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On s-Convexity of Dual Simpson Type Integral Inequalities

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Abstract: Integral inequalities are a powerful tool for estimating errors of quadrature formulas. In this study, some symmetric dual Simpson type integral inequalities for the classes of s-convex, bounded and Lipschitzian functions are proposed. The obtained results are based on a new identity and the use of some standard techniques such as Hölder as well as power mean inequalities. We give at the end some applications to the estimation of quadrature rules and to particular means.

Keywords: dual Simpson inequality; Newton–Cotes quadrature; s-convex functions; Lipschitzian functions; bounded functions

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1. Introduction

The concept of convexity and its variants plays a fundamental and important role in the development of various fields of science and engineering in a direct or indirect way. This concept has a closed relationship in the development of the theory of inequalities, which is an important tool in the study of some qualitative properties of solutions for differential and integral equations as well as in numerical analysis for estimating the errors in quadrature formulas. Noting that the most used methods for evaluating the integrals by a numerical approach is that of Newton–Cotes, which comprises a group of formulas involving a certain numbers of equally spaced points.

Definition 1 ([1]). *A function $\Lambda : I \to \mathbb{R}$ is said to be convex, if*

$$\Lambda(ie + (1-i)k) \ \leq i\Lambda(e) + (1-i)\Lambda(k)$$

holds for all $e, k \in I$ and all $i \in [0,1]$.

Bakula et al. [2], studied the following general form of three point Newton–Cotes formula via weighted Montgomery identities:

$$\int_e^f w(j)\aleph(j)dj = C(\chi)(\aleph(\chi) + \aleph(e+f-\chi)) + (1-2C(\chi))\aleph\left(\tfrac{e+f}{2}\right) + \mathcal{R}(w,\aleph,\chi), \qquad (1)$$

where $e < f$ and $\chi \in \left[e, \frac{e+f}{2}\right)$, $\mathcal{R}(w,\aleph,\chi)$ is the remainder term and $C(\chi)$ is an arbitrary real function defined on $\left[e, \frac{e+f}{2}\right)$ and gives the following results:

$$\left|\int_e^f w(j)\aleph(j)dj - C(\chi)(\aleph(\chi) + \aleph(e+f-\chi)) + (1-2C(\chi))\aleph\left(\tfrac{e+f}{2}\right) - g_n(\chi)\right|$$

$$\leq \frac{2B(\alpha+1,n-1)L}{(\alpha+n)(n-2)!}\left(|C(x)|\left((\chi-e)^{\alpha+n}+(f-\chi)^{\alpha+n}\right)+|1-2C(\chi)|\left(\frac{f-e}{2}\right)^{\alpha+n}\right),$$

where

$$\begin{aligned} g_n(\chi) =& C(\chi)\left(\sum_{i=0}^{n-1}\frac{\aleph^{(i+1)}(f)}{i!}\left(\int_x^f(1-W(j))(j-f)^i dj+\int_{e+f-\chi}^f(1-W(j))(j-f)^i dj\right)\right.\\ &\left.-\sum_{i=0}^{n-1}\frac{\aleph^{(i+1)}(e)}{i!}\left(\int_e^\chi(1-W(j))(j-e)^i dj+\int_a^{e+f-\chi}(1-W(j))(j-e)^i dj\right)\right)\\ &+(1-2C(\chi))\left(\sum_{i=0}^{n-1}\frac{\aleph^{(i+1)}(f)}{i!}\int_{\frac{e+f}{2}}^f(1-W(j))(j-f)^i dj\right.\\ &\left.-\sum_{i=0}^{n-1}\frac{\aleph^{(i+1)}(f)}{i!}\int_e^{\frac{e+f}{2}}(1-W(j))(j-e)^i dj\right), \end{aligned}$$

with $W(\chi)=\int_e^\chi w(j)dj$ for all $x\in[e,f]$ and $W(f)=1$. Additionally,

$$\begin{aligned} &\left|\int_a^b w(j)\aleph(j)dj - C(\chi)(\aleph(\chi)+\aleph(e+f-\chi))+(1-2C(\chi))\aleph\left(\frac{e+f}{2}\right)-r_n(\chi)\right|\\ &\leq \frac{2B(\alpha+1,n-1)L}{(f-e)(\alpha+n+1)(n-2)!}\\ &\quad\times\left(|C(\chi)|\left((\chi-e)^{\alpha+n+1}+(f-\chi)^{\alpha+n+1}\right)+|1-2C(\chi)|\left(\frac{f-e}{2}\right)^{\alpha+n+1}\right), \end{aligned}$$

where

$$\begin{aligned} r_n(\chi) =& C(\chi)\sum_{i=0}^{n-1}\frac{\left((-1)^i\aleph^{(i+1)}(f)-f^{(i+1)}(e)\right)\left((x-e)^{i+2}+(f-x)^{i+2}\right)}{i!(i+2)(f-e)}\\ &+(1-2C(\chi))\sum_{i=0}^{n-1}\frac{\left((-1)^i\aleph^{(i+1)}(f)-\aleph^{(i+1)}(e)\right)(f-e)^{i+2}}{i!(i+2)2^{i+2}}. \end{aligned}$$

Obviously, if $C\left(\frac{3e+f}{4}\right)=\frac{2}{3}$, then identity (1) gives the weighted version of the dual Simpson inequality. Moreover, if we choose $w(j)=\frac{1}{f-e}$, we obtain the classical dual Simpson type inequality for functions whose n^{th} derivatives are α-L-Hölderians.

