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Survival Analysis and Applications of Weighted NH Parameters Using Progressively Censored Data

Ahmed Elshahhat [1,*] and Heba S. Mohammed [2]

[1] Faculty of Technology and Development, Zagazig University, Zagazig 44519, Egypt
[2] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
* Correspondence: aelshahhat@ftd.zu.edu.eg

Abstract: A new weighted Nadarajah–Haghighi (WNH) distribution, as an alternative competitor model to gamma, standard half-logistic, generalized-exponential, Weibull, and other distributions, is considered. This paper explores both maximum likelihood and Bayesian estimation approaches for estimating the parameters, reliability, and hazard rate functions of the WNH distribution when the sample type is Type-II progressive censored order statistics. In the classical interval setup, both asymptotic and bootstrap intervals of each unknown parameter are constructed. Using independent gamma priors and symmetric squared-error loss, the Bayes estimators cannot be obtained theoretically. Thus, two approximation techniques, namely: Lindley and Markov-Chain Monte Carlo (MCMC) methods, are used. From MCMC variates, the Bayes credible and highest posterior density intervals of all unknown parameters are also created. Extensive Monte Carlo simulations are implemented to compare the performance of the proposed methodologies. Numerical evaluations showed that the estimates developed by the MCMC sampler performed better than the Lindley estimates, and both behaved significantly better than the frequentist estimates. To choose the optimal censoring scheme, several optimality criteria are considered. Three engineering applications, including vehicle fatalities, electronic devices, and electronic components data sets, are provided. These applications demonstrated how the proposed methodologies could be applied in real practice and showed that the proposed model provides a satisfactory fit compared to three new weighted models, namely: weighted exponential, weighted Gompertz, and new weighted Lindley distributions.

Keywords: weighted Nadarajah–Haghighi distribution; progressive censoring; symmetric Bayesian estimation; optimum censoring; bootstrapping; MCMC algorithms; Fisher information

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1. Introduction

The Nadarajah–Haghighi (NH) distribution, proposed by Nadarajah and Haghighi [1], has a density shape that can be growing or unimodal as well as a hazard rate that can be increasing, decreasing, or constant. Recently, as a new weighted version of the NH distribution, according to the idea of created weighted distributions, Khan et al. [2] proposed the weighted Nadarajah–Haghighi (WNH) distribution as a competitive model to other lifetime models, such as gamma, exponentiated half-logistic, generalized-exponential, Weibull, etc. They also stated that the WNH distribution is suitable for modeling data from different areas; for example, reliability, survival analysis, forest, ecological, etc. Suppose that X is a lifetime random variable of the test unit(s) following the two-parameter WNH(δ,μ) distribution. Then, its probability density function (PDF) $f(\cdot)$ and cumulative distribution function (CDF) $F(\cdot)$ are given, respectively, as

$$f(x;\delta,\mu)=\frac{2\delta\mu(1+\mu x)^{\delta-1}e^{1-(1+\mu x)^{\delta}}}{\left(1+e^{1-(1+\mu x)^{\delta}}\right)^{2}},\quad x>0, \tag{1}$$

and

$$F(x;\delta,\mu) = \frac{1 - e^{1-(1+\mu x)^{\delta}}}{1 + e^{1-(1+\mu x)^{\delta}}}, \quad x > 0, \tag{2}$$

where $\delta > 0$ and $\mu > 0$ are the shape and scale parameters, respectively. From (1), Khan et al. [2] showed that the WNH distribution could be considered an extended model of three common distributions, namely: NH, standard half-logistic and zero truncated weighted Weibull models, such as:

- Nadarajah–Haghighi distribution by multiplying (1) by $\frac{1}{2}\left[1 + e^{1-(1+\mu x)^{\delta}}\right]^{-2}$.
- Standard half-logistic distribution by setting $Y = (1 + \mu x)^{\delta} - 1$.
- Truncated weighted Weibull by setting $Y = X + \mu^{-1}$.

Further, the reliability function (RF) $R(\cdot)$ and hazard function (HF) $h(\cdot)$ at distinct time t are

$$R(t;\delta,\mu) = 2\left(1 + e^{(1+\mu t)^{\delta}-1}\right)^{-1}, \quad t > 0, \tag{3}$$

and

$$h(t;\delta,\mu) = \frac{\delta\mu(1+\mu t)^{\delta-1}}{1 + e^{1-(1+\mu t)^{\delta}}}, \quad t > 0, \tag{4}$$

respectively. From (2), where $X \sim \text{WNH}(\delta,\mu)$, two lifetime models can be obtained by making a useful transformation, namely (i) standard half-logistic distribution when $Y = (1+\mu X)^{\delta} - 1$ and (ii) zero truncated weighted Weibull distribution when $Y = X + \mu^{-1}$. Figure 1 depicts the density and hazard rate shapes of the WNH(δ,μ) distribution for several choices of its parameters. It reveals that the PDF (1) is positively skewed while the HF (4) has increasing, decreasing, and constant shapes.

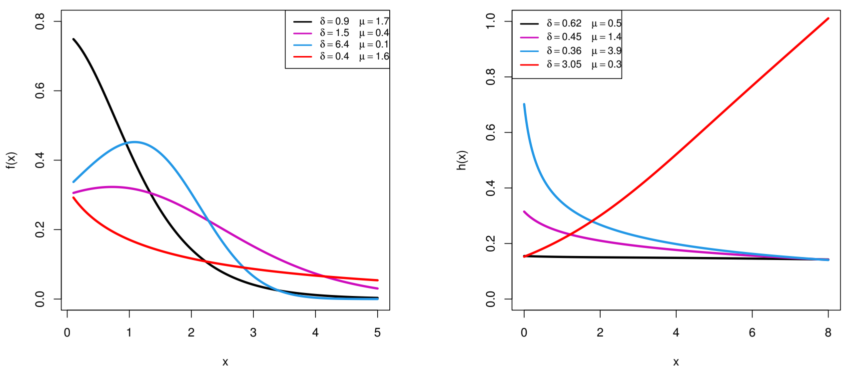


Figure 1. Density (**left**) and hazard rate (**right**) shapes of the WNH distribution.

In the past decade, researchers have favored dealing with progressive Type-II censoring (PCTII) over other censoring mechanisms because it (i) reduces the total testing time, (ii) reduces the cost of failed units, (iii) terminates the experiment rapidly, and (iv) allows the researcher to make use of the units removed during the experiment for use in other tests. Briefly, the PCTII can be described as: Suppose that n independent and identical units are subjected to a life-testing experiment at time zero, m is a fixed number of failures and the censoring scheme $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ is also pre-specified. At the time that the first failure is observed (say $X_{(1)}$), R_1 survival units are randomly removed from the experiment. Following the second observed failure (say $X_{(2)}$), R_2 of survival units are randomly removed from $n - R_1 - 1$ units, and so on. This mechanism continues until the $m - th$ failure occurs and the experiment is stopped. At the same time, all remaining $R_m = n - m - \sum_{i=1}^{m-1} R_i$ units are removed. Let $\mathbf{x} = \{x_{(1)} < x_{(2)} < \cdots < x_{(m)}\}$ be a PCTII sample of size m with censoring $\mathbf{R}$ from a continuous population.

Then, the likelihood function of $\mathbf{x}$, where Θ is the parameter vector, can be expressed as

$$L(\Theta|\mathbf{x}) = \mathcal{K}\prod\nolimits_{i=1}^{m} f(x_{(i)};\Theta)[1-F(x_{(i)};\Theta)]^{R_i}, \tag{5}$$

where $K = n(n-R_1-1)\ldots(n-\sum_{i=1}^{m}(R_i+1))$.

Various studies on the NH distribution using censored data have been made in the literature, readers may refer to Mohie El-Din et al. [3], Ashour et al. [4], Wu and Gui [5], Elshahhat et al. [6], Elshahhat and Abu El Azm [7], Dey et al. [8], Almarashi et al. [9], among others.

Although the NH distribution has received a lot of attention from several authors to the best of our knowledge, the problem of estimating the WNH parameters and/or the RF and HF via PCTII sampling has yet to be investigated. Therefore, this article offers an analysis of PCTII lifetime data when each test unit follows the WNH distribution. The maximum likelihood estimators (MLEs) with their Fisher information members are evaluated through the standard Newton–Raphson algorithm. The associated asymptotic normality of the MLE and log-MLE is used to construct the asymptotic confidence intervals (ACIs). Further, for all unknown quantities, bootstrap-p and bootstrap-t intervals are also obtained. Using the Lindley and Metropolis–Hastings (M-H) methods, the Bayes estimates against the squared-error loss (SEL) function can be easily approximated. Using Markov chain Monte Carlo (MCMC) steps, the Bayes credible interval (BCI) and the highest posterior density (HPD) interval for each unknown parameter are obtained. To compare two (or more) different progressive censoring plans, using their Fisher information, different optimality criteria are taken into account. To assess the performance of the proposed estimation methodologies, an extensive Monte Carlo simulation is performed. Lastly, three applications based on real-life engineering data are presented to show the superiority and flexibility of the WNH model over three weighted distributions (as competitors), namely: weighted exponential, weighted Gompertz, and weighted Lindley distributions. Additionally, the proposed applications aim to demonstrate the applicability of the acquired estimators in a real-life scenario.

The rest of the article is classified as: Section 2 provides the likelihood inference. Two kinds of bootstrapping are presented in Section 3. Bayes approximation techniques are discussed in Section 4. Simulation results are provided in Section 5. In Section 6, a brief explanation of the optimum progressive censoring is presented. Three real applications are presented in Section 7. In Section 8, some conclusions are offered.

2. Likelihood Inference

This section deals with finding the MLEs of the model parameters (δ,μ) and the reliability characteristics $(R(t),h(t))$ of the WNH distribution based on PCTII data. Two-sided $(1-\tau)100\%$ ACIs of the same unknown parameters are also obtained here.

2.1. Maximum Likelihood Estimators

Suppose $X_{(i)}$, $i = 1,2,\ldots,m$, $(1 \leqslant m < n)$ is a PCTII sample acquired from the WNH(δ,μ) having a fixed $\mathbf{R}$. Substituting (1) and (2) into (5), where x_i is used in the place of $X_{(i)}$, we receive

$$L(\delta,\mu|\mathbf{x}) \propto (\delta\mu)^m e^{\sum_{i=1}^{m}(R_i+1)\psi_i(\delta,\mu)}\prod_{i=1}^{m}(1+\mu x_i)^{\delta-1}\left(1+e^{\psi_i(\delta,\mu)}\right)^{-(R_i+2)}, \tag{6}$$

where $\psi_i(\delta,\mu) = 1-(1+\mu x_i)^{\delta}$, $i = 1,2,...,m$.

The log-likelihood function $(\mathcal{L}(\cdot) \propto \log L(\cdot))$ of (6) can be expressed (up to proportional) as

$$\begin{aligned}\mathcal{L}(\delta,\mu|\mathbf{x}) \propto\ & m\log(\delta\mu) + \sum\nolimits_{i=1}^{m}(R_i+1)\psi_i(\delta,\mu) \\ & + (\delta-1)\sum\nolimits_{i=1}^{m}\log(1+\mu x_i) - \sum\nolimits_{i=1}^{m}(R_i+2)\log\left(1+e^{\psi_i(\delta,\mu)}\right).\end{aligned} \tag{7}$$

Differentiating (7) partially with respect to δ and μ, two likelihood equations are obtained as

$$\frac{\partial\mathcal{L}}{\partial\delta} = \frac{m}{\delta} + \sum_{i=1}^{m}(R_i+1)\psi_i^{\delta'}(\delta,\mu) + \sum_{i=1}^{m}\log(1+\mu x_i) - \sum_{i=1}^{m}(R_i+2)\psi_i^{\delta'}(\delta,\mu)e^{\psi_i(\delta,\mu)}\left(1+e^{\psi_i(\delta,\mu)}\right)^{-1}, \tag{8}$$

and

$$\frac{\partial\mathcal{L}}{\partial\mu} = \frac{m}{\mu} + \sum_{i=1}^{m}(R_i+1)\psi_i^{\mu'}(\delta,\mu) + (\delta-1)\sum_{i=1}^{m}x_i(1+\mu x_i)^{-1} - \sum_{i=1}^{m}(R_i+2)\psi_i^{\mu'}(\delta,\mu)e^{\psi_i(\delta,\mu)}\left(1+e^{\psi_i(\delta,\mu)}\right)^{-1}, \tag{9}$$

where $\psi_i^{\eta'}(\cdot)$ is the first-partial derivative with respect to η, such as $\psi_i^{\delta'}(\delta,\mu) = -(1+\mu x_i)^{\delta}\log(1+\mu x_i)$ and $\psi_i^{\mu'}(\delta,\mu) = -\delta x_i(1+\mu x_i)^{\delta-1}$ for $i=1,2,\ldots,m$.

It is observed, from (8) and (9), that the MLEs $\hat{\delta}$ and $\hat{\mu}$ are derived in a system of nonlinear equations. Thus, a suitable iterative procedure, such as the Newton–Raphson iterative method, is used. For this reason, the 'maxLik' package suggested by Henningsen and Toomet [10] is considered. Next, by replacing δ and μ with their MLEs $\hat{\delta}$ and $\hat{\mu}$, respectively, the MLEs $\hat{R}(t)$ and $\hat{h}(t)$ can be easily derived.

Remark 1. *The results of Khan et al. [2] in the case of the WNH distribution based on complete sampling can be easily obtained as a special case from (7) by setting $n=m$ and $R_i=0$, $i=1,2,\ldots,m$.*

2.2. Asymptotic Interval Estimators

To create the $100(1-\tau)\%$ ACIs of δ, μ, $R(t)$, or $h(t)$, the variance-covariance (VCov) matrix of $\hat{\delta}$ and $\hat{\mu}$ must be obtained first as

$$\mathbf{I}^{-1}(\hat{\delta},\hat{\mu}) \cong \begin{bmatrix} -\mathcal{L}_{11} & -\mathcal{L}_{12} \\ -\mathcal{L}_{21} & -\mathcal{L}_{22} \end{bmatrix}^{-1}_{(\hat{\delta},\hat{\mu})} = \begin{bmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} \\ \hat{\sigma}_{21} & \hat{\sigma}_{22} \end{bmatrix}, \tag{10}$$

where, from (7), the elements $\mathcal{L}_{ij},\ i,j=1,2$, are obtained and presented in Appendix A.

Under certain regularity conditions, the asymptotic normality of $\hat{\Theta} = (\hat{\delta},\hat{\mu})^{\mathbf{T}}$ is approximately distributed as bivariate normal $\hat{\Theta} \sim N(\Theta,\mathbf{I}^{-1}(\hat{\Theta}))$. Thus, following the normal approximation (NA) of the MLEs $\hat{\delta}$ and $\hat{\mu}$, two bounds $100(1-\tau)\%$ ACI via NA (ACI-NA) of δ and μ can be obtained, respectively, by

$$\hat{\delta} \mp z_{\tau/2}\sqrt{\hat{\sigma}_{11}} \quad \text{and} \quad \hat{\mu} \mp z_{\tau/2}\sqrt{\hat{\sigma}_{22}},$$

where $z_{\tau/2}$ is the upper $(\tau/2)$th probability point of the standard normal distribution.

Further, to obtain the ACIs of $R(t)$ and $h(t)$, the delta method is used to approximate their variances, see Greene [11]. However, from the Fisher information (10), the approximated variances of $\hat{R}(t)$ and $\hat{h}(t)$ are given by

$$\hat{\sigma}^2_{\hat{R}(t)} \simeq \mathcal{C}_R\mathbf{I}^{-1}(\delta,\mu)\mathcal{C}_R^{\mathbf{T}}\Big|_{(\hat{\delta},\hat{\mu})} \text{ and } \hat{\sigma}^2_{\hat{h}(t)} \simeq \mathcal{C}_h\mathbf{I}^{-1}(\delta,\mu)\mathcal{C}_h^{\mathbf{T}}\Big|_{(\hat{\delta},\hat{\mu})},$$

respectively, where $\mathcal{C}_R = \begin{bmatrix} \frac{\partial R}{\partial\delta} & \frac{\partial R}{\partial\mu} \end{bmatrix}$ and $\mathcal{C}_h = \begin{bmatrix} \frac{\partial h}{\partial\delta} & \frac{\partial h}{\partial\mu} \end{bmatrix}$, such as

$$\frac{\partial R}{\partial \delta} = -2(1+\mu t)^{\delta}\log(1+\mu t)e^{(1+\mu t)^{\delta}-1}\left(1+e^{(1+\mu t)^{\delta}-1}\right)^{-2},$$

$$\frac{\partial R}{\partial \mu} = -2\delta t(1+\mu t)^{\delta-1}e^{(1+\mu t)^{\delta}-1}\left(1+e^{(1+\mu t)^{\delta}-1}\right)^{-2},$$

$$\frac{\partial h}{\partial \delta} = \frac{\mu(1+\mu t)^{\delta-1}}{\left(1+e^{\psi(t;\delta,\mu)}\right)}\left[1+\delta\log(1+\mu t)\left[1-(1+\mu t)^{\delta}e^{\psi(t;\delta,\mu)}\left(1+e^{\psi(t;\delta,\mu)}\right)^{-1}\right]\right],$$

and

$$\frac{\partial h}{\partial \mu} = \frac{\delta(1+\mu t)^{\delta-1}}{\left(1+e^{\psi(t;\delta,\mu)}\right)}\left[1+\frac{\mu t(\delta-1)}{(1+\mu t)}\left[1-(1+\mu t)^{\delta}e^{\psi(t;\delta,\mu)}\left(1+e^{\psi(t;\delta,\mu)}\right)^{-1}\right]\right],$$

where $\psi(t;\delta,\mu) = 1-(1+\mu t)^{\delta}$.

