\theta) \leq \omega(\theta\gamma\varphi) \leq \omega(\theta\gamma\varphi\zeta\phi) = \omega(0),$$

we find out that $\omega(\theta\gamma\varphi) = \omega(0)$; or in a similar manner, we can find that $\sqrt{\omega}(\theta\zeta\phi) = \omega(0)$ or $\sqrt{\omega}(\varphi\zeta\phi) = \omega(0)$. Therefore, ω is a 2-absorbing primary K-vague Γ-ideal of $\Re$. □

Corollary 4. *Every 2-absorbing primary vague weakly complete Γ-ideal of $\Re$ is a 2-absorbing primary K-vague Γ-ideal of $\Re$.*

Proof. Let ω be a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$. If $\omega(\theta\gamma\varphi\zeta\phi) = \omega(0)$ for each $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$, then we obtain

$$\begin{aligned} \omega(0) &= \omega(\theta\gamma\varphi\zeta\phi) \leq \omega(\theta\gamma\varphi) \leq \omega(0) \quad \text{or} \\ \omega(0) &= \omega(\theta\gamma\varphi\zeta\phi) \leq \sqrt{\omega}(\theta\zeta\phi) \leq \omega(0) \quad \text{or} \\ \omega(0) &= \omega(\theta\gamma\varphi\zeta\phi) \leq \sqrt{\omega}(\varphi\zeta\phi) \leq \omega(0), \end{aligned}$$

since ω is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$. The corresponding solution of the result is obtained as a logical consequence

$$\omega(\theta\gamma\varphi) = \omega(0) \text{ or } \sqrt{\omega}(\theta\zeta\phi) = \omega(0) \text{ or } \sqrt{\omega}(\varphi\zeta\phi) = \omega(0).$$

We find out that ω is a 2-absorbing primary K-vague Γ-ideal of $\Re$. □

In the next example, we see that a 2-absorbing primary K-vague Γ-ideal is not a 2-absorbing primary vague weakly complete Γ-ideal.

Example 3. *Let $\Re = \mathbb{Z}$ and $\Gamma = 2\mathbb{Z}$. We define the vague Γ-ideal ω of $\mathbb{Z}$ as*

$$\omega(\theta) = \left\{ \begin{array}{ll} (1,1) & ,\theta = 0 \\ \left(\frac{1}{2}, \frac{2}{3}\right) & ,\theta \in 420\mathbb{Z} - \{0\} \\ \left(\frac{1}{3}, \frac{1}{2}\right) & ,\theta \in \mathbb{Z} - 420\mathbb{Z} \end{array} \right\}.$$

Then, ω is a 2-absorbing primary K-vague Γ-ideal. However, since

$$\omega(3.2.5.2.7) = \left(\frac{1}{2}, \frac{2}{3}\right) > \bigvee\{\omega(3.2.5), \omega(3.2.7), \omega(5.2.7)\} = \left(\frac{1}{3}, \frac{1}{2}\right),$$

or

$$\omega(3.2.5.2.7) = \left(\frac{1}{2}, \frac{2}{3}\right) > \bigvee\left\{\sqrt{\omega}(3.2.5), \sqrt{\omega}(3.2.7), \sqrt{\omega}(5.2.7)\right\} = \left(\frac{1}{3}, \frac{1}{2}\right),$$

ω is not a 2-absorbing primary vague weakly complete Γ-ideal.

Corollary 5. *Every 2-absorbing K-vague Γ-ideal of $\Re$ is a 2-absorbing primary K-vague Γ-ideal of $\Re$.*

Proof. To show that ω is a 2-absorbing primary K-vague Γ-ideal of $\Re$, suppose that $\omega(\theta\gamma\varphi\zeta\phi) = \omega(0)$ for all $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$. Since ω is a 2-absorbing K-vague Γ-ideal of $\Re$, it implies that

$$\omega(\theta\gamma\varphi) = \omega(0) or \omega(\theta\zeta\phi) = \omega(0) or \omega(\varphi\zeta\phi) = \omega(0).$$

We can always consider $n = 1$ for a 2-absorbing K-vague Γ-ideal of $\Re$. Then, this implies that ω is a 2-absorbing primary K-vague Γ-ideal of $\Re$. $\square$

In the next example, we see that the converse of Corollary 5 is not consistently valid.