In [3], Pečarić and Vukelić used the Euler-type identities and gave some estimates of the general dual Simpson quadrature formula for functions as well as first derivatives are of bounded variation on $[0,1]$, L-Lipschitzian and R-integrable as follows.

In the case where $\aleph$ is L-Lipschitzian on $[0,1]$, we have

$$\left|\int_0^1\aleph(j)dj-\frac{1}{2u-v}\left(u\aleph\left(\frac{1}{4}\right)-v\aleph\left(\frac{1}{2}\right)+u\aleph\left(\frac{3}{4}\right)\right)\right|\leq\frac{2u+v}{8(2u-v)}L. \quad (2)$$

If $\aleph'$ is L-Lipschitzian on $[0,1]$, then

$$\begin{aligned} &\left|\int_0^1\aleph(j)dj-\frac{1}{2u-v}\left(u\aleph\left(\frac{1}{4}\right)-v\aleph\left(\frac{1}{2}\right)+u\aleph\left(\frac{3}{4}\right)\right)\right|\\ &\leq\frac{2u^2\left(3v+\sqrt{2uv}\right)+uv\left(5v-\sqrt{2uv}\right)+2v^2\left(v+3\sqrt{2uv}\right)}{48(2u-v)\left(v+\sqrt{2uv}\right)\left(2u+v+2\sqrt{2uv}\right)}L. \end{aligned} \quad (3)$$

If $\aleph$ is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 \aleph(j)dj - \frac{1}{2u-v}\left(u\aleph\left(\frac{1}{4}\right) - v\aleph\left(\frac{1}{2}\right) + u\aleph\left(\frac{3}{4}\right)\right)\right| \le \frac{2u+v}{4(2u-v)} \bigvee_0^1 (f). \tag{4}$$

If $\aleph'$ is a continuous function of bounded variation $[0,1]$, then

$$\left|\int_0^1 \aleph(j)dj - \frac{1}{2u-v}\left(u\aleph\left(\frac{1}{4}\right) - v\aleph\left(\frac{1}{2}\right) + u\aleph\left(\frac{3}{4}\right)\right)\right|$$
$$\le \frac{2u+3v+|2u-5v|}{64(2u-v)} \bigvee_0^1 (\aleph'). \tag{5}$$

By taking $u = 2$ and $v = 1$, inequalities (2)–(5) will be reduced to the classical dual Simpson inequality, of which the general form is as follows:

$$\left|\frac{1}{3}\left(2\aleph\left(\frac{3e+f}{4}\right) - \aleph\left(\frac{e+f}{2}\right) + 2\aleph\left(\frac{e+3f}{4}\right)\right) - \frac{1}{f-e}\int_e^f \aleph(j)dj\right| \le \frac{7(f-e)^4}{23040}\left\|\aleph^{(4)}\right\|_\infty, \tag{6}$$

where $\aleph$ is a four-times continuously differentiable function on (e,f), and $\left\|\aleph^{(4)}\right\|_\infty = \sup_{x\in(e,f)}\left|\aleph^{(4)}(x)\right|$, (see [4–6]).

In [7], Dragomir gave the following Simpson inequality for mapping of bounded variation:

$$\left|\frac{1}{6}\left(\aleph(e) + 4\aleph\left(\frac{e+f}{2}\right) + \aleph(f)\right) - \frac{1}{f-e}\int_e^f \aleph(j)dj\right| \le \frac{1}{3}(f-e) \bigvee_e^f (\aleph),$$

where $\bigvee_e^f\left(\aleph^{(n)}\right)$ is the total variation of function $\aleph$.

Pečarić and Varošanec [8] discussed the Simpson inequality for derivatives of bounded variation

$$\left|\frac{1}{6}\left(\aleph(e) + 4\aleph\left(\frac{e+f}{2}\right) + \aleph(f)\right) - \frac{1}{f-e}\int_e^f \aleph(j)dj\right| \le c_n(f-e)^n \bigvee_e^f \left(\aleph^{(n)}\right),$$

where $n \in \{0,1,2,3\}$ with $c_0 = \frac{1}{3}, c_1 = \frac{1}{24}, c_2 = \frac{1}{324}, c_3 = \frac{1}{1152}$ and $\bigvee_e^f\left(\aleph^{(n)}\right)$ is the total variation of function $\aleph^{(n)}$.

Regarding some papers dealing with three-point Newton–Cotes, we refer readers to [9–14] and references therein.

In this paper, by adopting a novel approach, we establish some dual Simpson-type inequalities for functions whose first derivatives are s-convex. The cases where the first derivatives are bounded as well as Lipschitzian functions are also discussed. Applications of the results are given.

2. Main Results

We recall that a non-negative function $\Lambda : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$, if

$$\Lambda(ie + (1-i)k) \le i^s\Lambda(e) + (1-i)^s\Lambda(k),$$

holds for all $e, k \in I$ and $i \in [0,1]$ (see [15]).

Now, we prove the following identity, which is basic to establish our main results.