Thus, at the confidence percent $100(1-\tau)\%$, the ACI-NA of $R(t)$ and $h(t)$ are obtained, respectively, as

$$\hat{R}(t) \mp z_{\tau/2}\sqrt{\hat{\sigma}^2_{\hat{R}(t)}} \quad \text{and} \quad \hat{h}(t) \mp z_{\tau/2}\sqrt{\hat{\sigma}^2_{\hat{h}(t)}}$$

where $z_{\tau/2}$ is the percentile of the standard normal distribution with right-tail probability $(\tau/2)-th$.

In practice, the main problem with the ACI-NA method is that it may give a negative lower bound for a lifetime parameter. For handling this disadvantage, Meeker and Escobar [12] developed the normal log-transformed (NL) approximation for the MLE. They also stated that the ACI has the highest CP based on the NL compared to the other NA. Then, the $100(1-\tau)\%$ ACI via NL (ACI-NL) of δ is given by

$$\hat{\delta}\exp\left(\mp z_{\tau/2}\frac{\sqrt{\hat{\sigma}_{11}}}{\hat{\delta}}\right),$$

where, in a similar fashion, the other $100(1-\tau)\%$ ACI-NL of μ, $R(t)$ or $h(t)$ can be easily computed.

3. Bootstrapping Estimators

If the sample size m is small, due to the law of large numbers, both ACIs developed by NA and NL methods cannot perform well. In this section, two bootstrapping techniques, namely: bootstrap-p and bootstrap-t algorithms, are obtained for δ, μ, $R(t)$, and $h(t)$. To obtain the $100(1-\tau)\%$ bootstrap-p (or bootstrap-t) of δ, μ, $R(t)$ or $h(t)$, we first need to receive a bootstrap sample, as shown in Algorithm 1.

Algorithm 1 Bootstrap Sampling:

Step 1: Compute the MLEs of δ and μ, using PCTII sample $X_1, X_2, \ldots, X_m$.
Step 2: Generate a bootstrap sample (say $X_1^*, X_2^*, \ldots, X_m^*$) from WNH$(\hat{\delta}, \hat{\mu})$.
Step 3: Calculate the bootstrap samples δ^*, μ^*, $R^*(t)$ and $h^*(t)$ based on $X_1^*, X_2^*, \ldots, X_m^*$.
Step 4: Redo Steps 2-3 $\mathcal{D}$ times.
Step 5: Set the bootstrap samples in ascending order as

$$\left(\delta^*_{(1)} < \cdots < \delta^*_{(\mathcal{D})}\right); \left(\mu^*_{(1)} < \cdots < \mu^*_{(\mathcal{D})}\right);$$
$$\left(R^*_{(1)}(t) < \cdots < R^*_{(\mathcal{D})}(t)\right); \left(h^*_{(1)}(t) < \cdots < h^*_{(\mathcal{D})}(t)\right).$$

Then, using the bootstrap-p and bootstrap-t methods, the $100(1-\tau)\%$ bootstrap intervals of δ, μ, $R(t)$ and $h(t)$ can be constructed as

(i) **Bootstrap-p (Boot-p) Method:** The $100(1-\tau)\%$ boot-p intervals of δ, μ, $R(t)$ and $h(t)$ (say ϕ) are given, respectively, by

$$\left(\hat{\phi}^{*}_{[\mathcal{D}(\frac{\tau}{2})]}, \hat{\phi}^{*}_{[\mathcal{D}(1-\frac{\tau}{2})]}\right).$$

(ii) **Bootstrap-t (Boot-t) Method:** The $100(1-\tau)\%$ boot-t intervals of δ, μ, $R(t)$ and $h(t)$ are given by

1. Define the T-statistic for δ, μ, $R(t)$ and $h(t)$ (say ϕ) as

$$T_j^{\phi} = (\widehat{var}(\hat{\phi}))^{-1/2}(\hat{\phi}_j^{*} - \hat{\phi}), \; j = 1, 2, \ldots, \mathcal{D},$$

where $\hat{\phi}$ is the MLE of ϕ and $\widehat{var}(\hat{\phi})$ is the observed estimated variance of ϕ.

2. Arrange the T-statistics as $\left(T^{\phi}_{(1)}, T^{\phi}_{(2)}, \ldots, T^{\phi}_{(\mathcal{D})}\right)$.
3. The $100(1-\tau)\%$ bootstrap-t interval of ϕ is given by

$$\left(\hat{\phi} + T^{\phi}_{[\mathcal{D}(\frac{\tau}{2})]}\sqrt{\widehat{var}(\hat{\phi})}, \hat{\phi} + T^{\phi}_{[\mathcal{D}(1-\frac{\tau}{2})]}\sqrt{\widehat{var}(\hat{\phi})}\right).$$

4. Bayes Estimators

The Bayes' paradigm provides the possibility of incorporating prior information about the unknown parameter(s) of interest. To establish this method, the WNH parameters δ and μ are assumed to be random variables with some extra information included in their prior distributions. One of the most flexible priors is called the gamma conjugate density prior; see Kundu [13]. Gamma density is fairly straightforward and flexible enough to cover a large variety of the experimenter's prior beliefs. We, therefore, assumed that δ and μ, having independent gamma PDFs as $\delta \sim Gamma(a_1, b_1)$ and $\mu \sim Gamma(a_2, b_2)$, where $a_i, b_i > 0, \; i = 1, 2$, are chosen to reflect prior knowledge about the WNH(δ, μ) parameters. Thus, the joint gamma PDF of δ and μ becomes

$$\pi(\delta, \mu) \propto \delta^{a_1-1}\mu^{a_2-1}e^{-(b_1\delta+b_2\mu)}, \quad \delta, \mu > 0, \tag{11}$$

where a_i and b_i, $i = 1, 2$, are assumed to be known. On the other hand, the most widely used loss is called the SEL function, $\ell(\cdot)$, and is defined as

$$\ell(\eta, \tilde{\eta}) = (\tilde{\eta} - \eta)^2, \tag{12}$$

where $\tilde{\eta}$ denotes the Bayes estimate of η.

From (12), the target Bayes estimate is given by the posterior expectation of η. The joint posterior PDF, by combining (11) with (6), of δ and μ is

$$\begin{aligned}\Omega(\delta, \mu|\mathbf{x}) = C^{-1}\delta^{m+a_1-1}\mu^{m+a_2-1}e^{-(b_1\delta+b_2\mu)}e^{\sum_{i=1}^{m}(R_i+1)\psi_i(\delta,\mu)} \\ \times \prod_{i=1}^{m}(1+\mu x_i)^{\delta-1}\left(1+e^{\psi_i(\delta,\mu)}\right)^{-(R_i+2)},\end{aligned} \tag{13}$$

where $C = \int_0^\infty \int_0^\infty \pi(\delta, \mu) \times L(\delta, \mu|\underline{\mathbf{x}})\mathrm{d}\delta\mathrm{d}\mu$. However, using (12), the Bayes estimator for any function of δ and μ, say $\rho(\delta, \mu)$ is given by

$$\tilde{\rho}(\delta, \mu) = E[\rho(\delta, \mu)|\mathbf{x}] = C^{-1}\int_0^\infty \int_0^\infty \rho(\delta, \mu) \times \pi(\delta, \mu) \times L(\delta, \mu|\underline{\mathbf{x}})\mathrm{d}\delta\mathrm{d}\mu. \tag{14}$$

It is clear, based on (14), that the Bayes estimate of $\rho(\delta, \mu)$ is derived in an expression of the ratio of double integrals.

Then, the Bayes solution for each unknown parameter is not available. We, therefore, propose to use both the Lindley and MCMC methods to approximate the Bayes estimates.

4.1. Lindley's Approximation

In this subsection, following Lindley [14], the closed expressions for the Bayes estimates $\rho(\delta,\mu)$ are obtained. However, setting $(\delta,\mu)=(\beta_1,\beta_2)$ for simplicity, the approximated posterior expectation (say $\tilde{\rho}_L(\beta_1,\beta_2)$) of $\rho(\beta_1,\beta_2)$ is given by

$$\tilde{\rho}_L(\beta_1,\beta_2)=\rho(\hat{\beta}_1,\hat{\beta}_2)+0.5(\mathcal{A}+\Upsilon_{30}\mathcal{B}_{12}+\Upsilon_{03}\mathcal{B}_{21}+\Upsilon_{21}\mathcal{C}_{12}+\Upsilon_{12}\mathcal{C}_{21})+\gamma_1\mathcal{A}_{12}+\gamma_2\mathcal{A}_{21}, \tag{15}$$

where all terms in (15) are evaluated at their $\hat{\alpha}$ and $\hat{\theta}$, $\mathcal{A}=\sum_{i=1}^{2}\sum_{j=1}^{2}v_{ij}\sigma_{ij}$, $\mathcal{A}_{ij}=v_i\sigma_{ii}+v_j\sigma_{ji}$, $\mathcal{B}_{ij}=(v_i\sigma_{ii}+v_j\sigma_{ij})\sigma_{ii}$, $\mathcal{C}_{ij}=3v_i\sigma_{ii}\sigma_{ij}+v_j(\sigma_{ii}\sigma_{jj}+2\sigma_{ij}^2)$, $v_i=\partial\rho(\beta_1,\beta_2)/\partial\beta_i$, $v_{ij}=\partial^2\rho(\beta_1,\beta_2)/\partial\beta_i\partial\beta_j$, $\Upsilon_{ij}=\partial^{i+j}\mathcal{L}(\beta_1,\beta_2)/\partial\beta_1^i\partial\beta_2^j$, $i,j=0,1,2,3$, $i+j=3$, $\gamma=\log\pi(\beta_1,\beta_2)$, $\gamma_i=\partial\gamma/\partial\beta_i$, and σ_{ij} in the $(i,j)-th$ element of the VCov matrix (10).

From (15), the Lindley's estimators $\tilde{\delta}_L$ and $\tilde{\mu}_L$ of δ and μ, respectively, as

$$\begin{aligned}\tilde{\delta}_L&=\hat{\delta}+\hat{\gamma}_1\hat{\sigma}_{11}+\hat{\gamma}_2\hat{\sigma}_{12}\\&\quad+0.5(\hat{\sigma}_{11}(\hat{\Upsilon}_{30}\hat{\sigma}_{11}+2\hat{\Upsilon}_{21}\hat{\sigma}_{12}+\hat{\Upsilon}_{12}\hat{\sigma}_{22})+\hat{\sigma}_{21}(\hat{\Upsilon}_{03}\hat{\sigma}_{22}+2\hat{\Upsilon}_{12}\hat{\sigma}_{21}+\hat{\Upsilon}_{21}\hat{\sigma}_{11})),\end{aligned}$$

and

$$\begin{aligned}\tilde{\mu}_L&=\hat{\mu}+\hat{\gamma}_1\hat{\sigma}_{21}+\hat{\gamma}_2\hat{\sigma}_{22}\\&\quad+0.5(\hat{\sigma}_{12}(\hat{\Upsilon}_{30}\hat{\sigma}_{11}+2\hat{\Upsilon}_{21}\hat{\sigma}_{12}+\hat{\Upsilon}_{12}\hat{\sigma}_{22})+\hat{\sigma}_{22}(\hat{\Upsilon}_{03}\hat{\sigma}_{22}+2\hat{\Upsilon}_{12}\hat{\sigma}_{21}+\hat{\Upsilon}_{21}\hat{\sigma}_{11})),\end{aligned}$$

where $\gamma_1=\frac{1}{\delta}(a_1-1)-b_1$, $\gamma_2=\frac{1}{\mu}(a_2-1)-b_2$, and the items $\Upsilon_{ij},\ i,j=0,1,2,3$ (for $i+j=3$) are obtained and reported in Appendix B.

Similarly, using (15), the Lindley's estimators of the reliability characteristic functions $\tilde{R}_L(\cdot)$ and $\tilde{h}_L(\cdot)$ of $R(\cdot)$ and $h(\cdot)$ are given (for $t>0$), respectively, by

$$\begin{aligned}\tilde{R}_L(t;\delta,\mu)&=R(t;\hat{\delta},\hat{\mu})+\hat{\gamma}_1(\hat{v}_1\hat{\sigma}_{11}+\hat{v}_2\hat{\sigma}_{21})+\hat{\gamma}_2(\hat{v}_2\hat{\sigma}_{22}+\hat{v}_1\hat{\sigma}_{12})\\&\quad+0.5((\hat{v}_{11}\hat{\sigma}_{11}+2\hat{v}_{12}\hat{\sigma}_{12}+\hat{v}_{22}\hat{\sigma}_{22})+\hat{\Upsilon}_{30}\hat{\sigma}_{11}(\hat{v}_1\hat{\sigma}_{11}+\hat{v}_2\hat{\sigma}_{12})+\hat{\Upsilon}_{03}\hat{\sigma}_{22}(\hat{v}_2\hat{\sigma}_{22}+\hat{v}_1\hat{\sigma}_{21})\\&\quad+\hat{\Upsilon}_{12}(3\hat{v}_2\hat{\sigma}_{22}\hat{\sigma}_{12}+\hat{v}_1(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\hat{\sigma}_{12}^2))+\hat{\Upsilon}_{21}(3\hat{v}_1\hat{\sigma}_{11}\hat{\sigma}_{12}+\hat{v}_2(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\hat{\sigma}_{12}^2))),\end{aligned}$$

and

$$\begin{aligned}\tilde{h}_L(t;\delta,\mu)&=h(t;\hat{\delta},\hat{\mu})+\hat{\gamma}_1(\hat{v}_1\hat{\sigma}_{11}+\hat{v}_2\hat{\sigma}_{21})+\hat{\gamma}_2(\hat{v}_2\hat{\sigma}_{22}+\hat{v}_1\hat{\sigma}_{12})\\&\quad+0.5((\hat{v}_{11}\hat{\sigma}_{11}+2\hat{v}_{12}\hat{\sigma}_{12}+\hat{v}_{22}\hat{\sigma}_{22})+\hat{\Upsilon}_{30}\hat{\sigma}_{11}(\hat{v}_1\hat{\sigma}_{11}+\hat{v}_2\hat{\sigma}_{12})+\hat{\Upsilon}_{03}\hat{\sigma}_{22}(\hat{v}_2\hat{\sigma}_{22}+\hat{v}_1\hat{\sigma}_{21})\\&\quad+\hat{\Upsilon}_{12}(3\hat{v}_2\hat{\sigma}_{22}\hat{\sigma}_{12}+\hat{v}_1(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\hat{\sigma}_{12}^2))+\hat{\Upsilon}_{21}(3\hat{v}_1\hat{\sigma}_{11}\hat{\sigma}_{12}+\hat{v}_2(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\hat{\sigma}_{12}^2))),\end{aligned}$$

where $v_i,\ i=1,2$ is given in Section 2.2 and $v_{ij},\ i=1,2$ of $R(t)$, and $h(t)$ are obtained and presented in Appendix C.

Although the Lindley estimators of δ, μ, $R(t)$, and $h(t)$ are expressed explicitly, the main problem of this approach is its disability to obtain an interval estimate. Consequently, to approximate the Bayes estimates and to construct their BCI and HPD interval estimates, the M-H algorithm is considered.

4.2. M-H Sampler

Since the joint posterior PDF cannot be reduced by analytical steps to any familiar form, the M-H algorithm (which is one of the effective MCMC techniques) is used, see Gelman et al. [15].

First, from (13), the conditional distribution of δ and μ must be obtained, respectively, as

$$\Omega_1(\delta|\mu,\mathbf{x}) \propto \delta^{m+a_1-1}e^{-b_1\delta}e^{\sum_{i=1}^{m}(R_i+1)\psi_i(\delta,\mu)}\prod_{i=1}^{m}(1+\mu x_i)^{\delta}\left(1+e^{\psi_i(\delta,\mu)}\right)^{-(R_i+2)}, \quad (16)$$

and

$$\Omega_2(\mu|\delta,\mathbf{x}) \propto \mu^{m+a_2-1}e^{-b_2\mu}e^{\sum_{i=1}^{m}(R_i+1)\psi_i(\delta,\mu)}\prod_{i=1}^{m}(1+\mu x_i)^{\delta-1}\left(1+e^{\psi_i(\delta,\mu)}\right)^{-(R_i+2)}. \quad (17)$$

As we anticipated, the conditional distributions (16) and (17) of δ and μ, respectively, cannot be reduced to any standard distribution. Figure 2 shows that the distributions (16) and (17) behave like a normal density. Therefore, to obtain the Bayes MCMC estimates along with their BCI and HPD intervals, we perform the generation process described in Algorithm 2.