Example 4. *Let $\Gamma = 2\mathbb{Z}$ and consider the vague Γ-ideal ω of $\mathbb{Z}$ as*

$$\omega(\theta) = \left\{ \begin{array}{ll} (1,1) & ,\theta \in 27\mathbb{Z} \\ (0,0) & ,\theta \notin 27\mathbb{Z} \end{array} \right\}.$$

Then, ω is a 2-absorbing primary K-vague Γ-ideal. However, since

$$\omega(3.2.3.2.3) = \sqrt{\omega}(3.2.3) = \bigvee_{k\geq 1} \omega\left(\theta^k\right) = (1,1) = \omega(0),$$

and

$$\omega(3.2.3.2.3) = (1,1) = \omega(0) \neq \omega(3.2.3) = \omega(18) = (0,0),$$

it follows that ω is not a 2-absorbing K-vague Γ-ideal of $\Re$.

Theorem 16. *Consider an onto Γ-ring homomorphism $\psi : \Re \to \mathcal{S}$. If ω is a 2-absorbing primary K-vague Γ-ideal of $\Re$ which is constant on Kerψ, then $\psi(\omega)$ is a 2-absorbing primary K-vague Γ-ideal of $\mathcal{S}$.*

Proof. The proof is similar to that of Theorem 13, and so, the proof is omitted. $\square$

Theorem 17. *Consider a Γ-ring homomorphism $\psi : \Re \to \mathcal{S}$. If ω_1 is a 2-absorbing primary K-vague Γ-ideal of $\mathcal{S}$, then $\psi^{-1}(\omega_1)$ is a 2-absorbing primary K-vague Γ-ideal of $\Re$.*

Proof. The proof is similar to that of Theorem 14, and so, the proof is omitted. $\square$

Corollary 6. *Let ψ be a Γ-ring homomorphism from $\Re$ onto $\mathcal{S}$. ψ induces a 1-1 inclusion preserving correspondence between 2-absorbing primary K-vague Γ-ideal of $\mathcal{S}$ in such a way that if ω is a 2-absorbing primary K-vague Γ-ideal of $\Re$ which is constant on Kerψ, then $\psi(\omega)$ is the corresponding 2-absorbing primary K-vague Γ-ideal of $\mathcal{S}$, and if ω_1 is a 2-absorbing primary K-vague Γ-ideal of $\mathcal{S}$, then $\psi^{-1}(\omega_1)$ is the corresponding 2-absorbing primary K-vague Γ-ideal of $\Re$.*

7. Vague Quotient Γ-Ring of $\Re$ Induced by a 2-Absorbing Vague Weakly Complete Γ-Ideal

Now, we investigate the vague quotient Γ-ring (VQ-Γ-ring) of $\Re$ induced by a 2-absorbing vague weakly complete Γ-ideal. We recall the notion of the vague quotient Γ-ring induced by a vague Γ-ideal of $\Re$. Let ω be a vague Γ-ideal of $\Re$. For each $\theta, \varphi \in \Re$, we describe a binary relation $\sim$ on $\Re$ (which is a congruence relation of $\Re$) by $\theta \sim \varphi$ if and only if

$$\omega(\theta - \varphi) = \omega(0).$$

Let $\omega[\theta] = \{\varphi \in \Re \mid \varphi \sim \theta\}$ be an equivalence class containing θ and $\Re/\omega = \{\omega[\theta] \mid \theta \in \Re\}$ be a set of all equivalence classes of $\Re$. Define

$$\omega[\theta] + \omega[\varphi] = \omega[\theta + \varphi],$$

and

$$\omega[\theta]\gamma\sigma[\varphi] = \omega[\theta\gamma\varphi],$$

for each $\theta, \varphi \in \Re, \gamma \in \Gamma$. Then, $\Re/\omega$ is a vague Γ-ring with two operations and we call it as a vague quotient Γ-ring of $\Re$ induced by the vague Γ-ideal ω.