Lemma 1. *Let $\aleph : [\vartheta,\kappa] \to \mathbb{R}$ be a differentiable function on $[\vartheta,\kappa]$, with $\vartheta < \kappa$ and $\aleph' \in L^1[\vartheta,\kappa]$, then the following equality holds*

$$\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\aleph\left(\frac{\vartheta+\kappa}{2}\right)+2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right)-\frac{1}{\kappa-\vartheta}\int_{\vartheta}^{\kappa}\aleph(j)dj$$
$$=\frac{\kappa-\vartheta}{16}\left(\int_0^1 i\aleph'\left((1-i)\vartheta+i\frac{3\vartheta+\kappa}{4}\right)di+\int_0^1\left(i-\frac{5}{3}\right)\aleph'\left((1-i)\frac{3\vartheta+\kappa}{4}+i\frac{\vartheta+\kappa}{2}\right)di\right.$$
$$\left.+\int_0^1\left(i+\frac{2}{3}\right)\aleph'\left((1-i)\frac{\vartheta+\kappa}{2}+i\frac{\vartheta+3\kappa}{4}\right)di+\int_0^1(i-1)\aleph'\left((1-i)\frac{\vartheta+3\kappa}{4}+i\kappa\right)di\right).$$

Proof. Let

$$I_1=\int_0^1 i\aleph'\left((1-i)\vartheta+i\frac{3\vartheta+\kappa}{4}\right)di,$$
$$I_2=\int_0^1\left(i-\frac{5}{3}\right)\aleph'\left((1-i)\frac{3\vartheta+\kappa}{4}+i\frac{\vartheta+\kappa}{2}\right)di,$$
$$I_3=\int_0^1\left(i+\frac{2}{3}\right)\aleph'\left((1-i)\frac{\vartheta+\kappa}{2}+i\frac{\vartheta+3\kappa}{4}\right)di,$$
$$I_4=\int_0^1(i-1)\aleph'\left((1-i)\frac{\vartheta+3\kappa}{4}+i\kappa\right)di.$$

Integrating by parts I_1, we obtain

$$\begin{aligned}I_1&=\frac{4}{\kappa-\vartheta}i\aleph\left((1-i)\vartheta+i\frac{3\vartheta+\kappa}{4}\right)\Big|_{i=0}^{i=1}\\&\quad-\frac{4}{\kappa-\vartheta}\int_0^1\aleph\left((1-i)\vartheta+i\frac{3\vartheta+\kappa}{4}\right)di\\&=\frac{4}{\kappa-\vartheta}\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\frac{4}{\kappa-\vartheta}\int_0^1\aleph\left((1-i)\vartheta+i\frac{3\vartheta+\kappa}{4}\right)di\\&=\frac{4}{\kappa-\vartheta}\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\frac{16}{(\kappa-\vartheta)^2}\int_{\vartheta}^{\frac{3\vartheta+\kappa}{4}}\aleph(j)dj.\end{aligned}\tag{7}$$

Similarly, we obtain

$$I_2=-\frac{8}{3(\kappa-\vartheta)}\aleph\left(\frac{\vartheta+\kappa}{2}\right)+\frac{20}{3(\kappa-\vartheta)}\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\frac{16}{(\kappa-\vartheta)^2}\int_{\frac{3\vartheta+\kappa}{4}}^{\frac{\vartheta+\kappa}{2}}\aleph(j)dj,\tag{8}$$

$$I_3=\frac{20}{3(\kappa-\vartheta)}\aleph\left(\frac{\vartheta+3\kappa}{4}\right)-\frac{8}{3(\kappa-\vartheta)}\aleph\left(\frac{\vartheta+\kappa}{2}\right)-\frac{16}{(\kappa-\vartheta)^2}\int_{\frac{\vartheta+\kappa}{2}}^{\frac{\vartheta+3\kappa}{4}}\aleph(j)dj,\tag{9}$$

and

$$I_4=\frac{4}{\kappa-\vartheta}\aleph\left(\frac{\vartheta+3\kappa}{4}\right)-\frac{16}{(\kappa-\vartheta)^2}\int_{\frac{\vartheta+3\kappa}{4}}^{\kappa}\aleph(j)dj.\tag{10}$$

Adding (7)–(10), multiplying the result by $\frac{\kappa-\vartheta}{16}$, we obtain the desired result. □

Theorem 1. *Let $\aleph$ be as in Lemma 1 with $0\le\vartheta<\kappa$. If $|\aleph'|$ is s-convex in the second sense for some fixed $s\in(0,1]$, then we have*

$$\left|\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\aleph\left(\frac{\vartheta+\kappa}{2}\right)+2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right)-\frac{1}{\kappa-\vartheta}\int_{\vartheta}^{\kappa}\aleph(j)dj\right|$$
$$\le\frac{\kappa-\vartheta}{16}\left(\frac{1}{(s+1)(s+2)}\left(|\aleph'(\vartheta)|+|\aleph'(\kappa)|\right)+\frac{4s+14}{3(s+1)(s+2)}\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|\right.$$

$$+\frac{8s+10}{3(s+1)(s+2)}\left(\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|+\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|\right)\Big).$$

Proof. From Lemma 1, properties of modulus, and s-convexity in the second sense of $|\aleph'|$, we have