Algorithm 2 Markov Chain Monte Carlo Sampling:

Step 1: Start with initial guesses $\delta^{(0)} = \hat{\delta}$ and $\mu^{(0)} = \hat{\mu}$.
Step 2: Set $j = 1$
Step 3: Generate δ^* and μ^* from $N(\hat{\delta},\hat{\sigma}_{11})$ and $N(\hat{\mu},\hat{\sigma}_{22})$, respectively.
Step 4: Obtain $\Delta_\delta = \frac{\Omega_1(\delta^*|\mu^{(j-1)},\mathbf{x})}{\Omega_1(\delta^{(j-1)}|\mu^{(j-1)},\mathbf{x})}$ and $\Delta_\mu = \frac{\Omega_2(\mu^*|\delta^{(j)},\mathbf{x})}{\Omega_2(\mu^{(j-1)}|\delta^{(j)},\mathbf{x})}$.
Step 5: Generate sample variates u_1 and u_2 from the uniform $U(0,1)$ distribution and set

$$\delta^{(j)} = \begin{cases} \delta^*, & \text{if } u_1 \leq \min\{1,\Delta_\delta\}, \\ \delta^{(j-1)}, & \text{otherwise.} \end{cases} \quad \text{and} \quad \mu^{(j)} = \begin{cases} \mu^*, & \text{if } u_2 \leq \min\{1,\Delta_\mu\}, \\ \mu^{(j-1)}, & \text{otherwise.} \end{cases}$$

Step 6: Obtain $R^{(j)}(t)$ and $h^{(j)}(t)$, for $t > 0$ by replacing δ and μ with their $\delta^{(j)}$ and $\mu^{(j)}$, respectively.
Step 7: Put $j = j + 1$.
Step 8: Repeat Steps 3-7 $\mathcal{B}$ times and obtain $\delta^{(j)}$ and $\mu^{(j)}$ for $j = 1,2,\ldots,\mathcal{B}$.
Step 9: Compute the Bayes estimates of δ, μ, $R(t)$ or $h(t)$ (say $\tilde{\rho}_{MH}$) under the SEL (12) as

$$\tilde{\rho}_{MH} = \sum_{j=\mathcal{B}_0+1}^{\mathcal{B}}\frac{\rho^{(j)}}{\mathcal{B}-\mathcal{B}_0},$$

where $\mathcal{B}_0$ is burn-in.
Step 10: Compute the $100(1-\tau)\%$ BCI of ρ by ordering $\rho^{(j)}, j = \mathcal{B}_0+1,\mathcal{B}_0+2,\ldots,\mathcal{B}$ as $\rho_{(\mathcal{B}_0+1)},\rho_{(\mathcal{B}_0+2)},\ldots,\rho_{(\mathcal{B})}$. Thus the $100(1-\tau)\%$ BCI of ρ is obtained as

$$\left\{\rho_{((\mathcal{B}-\mathcal{B}_0)\frac{\tau}{2})},\ \rho_{((\mathcal{B}-\mathcal{B}_0)(1-\frac{\tau}{2}))}\right\}.$$

Step 11: Compute the $100(1-\tau)\%$ HPD interval of ρ as

$$\left\{\rho_{(j^*)},\rho_{(j^*+(\mathcal{B}-\mathcal{B}_0)(1-\tau))}\right\},$$

where $j^* = \mathcal{B}_0+1,\mathcal{B}_0+2,\ldots,\mathcal{B}$ is selected so that

$$\rho_{(j^*+[(\mathcal{B}-\mathcal{B}_0)(1-\tau)])}-\rho_{(j^*)} = \min_{1\leqslant j\leqslant\tau(\mathcal{B}-\mathcal{B}_0)}\left[\rho_{(j+[(\mathcal{B}-\mathcal{B}_0)(1-\tau)])}-\rho_{(j)}\right].$$

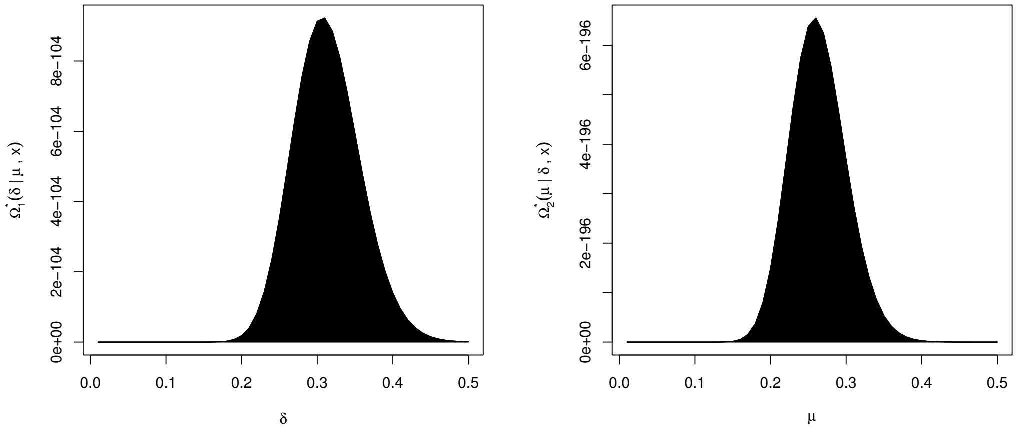


Figure 2. Conditional PDFs of δ and μ.

5. Numerical Comparisons

To compare the efficiency of the proposed estimators of δ, μ, $R(t)$, and $h(t)$ described in the preceding sections, based on 2000 PCTII samples generated from WNH$(0.3, 0.1)$ via Algorithm 3, an extensive Monte Carlo simulation is conducted. At distinct time $t = 5$, the corresponding actual values of $R(t)$ and $h(t)$ are taken as 0.935 and 0.012, respectively. Various combinations of the total sample size n and effective sample size m are also considered, such as $n(= 50, 100, 150)$, and m is taken as 50% and 80% for each n. Three different progressive designs, where $0^*(m-1)$ means that 0 is repeated $(m-1)$ times, are used, namely:

$$
\begin{aligned}
&\text{S1}: \quad \mathbf{R} = (n-m, 0^*(m-1)),\\
&\text{S2}: \begin{cases} \mathbf{R} = \left(0^*\left(\left(\frac{m}{2}\right)-1\right), n-m, 0^*\left(\frac{m}{2}\right)\right), & \text{if } m \text{ is even;}\\ \mathbf{R} = \left(0^*\frac{(m-1)}{2}, n-m, 0^*\frac{(m-1)}{2}\right), & \text{if } m \text{ is odd,} \end{cases}\\
&\text{S3}: \begin{cases} \mathbf{R} = \left(\frac{(n-m)}{2}, 0^*(m-2), \frac{(n-m)}{2}\right), & \text{if } m \text{ is even;}\\ \mathbf{R} = \left(\frac{(m-1)}{2}, 0^*(m-2), \frac{(m+1)}{2}\right), & \text{if } m \text{ is odd.} \end{cases}
\end{aligned}
$$

To draw a PCTII sample from the WNH distribution for pre-specified values of n, m, and $\mathbf{R}$, we perform the following steps:

Algorithm 3 Progressive Type-II Censored Sampling:

Step 1: Create ϵ independent observations with size m as $\epsilon_1, \epsilon_2, \ldots, \epsilon_m$ from uniform $U(0,1)$ distribution.

Step 2: For specific n, m, and $\mathbf{R}$, set $\nu_i = \epsilon_i^{\left(i+\sum_{j=m-i+1}^{m} R_j\right)^{-1}}$, $i = 1, 2, \ldots, m$.

Step 3: Set $u_i = 1 - \nu_m \nu_{m-1} \cdots \nu_{m-i+1}$ for $i = 1, 2, \ldots, m$.

Step 4: Set $x_i = \mu^{-1}\left[\left(1 - \log\left[\frac{1-u_i}{1+u_i}\right]\right)^{\frac{1}{\delta}} - 1\right]$, $i = 1, 2, \ldots, m$, is a PCTII sample from WNH(δ, μ).

Using each simulated sample, the point estimators, such as maximum likelihood, Lindley, and MCMC, are compared using their mean absolute bias (MAB), and root mean squared error (RMSE) simulated values. Further, the 95% interval estimators, called ACI-NA, ACI-NL, Boot-p, Boot-t, BCI, and HPD intervals, are examined via their simulated average confidence length (ACL) and coverage probability (CP) values. Following Algorithm 1, the bootstrapping procedure of δ, μ, $R(t)$, and $h(t)$ is repeated 10,000 times.

To compute the Bayes estimates and BCI/HPD intervals of δ, μ, $R(t)$, or $h(t)$, we repeat Algorithm 2 12,000 times and ignore the first 2000 times as burn-in. For both WNH parameters δ and μ, two different informative sets of the hyperparameters $(a_i, b_i),\ i = 1, 2$, are used, namely: Prior-1 (P1): $(a_1, a_2, b_1, b_2) = (1.5, 0.5, 5, 5)$ and Prior-2 (P2): $(a_1, a_2, b_1, b_2) = (3, 1, 10, 10)$. These prior sets are specified based on two criteria called prior mean and prior variance in such a way that the prior mean becomes the plausible value of the corresponding unknown parameter. All numerical computations are carried out via R 4.1.2 software utilizing two recommended packages, namely: 'coda' and 'maxLik' packages, as suggested by Plummer et al. [16] and Henningsen and Toomet [10], respectively.

A heatmap is a graphical representation of numerical data used to show relationships between two variables of interest so that individual data points contained in a data set are represented via different colors. Therefore, by R software, the heatmap data tool is used to display the simulated results, including: MAB, RMSE, ACL, and CP values of δ, μ, $R(t)$, and $h(t)$, see Figures 3–6, respectively. Each heatmap displays the proposed estimation methods and the specified test settings on the "x-axis" and "y-axis" lines, respectively. On the other hand, all simulation results of δ, μ, $R(t)$, and $h(t)$ are also available in Tables 1–4.

From Figures 3–6, we can draw the following observations:

- Overall, the Bayes MCMC estimates of all unknown parameters perform better than the MLEs in terms of the smallest values of MABs, RMSEs, and ACLs, as well as the highest CP values.
- As n (or m) increases, the MABs, RMSEs, and ACLs tend to decrease (except CPs increase), which infers that the proposed estimates are asymptotically unbiased and consistent. This result is also reached when the total number of progressive items, $\sum_{i=1}^{m} R_i$, decreases.
- Comparing the proposed point estimation methods on the basis of the smallest MABs and RMSEs, it is clear that the performance of MCMC estimates of all unknown parameters is highly recognizable compared to those obtained based on the Lindley approach, and both exhibit reasonably good performance compared to the likelihood method. It is an anticipated result because the calculated MCMC estimates included additional prior information.
- Comparing the proposed interval estimation methods in terms of the smallest ACLs and highest CPs, it is observed that (i) ACI-NA performs better than ACI-NL, (ii) Boot-t performs better than Boot-p, and (iii) the HPD interval performs better than BCI. This conclusion holds for all unknown parameters. It is also clear, in most cases, that the CPs of classical (or Bayes) interval estimates are mostly close to (or below) the specified nominal level.
- In particular, based on the ACL and CP criteria, it is also noted that the HPD interval of all unknown parameters provides the best bounds compared to its competitive intervals.
- Since the variance in P2 is smaller than the variance in P1, consequently, the Bayes estimates (or BCI/HPD intervals) perform better based on P2 than the others.
- Comparing the proposed censoring schemes, both point and interval estimation methods of δ provide good behavior under S1 (where the survival $n - m$ units are removed in the first stage) compared with other competing schemes. Moreover, both point and interval estimation methods of μ, $R(t)$, and $h(t)$ provide good behavior under S2 (where the survival $n - m$ units are removed in the middle stage) compared to others.
- Lastly, to evaluate the unknown parameter(s) of the WNH lifetime model in the presence of data collected from Type-II progressive censoring, the Bayes MCMC methodology is recommended.

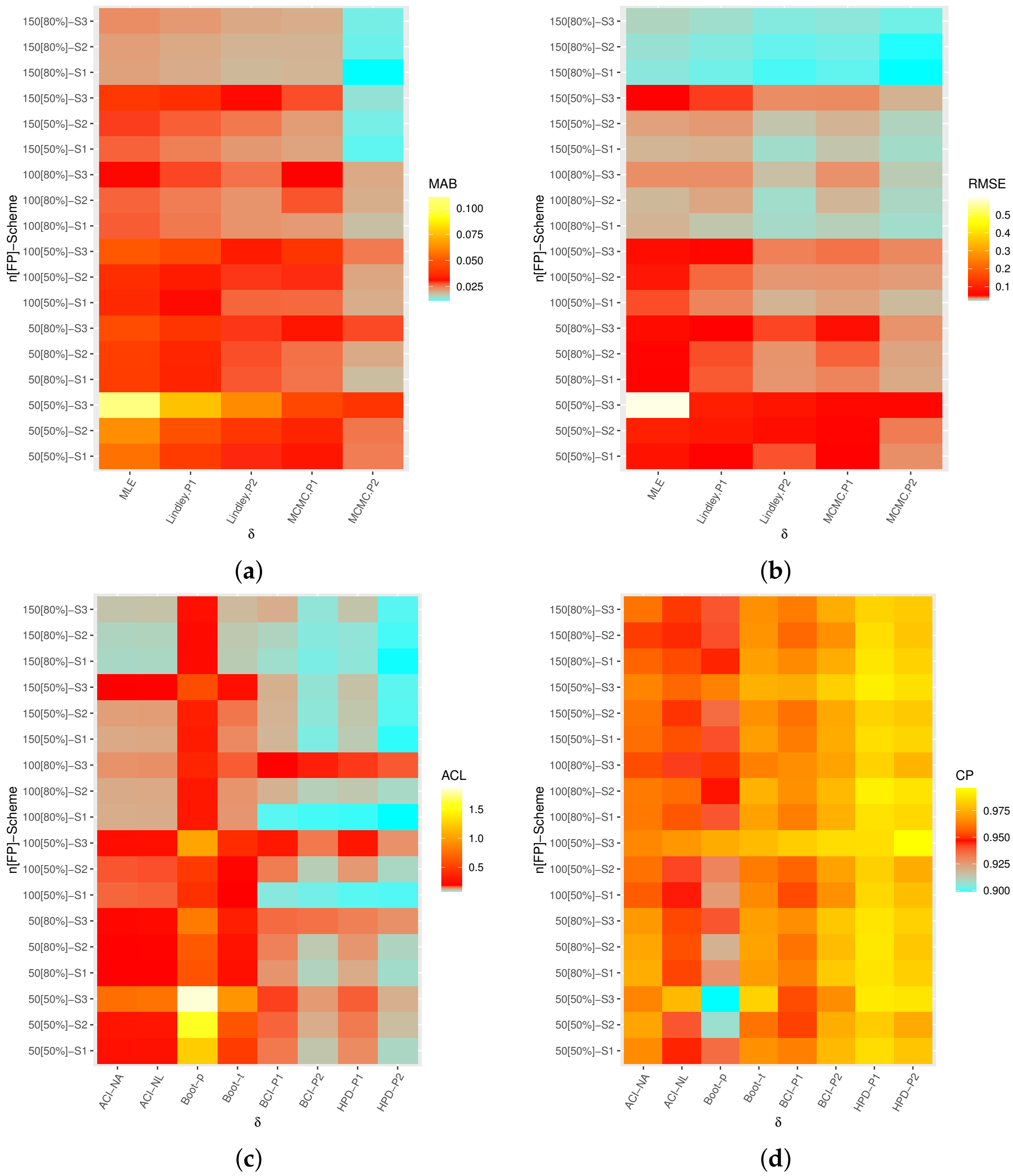


Figure 3. Heatmaps for the simulation results of MAB (**a**), RMSE (**b**), ACL (**c**), and CP (**d**) for δ.

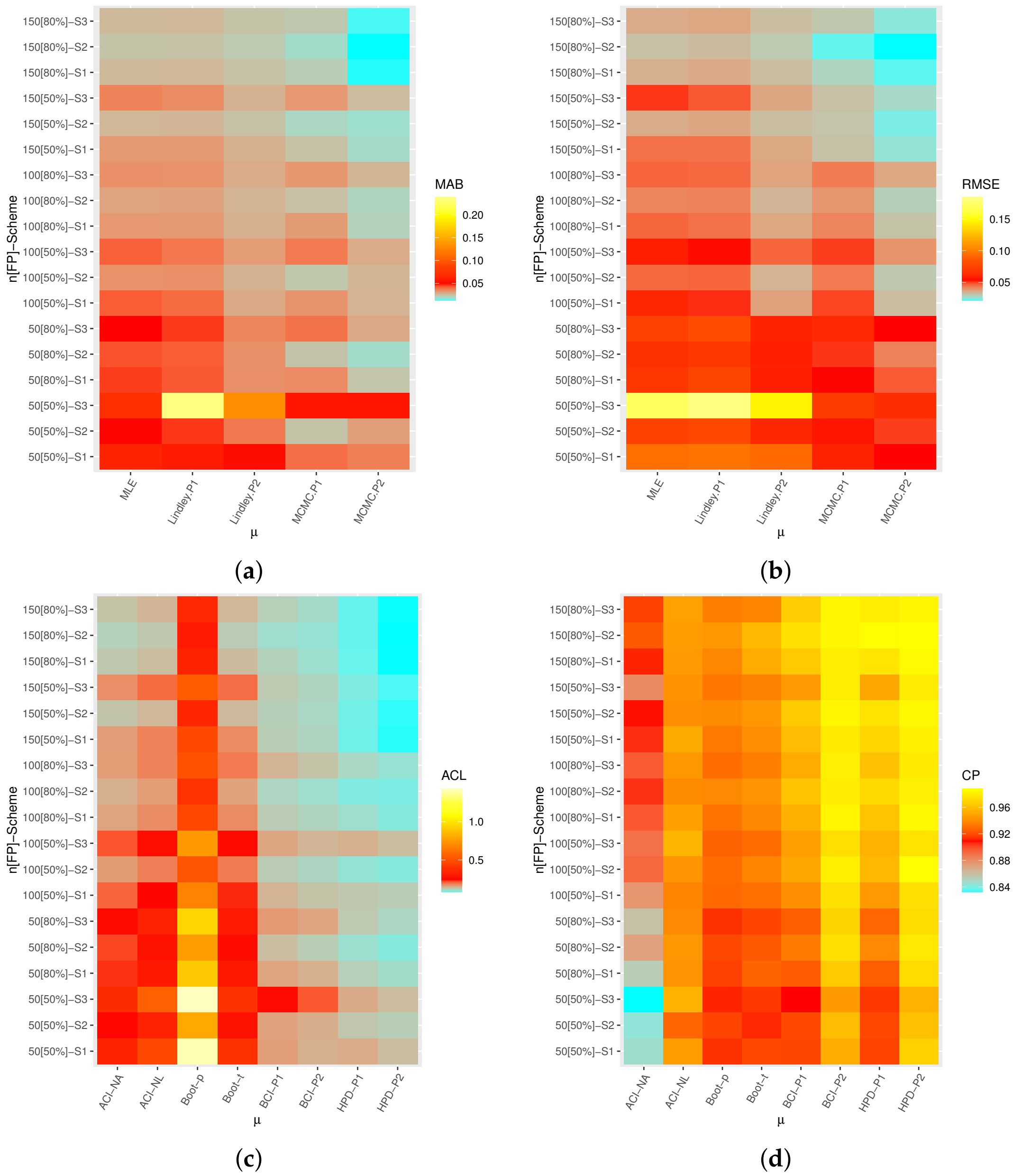


Figure 4. Heatmaps for the simulation results of MAB (a), RMSE (b), ACL (c), and CP (d) for μ.