Theorem 18. *Let ω be a non-constant vague Γ-ideal of $\Re$. Then ω is a 2-absorbing K-vague Γ-ideal of $\Re$ if and only if $\Re/\omega$ is a vague quotient Γ-ring induced by a 2-absorbing K-vague Γ-ideal of $\Re$.*

Proof. Suppose that ω is a 2-absorbing K-vague Γ-ideal of $\Re$ and let $\omega[\theta], \omega[\varphi], \omega[\phi] \in \Re/\omega$ be such that $\omega[\theta]\gamma\sigma[\varphi]\zeta\sigma[\phi] = \omega[0]$. By $\omega[\theta]\gamma\sigma[\varphi]\zeta\sigma[\phi] = \omega[\theta\gamma\varphi\zeta\phi]$, we have

$$\omega(\theta\gamma\varphi\zeta\phi) = \omega(\theta\gamma\varphi\zeta\phi - 0) = (1,1) = \omega(0).$$

Since ω is considered to be a 2-absorbing K-vague Γ-ideal of $\Re$,

$$\omega(\theta\gamma\varphi) = \omega(0) = (1,1) \text{ or } \omega(\theta\zeta\phi) = \omega(0) = (1,1) \text{ or } \omega(\varphi\zeta\phi) = \omega(0) = (1,1).$$

It follows that

$$\begin{aligned} \omega[\theta\gamma\varphi] &= \omega[\theta]\gamma\sigma[\varphi] = \omega[0] \text{ or} \\ \omega[\theta\zeta\phi] &= \omega[\theta]\zeta\sigma[\phi] = \omega[0] \text{ or} \\ \omega[\varphi\zeta\phi] &= \omega[\varphi]\zeta\sigma[\phi] = \omega[0]. \end{aligned}$$

So, $\Re/\omega$ is a vague quotient Γ-ring induced by a 2-absorbing K-vague Γ-ideal of $\Re$. Otherwise, assume that $\Re/\omega$ is a vague quotient Γ-ring induced by a 2-absorbing K-vague Γ-ideal of $\Re$, and let $\omega(\theta\gamma\varphi\zeta\phi) = \omega(0) = (1,1)$ for each $\theta, \varphi, \phi \in \Re$ and $\gamma, \zeta \in \Gamma$. Then, we have

$$\omega[\theta]\gamma\sigma[\varphi]\zeta\sigma[\phi] = \omega[\theta\gamma\varphi\zeta\phi] = \omega[0].$$

Since $\Re/\omega$ is a vague quotient Γ-ring induced by a 2-absorbing K-vague Γ-ideal of $\Re$, then

$$\omega[\theta\gamma\varphi] = \omega[0] \text{ or } \omega[\theta\zeta\phi] = \omega[0] \text{ or } \omega[\varphi\zeta\phi] = \omega[0],$$

which implies that ω is a 2-absorbing K-vague Γ-ideal of $\Re$. $\square$

Corollary 7. *Let ω be a 2-absorbing vague weakly complete Γ-ideal of $\Re$. Then $\Re/\omega$ is a vague quotient Γ-ring induced by a 2-absorbing vague weakly complete Γ-ideal of $\Re$.*

Theorem 19. *Let ω be a non-constant vague Γ-ideal of $\Re$. ω is a 2-absorbing primary K-vague Γ-ideal of $\Re$ if and only if $\Re/\omega$ is a vague quotient Γ-ring induced by a 2-absorbing primary K-vague Γ-ideal of $\Re$.*

Proof. The proof is similar to the proof of Theorem 18. $\square$

Corollary 8. *If ω is a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$, then $\Re/\omega$ is a vague quotient Γ-ring induced by a 2-absorbing primary vague weakly complete Γ-ideal of $\Re$.*

Remark 1. *The followings scheme simplifies all implications about 2-absorbing vague weakly complete Γ-ideals of $\Re$.*

$$\boxed{\begin{array}{ccccccc} PVWC\text{-}\Gamma\text{-}ideal & \Longrightarrow & PRVWC\text{-}\Gamma\text{-}ideal & \Longrightarrow & 2AVWC\text{-}\Gamma\text{-}ideal & \Longrightarrow & 2APVW\text{-}\Gamma\text{-}ideal \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ PKV\text{-}\Gamma\text{-}ideal & \Longrightarrow & PRKV\text{-}\Gamma\text{-}ideal & \Longrightarrow & 2AKV\text{-}\Gamma\text{-}ideal & \Longrightarrow & 2APKV\text{-}\Gamma\text{-}ideal \end{array}}$$

8. Conclusions

When we process fuzzy information, vague sets perform slightly better than fuzzy sets. Human perception is typically gradual. As a result, the question of how to define a fuzzy concept and measure its level of uncertainty proves intriguing. However, due to a lack of information, the idea of a simple vague set is insufficient to accurately describe the occurrence of ratings or grades. Furthermore, it is not enough to adequately describe the occurrence of uncertainty and vagueness in tricky situations with difficult decisions.