$$\begin{aligned}
&\left|\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\aleph\left(\frac{\vartheta+\kappa}{2}\right)+2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right)-\frac{1}{\kappa-\vartheta}\int_{\vartheta}^{\kappa}\aleph(j)dj\right| \\
&\leq\frac{\kappa-\vartheta}{16}\left(\int_0^1 t\left|\aleph'\left((1-i)\vartheta+i\frac{3\vartheta+\kappa}{4}\right)\right|di+\int_0^1\left(\frac{5}{3}-i\right)\left|\aleph'\left((1-i)\frac{3\vartheta+\kappa}{4}+i\frac{\vartheta+\kappa}{2}\right)\right|di\right. \\
&\quad\left.+\int_0^1\left(i+\frac{2}{3}\right)\left|\aleph'\left((1-i)\frac{\vartheta+\kappa}{2}+i\frac{\vartheta+3\kappa}{4}\right)\right|di+\int_0^1(1-i)\left|\aleph'\left((1-i)\frac{\vartheta+3\kappa}{4}+i\kappa\right)\right|di\right) \\
&\leq\frac{\kappa-\vartheta}{16}\left(\int_0^1 i\left((1-i)^s|\aleph'(\vartheta)|+i^s\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|\right)di\right. \\
&\quad+\int_0^1\left(\frac{5}{3}-i\right)\left((1-i)^s\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|+i^s\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|\right)di \\
&\quad+\int_0^1\left(i+\frac{2}{3}\right)\left((1-i)^s\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|+i^s\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|\right)di \\
&\quad\left.+\int_0^1(1-i)\left((1-i)^s\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|+i^s|\aleph'(\kappa)|\right)di\right) \\
&=\frac{\kappa-\vartheta}{16}\left(\left(\int_0^1 i(1-i)^s di\right)|\aleph'(\vartheta)|+\left(\int_0^1 i^{s+1}di+\int_0^1\left(\frac{5}{3}-i\right)(1-i)^s di\right)\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|\right. \\
&\quad+\left(\int_0^1\left(\frac{5}{3}-i\right)i^s di+\int_0^1\left(i+\frac{2}{3}\right)(1-i)^s di\right)\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right| \\
&\quad\left.+\left(\int_0^1\left(i+\frac{2}{3}\right)i^s di+\int_0^1(1-i)^{s+1}di\right)\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|+\left(\int_0^1(1-i)i^s di\right)|\aleph'(\kappa)|\right) \\
&=\frac{\kappa-\vartheta}{16}\left(\frac{1}{(s+1)(s+2)}\left(|\aleph'(\vartheta)|+|\aleph'(\kappa)|\right)+\frac{4s+14}{3(s+1)(s+2)}\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|\right. \\
&\quad\left.+\frac{8s+10}{3(s+1)(s+2)}\left(\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|+\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|\right)\right),
\end{aligned}$$

where we have used the fact that

$$\int_0^1 i(1-i)^s di=\int_0^1 i^s(1-i)di=\frac{1}{(s+1)(s+2)}, \tag{11}$$

$$\int_0^1 i^{s+1}di=\int_0^1(1-i)^{s+1}di=\frac{1}{s+2}, \tag{12}$$

$$\int_0^1\left(\frac{5}{3}-i\right)(1-i)^s di=\int_0^1 i^s\left(i+\frac{2}{3}\right)di=\frac{5s+7}{3(s+1)(s+2)}, \tag{13}$$

and

$$\int_0^1 i^s\left(\frac{5}{3}-i\right)di=\int_0^1\left(i+\frac{2}{3}\right)(1-i)^s di=\frac{2s+7}{3(s+1)(s+2)}. \tag{14}$$

The proof is completed. □

Corollary 1. *For $s=1$, Theorem 1 gives*

$$\left|\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\aleph\left(\frac{\vartheta+\kappa}{2}\right)+2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right)-\frac{1}{\kappa-\vartheta}\int_{\vartheta}^{\kappa}\aleph(j)dj\right|$$

$$\leq \frac{5(\kappa-\vartheta)}{24}\left(\frac{|\aleph'(\vartheta)|+6\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|+6\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|+6\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|+|\aleph'(\kappa)|}{20}\right).$$

Theorem 2. *Let $\aleph$ be as in Lemma 1 with $0 \leq \vartheta < \kappa$. If $|\aleph'|^q$ is s-convex in the second sense for some fixed $s \in (0,1]$ where $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\begin{aligned}
&\left|\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\aleph\left(\frac{\vartheta+\kappa}{2}\right)+2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right)-\frac{1}{\kappa-\vartheta}\int_{\vartheta}^{\kappa}\aleph(j)dj\right| \\
&\leq \frac{\kappa-\vartheta}{16(p+1)^{\frac{1}{p}}}\left(\left(\frac{|\aleph'(\vartheta)|^q+\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q}{1+s}\right)^{\frac{1}{q}}+\left(\frac{\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q+|\aleph'(\kappa)|^q}{1+s}\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\frac{5^{p+1}-2^{p+1}}{3^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q+\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q}{1+s}\right)^{\frac{1}{q}}+\left(\frac{\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q+\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q}{1+s}\right)^{\frac{1}{q}}\right)\right).
\end{aligned}$$