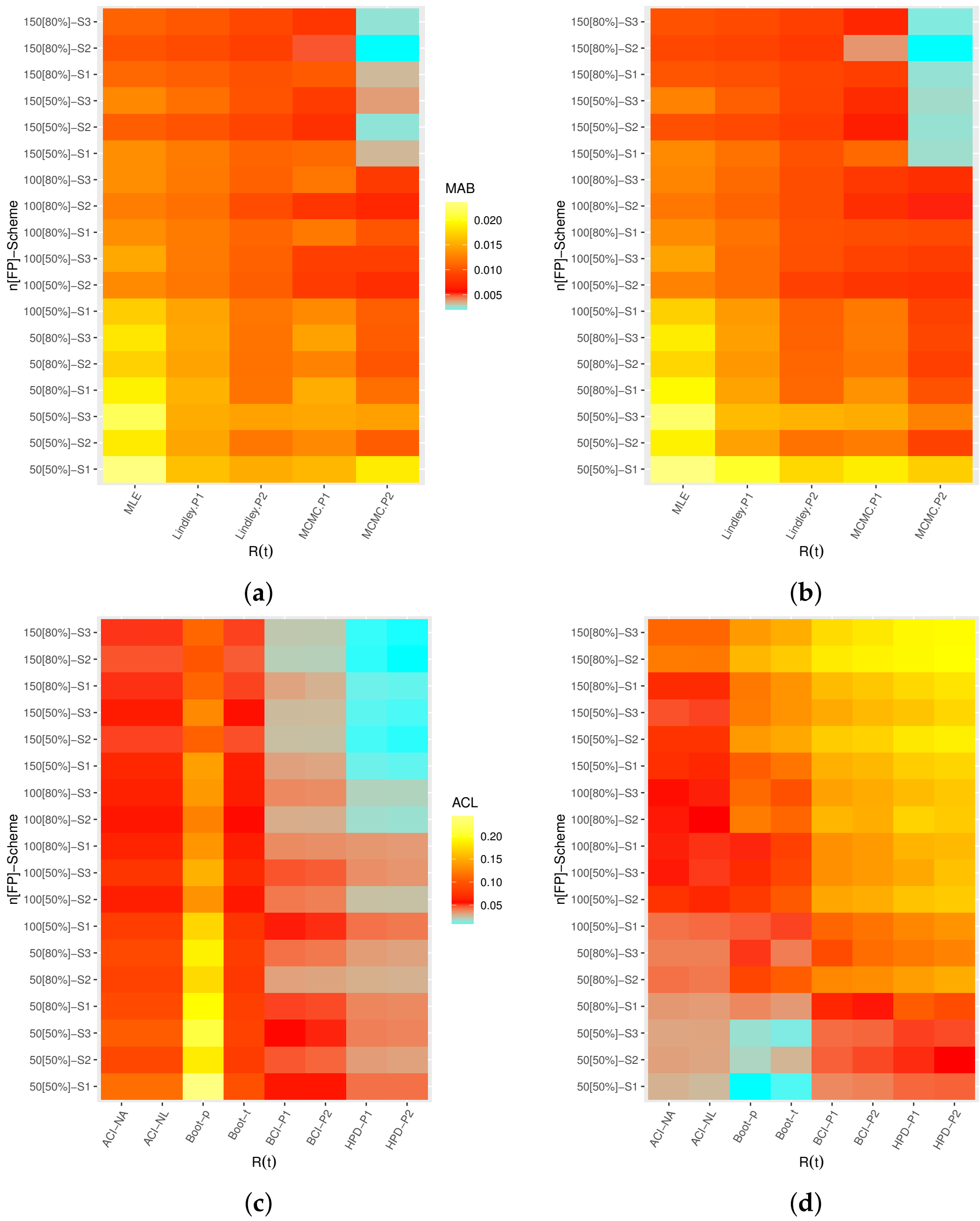


Figure 5. Heatmaps for the simulation results of MAB (**a**), RMSE (**b**), ACL (**c**), and CP (**d**) for $R(t)$.

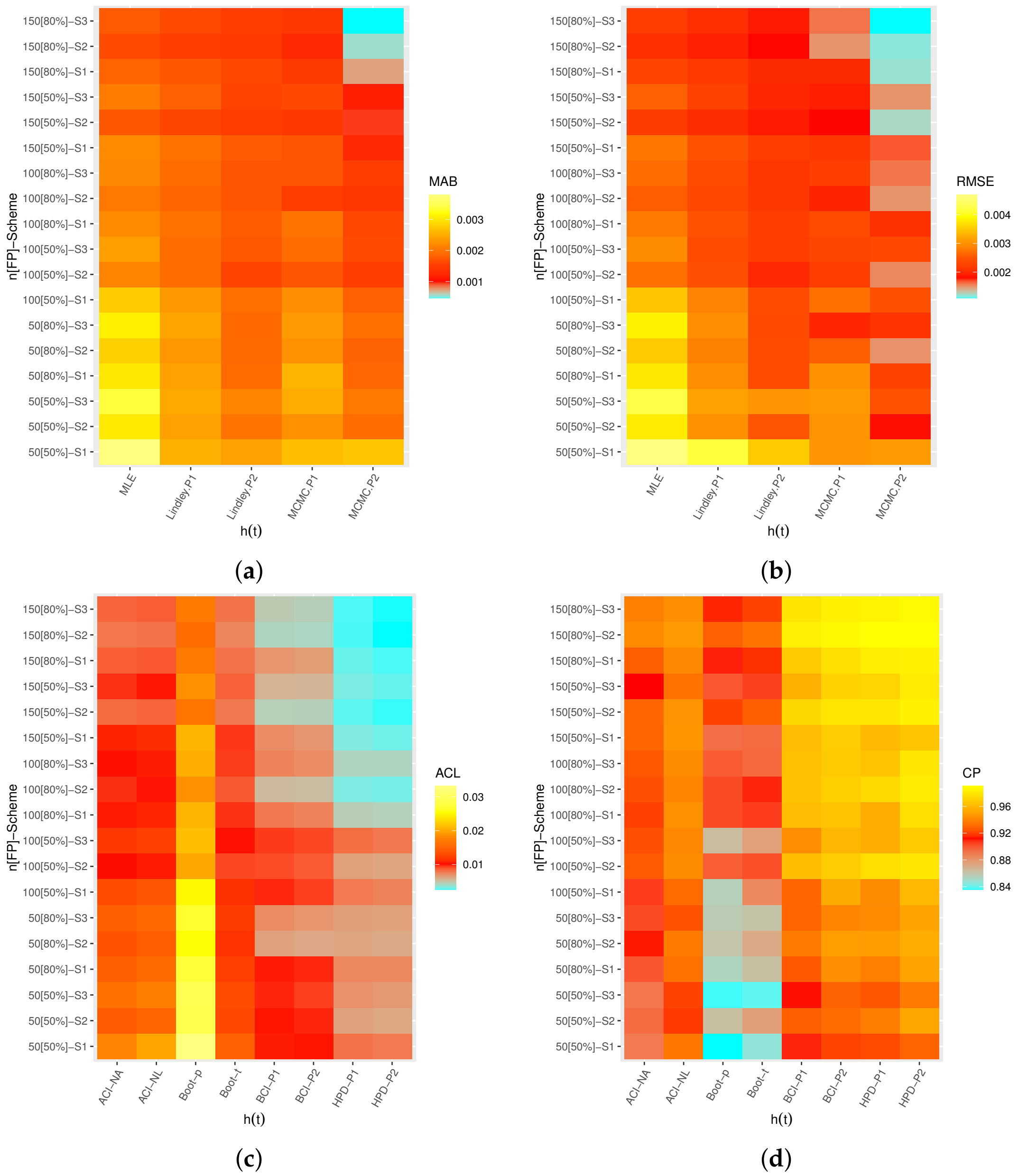


Figure 6. Heatmaps for the simulation results of MAB (**a**), RMSE (**b**), ACL (**c**), and CP (**d**) for $h(t)$.

Table 1. The point estimation results of δ and μ.

Par.	(n,m)	**Scheme**	**MLE**		**Lindley**				**MCMC**			
Prior →					**1**		**2**		**1**		**2**	
			MAB	**RMSE**	**MAB**	**RMSE**	**MAB**	**RMSE**	**MAB**	**RMSE**	**MAB**	**RMSE**
δ	(50,25)	S1	0.0570	0.0832	0.0416	0.0571	0.0309	0.0489	0.0333	0.0561	0.0255	0.0417
		S2	0.0640	0.1025	0.0491	0.0894	0.0401	0.0745	0.0357	0.0583	0.0261	0.0442
		S3	0.1107	0.5980	0.0766	0.0992	0.0630	0.0841	0.0456	0.0644	0.0395	0.0630
	(50,40)	S1	0.0423	0.0575	0.0355	0.0481	0.0284	0.0409	0.0263	0.0432	0.0192	0.0380
		S2	0.0426	0.0592	0.0360	0.0492	0.0290	0.0411	0.0265	0.0473	0.0213	0.0387
		S3	0.0478	0.0687	0.0399	0.0559	0.0303	0.0500	0.0333	0.0527	0.0293	0.0412

Table 1. *Cont.*

Par.	(n,m)	Scheme	MLE		Lindley				MCMC			
Prior →					1		2		1		2	
			MAB	RMSE	MAB	RMSE	MAB	RMSE	MAB	RMSE	MAB	RMSE
	(100,50)	S1	0.0367	0.0493	0.0323	0.0433	0.0274	0.0365	0.0272	0.0388	0.0209	0.0354
		S2	0.0381	0.0525	0.0342	0.0467	0.0303	0.0406	0.0307	0.0409	0.0216	0.0397
		S3	0.0518	0.0740	0.0460	0.0631	0.0341	0.0437	0.0399	0.0455	0.0258	0.0427
	(100,80)	S1	0.0281	0.0364	0.0259	0.0335	0.0235	0.0304	0.0228	0.0321	0.0189	0.0300
		S2	0.0276	0.0357	0.0255	0.0384	0.0234	0.0301	0.0286	0.0360	0.0207	0.0309
		S3	0.0320	0.0419	0.0295	0.0418	0.0264	0.0345	0.0317	0.0414	0.0212	0.0327
	(150,75)	S1	0.0276	0.0362	0.0253	0.0369	0.0229	0.0300	0.0218	0.0337	0.0129	0.0302
		S2	0.0299	0.0392	0.0279	0.0403	0.0259	0.0337	0.0226	0.0364	0.0137	0.0314
		S3	0.0405	0.0543	0.0378	0.0507	0.0322	0.0421	0.0291	0.0423	0.0149	0.0366
	(150,120)	S1	0.0221	0.0282	0.0209	0.0267	0.0197	0.0252	0.0200	0.0260	0.0118	0.0242
		S2	0.0224	0.0292	0.0213	0.0277	0.0201	0.0262	0.0201	0.0268	0.0133	0.0244
		S3	0.0240	0.0313	0.0228	0.0298	0.0213	0.0277	0.0208	0.0285	0.0137	0.0267
μ	(50,25)	S1	0.0628	0.0950	0.0586	0.0965	0.0531	0.0931	0.0410	0.0605	0.0384	0.0542
		S2	0.0511	0.0741	0.0477	0.0787	0.0393	0.0618	0.0255	0.0571	0.0328	0.0513
		S3	0.0716	0.1710	0.2377	0.1840	0.1335	0.1471	0.0561	0.0708	0.0555	0.0641
	(50,40)	S1	0.0470	0.0680	0.0440	0.0769	0.0354	0.0595	0.0362	0.0545	0.0251	0.0484
		S2	0.0449	0.0655	0.0433	0.0686	0.0352	0.0597	0.0253	0.0520	0.0204	0.0434
		S3	0.0507	0.0737	0.0474	0.0805	0.0372	0.0607	0.0403	0.0627	0.0308	0.0542
	(100,50)	S1	0.0434	0.0618	0.0415	0.0525	0.0305	0.0387	0.0347	0.0507	0.0287	0.0336
		S2	0.0350	0.0468	0.0354	0.0473	0.0290	0.0353	0.0246	0.0445	0.0283	0.0317
		S3	0.0429	0.0591	0.0396	0.0554	0.0332	0.0472	0.0391	0.0513	0.0302	0.0405
	(100,80)	S1	0.0340	0.0471	0.0337	0.0459	0.0293	0.0375	0.0334	0.0430	0.0226	0.0326
		S2	0.0316	0.0428	0.0320	0.0433	0.0285	0.0349	0.0261	0.0402	0.0218	0.0305
		S3	0.0354	0.0475	0.0351	0.0468	0.0302	0.0382	0.0340	0.0443	0.0283	0.0375
	(150,75)	S1	0.0336	0.0458	0.0337	0.0456	0.0291	0.0373	0.0257	0.0329	0.0207	0.0266
		S2	0.0279	0.0367	0.0282	0.0375	0.0258	0.0335	0.0215	0.0324	0.0196	0.0249
		S3	0.0376	0.0519	0.0361	0.0487	0.0289	0.0375	0.0339	0.0331	0.0268	0.0286
	(150,120)	S1	0.0275	0.0360	0.0278	0.0369	0.0257	0.0334	0.0234	0.0295	0.0129	0.0232
		S2	0.0254	0.0331	0.0257	0.0341	0.0240	0.0313	0.0205	0.0236	0.0126	0.0215
		S3	0.0276	0.0368	0.0277	0.0374	0.0254	0.0334	0.0244	0.0318	0.0139	0.0261

Table 2. The point estimation results of $R(t)$ and $h(t)$.