A novel concept of a vague ideal, namely the 2-absorbing vague weakly complete Γ-ideal, is introduced by integrating the features of a vague weakly complete ideal and a 2-absorbing ideal. In this paper, we have discussed the concepts of 2-absorbing vague weakly complete Γ-ideals and 2-absorbing K-vague Γ-ideals of $\Gamma-$ring $\Re$ and also the associated algebraic properties using examples. Furthermore, we have theorized the 2-absorbing primary vague weakly complete Γ-ideal and 2-absorbing primary K-vague Γ-ideal of Γ ring $\Re$. We have also shown that the image and inverse image of 2-absorbing primary vague weakly complete Γ-ideal are again 2-absorbing primary vague weakly complete Γ-ideals. Moreover, we provided a 1-1 inclusion-preserving correspondence theorem about these algebraic structures. Furthermore, we examined a 2-absorbing vague weakly complete Γ-ideal induced by a vague quotient Γ-ring of $\Re$ and proved that if ω is a 2-absorbing vague weakly complete Γ-ideal, then the vague quotient Γ-ring of $\Re$ is induced by the vague Γ-ideal of $\Re$. Finally, we gave a diagram of the transition between these algebraic structures.

Scientists have combined this coherent approach to produce a variety of important results across 2-absorbing primary vague weakly complete Γ-ideals and 2-absorbing primary K-vague Γ-ideals. Based on our work, we suggest some idea-generating questions for researchers:

(1) Can we represent 2-absorbing semi-primary vague weakly complete Γ-ideals?
(2) Can we suggest 2-absorbing δ-primary vague weakly complete Γ-ideals?,
(3) Can we identify 2-absorbing δ-semiprimary vague weakly complete Γ-ideals?
(4) Can we study 2-absorbing primary complex vague weakly complete Γ-ideals?
(5) Can we characterize 2-absorbing vague weakly complete Γ-hyperideals?
(5) Can we describe the 1-absorbing vague weakly complete Γ-ideal of a Γ-ring?

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References

1. Badawi, A. On 2-absorbing ideals of commutative rings. *Bull. Austral. Math. Soc.* **2007**, *75*, 417–429. [CrossRef]
2. Issoual, M.; Mahdou, N.; ; Moutui, M.A.S. On n-absorbing prime ideals of commutative rings. *Hacet. J. Math. Stat.* **2022**, *20*, 2150174. [CrossRef]
3. Sawalmeh, L.; Saleh, M. On 2-absorbing ideals of commutative semirings. *J. Algebra Its Appl.* **2021**, *22*, 2350063. [CrossRef]
4. Alshehry, A.S.; Habeb, J.M.; Abu-Dawwas, R.; Alrawabdeh, A. Graded Weakly 2-Absorbing Ideals over Non-Commutative Graded Rings. *Symmetry* **2022**, *14*, 1472. [CrossRef]
5. Teshale Amare, N. Weakly 2-Absorbing Ideals in Almost Distributive Lattices. *J. Math.* **2022**, *2022*, 9252860. [CrossRef]
6. Badawi, A.; El Khalfi, A.; Mahdou, N. On (m,n)-absorbing prime ideals and (m,n)-absorbing ideals of commutative rings. *São Paulo J. Math. Sci.* **2023**, 1–14. [CrossRef]
7. Badawi, A.; Fahid, B. On weakly 2-absorbing δ-primary ideals of commutative rings. *Georgian Math. J.* **2020**, *27*, 503–516. [CrossRef]