Proof. From Lemma 1, properties of modulus, Hölder's inequality, and s-convexity in the second sense of $|\aleph'|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\aleph\left(\frac{\vartheta+\kappa}{2}\right)+2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right)-\frac{1}{\kappa-\vartheta}\int_{\vartheta}^{\kappa}\aleph(j)dj\right| \\
&\leq \frac{\kappa-\vartheta}{16}\left(\left(\int_0^1 i^p di\right)^{\frac{1}{p}}\left(\int_0^1\left|\aleph'\left((1-i)\vartheta+i\frac{3\vartheta+\kappa}{4}\right)\right|^q di\right)^{\frac{1}{q}}\right. \\
&\quad +\left(\int_0^1\left(\frac{5}{3}-i\right)^p di\right)^{\frac{1}{p}}\left(\int_0^1\left|\aleph'\left((1-i)\frac{3\vartheta+\kappa}{4}+i\frac{\vartheta+\kappa}{2}\right)\right|^q di\right)^{\frac{1}{q}} \\
&\quad +\left(\int_0^1\left(i+\frac{2}{3}\right)^p di\right)^{\frac{1}{p}}\left(\int_0^1\left|\aleph'\left((1-i)\frac{\vartheta+\kappa}{2}+i\frac{\vartheta+3\kappa}{4}\right)\right|^q di\right)^{\frac{1}{q}} \\
&\quad \left.+\left(\int_0^1(1-i)^p di\right)^{\frac{1}{p}}\left(\int_0^1\left|\aleph'\left((1-i)\frac{\vartheta+3\kappa}{4}+i\kappa\right)\right|^q di\right)^{\frac{1}{q}}\right) \\
&\leq \frac{\kappa-\vartheta}{16(p+1)^{\frac{1}{p}}}\left(\left(\int_0^1\left((1-i)^s|\aleph'(\vartheta)|^q+i^s\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q\right)di\right)^{\frac{1}{q}}\right. \\
&\quad +\left(\frac{5^{p+1}-2^{p+1}}{3^{p+1}}\right)^{\frac{1}{p}}\left(\left(\int_0^1\left((1-i)^s\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q+i^s\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q\right)di\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_0^1\left((1-i)^s\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q+i^s\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q\right)di\right)^{\frac{1}{q}}\right) \\
&\quad \left.+\left(\int_0^1\left((1-i)^s\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q+i^s|\aleph'(\kappa)|^q\right)di\right)^{\frac{1}{q}}\right) \\
&= \frac{\kappa-\vartheta}{16(p+1)^{\frac{1}{p}}}\left(\left(\frac{|\aleph'(\vartheta)|^q+\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q}{1+s}\right)^{\frac{1}{q}}+\left(\frac{\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q+|\aleph'(\kappa)|^q}{1+s}\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\frac{5^{p+1}-2^{p+1}}{3^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q+\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q}{1+s}\right)^{\frac{1}{q}}+\left(\frac{\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q+\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q}{1+s}\right)^{\frac{1}{q}}\right)\right).
\end{aligned}$$

The proof is completed. □

Corollary 2. *For $s = 1$, Theorem 2 gives*

$$\begin{aligned}
&\left|\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right) - \aleph\left(\frac{\vartheta+\kappa}{2}\right) + 2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right) - \frac{1}{\kappa-\vartheta}\int_{\vartheta}^{\kappa}\aleph(j)dj\right| \\
&\leq \frac{\kappa-\vartheta}{16(p+1)^{\frac{1}{p}}}\left(\left(\left(\frac{|\aleph'(\vartheta)|^q + \left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q + |\aleph'(\kappa)|^q}{2}\right)^{\frac{1}{q}}\right.\right. \\
&\quad + \left.\left(\frac{5^{p+1}-2^{p+1}}{3^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q + \left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q + \left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right)\right).
\end{aligned}$$

Theorem 3. *Let $\aleph$ be as in Lemma 1 with $0 \leq \vartheta < \kappa$. If $|\aleph'|^q$ is s-convex in the second sense for some fixed $s \in (0,1]$ where $q \geq 1$, then we have*

$$\begin{aligned}
&\left|\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right) - \aleph\left(\frac{\vartheta+\kappa}{2}\right) + 2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right) - \frac{1}{\kappa-\vartheta}\int_{\vartheta}^{\kappa}\aleph(j)dj\right| \\
&\leq \frac{\kappa-\vartheta}{16}\left(\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left(\frac{|\aleph'(\vartheta)|^q + (s+1)\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}} + \left(\frac{(s+1)\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q + |\aleph'(\kappa)|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}}\right)\right. \\
&\quad + \left(\frac{7}{6}\right)^{1-\frac{1}{q}}\left(\frac{(5s+7)\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q + (2s+7)\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q}{3(s+1)(s+2)}\right)^{\frac{1}{q}} \\
&\quad + \left.\left(\frac{7}{6}\right)^{1-\frac{1}{q}}\left(\frac{(2s+7)\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q + (5s+7)\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q}{3(s+1)(s+2)}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Proof. From Lemma 1, properties of modulus, power mean inequality, and s-convexity in the second sense of $|\aleph'|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right) - \aleph\left(\frac{\vartheta+\kappa}{2}\right) + 2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right) - \frac{1}{\kappa-\vartheta}\int_{\vartheta}^{\kappa}\aleph(j)dj\right| \\
&\leq \frac{\kappa-\vartheta}{16}\left(\left(\int_0^1 i\,di\right)^{1-\frac{1}{q}}\left(\int_0^1 i\left|\aleph'\left((1-i)\vartheta + i\frac{3\vartheta+\kappa}{4}\right)\right|^q di\right)^{\frac{1}{q}}\right. \\
&\quad + \left(\int_0^1 \left(\frac{5}{3}-i\right)di\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\frac{5}{3}-i\right)\left|\aleph'\left((1-i)\frac{3\vartheta+\kappa}{4} + i\frac{\vartheta+\kappa}{2}\right)\right|^q di\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 \left(i+\frac{2}{3}\right)di\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(i+\frac{2}{3}\right)\left|\aleph'\left((1-i)\frac{\vartheta+\kappa}{2} + i\frac{\vartheta+3\kappa}{4}\right)\right|^q di\right)^{\frac{1}{q}} \\
&\quad + \left.\left(\int_0^1 (1-i)di\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-i)\left|\aleph'\left((1-i)\frac{\vartheta+3\kappa}{4} + i\kappa\right)\right|^q di\right)^{\frac{1}{q}}\right) \\
&\leq \frac{\kappa-\vartheta}{16}\left(\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 i\left((1-i)^s|\aleph'(\vartheta)|^q + i^s\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q\right)di\right)^{\frac{1}{q}}\right. \\
&\quad + \left(\frac{7}{6}\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\frac{5}{3}-i\right)\left((1-i)^s\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q + i^s\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q\right)di\right)^{\frac{1}{q}} \\
&\quad + \left(\frac{7}{6}\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(i+\frac{2}{3}\right)\left((1-i)^s\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q + i^s\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q\right)di\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-i)\left((1-i)^s\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q+i^s|\aleph'(\kappa)|^q\right)di\right)^{\frac{1}{q}}\Bigg)$$