Par.	(n,m)	Scheme	MLE		Lindley				MCMC			
Prior →					1		2		1		2	
			MAB	RMSE	MAB	RMSE	MAB	RMSE	MAB	RMSE	MAB	RMSE
$R(t)$	(50,25)	S1	0.0235	0.0301	0.0164	0.0255	0.0152	0.0224	0.0159	0.0237	0.0188	0.0217
		S2	0.0188	0.0240	0.0148	0.0189	0.0122	0.0158	0.0132	0.0164	0.0106	0.0122
		S3	0.0219	0.0287	0.0152	0.0206	0.0147	0.0201	0.0149	0.0196	0.0145	0.0167
	(50,40)	S1	0.0192	0.0245	0.0156	0.0190	0.0120	0.0152	0.0153	0.0178	0.0117	0.0139
		S2	0.0173	0.0222	0.0147	0.0183	0.0118	0.0151	0.0129	0.0160	0.0102	0.0120
		S3	0.0186	0.0236	0.0150	0.0188	0.0119	0.0149	0.0146	0.0163	0.0106	0.0126
	(100,50)	S1	0.0171	0.0218	0.0148	0.0184	0.0121	0.0148	0.0133	0.0162	0.0107	0.0121
		S2	0.0133	0.0169	0.0122	0.0154	0.0107	0.0121	0.0087	0.0111	0.0075	0.0109
		S3	0.0150	0.0190	0.0124	0.0155	0.0108	0.0136	0.0090	0.0122	0.0090	0.0117
	(100,80)	S1	0.0135	0.0174	0.0124	0.0159	0.0111	0.0138	0.0122	0.0133	0.0102	0.0128
		S2	0.0126	0.0161	0.0117	0.0149	0.0098	0.0134	0.0082	0.0107	0.0071	0.0088
		S3	0.0136	0.0170	0.0123	0.0154	0.0108	0.0133	0.0121	0.0113	0.0087	0.0106

Table 2. *Cont.*

Par.	(n,m)	**Scheme**	**MLE**		**Lindley**				**MCMC**			
Prior →					**1**		**2**		**1**		**2**	
			MAB	**RMSE**	**MAB**	**RMSE**	**MAB**	**RMSE**	**MAB**	**RMSE**	**MAB**	**RMSE**
	(150,75)	S1	0.0135	0.0173	0.0124	0.0158	0.0110	0.0137	0.0113	0.0153	0.0035	0.0047
		S2	0.0107	0.0135	0.0101	0.0128	0.0094	0.0118	0.0077	0.0097	0.0026	0.0046
		S3	0.0132	0.0168	0.0117	0.0147	0.0101	0.0125	0.0087	0.0105	0.0040	0.0047
	(150,120)	S1	0.0112	0.0142	0.0107	0.0135	0.0100	0.0125	0.0105	0.0118	0.0035	0.0045
		S2	0.0101	0.0127	0.0097	0.0122	0.0091	0.0114	0.0051	0.0067	0.0021	0.0037
		S3	0.0109	0.0137	0.0103	0.0130	0.0095	0.0120	0.0083	0.0102	0.0026	0.0043
$h(t)$	(50,25)	S1	0.0038	0.0047	0.0026	0.0042	0.0024	0.0035	0.0027	0.0031	0.0027	0.0031
		S2	0.0031	0.0038	0.0024	0.0030	0.0020	0.0025	0.0023	0.0030	0.0019	0.0018
		S3	0.0034	0.0042	0.0025	0.0032	0.0022	0.0031	0.0025	0.0031	0.0021	0.0025
	(50,40)	S1	0.0030	0.0038	0.0024	0.0030	0.0019	0.0024	0.0026	0.0030	0.0019	0.0023
		S2	0.0028	0.0035	0.0023	0.0029	0.0019	0.0024	0.0023	0.0026	0.0018	0.0015
		S3	0.0031	0.0039	0.0024	0.0030	0.0019	0.0024	0.0024	0.0020	0.0020	0.0021
	(100,50)	S1	0.0028	0.0035	0.0023	0.0029	0.0020	0.0024	0.0022	0.0027	0.0018	0.0025
		S2	0.0022	0.0027	0.0019	0.0024	0.0016	0.0020	0.0017	0.0022	0.0015	0.0016
		S3	0.0024	0.0030	0.0019	0.0024	0.0018	0.0022	0.0019	0.0023	0.0016	0.0023
	(100,80)	S1	0.0022	0.0028	0.0020	0.0025	0.0018	0.0022	0.0020	0.0024	0.0016	0.0021
		S2	0.0021	0.0026	0.0019	0.0024	0.0017	0.0021	0.0015	0.0020	0.0014	0.0015
		S3	0.0022	0.0027	0.0019	0.0024	0.0017	0.0021	0.0017	0.0022	0.0015	0.0016
	(150,75)	S1	0.0022	0.0028	0.0020	0.0025	0.0018	0.0022	0.0017	0.0021	0.0013	0.0017
		S2	0.0017	0.0022	0.0016	0.0020	0.0015	0.0019	0.0014	0.0018	0.0010	0.0013
		S3	0.0021	0.0026	0.0018	0.0023	0.0016	0.0020	0.0016	0.0019	0.0012	0.0015
	(150,120)	S1	0.0019	0.0023	0.0017	0.0022	0.0016	0.0020	0.0015	0.0020	0.0008	0.0012
		S2	0.0016	0.0021	0.0015	0.0019	0.0015	0.0018	0.0013	0.0015	0.0006	0.0012
		S3	0.0018	0.0022	0.0016	0.0021	0.0015	0.0019	0.0015	0.0016	0.0005	0.0011

Table 3. The interval estimation results of δ and μ.

(n,m)	Scheme	ACI-NA ACI-NL		Boot-*p* Boot-*t*		BCI-P1 HPD-P1		BCI-P2 HPD-P2		ACI-NA ACI-NL		Boot-*p* Boot-*t*		BCI-P1 HPD-P1		BCI-P2 HPD-P2	
Par. →		δ								μ							
		ACL	CP	ACL	CP	ACL	CP	ACL	CP	ACL	CP	ACL	CP	ACL	CP	ACL	CP
(50,25)	S1	0.269	0.967	1.242	0.936	0.165	0.964	0.119	0.978	0.338	0.848	1.411	0.915	0.173	0.919	0.162	0.919
		0.278	0.949	0.494	0.939	0.155	0.987	0.108	0.980	0.481	0.947	0.408	0.920	0.158	0.954	0.146	0.968
	S2	0.292	0.972	1.583	0.908	0.175	0.952	0.135	0.975	0.260	0.845	0.806	0.919	0.170	0.920	0.137	0.919
		0.304	0.940	0.624	0.961	0.164	0.982	0.125	0.974	0.333	0.928	0.281	0.914	0.160	0.960	0.131	0.962
	S3	0.756	0.965	1.854	0.899	0.188	0.954	0.146	0.967	0.376	0.833	1.432	0.913	0.251	0.910	0.163	0.916
		0.786	0.978	0.956	0.984	0.177	0.990	0.133	0.989	0.566	0.956	0.407	0.916	0.224	0.946	0.147	0.956
(50,40)	S1	0.203	0.975	0.616	0.928	0.149	0.964	0.111	0.982	0.243	0.855	0.916	0.918	0.169	0.925	0.129	0.925
		0.206	0.953	0.263	0.970	0.136	0.988	0.104	0.983	0.303	0.944	0.300	0.927	0.158	0.965	0.117	0.973
	S2	0.207	0.973	0.649	0.920	0.160	0.961	0.117	0.978	0.234	0.873	0.770	0.920	0.145	0.935	0.114	0.940
		0.211	0.955	0.281	0.972	0.149	0.989	0.110	0.981	0.286	0.946	0.269	0.924	0.132	0.975	0.104	0.979
	S3	0.230	0.970	0.825	0.940	0.172	0.967	0.168	0.981	0.265	0.860	0.956	0.915	0.178	0.926	0.135	0.929
		0.236	0.953	0.347	0.971	0.161	0.989	0.152	0.983	0.335	0.940	0.309	0.919	0.168	0.969	0.122	0.973
(100,50)	S1	0.174	0.957	0.443	0.926	0.096	0.953	0.091	0.967	0.218	0.878	0.677	0.929	0.154	0.942	0.136	0.946
		0.176	0.947	0.200	0.966	0.086	0.985	0.086	0.979	0.258	0.938	0.245	0.931	0.140	0.975	0.132	0.975
	S2	0.181	0.961	0.482	0.931	0.163	0.958	0.114	0.972	0.176	0.894	0.519	0.930	0.133	0.951	0.110	0.959
		0.183	0.943	0.219	0.963	0.147	0.983	0.107	0.974	0.198	0.945	0.202	0.938	0.124	0.982	0.104	0.987
	S3	0.258	0.966	1.034	0.974	0.317	0.982	0.165	0.987	0.227	0.891	0.757	0.927	0.166	0.948	0.157	0.956
		0.266	0.969	0.420	0.978	0.300	0.987	0.151	0.996	0.275	0.957	0.266	0.930	0.152	0.974	0.146	0.976
(100,80)	S1	0.134	0.962	0.316	0.940	0.086	0.963	0.084	0.975	0.169	0.898	0.476	0.932	0.133	0.954	0.116	0.963
		0.136	0.956	0.148	0.972	0.083	0.988	0.080	0.985	0.188	0.949	0.188	0.939	0.126	0.984	0.104	0.985
	S2	0.136	0.963	0.319	0.948	0.132	0.968	0.113	0.978	0.159	0.906	0.413	0.940	0.123	0.963	0.103	0.972
		0.137	0.960	0.150	0.976	0.118	0.992	0.107	0.988	0.175	0.941	0.170	0.944	0.116	0.980	0.101	0.981
	S3	0.151	0.954	0.382	0.951	0.207	0.967	0.196	0.972	0.174	0.897	0.512	0.930	0.154	0.953	0.121	0.962
		0.152	0.944	0.178	0.964	0.190	0.984	0.180	0.976	0.196	0.945	0.202	0.936	0.140	0.981	0.111	0.979
(150,75)	S1	0.137	0.960	0.331	0.941	0.129	0.963	0.093	0.974	0.174	0.907	0.477	0.934	0.132	0.960	0.097	0.970
		0.139	0.955	0.155	0.971	0.116	0.987	0.082	0.985	0.195	0.953	0.189	0.942	0.124	0.979	0.085	0.982
	S2	0.143	0.961	0.352	0.936	0.131	0.961	0.098	0.973	0.139	0.911	0.351	0.941	0.130	0.966	0.098	0.976
		0.145	0.951	0.166	0.967	0.119	0.984	0.086	0.982	0.151	0.942	0.149	0.945	0.122	0.984	0.085	0.985
	S3	0.199	0.965	0.586	0.964	0.133	0.974	0.099	0.983	0.186	0.882	0.548	0.933	0.135	0.947	0.102	0.951
		0.202	0.959	0.262	0.975	0.122	0.991	0.087	0.988	0.212	0.943	0.211	0.937	0.125	0.980	0.089	0.980
(150,120)	S1	0.107	0.958	0.240	0.949	0.104	0.966	0.093	0.974	0.136	0.913	0.343	0.939	0.128	0.966	0.095	0.977
		0.108	0.954	0.116	0.971	0.098	0.989	0.081	0.984	0.146	0.947	0.147	0.953	0.115	0.981	0.083	0.986
	S2	0.110	0.952	0.243	0.941	0.110	0.959	0.096	0.967	0.128	0.924	0.307	0.945	0.116	0.975	0.094	0.987
		0.110	0.950	0.117	0.968	0.099	0.987	0.084	0.981	0.136	0.948	0.133	0.959	0.111	0.984	0.083	0.988
	S3	0.120	0.961	0.265	0.940	0.134	0.963	0.099	0.974	0.141	0.918	0.361	0.936	0.130	0.967	0.095	0.980
		0.121	0.951	0.127	0.968	0.120	0.984	0.086	0.982	0.152	0.949	0.152	0.938	0.119	0.984	0.083	0.984

Table 4. The interval estimation results of $R(t)$ and $h(t)$.

(n,m)	Scheme	ACI-NA ACI-NL		Boot-p Boot-t		BCI-P1 HPD-P1		BCI-P2 HPD-P2		ACI-NA ACI-NL		Boot-p Boot-t		BCI-P1 HPD-P1		BCI-P2 HPD-P2	
Par. →		δ								μ							
		ACL	CP	ACL	CP	ACL	CP	ACL	CP	ACL	CP	ACL	CP	ACL	CP	ACL	CP
(50,25)	S1	0.115	0.887	0.243	0.860	0.061	0.900	0.045	0.909	0.018	0.891	0.033	0.836	0.011	0.911	0.008	0.923
		0.115	0.885	0.098	0.863	0.061	0.902	0.045	0.910	0.020	0.936	0.015	0.848	0.011	0.921	0.008	0.930
	S2	0.090	0.893	0.186	0.875	0.050	0.910	0.036	0.919	0.014	0.896	0.029	0.864	0.011	0.928	0.007	0.938
		0.090	0.891	0.081	0.886	0.047	0.915	0.034	0.922	0.015	0.919	0.013	0.878	0.010	0.932	0.006	0.953
	S3	0.106	0.891	0.211	0.871	0.058	0.907	0.042	0.917	0.016	0.892	0.029	0.838	0.010	0.912	0.007	0.926
		0.106	0.892	0.086	0.867	0.055	0.909	0.041	0.915	0.018	0.921	0.013	0.841	0.010	0.929	0.007	0.937
(50,40)	S1	0.093	0.895	0.195	0.900	0.052	0.924	0.041	0.934	0.015	0.902	0.027	0.855	0.010	0.927	0.008	0.939
		0.093	0.894	0.084	0.895	0.051	0.921	0.040	0.930	0.016	0.935	0.012	0.864	0.010	0.944	0.008	0.952
	S2	0.086	0.907	0.176	0.929	0.035	0.945	0.031	0.952	0.014	0.915	0.026	0.862	0.007	0.937	0.006	0.950
		0.086	0.905	0.078	0.934	0.034	0.947	0.031	0.956	0.015	0.938	0.012	0.873	0.006	0.950	0.006	0.956
	S3	0.091	0.903	0.189	0.918	0.045	0.930	0.035	0.941	0.015	0.904	0.027	0.859	0.007	0.930	0.007	0.943
		0.091	0.903	0.081	0.903	0.044	0.938	0.034	0.944	0.016	0.925	0.012	0.863	0.007	0.941	0.007	0.952
(100,50)	S1	0.082	0.906	0.174	0.911	0.056	0.936	0.045	0.945	0.013	0.907	0.025	0.857	0.010	0.931	0.008	0.943
		0.082	0.908	0.076	0.916	0.055	0.939	0.043	0.949	0.014	0.932	0.012	0.886	0.010	0.955	0.008	0.961
	S2	0.065	0.926	0.135	0.927	0.044	0.954	0.027	0.964	0.010	0.927	0.020	0.899	0.009	0.963	0.007	0.976
		0.065	0.925	0.062	0.933	0.042	0.957	0.027	0.966	0.011	0.943	0.009	0.903	0.009	0.969	0.006	0.980
	S3	0.076	0.921	0.153	0.925	0.050	0.948	0.039	0.955	0.012	0.924	0.021	0.865	0.010	0.944	0.008	0.956
		0.076	0.918	0.069	0.929	0.049	0.951	0.037	0.963	0.012	0.942	0.010	0.877	0.009	0.960	0.008	0.968
(100,80)	S1	0.066	0.921	0.138	0.920	0.040	0.948	0.037	0.959	0.011	0.921	0.021	0.904	0.008	0.962	0.005	0.955
		0.066	0.919	0.064	0.928	0.039	0.951	0.036	0.960	0.011	0.945	0.010	0.907	0.008	0.965	0.005	0.976
	S2	0.061	0.921	0.125	0.942	0.032	0.959	0.019	0.969	0.010	0.924	0.019	0.904	0.006	0.964	0.004	0.974
		0.061	0.922	0.058	0.936	0.031	0.957	0.019	0.966	0.010	0.941	0.009	0.911	0.005	0.970	0.003	0.982
	S3	0.065	0.923	0.139	0.937	0.040	0.953	0.022	0.961	0.010	0.927	0.021	0.900	0.008	0.963	0.005	0.965
		0.065	0.921	0.064	0.931	0.039	0.956	0.022	0.965	0.011	0.943	0.010	0.896	0.007	0.968	0.005	0.980
(150,75)	S1	0.068	0.926	0.143	0.934	0.034	0.958	0.014	0.967	0.011	0.931	0.021	0.894	0.007	0.962	0.004	0.961
		0.068	0.925	0.064	0.940	0.034	0.960	0.013	0.970	0.011	0.947	0.010	0.896	0.007	0.969	0.003	0.966
	S2	0.052	0.926	0.108	0.951	0.027	0.967	0.012	0.976	0.009	0.931	0.017	0.921	0.005	0.973	0.003	0.980
		0.052	0.926	0.051	0.955	0.027	0.969	0.011	0.978	0.009	0.947	0.008	0.928	0.005	0.979	0.003	0.984
	S3	0.063	0.914	0.130	0.942	0.028	0.956	0.013	0.964	0.010	0.913	0.019	0.901	0.006	0.957	0.003	0.972
		0.063	0.916	0.060	0.949	0.028	0.960	0.012	0.970	0.010	0.935	0.009	0.906	0.006	0.971	0.003	0.982
(150,120)	S1	0.054	0.925	0.110	0.941	0.034	0.961	0.014	0.971	0.009	0.928	0.017	0.911	0.007	0.968	0.003	0.982
		0.054	0.925	0.052	0.949	0.031	0.965	0.013	0.975	0.009	0.942	0.008	0.918	0.007	0.977	0.003	0.984
	S2	0.050	0.942	0.103	0.960	0.023	0.977	0.011	0.982	0.008	0.943	0.016	0.929	0.005	0.983	0.003	0.989
		0.050	0.941	0.049	0.966	0.023	0.980	0.010	0.984	0.008	0.949	0.008	0.935	0.004	0.987	0.003	0.990
	S3	0.054	0.936	0.111	0.951	0.025	0.972	0.011	0.982	0.009	0.939	0.017	0.916	0.005	0.979	0.003	0.985
		0.054	0.936	0.053	0.957	0.025	0.976	0.010	0.983	0.009	0.945	0.008	0.921	0.005	0.984	0.003	0.988

6. Optimum PCTII Fashions

The issue of determining the optimum progressive censoring (removal pattern design **R**) from a group of all possible plans is a great challenge for any practitioner. In this context, independently, Balakrishnan and Aggarwala [17] and Ng et al. [18] first studied this issue through various setups. The main idea of optimal progressive censoring is, for fixed n and m, that the experimenter chooses the scheme $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ in the sense it provides a significant amount of information of the unknown parameter(s) under study. According to the experimenter's knowledge about the availability of test items, experimental sources, and test cost, the best censoring plan is selected.

In the literature, for different lifetime models, various works have investigated the problem of optimum censoring; see, for example, Pradhan and Kundu [19], Elshahhat and Nassar [20], Elshahhat and Abu El Azm [7], among others. In Table 5, to select the optimal PCTII plan, several criteria are listed.

Table 5. Optimum criteria of PCTII fashions.