8. Sahoo, T.; Deepak, Shetty, M.; Groenewald, N.J.; Harikrishnan, P.K.; Kuncham, S.P. On completely 2-absorbing ideals of N-groups. *J. Discret. Math. Sci. Cryptogr.* **2021**, *24*, 541–556. [CrossRef]
9. Celikel, E.Y. 2-absorbing δ-semiprimary Ideals of Commutative Rings. *Kyungpook Math. J.* **2021**, *61*, 711–725.
10. Yavuz, S.; Onar, S.; Ersoy, B.A.; Tekir, Ü.; Koc, S. 2-absorbing ϕ-δ-primary ideals. *Turk. J. Math.* **2021**, *45*, 1927–1939. [CrossRef]
11. Alhazmy, K.; Almahdi, F.A.A.; Bouba, E.M.; Tamekkante, M. On (1, 2)-absorbing primary ideals and uniformly primary ideals with order ≤ 2. *Analele Stiinţifice ale Universităţii "Ovidius" Constanţa. Seria Matematică*, **2023**, *31*, 5–21.
12. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
13. Rosenfeld, A. Fuzzy groups. *J. Math. Anal. Appl.* **1971**, *35*, 512–517. [CrossRef]
14. Liu, W.J. Operation on fuzzy ideals. *Fuzzy Sets Syst.* 1983, *11*, 31–41.
15. Nobusawa, N. On a generalization of the ring theory. *Osaka Math. J.* **1964**, *1*, 81–89.
16. Barnes, W.E. On the Γ-rings of Nobusawa. *Pac. J. Math.* **1966**, *18*, 411–422. [CrossRef]
17. Kyuno, S. A gamma ring with the right and left unities. *Math. Jpn.* **1979**, *24*, 191–193.
18. Kyuno, S. On prime gamma rings. *Pac. J. Math.* **1978**, *75*, 185–190. [CrossRef]
19. Kyuno, S. Prime ideals in gamma rings. *Pac. J. Math.* **1982**, *98*, 375–379. [CrossRef]
20. Luh, J. On the theory of simple Γ-rings. *Mich. Math. J.* **1969**, *16*, 65–75. [CrossRef]
21. Dutta, T.K.; Chanda, T. Fuzzy prime ideals in Γ-rings. *Bull. Malays. Math. Sci. Soc.* **2007**, *30*, 65–73.
22. Ersoy, B.A. Fuzzy semiprime ideals in Γ-rings. *Int. J. Phys. Sci.* **2010**, *5*, 308–312.
23. Darani, A.Y. On L-fuzzy 2-absorbing ideals. *Ital. J. Pure Appl. Math.* **2016**, *36*, 147–154.
24. Darani, A.Y.; Hashempoor, A. L-fuzzy 0-(1- or 2- or 3-) 2-absorbing ideals in semiring. *Ann. Fuzzy Math. Inform.* **2014**, *7*, 303–311.
25. Elkettani, M.Y.; Kasem, A. On 2-absorbing δ-primary gamma ideal of gamma ring. *Int. J. Pure Appl. Math.* **2016**, *106*, 543–550. [CrossRef]
26. Sönmez, D.; Yeşilot, G.; Onar, S.; Ersoy, B.A.; Davvaz, B. On 2-absorbing primary fuzzy ideals of commutative rings. *Math. Probl. Eng.* **2017**, *2017*, 5485839. [CrossRef]
27. Hanoon, W.H.; Aljassa, S.M.; Fiadh, M.S. Nearly 2-absorbing and weakly nearly 2-absorbing fuzzy submodules. *J. Discret. Sci. Cryptogr.* **2020**, *23*, 1025–1037. [CrossRef]
28. Yiarayong, P. On 2-absorbing bipolar fuzzy ideals over LA-semigroups. *J. Intell. Fuzzy Syst.* **2021**, *41*, 3173–3181. [CrossRef]
29. Sharma, P. K.; Lata, H.; Bharadwaj, N. A study on intuitionistic fuzzy 2-absorbing primary ideals in γ-ring. *Notes Intuitionistic Fuzzy Sets* **2022**, *28*, 280–292. [CrossRef]
30. Mandal, D. On 2-absorbing fuzzy ideals of commutative semirings. *TWMS J. Appl. Eng. Math.* **2021**, *11*, 368–373.
31. Nimbhorkar, S.K.; Patil, Y.S. Fuzzy Weakly 2-Absorbing Ideals of a Lattice. *Discuss.-Math.-Gen. Algebra Appl.* **2022**, *42*, 255–277. [CrossRef]
32. Hu, K.H.; Chen, F.H.; Hsu, M.F.; Tzeng, G.H. Identifying key factors for adopting artificial intelligence-enabled auditing techniques by joint utilization of fuzzy-rough set theory and MRDM technique. *Technol. Econ. Dev. Econ.* **2021**, *27*, 459–492. [CrossRef]