$$=\frac{\kappa-\vartheta}{16}\left(\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(|\aleph'(\vartheta)|^q\int_0^1 i(1-i)^s di+\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q\int_0^1 i^{s+1}di\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{7}{6}\right)^{1-\frac{1}{q}}\left(\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q\int_0^1\left(\frac{5}{3}-i\right)(1-i)^s di+\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q\int_0^1\left(\frac{5}{3}-i\right)i^s di\right)^{\frac{1}{q}}$$

$$+\left(\frac{7}{6}\right)^{1-\frac{1}{q}}\left(\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q\int_0^1\left(i+\frac{2}{3}\right)(1-i)^s di+\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q\int_0^1\left(i+\frac{2}{3}\right)i^s di\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q\int_0^1(1-i)^{s+1}di+|\aleph'(b)|^q\int_0^1(1-i)i^s di\right)^{\frac{1}{q}}\right)$$

$$=\frac{\kappa-\vartheta}{16}\left(\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left(\frac{|\aleph'(\vartheta)|^q+(s+1)\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}}+\left(\frac{(s+1)\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q+|\aleph'(\kappa)|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}}\right)\right.$$

$$+\left(\frac{7}{6}\right)^{1-\frac{1}{q}}\left(\frac{(5s+7)\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q+(2s+7)\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q}{3(s+1)(s+2)}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{7}{6}\right)^{1-\frac{1}{q}}\left(\frac{(2s+7)\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q+(5s+7)\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q}{3(s+1)(s+2)}\right)^{\frac{1}{q}}\right),$$

where we have used (11)–(14). The proof is achieved. □

Corollary 3. *For $s=1$, Theorem 3 gives*

$$\left|\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\aleph\left(\frac{\vartheta+\kappa}{2}\right)+2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right)-\frac{1}{\kappa-\vartheta}\int_\vartheta^\kappa\aleph(j)dj\right|$$

$$\le\frac{\kappa-\vartheta}{32}\left(\left(\frac{|\aleph'(\vartheta)|^q+2\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q+|\aleph'(\kappa)|^q}{3}\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{7}{3}\left(\left(\frac{4\left|\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right|^q+3\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q}{7}\right)^{\frac{1}{q}}+\left(\frac{3\left|\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right|^q+4\left|\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right|^q}{7}\right)^{\frac{1}{q}}\right)\right).$$

3. Further Results

In the following results, we will discuss the cases where $\aleph'(x)$ is bounded as well as $\aleph'(x)$ of L-Lipschitzian functions.

Theorem 4. *Let $\aleph$ be as in Lemma 1. If there exist constants $-\infty<m<M<+\infty$ such that $m\le\aleph'(x)\le M$ for all $x\in[\vartheta,\kappa]$, then we have*

$$\left|\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\aleph\left(\frac{\vartheta+\kappa}{2}\right)+2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right)-\frac{1}{\kappa-\vartheta}\int_\vartheta^\kappa\aleph(j)dj\right|\le\frac{5(\kappa-\vartheta)(M-m)}{48}.$$

Proof. From Lemma 1, we have

$$\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\aleph\left(\frac{\vartheta+\kappa}{2}\right)+2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right)-\frac{1}{\kappa-\vartheta}\int_\vartheta^\kappa\aleph(j)dj$$

$$=\frac{\kappa-\vartheta}{16}\left(\int_0^1 i\aleph'\left((1-i)\vartheta+i\frac{3\vartheta+\kappa}{4}\right)di+\int_0^1\left(i-\frac{5}{3}\right)\aleph'\left((1-i)\frac{3\vartheta+\kappa}{4}+i\frac{\vartheta+\kappa}{2}\right)di\right.$$

$$
\begin{aligned}
&+\int_0^1 \left(i+\tfrac{2}{3}\right)\aleph'\left((1-i)\tfrac{\vartheta+\kappa}{2}+i\tfrac{\vartheta+3\kappa}{4}\right)di+\int_0^1 (i-1)\aleph'\left((1-i)\tfrac{\vartheta+3\kappa}{4}+i\kappa\right)di\Bigg)\\
=&\tfrac{\kappa-\vartheta}{16}\Bigg(\int_0^1 i\left(\aleph'\left((1-i)\vartheta+i\tfrac{3\vartheta+\kappa}{4}\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)di\\
&+\int_0^1 \left(i-\tfrac{5}{3}\right)\left(\aleph'\left((1-i)\tfrac{3\vartheta+\kappa}{4}+i\tfrac{\vartheta+\kappa}{2}\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)di\\
&+\int_0^1 \left(i+\tfrac{2}{3}\right)\left(\aleph'\left((1-i)\tfrac{\vartheta+\kappa}{2}+i\tfrac{\vartheta+3\kappa}{4}\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)di\\
&+\int_0^1 (i-1)\left(\aleph'\left((1-i)\tfrac{\vartheta+3\kappa}{4}+i\kappa\right)-\tfrac{m+M}{2}+\tfrac{m+M}{2}\right)di\Bigg)\\
=&\tfrac{\kappa-\vartheta}{16}\Bigg(\int_0^1 i\left(\aleph'\left((1-i)\vartheta+i\tfrac{3\vartheta+\kappa}{4}\right)-\tfrac{m+M}{2}\right)di\\
&+\int_0^1 \left(i-\tfrac{5}{3}\right)\left(\aleph'\left((1-i)\tfrac{3\vartheta+\kappa}{4}+i\tfrac{\vartheta+\kappa}{2}\right)-\tfrac{m+M}{2}\right)di\\
&+\int_0^1 \left(i+\tfrac{2}{3}\right)\left(\aleph'\left((1-i)\tfrac{\vartheta+\kappa}{2}+i\tfrac{\vartheta+3\kappa}{4}\right)-\tfrac{m+M}{2}\right)di\\
&+\int_0^1 (i-1)\left(\aleph'\left((1-i)\tfrac{\vartheta+3\kappa}{4}+i\kappa\right)-\tfrac{m+M}{2}\right)di,
\end{aligned}
\tag{15}
$$