Criterion	Goal	Formula
1	*Maximize* $\text{Trace}\{\mathbf{I}(\hat{\delta}, \hat{\mu})\}$	*Maximize* $\{-(\mathcal{L}_{11} + \mathcal{L}_{22})\}$
2	*Minimize* $\text{Trace}\{\mathbf{I}^{-1}(\hat{\delta}, \hat{\mu})\}$	*Minimize* $\{\hat{\sigma}_{11} + \hat{\sigma}_{22}\}$
3	*Minimize* $\text{Det}\{\mathbf{I}^{-1}(\hat{\delta}, \hat{\mu})\}$	*Minimize* $\{\hat{\sigma}_{11}\hat{\sigma}_{22} - \hat{\sigma}_{12}\hat{\sigma}_{21}\}$
4	*Minimize* $\widehat{var}\{\log(\hat{\mathcal{T}}_{\varrho})\}$	*Minimize* $\widehat{var}\left\{\log\left(\mu^{-1}\left[\left(1-\log\left[\frac{1-\varrho}{1+\varrho}\right]\right)^{\frac{1}{\delta}}-1\right]\right)\right\}$

From Table 5; regarding criterion 1, the experimenter aims to increase the main diagonal values of the observed Fisher information. Further, regarding criteria 2 and 3, the experimenter aims to reduce the trace and determinant values of the estimated VCov matrix. Under criterion 4, the experimenter aims to minimize the estimated variance in the logarithm of the MLE of ϱth quantile, $\log(\hat{\mathcal{T}}_{\varrho})$ for $0 < \varrho < 1$.

The optimized censoring should coincide with the largest value of criterion 1 and the smallest value of the other criteria. According to criterion 4, using the delta method, the variance of $\log(\hat{\mathcal{T}}_{\varrho})$ is given by

$$\widehat{var}(\log(\hat{\mathcal{T}}_{\varrho})) = [\nabla \log(\hat{\mathcal{T}}_{\varrho})]^{\mathsf{T}} \mathbf{I}^{-1}(\hat{\delta}, \hat{\mu}) [\nabla \log(\hat{\mathcal{T}}_{\varrho})],$$

where $[\nabla \log(\hat{\mathcal{T}}_{\varrho})]^{\mathsf{T}} = \left[\frac{\partial}{\partial \delta} \log(\hat{\mathcal{T}}_{\varrho}) \qquad \frac{\partial}{\partial \mu} \log(\hat{\mathcal{T}}_{\varrho}) \right]_{(\delta=\hat{\delta}, \mu=\hat{\mu})}$.

7. Three Engineering Applications

To show the applicability of the proposed estimation methodologies to real phenomena, we will analyze three different actual data sets. The first data set (say Data A), reported by Mann [21] and re-analyzed by Alotaibi et al. [22], consists of the number of vehicle fatalities for 39 counties in South Carolina in 2012. The second data set (say Data B), given by Wang [23] and discussed earlier by Elshahhat and Abu El Azm [7], represents the times to failure of eighteen electronic devices. The last data set (say Data C), taken from Lawless [24] and Mohammed et al. [25], represents the failure times (in minutes) of fifteen electronic components in an accelerated life test. These data sets are reported in Table 6.

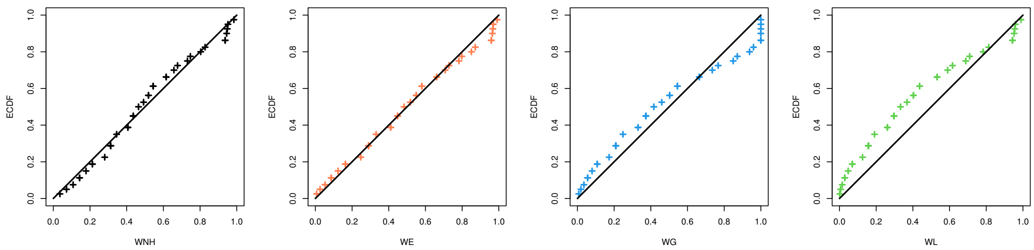
Table 6. The failure times of three real-life data sets.

Data	Times													
A	1	2	3	4	4	5	6	6	8	9	9	9	9	10	12
	12	13	13	13	14	15	16	16	17	17	20	20	22	23	26
	27	31	33	48	48	50	51	52	68						
B	5	11	21	31	46	75	98	122	145	165	196	224	245	293	321
	330	350	420												
C	1.4	5.1	6.3	10.8	12.1	18.5	19.7	22.2	23.0	30.6	37.3	46.3	53.9	59.8	66.2

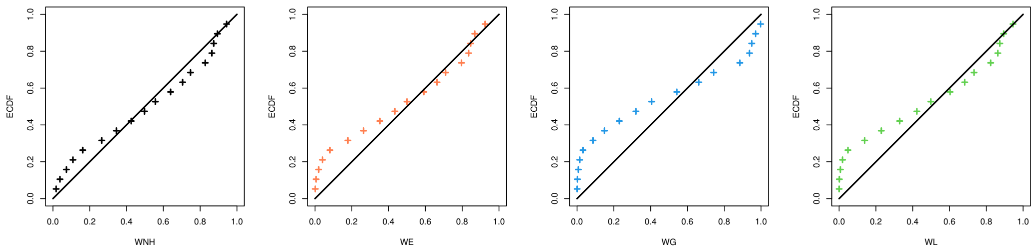
To highlight the flexibility and adaptability of the proposed model, based on different criteria, the WNH distribution is compared with three weighted distributions as competitors, namely: weighted exponential (WE) by Gupta and Kundu [26], weighted Gompertz (WG) by Bakouch et al. [27] and new weighted Lindley (WL) by Asgharzadeh et al. [28] distributions. The respective PDFs of the compared WE, WG, and WL distributions (for $x > 0$ and $\delta, \mu > 0$) are:

- $f_{WE}(x) = \frac{\delta+1}{\delta} \mu e^{-\mu x} (1 - e^{-\delta \mu x})$
- $f_{WG}(x) = \frac{\mu \delta^2}{1+\mu\delta} (e^{\mu x} + \mu - 1) e^{\mu x - \delta(e^{\mu x} - 1)}$
- $f_{WL}(x) = \frac{\mu^2 (\delta+1)^2 (1+x) e^{-\mu x}}{\delta^2 + 2\delta + \delta\mu(\delta+1)} (1 - e^{-\delta \mu x})$

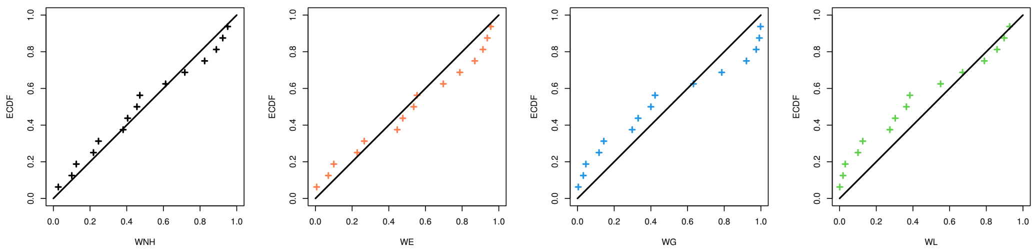
For each data set, the parameter estimates of δ and μ with their standard errors (SEs) of each competing model are evaluated by the ML method. Moreover, the estimated values of the negative log-likelihood (NL), Akaike information (AI), consistent Akaike information (CAI), Bayesian information (BI), Hannan–Quinn information (HQI) and the Kolmogorov–Smirnov statistic with its P-value are computed and reported in Table 7. Figure 7 represents the probability-probability (PP) plots of the WNH, WE, WG and WL distributions. It indicates that the plotted points of the fitted WNH distribution based on each data set are quite close to the empirical distribution line. It also confirms the same findings in Table 7. Moreover, Figure 8 displays the estimated PDFs with their histograms as well as the estimated RFs for the compared distributions under the three data sets. It is clear that the fitted WNH density captures the empirical histograms well. To sum up, we concluded that the WNH distribution has a favorable superiority over other weighted distributions because it has the smallest values of the fitting criteria based on the given data sets. For more illustration, for each real data set, the empirical/estimated TTT transform plot of the WNH distribution is provided in Figure 9. It exhibits that the given real data sets have a constant hazard rate function.



(**a**) Data A



(**b**) Data B



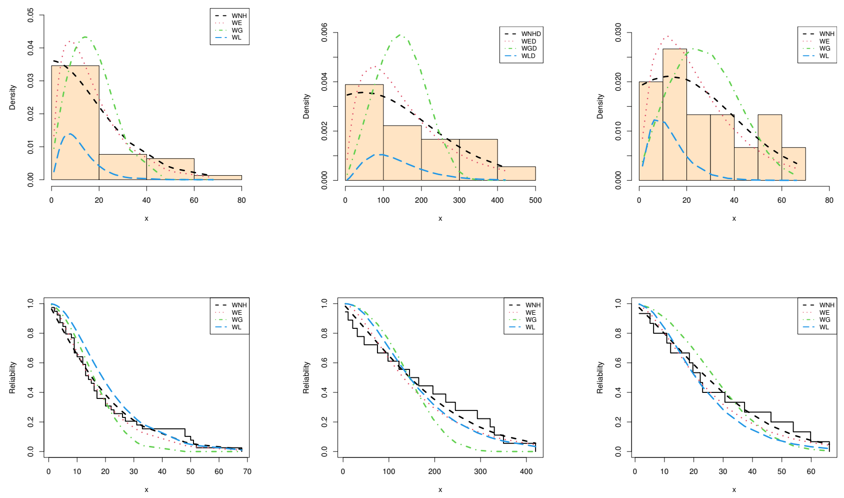
(**c**) Data C

Figure 7. The PP plots of the WNH distribution and its competitive models under three real data sets.

Table 7. Parameter estimates (SEs) and summary fit for the three data sets.

Data	Model	$\hat{\delta}$	$\hat{\mu}$	NL	AI	CAI	BI	HQI	KS(*P*-value)
A	WNH	0.9853(0.3123)	0.0734(0.0415)	154.062	312.125	312.458	315.452	313.318	0.0967(0.859)
	WE	0.9438(0.5143)	0.0978(0.0820)	155.563	315.126	315.459	318.453	316.319	0.1351(0.475)
	WG	1.9629(0.8424)	0.0379(0.0441)	183.312	370.623	370.957	373.950	371.817	0.1534(0.318)
	WL	1.0323(0.7126)	0.1168(0.0886)	245.940	495.880	496.213	499.207	497.073	0.1534(0.318)
B	WNH	1.2847(0.5563)	0.0054(0.0035)	110.007	224.015	224.815	225.796	224.260	0.1157(0.947)
	WE	0.9656(0.4975)	0.0080(0.0044)	112.999	229.997	230.797	231.778	230.243	0.1843(0.516)
	WG	1.5689(0.9971)	0.0055(0.0037)	126.963	255.926	256.726	257.707	256.171	0.2807(0.095)
	WL	1.5683(0.9768)	0.0141(0.1129)	160.595	325.191	325.991	326.971	325.436	0.2015(0.404)
C	WNH	1.6947(1.6346)	0.0224(0.0309)	63.9266	131.853	132.853	133.269	131.383	0.1284(0.939)
	WE	1.7791(1.4654)	0.0514(0.0694)	64.6491	133.298	134.298	134.714	133.283	0.1230(0.956)
	WG	1.5548(1.5448)	0.0288(0.0536)	68.6265	141.253	142.253	142.669	141.238	0.2000(0.522)
	WL	1.8443(1.5011)	0.0969(0.0972)	106.630	217.261	218.261	218.677	217.246	0.1996(0.524)

Next, using various choices of m and $\mathbf{R}$, three different PCTII samples from each complete real data set are generated and listed in Table 8. For each PCTII sample in Table 8, the theoretical results of δ, μ, $R(t)$, and $h(t)$ are evaluated numerically via R software. The point estimates (including: MLEs, Lindley, and MCMC) with their SEs as well as the 95% interval estimates (including: ACI-NA, ACI-NL, Boot-p, Boot-t, BCI, and HPD) of δ, μ, $R(t)$, and $h(t)$ (at $t = 10$) are presented in Tables 9 and 10, respectively. To examine the uniqueness and existence of the MLEs of δ and μ for Ai, Bi, and Ci for $i = 1, 2, 3$ samples, the contour plots of the log-likelihood function (7) are shown in Figure 10. It shows that all acquired estimates of $\hat{\delta}$ and $\hat{\mu}$ exist and are unique. For obtaining the Bayes MCMC estimates along with their BCI/HPD intervals based on noninformative gamma prior, the M-H algorithm proposed in Section 4.2 is repeated 30,000 times, and the first 5000 draws are thrown out. From Tables 9 and 10, in terms of minimum standard errors and interval lengths, it can be seen that the MCMC estimates are more precise than the classical estimates. Moreover, the two interval bounds obtained by asymptotic, bootstrap, and credible intervals are close to each other.



(i) Data A (ii) Data B (iii) Data C

Figure 8. The estimated PDFs (left) and estimated RFs (right) of the competing models under the three real data sets.

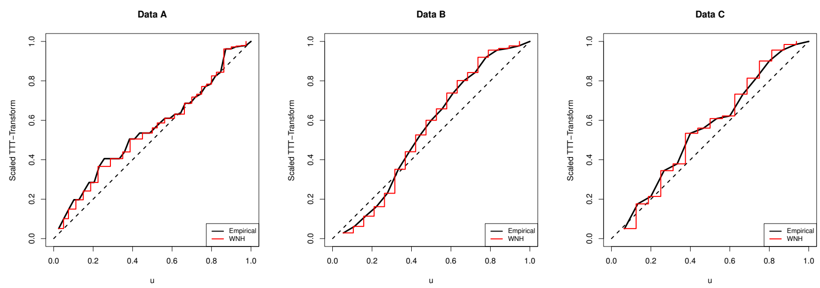
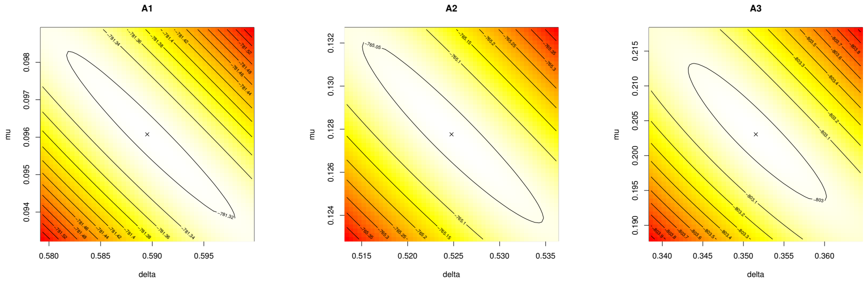


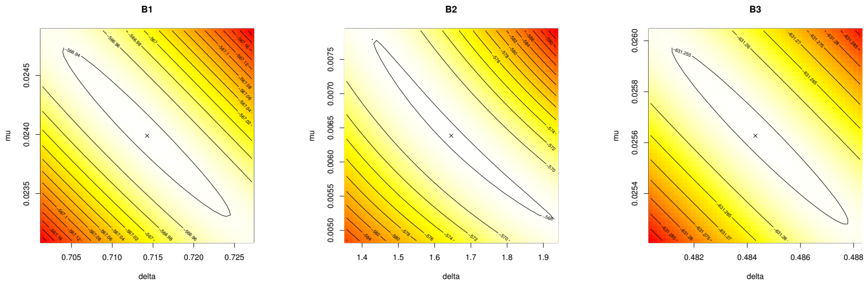
Figure 9. TTT-Transform plots of the WNH distribution under the three real data sets.

Table 8. Artificial PCTII samples from the real data sets.

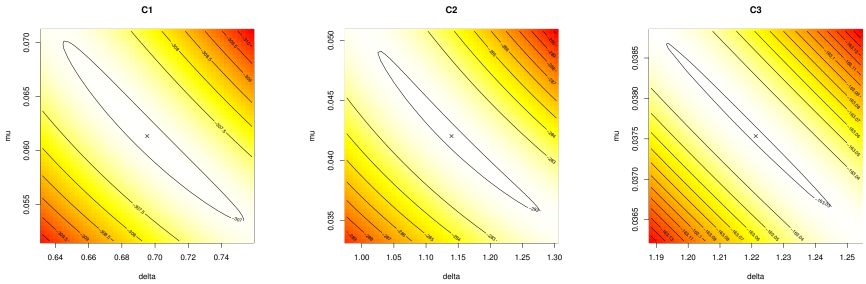
Data	(n, m)	Sample	Scheme	PCTII Data
A	(39,13)	A1	$(26, 0^{*}12)$	1, 2, 4, 5, 6, 9, 10, 12, 12, 13, 16, 22, 27
		A2	$(0^{*}6, 26, 0^{*}6)$	1, 2, 3, 4, 4, 5, 6, 6, 9, 12, 15, 26, 33
		A3	$(0^{*}12, 26)$	1, 2, 3, 4, 4, 5, 6, 6, 8, 9, 9, 9, 9
B	(18,10)	B1	$(8, 0^{*}9)$	5, 11, 21, 31, 46, 98, 122, 165, 224, 293
		B2	$(0^{*}4, 4, 4, 0^{*}4)$	5, 11, 21, 31, 46, 75, 98, 122, 165, 245
		B3	$(0^{*}9, 8)$	5, 11, 21, 31, 46, 75, 98, 122, 145, 165
C	(15,8)	C1	$(7, 0^{*}7)$	1.4, 5.1, 10.8, 12.1, 19.7, 23.0, 37.3, 46.3
		C2	$(0^{*}4, 7, 0^{*}3)$	1.4, 5.1, 6.3, 10.8, 12.1, 19.7, 22.2, 37.3
		C3	$(0^{*}7, 7)$	1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2



(**a**) Data A



(**b**) Data B



(**c**) Data C

Figure 10. Contour plots of the log-likelihood function of δ and μ from the three data sets

In Figure 11, based on samples A1, B1, and C1 (as an example), the simulated estimates with their histograms for the marginal posterior distribution functions of δ, μ, $R(t)$, and $h(t)$ are plotted with the Gaussian line. It is evident that the MCMC estimates behave appropriately with the normal distribution, where (:) represents the mean sample line. In addition, trace plots using A1, B1, and C1 of the same parameters are displayed in Figure 11 where the sample mean, and 95% BCI/HPD intervals are shown with solid (—), dotted ($\cdots$), and dashed (- - -) horizontal lines, respectively. It shows that the MCMC iterations based on each generated PCTII sample are mixed well.