33. Lu, K.; Liao, H.; Zavadskas, E. K. An overview of fuzzy techniques in supply chain management: bibliometrics, methodologies, applications and future directions. *Technol. Econ. Dev. Econ.* **2021**, *27*, 402–458. [CrossRef]
34. Tang, Y.M.; Zhang, L.; Bao, G.Q.; Ren, F.J.; Pedrycz, W. Symmetric implicational algorithm derived from intuitionistic fuzzy entropy. *Iran. J. Fuzzy Syst.* **2022**, *19*, 27–44.
35. Raj, J.S. A comprehensive survey on the computational intelligence techniques and its applications. *J. ISMAC* **2019**, *1*, 147–159. [CrossRef]
36. Guan, H.; Yousafzai, F.; Zia, M.D.; Khan, M.-u.-I.; Irfan, M.; Hila, K. Complex Linear Diophantine Fuzzy Sets over AG-Groupoids with Applications in Civil Engineering. *Symmetry* **2023**, *15*, 74. [CrossRef]
37. Al Tahan, M.; Hoskova-Mayerova, S.; Davvaz, B. Some Results on (Generalized) Fuzzy Multi-H_v-Ideals of H_v-Rings. *Symmetry* **2019**, *11*, 1376. [CrossRef]
38. Xin, X.; Borzooei, R.A.; Bakhshi, M.; Jun, Y.B. Intuitionistic Fuzzy Soft Hyper BCK Algebras. *Symmetry* **2019**, **11**, 399. [CrossRef]
39. Su, S.; Li, Q.; Li, Q. Z_L-Completions for Z_L-Semigroups. *Symmetry* **2022**, *14*, 578. [CrossRef]
40. Gau, W.L.; Buehrer, D.J. Vague sets. *IEEE Trans. Syst. Manand Cybern.* **1993**, *23*, 610–614. [CrossRef]
41. Ren, Q.; Zhang, D.; Ma, Z. On vague subring and its structure. In *Fuzzy Information and Engineering: Proceedings of the Second International Conference of Fuzzy Information and Engineering (ICFIE)*; Springer: Berlin/Heidelberg, Germany, 2007; pp. 138–143.
42. Sezer, S. Vague rings and vague ideals. *Iran. J. Fuzzy Syst.* **2011**, *8*, 145–157.
43. Yin, Y.; Jun, Y.B.; Zhan, J. Vague soft hemirings. *Comput. Math. Appl.* **2011**, *62*, 199–213. [CrossRef]
44. Davvaz, B.; Majumder, S.K. On Vague Bi-ideals and Vague Weakly Completely Prime Ideals in Γ-semigroups. *TWMS J. Pure Appl. Math.* 2012, *3*, 62–74.
45. Bhaskar, L.; Swamy, P.N.; Nagaiah, T.; Srinivas, T. Sum of vague ideals of a near-ring. *AIP Conf. Proc.* **2019**, *2177*, 020017.
46. Baghernejad, M.; Borzooei, R.A. Vague multigraphs. *Soft Comput.* **2019**, *23*, 12607–12619. [CrossRef]
47. Ragamayi, S.; Reddy, N.K.; Kumar, B.S. Results on L-Vague ideal of a Γ-near ring. *AIP Conf. Proc.* **2021**, *2375*, 020014.
48. Bhargavi, Y. A study on translational invariant vague set of a Γ-semiring. *Afr. Mat.* **2020**, *31*, 1273–1282. [CrossRef]
49. Gahlot, N.; Dasari, N. Interval valued vague ideals in Γ-nearrings. *Ital. J. Pure Appl. Math.* **2022**, *48*, 595–604.
50. Alkhazaleh, S. Neutrosophic Vague Set Theory. *Crit. Rev.* **2015**, *10*, 29–39.

51. Bustince, H.; Burillo, P. Vague sets are intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1996**, *79*, 403–405. [CrossRef]
52. Chen, S.M. Similarity measures between vague sets and between elements. *IEEE Trans. Syst. Man Cybern.* **1997**, *27*, 153–158. [CrossRef] [PubMed]
53. Kumar, A.; Yadav, S.P.; Kumar, S. Fuzzy system reliability analysis using based arithmetic operations on L–$\Re$ type interval valued vague sets. *Inter. J. Qual. Reliab. Manag.* **2007**, *24*, 846–860. [CrossRef]
54. Xu, W.; Ma, J.; Wang, S.; Hao, G. Vague soft sets and their properties. *Comput. Math. Appl.* **2010**, *59*, 787–794. [CrossRef]
55. Dutta, T.K.; Chanda, T. Structures of fuzzy ideals of Γ-ring. *Bull. Malays. Math. Sci. Soc.* **2005**, *28*, 9–18.