where we have used the fact that

$$
\int_0^1 i\,di+\int_0^1\left(i-\tfrac{5}{3}\right)di+\int_0^1\left(i+\tfrac{2}{3}\right)di+\int_0^1 (i-1)di=\int_0^1 (4i-2)di=0.
$$

Applying the absolute value in both sides of (15), we obtain

$$
\begin{aligned}
&\left|\tfrac{1}{3}\left(2\aleph\left(\tfrac{3\vartheta+\kappa}{4}\right)-\aleph\left(\tfrac{\vartheta+\kappa}{2}\right)+2\aleph\left(\tfrac{\vartheta+3\kappa}{4}\right)\right)-\tfrac{1}{\kappa-\vartheta}\int_\vartheta^\kappa \aleph(j)dj\right|\\
\leq&\tfrac{\kappa-\vartheta}{16}\Bigg(\int_0^1 i\left|\aleph'\left((1-i)\vartheta+i\tfrac{3\vartheta+\kappa}{4}\right)-\tfrac{m+M}{2}\right|di\\
&+\int_0^1 \left(\tfrac{5}{3}-i\right)\left|\aleph'\left((1-i)\tfrac{3\vartheta+\kappa}{4}+i\tfrac{\vartheta+\kappa}{2}\right)-\tfrac{m+M}{2}\right|di\\
&+\int_0^1 \left(i+\tfrac{2}{3}\right)\left|\aleph'\left((1-i)\tfrac{\vartheta+\kappa}{2}+i\tfrac{\vartheta+3\kappa}{4}\right)-\tfrac{m+M}{2}\right|di\\
&+\int_0^1 (1-i)\left(\left|\aleph'\left((1-i)\tfrac{\vartheta+3\kappa}{4}+i\kappa\right)-\tfrac{m+M}{2}\right|\right)di\Bigg).
\end{aligned}
\tag{16}
$$

Since $m \leq \aleph'(x) \leq M$ for all $x \in [\vartheta,\kappa]$, we have

$$
\left|\aleph'\left((1-i)\vartheta+i\tfrac{3\vartheta+\kappa}{4}\right)-\tfrac{m+M}{2}\right| \leq \tfrac{M-m}{2}, \tag{17}
$$

$$
\left|\aleph'\left((1-i)\tfrac{3\vartheta+\kappa}{4}+i\tfrac{\vartheta+\kappa}{2}\right)-\tfrac{m+M}{2}\right| \leq \tfrac{M-m}{2}, \tag{18}
$$

$$
\left|\aleph'\left((1-i)\tfrac{\vartheta+\kappa}{2}+i\tfrac{\vartheta+3\kappa}{4}\right)-\tfrac{m+M}{2}\right| \leq \tfrac{M-m}{2}, \tag{19}
$$

and

$$
\left|\aleph'\left((1-i)\tfrac{\vartheta+3\kappa}{4}+i\kappa\right)-\tfrac{m+M}{2}\right| \leq \tfrac{M-m}{2}. \tag{20}
$$

Using (17)–(20) in (16), we obtain

$$\begin{aligned}
&\left|\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\aleph\left(\frac{\vartheta+\kappa}{2}\right)+2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right)-\frac{1}{\kappa-\vartheta}\int_{\vartheta}^{\kappa}\aleph(j)dj\right| \\
\le&\frac{(\kappa-\vartheta)(M-m)}{32}\left(\int_0^1 idi+\int_0^1\left(\frac{5}{3}-i\right)dt+\int_0^1\left(i+\frac{2}{3}\right)dt+\int_0^1(1-i)dt\right) \\
=&\frac{5(\kappa-\vartheta)(M-m)}{48}.
\end{aligned}$$

The proof is completed. □

Theorem 5. *Let $\aleph$ be as in Lemma 1. If $\aleph'$ is L-Lipschitzian function on $[\vartheta,\kappa]$, then we have*

$$\left|\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\aleph\left(\frac{\vartheta+\kappa}{2}\right)+2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right)-\frac{1}{\kappa-\vartheta}\int_{\vartheta}^{\kappa}\aleph(j)dj\right|\le\frac{13(\kappa-\vartheta)^2}{192}L.$$