Using the generated PCTII samples in Table 8, the optimal progressive censoring plans, with respect to different criteria reported in Table 5, are obtained and reported in Table 11. It is noted, from Table 11, that

- For Data A; the censoring schemes used in samples A1 (with respect to criteria 1 and 3) and A3 (with respect to criteria 2 and 4) are the optimum plans.
- For Data B; the censoring schemes used in samples B1 (with respect to criterion 3), B2 (with respect to criterion 1), and B3 (with respect to criteria 2 and 4) are the optimum plans.
- For Data C; the censoring schemes used in samples C1 (with respect to criteria 2 and 3), C2 (with respect to criterion 1) and C3 (with respect to criterion 4) are the optimum plans.

Table 11 also supports the same results of the proposed censoring plans (Si, $i = 1,2,3,$) discussed in the simulation section. Briefly, we concluded that the WNH lifetime model provides a good and adequate explanation of the proposed censoring plan for all given real data sets.

Table 9. Point estimates (SEs) of δ, μ, $R(t)$, and $h(t)$ from the three data sets.

Data	Sample	Parameter	MLE	Lindley	MCMC
A	A1	δ	0.5897(0.3913)	0.6326	0.5887(0.0098)
		μ	0.0960(0.1079)	0.1052	0.0944(0.0095)
		$R(5)$	0.8706(0.0480)	0.8842	0.8730(0.0119)
		$h(5)$	0.0272(0.0087)	0.0238	0.0267(0.0027)
	A2	δ	0.5248(0.2854)	0.5721	0.5237(0.0099)
		μ	0.1278(0.1221)	0.1579	0.1264(0.0099)
		$R(5)$	0.8531(0.0483)	0.8793	0.8549(0.0188)
		$h(5)$	0.0304(0.0089)	0.0244	0.0300(0.0036)
	A3	δ	0.3515(0.1614)	0.4014	0.3503(0.0099)
		μ	0.2031(0.1882)	0.2776	0.2017(0.0100)
		$R(5)$	0.8612(0.0467)	0.8740	0.8625(0.0106)
		$h(5)$	0.0258(0.0072)	0.0253	0.0255(0.0022)
B	B1	δ	0.7141(0.4439)	0.8526	0.7138(0.0050)
		μ	0.0240(0.0308)	0.0327	0.0234(0.0042)
		$R(5)$	0.9579(0.0276)	0.9699	0.9589(0.0886)
		$h(5)$	0.0086(0.0056)	0.0072	0.0084(0.0189)
	B2	δ	1.6471(1.9045)	1.9170	1.6471(0.0010)
		μ	0.0064(0.0103)	0.0041	0.0063(0.0008)
		$R(5)$	0.9735(0.0128)	0.9766	0.9738(0.1032)
		$h(5)$	0.0055(0.0027)	0.0060	0.0054(0.0218)
	B3	δ	0.4845(0.3724)	0.4516	0.4845(0.0010)
		μ	0.0256(0.0391)	0.0274	0.0256(0.0010)
		$R(5)$	0.9699(0.0221)	0.9704	0.9700(0.0994)
		$h(5)$	0.0060(0.0042)	0.0070	0.0060(0.0212)
C	C1	δ	0.6951(0.6325)	0.7548	0.6950(0.0010)
		μ	0.0614(0.0973)	0.0685	0.0613(0.0010)
		$R(5)$	0.8981(0.0599)	0.8954	0.8982(0.0276)
		$h(5)$	0.0217(0.0116)	0.0229	0.0216(0.0056)
	C2	δ	1.1384(1.3816)	1.1471	1.1384(0.0010)
		μ	0.0421(0.0741)	0.0426	0.0421(0.0010)
		$R(5)$	0.8791(0.0599)	0.8720	0.8792(0.0090)
		$h(5)$	0.0276(0.0130)	0.0294	0.0276(0.0008)
	C3	δ	1.2189(3.1401)	1.2281	1.2189(0.0010)
		μ	0.0376(0.1245)	0.0330	0.0376(0.0010)
		$R(5)$	0.8836(0.0693)	0.8824	0.8837(0.0031)
		$h(5)$	0.0266(0.0136)	0.0259	0.0266(0.0008)

Table 10. Interval estimates of δ, μ, $R(t)$, and $h(t)$ from the three real data sets.

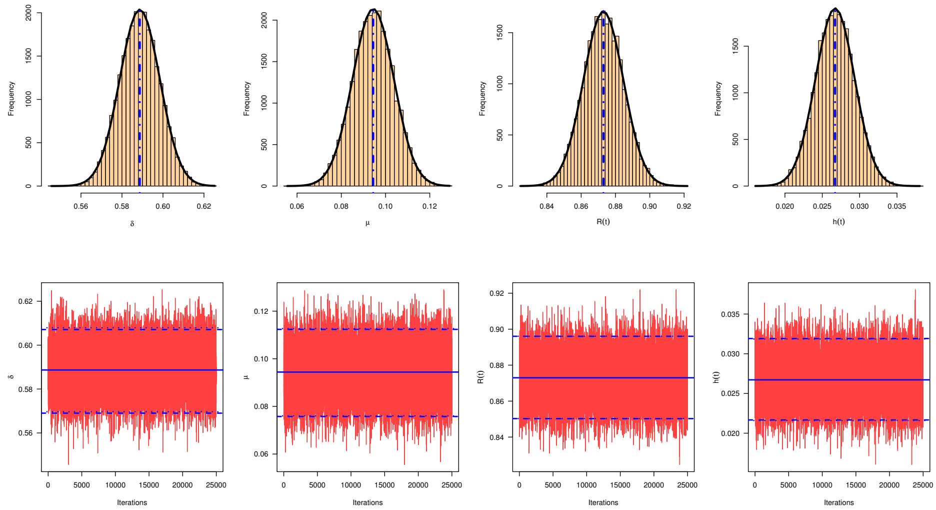
Data	Sample	Par.	ACI-NA ACI-NL			Boot-p Boot-t			BCI HPD		
			Lower	Upper	Length	Lower	Upper	Length	Lower	Upper	Length
A	A1	δ	0.0000	1.3567	1.3567	0.2561	1.0981	0.8420	0.5697	0.6079	0.0382
			0.1606	2.1650	2.0043	0.2555	1.0967	0.8412	0.5690	0.6071	0.0381
		μ	0.0000	0.3076	0.3076	0.0417	0.1788	0.1371	0.0760	0.1127	0.0366
			0.0106	0.8694	0.8588	0.0416	0.1786	0.1370	0.0758	0.1123	0.0366
		$R(5)$	0.7766	0.9647	0.1882	0.8145	0.9215	0.1070	0.8503	0.8961	0.0458
			0.7815	0.9700	0.1885	0.8127	0.9203	0.1076	0.8503	0.8960	0.0458
		$h(5)$	0.0101	0.0443	0.0342	0.0218	0.0507	0.0289	0.0216	0.0319	0.0103
			0.0145	0.0510	0.0365	0.0215	0.0509	0.0294	0.0217	0.0319	0.0102
	A2	δ	0.0000	1.0841	1.0841	0.2279	0.9771	0.7493	0.5044	0.5426	0.0382
			0.1807	1.5237	1.3429	0.2274	0.9758	0.7485	0.5040	0.5421	0.0381
		μ	0.0000	0.3671	0.3671	0.0555	0.2380	0.1825	0.1073	0.1456	0.0383
			0.0197	0.8310	0.8113	0.0558	0.2374	0.1816	0.1074	0.1457	0.0383
		$R(5)$	0.7584	0.9478	0.1894	0.7814	0.8886	0.1072	0.8345	0.8755	0.0409
			0.7634	0.9532	0.1898	0.7805	0.8874	0.1069	0.8339	0.8746	0.0407
		$h(5)$	0.0130	0.0478	0.0348	0.0232	0.0516	0.0284	0.0255	0.0345	0.0090
			0.0172	0.0539	0.0367	0.0229	0.0510	0.0281	0.0257	0.0346	0.0090
	A3	δ	0.0352	0.6679	0.6327	0.1526	0.6546	0.5019	0.3315	0.3696	0.0381
			0.1430	0.8645	0.7216	0.1523	0.6537	0.5014	0.3317	0.3696	0.0379
		μ	0.0000	0.5719	0.5719	0.0882	0.3781	0.3431	0.1823	0.2212	0.0388
			0.0330	1.2484	1.2154	0.0880	0.3776	0.2896	0.1831	0.2218	0.0387
		$R(5)$	0.7696	0.9528	0.1832	0.8439	0.9234	0.0795	0.8489	0.8757	0.0268
			0.7743	0.9579	0.1836	0.8432	0.9220	0.0788	0.8486	0.8754	0.0267
		$h(5)$	0.0116	0.0400	0.0283	0.0171	0.0426	0.0255	0.0230	0.0283	0.0054
			0.0149	0.0447	0.0298	0.0178	0.0425	0.0247	0.0227	0.0281	0.0053

Table 10. *Cont*.

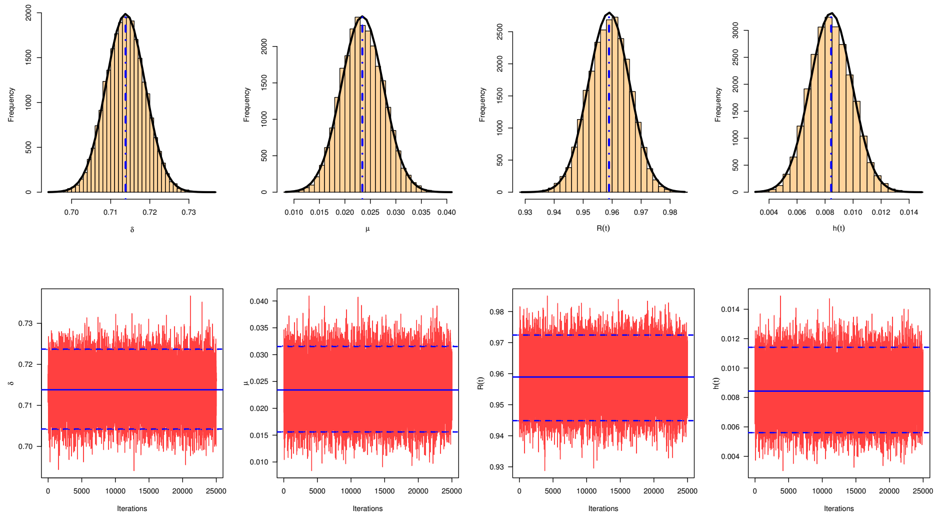
Data	Sample	Par.	ACI-NA ACI-NL			Boot-*p* Boot-*t*			BCI HPD		
			Lower	**Upper**	**Length**	**Lower**	**Upper**	**Length**	**Lower**	**Upper**	**Length**
B	B1	δ	0.0000	1.5840	1.5840	0.3100	1.3296	1.0195	0.7041	0.7236	0.0195
			0.2112	2.4147	2.2035	0.3094	1.3278	1.0185	0.7042	0.7237	0.0195
		μ	0.0000	0.0844	0.0844	0.0104	0.0447	0.0343	0.0156	0.0316	0.0160
			0.0019	0.2975	0.2955	0.0108	0.0447	0.0339	0.0156	0.0315	0.0159
		$R(5)$	0.9038	0.9978	0.0940	0.8675	0.9215	0.0540	0.9447	0.9724	0.0277
			0.9053	0.9935	0.0882	0.8656	0.9193	0.0537	0.9449	0.9724	0.0276
		$h(5)$	0.0012	0.0195	0.0183	0.0018	0.0196	0.0178	0.0055	0.0114	0.0059
			0.0025	0.0305	0.0280	0.0017	0.0194	0.0177	0.0056	0.0114	0.0058
	B2	δ	0.0000	5.3800	5.3800	1.0152	4.1225	3.1073	1.6452	1.6491	0.0040
			0.1708	15.884	15.713	1.0130	4.1171	3.1042	1.6451	1.6491	0.0039
		μ	0.0000	0.0265	0.0265	0.0028	0.0199	0.0171	0.0047	0.0082	0.0035
			0.0003	0.1495	0.1493	0.0026	0.0194	0.0168	0.0047	0.0080	0.0033
		$R(5)$	0.9485	0.9985	0.0500	0.9544	0.9989	0.0445	0.9666	0.9805	0.0139
			0.9488	0.9988	0.0500	0.9532	0.9972	0.0440	0.9668	0.9805	0.0137
		$h(5)$	0.0003	0.0107	0.0104	0.0013	0.0106	0.0093	0.0039	0.0069	0.0030
			0.0021	0.0142	0.0120	0.0014	0.0104	0.0090	0.0040	0.0069	0.0029
	B3	δ	0.0054	1.2143	1.2089	0.2104	0.9021	0.6917	0.4823	0.4864	0.0041
			0.1074	2.1853	2.0779	0.2099	0.9009	0.6910	0.4824	0.4864	0.0040
		μ	0.0000	0.1023	0.1023	0.0078	0.0826	0.0748	0.0237	0.0276	0.0041
			0.0013	0.5099	0.5086	0.0076	0.0821	0.0745	0.0236	0.0275	0.0039
		$R(5)$	0.9266	0.9934	0.0668	0.9439	0.9923	0.0484	0.9678	0.9722	0.0044
			0.9275	0.9917	0.0642	0.9412	0.9889	0.0477	0.9679	0.9723	0.0044
		$h(5)$	0.0002	0.0142	0.0140	0.0017	0.0126	0.0109	0.0054	0.0064	0.0010
			0.0015	0.0235	0.0220	0.0016	0.0124	0.0108	0.0055	0.0064	0.0009
C	C1	δ	0.0000	1.9346	1.9346	0.3018	1.2942	0.9924	0.6931	0.6971	0.0040
			0.1168	4.1357	4.0189	0.3011	1.2925	0.9914	0.6931	0.6970	0.0039
		μ	0.0000	0.2521	0.2521	0.0026	0.1929	0.1903	0.0592	0.0633	0.0041
			0.0027	1.3734	1.3707	0.0024	0.1922	0.1898	0.0594	0.0633	0.0039
		$R(5)$	0.7808	0.9841	0.2033	0.8175	0.9515	0.1340	0.8950	0.9014	0.0063
			0.7881	0.9897	0.2016	0.8167	0.9503	0.1336	0.8951	0.9013	0.0062
		$h(5)$	0.0010	0.0443	0.0433	0.0024	0.0276	0.0252	0.0209	0.0223	0.0014
			0.0076	0.0616	0.0540	0.0023	0.0271	0.0248	0.0209	0.0223	0.0014
	C2	δ	0.0000	3.8464	3.8464	0.4515	3.1225	2.6710	1.1363	1.1403	0.0040
			0.1055	12.285	12.179	0.4505	3.1184	2.6679	1.1365	1.1404	0.0039
		μ	0.0000	0.1874	0.1874	0.0183	0.1299	0.1116	0.0398	0.0440	0.0042
			0.0013	1.3260	1.3246	0.0181	0.1292	0.1111	0.0401	0.0440	0.0039
		$R(5)$	0.7617	0.9966	0.2349	0.8344	0.9786	0.1442	0.8735	0.8849	0.0114
			0.7692	0.9848	0.2156	0.8330	0.9766	0.1436	0.8737	0.8849	0.0112
		$h(5)$	0.0020	0.0531	0.0511	0.0013	0.0274	0.0261	0.0261	0.0290	0.0029
			0.0109	0.0696	0.0587	0.0013	0.0274	0.0260	0.0261	0.0290	0.0029
	C3	δ	0.0000	7.3734	7.3734	0.0257	4.0248	3.9991	1.2168	1.2208	0.0041
			0.0078	190.04	190.03	0.0257	3.9940	3.9683	1.2169	1.2209	0.0040
		μ	0.0000	0.2816	0.2816	0.0178	0.1786	0.1608	0.0355	0.0396	0.0041
			0.0005	24.568	24.568	0.0177	0.1780	0.1603	0.0357	0.0396	0.0039
		$R(5)$	0.7478	0.9897	0.2419	0.7819	0.9628	0.1809	0.8774	0.8897	0.0124
			0.7577	0.9867	0.2290	0.7825	0.9608	0.1783	0.8775	0.8897	0.0122
		$h(5)$	0.0016	0.0532	0.0516	0.0217	0.0646	0.0429	0.0248	0.0282	0.0033
			0.0098	0.0722	0.0625	0.0215	0.0635	0.0420	0.0250	0.0282	0.0031

Table 11. Optimum PCTII plans from the real data sets.

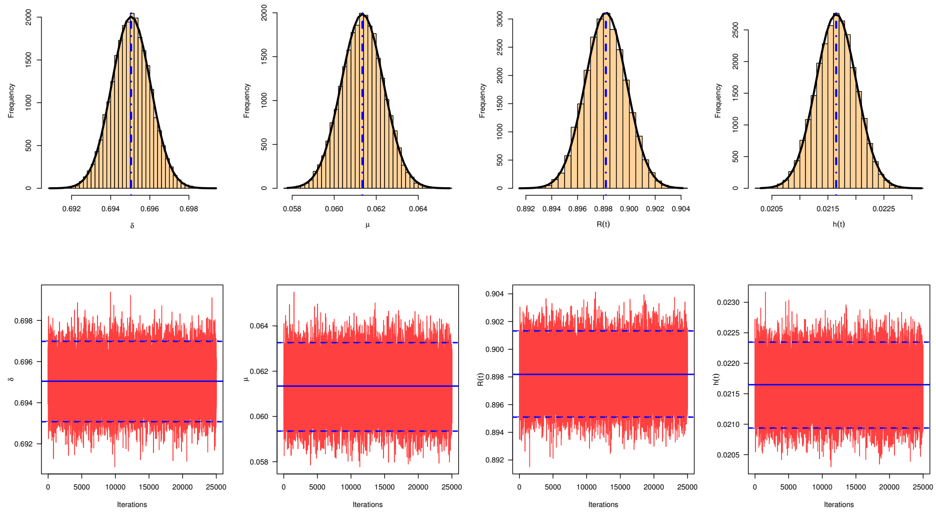
Data	Sample	Criterion					
		1	**2**	**3**	**4**		
$\varrho \rightarrow$					**0.3**	**0.6**	**0.9**
A	A1	1403.933	0.164772	0.000117	16.70746	148.9399	4784.368
	A2	789.1572	0.096350	0.000122	27.44207	805.4266	62029.04
	A3	436.6285	0.061455	0.000141	13.49689	143.3824	4387.135
B	B1	17524.80	0.197976	0.000008	405.2305	1260.421	7284.561
	B2	477786.5	0.627394	0.000012	907.6868	6271.593	321362.3
	B3	11544.30	0.140186	0.000011	337.3565	742.7814	2647.492
C	C1	2303.025	0.409467	0.000178	53.18096	198.7051	4162.369
	C2	6946.600	3.875724	0.000573	23.53281	161.6642	3427.930
	C3	6278.967	1.914403	0.000276	22.39803	60.40023	632.9782



(**a**) Sample A1



(**b**) Sample B1



(**c**) Sample C1

Figure 11. Histograms (top) and Trace (bottom) plots of δ, μ, $R(t)$, and $h(t)$ from the real data sets.

8. Concluding Remarks

This article derives both point and interval estimators of the parameters, reliability, and hazard functions of the new, two-parameter weighted Nadarajah–Haghighi distribution based on Type-II progressive censored sample via the maximum likelihood and Bayesian estimation methods. Further, two parametric bootstrap confidence intervals of the same unknown parameters have been derived. Since the Bayes estimators cannot be written in closed forms, Lindley and MCMC approximations have been investigated. Numerical comparisons have been performed to judge the behavior of the proposed estimators, and they showed that the Bayes estimates using the Metropolis–Hastings algorithm behave well compared to Lindley's procedure, and both behave much better than the frequentist estimates. Three real data sets taken from several areas, namely: insurance, reliability, and accelerated life testing, have been analyzed to show the usefulness of the proposed model in real practice. All theoretical results have been evaluated utilizing two useful packages in R software, namely: 'maxLik' and 'coda' packages. As a result, the Bayes MCMC paradigm is recommended to estimate the model parameters and/or the reliability characteristic of the weighted Nadarajah–Haghighi model in the presence of a sample obtained from Type-II progressive censoring. As a future work, one can extend the results and methodologies discussed here to accelerate life tests or other censoring plans. We also hope that the proposed methodologies will be beneficial to researchers and reliability practitioners.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Fisher elements $\mathcal{L}_{ij},\ i,j=1,2$ from (7) with respect to δ and μ are

$$\begin{aligned}\mathcal{L}_{11}=&-\frac{m}{\delta^2}+\textstyle\sum_{i=1}^{m}(R_i+1)\psi_i^{\delta''}(\delta,\mu)\\&-\textstyle\sum_{i=1}^{m}(R_i+2)\kappa(x_i;\delta,\mu)\left[\psi_i^{\delta''}(\delta,\mu)-\left(\psi_i^{\delta'}(\delta,\mu)\right)^2[\kappa(x_i;\delta,\mu)-1]\right],\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{22}=&-\frac{m}{\mu^2}+\textstyle\sum_{i=1}^{m}(R_i+1)\psi_i^{\mu''}(\delta,\mu)-(\delta-1)\sum_{i=1}^{m}x_i^2(1+\mu x_i)^{-2}\\&-\textstyle\sum_{i=1}^{m}(R_i+2)\kappa(x_i;\delta,\mu)\left[\psi_i^{\mu''}(\delta,\mu)-\left(\psi_i^{\mu'}(\delta,\mu)\right)^2[\kappa(x_i;\delta,\mu)-1]\right],\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_{12}=&\textstyle\sum_{i=1}^{m}(R_i+1)\psi_i^{\delta'\mu'}(\delta,\mu)+\sum_{i=1}^{m}x_i(1+\mu x_i)^{-1}\\&-\textstyle\sum_{i=1}^{m}(R_i+2)\kappa(x_i;\delta,\mu)\left[\psi_i^{\delta'\mu'}(\delta,\mu)-\psi_i^{\delta'}(\delta,\mu)\psi_i^{\mu'}(\delta,\mu)[\kappa(x_i;\delta,\mu)-1]\right],\end{aligned}$$

where $\psi_i^{\eta''}$ is the second-partial derivative with respect η to for $i = 1, 2, \ldots, m$ as $\psi_i^{\delta''}(\delta, \mu) = -(1 + \mu x_i)^{\delta}(\log(1 + \mu x_i))^2$, $\psi_i^{\mu''}(\delta, \mu) = -\delta(\delta - 1)x_i^2(1 + \mu x_i)^{\delta-2}$, $\psi_i^{\delta'\mu'}(\delta, \mu) = -x_i(1 + \mu x_i)^{\delta-1}(1 + \delta\log(1 + \mu x_i))$ and $\kappa(x_i; \delta, \mu) = e^{\psi(x_i;\delta,\mu)}\left(1 + e^{\psi(x_i;\delta,\mu)}\right)^{-1}$.

Appendix B

Lindley's elements $Y_{ij},\ i, j = 0, 1, 2, 3,\ i + j = 3$ from (7) with respect to δ and μ are

$$
\begin{aligned}
Y_{12} = {} & \sum_{i=1}^{m} (R_i + 1)x_i^2(1 + \mu x_i)^{\delta-2}[1 - 2\delta - \delta(\delta - 1)\log(\mu x_i)] - \sum_{i=1}^{m} x_i^2(1 + \mu x i^{-2}) \\
& + \sum_{i=1}^{m} (R_i + 2)x_i^2\kappa(x_i; \delta, \mu)\{\delta(1 + \mu x_i)^{\delta}\kappa(x_i; \delta, \mu) \\
& \times [\{(\delta - 1)(1 + \mu x_i)^{\delta-2} - \delta(1 + \mu x_i)^{2(\delta-1)}\}\log(\mu x_i) \\
& + (1 + \mu x_i)^{\delta-1}[(1 + \mu x_i)^{-1} + \delta\log(\mu x_i)(1 + \mu x_i)^{\delta-1}\kappa(x_i; \delta, \mu)]] - (1 + \mu x_i)^{\delta-2} \\
& + \delta(1 + \mu x_i)^{-1}[(1 + \mu x_i)^{\delta-1} - (1 + \mu x_i)^{2\delta-1}] \\
& + \delta\log(\mu x_i)[(\delta - 1)(1 + \mu x_i)^{\delta-2} - (2\delta - 1)(1 + \mu x_i)^{2(\delta-1)}] \\
& + \delta(1 + \mu x_i)^{\delta-1}[(1 + \mu x_i)^{-1} + \delta\log(\mu x_i)(1 + \mu x_i)^{\delta-1}\kappa(x_i; \delta, \mu)] \\
& - \delta(1 + \mu x_i)^{\delta-1}(1 - \kappa(x_i; \delta, \mu))\{(1 + \mu x_i)^{\delta}[(1 + \mu x_i)^{-1} \\
& + \delta\log(\mu x_i)(1 + \mu x_i)^{\delta-1}\kappa(x_i; \delta, \mu)] \\
& + \delta\log(\mu x_i)[(1 + \mu x_i)^{\delta-1} - (1 + \mu x_i)^{2\delta-1}]\}\},
\end{aligned}
$$

$$
\begin{aligned}
Y_{21} = {} & -\sum_{i}^{m} (R_i + 1)x_i\log(\mu x_i)(1 + \mu x_i)^{\delta-1}(2 + \delta\log(\mu x_i)) \\
& + \sum_{i}^{m} (R_i + 2)x_i\log(\mu x_i)\kappa(x_i; \delta, \mu) \\
& \{\times(1 + \mu x_i)^{2\delta}\kappa(x_i; \delta, \mu)[1 + \delta\log(\mu x_i)\{1 - (1 + \mu x_i)^{\delta}(1 - \kappa(x_i; \delta, \mu))\}] \\
& + (1 + \mu x_i)^{\delta-1}[2 + \delta\log(\mu x_i)(1 + \mu x_i)^{\delta}\kappa(x_i; \delta, \mu)] \\
& + \delta\log(\mu x_i)(1 + \mu x_i)^{\delta-1}[1 - 2(1 + \mu x_i)^{\delta}] - (1 - \kappa(x_i; \delta, \mu))(1 + \mu x_i)^{\delta}\{ \\
& (1 + \mu x_i)^{\delta-1}[1 + \delta\log(\mu x_i)(1 + \mu x_i)^{\delta}\kappa(x_i; \delta, \mu)] \\
& + \delta\log(\mu x_i)(1 + \mu x_i)^{\delta-1}[1 - (1 + \mu x_i)^{\delta}] - (1 + \mu x_i)^{2\delta-1}\}\},
\end{aligned}
$$

$$
\begin{aligned}
Y_{30} = {} & 2m\delta^{-3} + \sum_{i=1}^{m}(R_i + 1)\psi_i^{\delta'''} + \sum_{i=1}^{m} (R_i + 2)\kappa(x_i; \delta, \mu) \\
& \times \{[(1 + \mu x_i)^{2\delta}(\kappa(x_i; \delta, \mu) + (1 + \mu x_i)^{-\delta} - 1)][(1 + \mu x_i)^{\delta}(\kappa(x_i; \delta, \mu) - 1)] \\
& + [(1 + \mu x_i)^{3\delta}(\kappa(x_i; \delta, \mu) + 2(1 + \mu x_i)^{-\delta} - 1)]\kappa(x_i; \delta, \mu) + (1 + \mu x_i)^{\delta} - 2(1 + \mu x_i)^{2\delta}\},
\end{aligned}
$$

$$
\begin{aligned}
Y_{03} = {} & 2m\mu^{-3} + \sum_{i=1}^{m}(R_i + 1)\psi_i^{\mu'''} + \delta\sum_{i=1}^{m} x_i^3(R_i + 2)\kappa(x_i; \delta, \mu) \\
& \times \{(\delta - 1)(\delta - 2)(1 + \mu x_i)^{\delta-3} + \delta\{[(1 + \mu x_i)^{\delta-1}(\kappa(x_i; \delta, \mu) - 1)] \\
& \times [(1 + \mu x_i)^{\delta-2}(\delta - 1) + \delta(1 + \mu x_i)^{2(\delta-1)}(\kappa(x_i; \delta, \mu) - 1)] \\
& + [2(\delta - 1)(1 + \mu x_i)^{2(\delta-1)-1} + \delta(1 + \mu x_i)^{2(\delta-1)+\delta-1}(\kappa(x_i; \delta, \mu) - 1)] \\
& \times \kappa(x_i; \delta, \mu) - 2(\delta - 1)(1 + \mu x_i)^{2(\delta-1)-1})\}\},
\end{aligned}
$$

where $\psi_i^{\eta'''}$ is the third-partial derivative with respect η to for $i = 1, 2, \ldots, m$ as $\psi_i^{\delta'''}(\delta, \mu) = -(1 + \mu x_i)^{\delta}(\log(1 + \mu x_i))^3$ and $\psi_i^{\mu'''}(\delta, \mu) = -\delta(\delta - 1)(\delta - 2)x_i^3(1 + \mu x_i)^{\delta-3}$.

Appendix C

The second-order partial derivatives of $R(t)$ from (3) with respect to δ and μ are

$$v_{11} = -2(\log(\mu t))^2(1+\mu t)^{2\delta}e^{(1+\mu t)^{\delta}-1}(1+e^{(1+\mu t)^{\delta}-1})^{-2}[1+(1+\mu t)^{-\delta}-2\vartheta(t;\delta,\mu)],$$

$$\begin{aligned} v_{22} &= -2\delta t^2 e^{(1+\mu t)^{\delta}-1}(1+e^{(1+\mu t)^{\delta}-1})^{-2} \\ &\times \{(\delta-1)(1+\mu t)^{\delta-2}+\delta((1+\mu t)^{2(\delta-1)}-2(((1+\mu t)^{2(\delta-1)}\vartheta(t;\delta,\mu))))\}, \end{aligned}$$

and

$$\begin{aligned} v_{12} &= -2te^{(1+\mu t)^{\delta}-1}(1+e^{(1+\mu t)^{\delta}-1})^{-2}[(1+\mu t)^{\delta}((1+\mu t)^{-1} \\ &-2(\delta\log(\mu t)(1+\mu t)^{\delta-1}\vartheta(t;\delta,\mu))) \\ &+\delta\log(\mu t)((1+\mu t)^{2\delta-1}+(1+\mu t)^{\delta-1})]. \end{aligned}$$

where $\vartheta(t;\delta,\mu) = e^{(1+\mu t)^{\delta}-1}(1+e^{(1+\mu t)^{\delta}-1})^{-1}$.

Similarly, the second-order partial derivatives of $h(t)$ from (4) with respect to δ and μ are

$$\begin{aligned} v_{11} &= \mu\log(t\mu)\left(1+e^{\psi(t;\delta,\mu)}\right)^{-1}\{2(1+\mu t)^{\delta-1}+\kappa(t;\delta,\mu)[(1+\mu t)^{2\delta-1}+(1+\mu t)^{3\delta-1} \\ &+\delta\log(t\mu)(1+\mu t)^{2\delta-1}\{(1+\mu t)^{\delta}\kappa(t;\delta,\mu)+1\}]+\delta\log(t\mu)\{(1+\mu t)^{\delta-1} \\ &+(1+\mu t)^{3\delta-1}\kappa(t;\delta,\mu)[\kappa(t;\delta,\mu)+2(1+\mu t)^{-\delta}-1]\}\}, \end{aligned}$$

$$\begin{aligned} v_{22} &= \delta\{((1+\mu t)^{\delta-1}+t\mu[(\delta-1)(1+\mu t)^{\delta-2}+t\delta^2(1+\mu t)^{2(\delta-1)}\kappa(t;\delta,\mu)])(1+\mu t)^{\delta-1} \\ &+(1+\mu t)^{2(\delta-1)}\}+\mu\delta^2t^3\left(1+e^{\psi(t;\delta,\mu)}\right)^{-1}\{(\delta-1)(\delta-2)(1+\mu t)^{\delta-3} \\ &+\delta\kappa(t;\delta,\mu)(2(\delta-1)(1+\mu t)^{2(\delta-1)-1} \\ &+\delta[(1+\mu t)^{2(\delta-1)+\delta-1}\kappa(t;\delta,\mu)-(1+\mu t)^{2(\delta-1)+\delta-1}])\}+2t\delta(\delta-1)(1+\mu t)^{\delta-2}, \end{aligned}$$

and

$$\begin{aligned} v_{12} &= ((1+\mu t)^{\delta-1}+\delta\log(\mu t)((1+\mu t)^{2\delta-1}\kappa(t;\delta,\mu)+(1+\mu t)^{\delta-1})) \\ &+\mu t(1+e^{1-(1+\mu t)^{\delta}})\{1+\delta(1+e^{1-(1+\mu t)^{\delta}})[\log(\mu t)\{((1+\mu t)^{2(\delta-1)}(2\delta-1) \\ &-\delta(1+\mu t)^{\delta-1}\kappa(t;\delta,\mu)((1+\mu t)^{2\delta-1}-(1+\mu t)^{2\delta-1}\kappa(t;\delta,\mu)))+(\delta-1)(1+\mu t)^{\delta-2}\} \\ &+(1+\mu t)^{\delta-1}((1+\mu t)^{\delta}\kappa(t;\delta,\mu)+1)(1+\mu t)^{-1} \\ &+((1+\mu t)^{\delta-1}+\delta\log(\mu t)((1+\mu t)^{2\delta-1}\kappa(t;\delta,\mu)+(1+\mu t)^{\delta-1}))(1+\mu t)^{\delta-1}\kappa(t;\delta,\mu)]\}, \end{aligned}$$

where $\kappa(t;\delta,\mu) = e^{\psi(t;\delta,\mu)}\left(1+e^{\psi(t;\delta,\mu)}\right)^{-1}$.

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