Proof. From Lemma 1, we have

$$\begin{aligned}
&\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\aleph\left(\frac{\vartheta+\kappa}{2}\right)+2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right)-\frac{1}{\kappa-\vartheta}\int_{\vartheta}^{\kappa}\aleph(j)dj \\
=&\frac{\kappa-\vartheta}{16}\left(\int_0^1 i\aleph'\left((1-i)\vartheta+i\frac{3\vartheta+\kappa}{4}\right)di+\int_0^1\left(i-\frac{5}{3}\right)\aleph'\left((1-i)\frac{3\vartheta+\kappa}{4}+i\frac{\vartheta+\kappa}{2}\right)di\right. \\
&\left.+\int_0^1\left(i+\frac{2}{3}\right)\aleph'\left((1-i)\frac{\vartheta+\kappa}{2}+i\frac{\vartheta+3\kappa}{4}\right)di+\int_0^1(i-1)\aleph'\left((1-i)\frac{\vartheta+3\kappa}{4}+i\kappa\right)di\right) \\
=&\frac{\kappa-\vartheta}{16}\left(\int_0^1 i\left(\aleph'\left((1-i)\vartheta+i\frac{3\vartheta+\kappa}{4}\right)-\aleph'(\vartheta)+\aleph'(\vartheta)\right)di\right. \\
&+\int_0^1\left(i-\frac{5}{3}\right)\left(\aleph'\left((1-i)\frac{3\vartheta+\kappa}{4}+i\frac{\vartheta+\kappa}{2}\right)-\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)+\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right)di \\
&+\int_0^1\left(i+\frac{2}{3}\right)\left(\aleph'\left((1-i)\frac{\vartheta+\kappa}{2}+i\frac{\vartheta+3\kappa}{4}\right)-\aleph'\left(\frac{\vartheta+\kappa}{2}\right)+\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right)di \\
&\left.+\int_0^1(i-1)\left(\aleph'\left((1-i)\frac{\vartheta+3\kappa}{4}+i\kappa\right)-\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)+\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right)di\right) \\
=&\frac{\kappa-\vartheta}{16}\left(\int_0^1 i\left(\aleph'\left((1-i)\vartheta+i\frac{3\vartheta+\kappa}{4}\right)-\aleph'(\vartheta)\right)di\right. \\
&+\int_0^1\left(i-\frac{5}{3}\right)\left(\aleph'\left((1-i)\frac{3\vartheta+\kappa}{4}+i\frac{\vartheta+\kappa}{2}\right)-\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right)di \\
&+\int_0^1\left(i+\frac{2}{3}\right)\left(\aleph'\left((1-i)\frac{\vartheta+\kappa}{2}+i\frac{\vartheta+3\kappa}{4}\right)-\aleph'\left(\frac{\vartheta+\kappa}{2}\right)\right)di \\
&+\int_0^1(i-1)\left(\aleph'\left((1-i)\frac{\vartheta+3\kappa}{4}+i\kappa\right)-\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right)di \\
&\left.+\frac{1}{2}\left(\aleph'(\vartheta)-\aleph'\left(\frac{\vartheta+3\kappa}{4}\right)\right)+\frac{7}{6}\left(\aleph'\left(\frac{\vartheta+\kappa}{2}\right)-\aleph'\left(\frac{3\vartheta+\kappa}{4}\right)\right)\right).
\end{aligned}\tag{21}$$

Applying the absolute value in both sides of (21), and by using the fact that $\aleph'$ is L-Lipschitzian on $[\vartheta,\kappa]$, we obtain

$$\left|\frac{1}{3}\left(2\aleph\left(\frac{3\vartheta+\kappa}{4}\right)-\aleph\left(\frac{\vartheta+\kappa}{2}\right)+2\aleph\left(\frac{\vartheta+3\kappa}{4}\right)\right)-\frac{1}{\kappa-\vartheta}\int_{\vartheta}^{\kappa}\aleph(j)dj\right|$$

$$
\begin{aligned}
&\leq \tfrac{\kappa-\vartheta}{16}\Bigg(\int_0^1 i\left|\aleph'\left((1-i)\vartheta+i\tfrac{3\vartheta+\kappa}{4}\right)-\aleph'(\vartheta)\right|di \\
&\quad+\int_0^1\left(\tfrac{5}{3}-i\right)\left|\aleph'\left((1-i)\tfrac{3\vartheta+\kappa}{4}+i\tfrac{\vartheta+\kappa}{2}\right)-\aleph'\left(\tfrac{3\vartheta+\kappa}{4}\right)\right|di \\
&\quad+\int_0^1\left(i+\tfrac{2}{3}\right)\left|\aleph'\left((1-i)\tfrac{\vartheta+\kappa}{2}+i\tfrac{\vartheta+3\kappa}{4}\right)-\aleph'\left(\tfrac{\vartheta+\kappa}{2}\right)\right|di \\
&\quad+\int_0^1(1-i)\left|\aleph'\left((1-i)\tfrac{\vartheta+3\kappa}{4}+i\kappa\right)-\aleph'\left(\tfrac{\vartheta+3\kappa}{4}\right)\right|di \\
&\quad+\tfrac{1}{2}\left|\aleph'(\vartheta)-\aleph'\left(\tfrac{\vartheta+3\kappa}{4}\right)\right|+\tfrac{7}{6}\left|\aleph'\left(\tfrac{\vartheta+\kappa}{2}\right)-\aleph'\left(\tfrac{3\vartheta+\kappa}{4}\right)\right|\Bigg) \\
&\leq \tfrac{\kappa-\vartheta}{16}\Bigg(\tfrac{(\kappa-\vartheta)L}{4}\int_0^1 i^2di+\tfrac{(\kappa-\vartheta)L}{4}\int_0^1\left(\tfrac{5}{3}-i\right)idi+\tfrac{(\kappa-\vartheta)L}{4}\int_0^1\left(i+\tfrac{2}{3}\right)idi \\
&\quad+\tfrac{(\kappa-\vartheta)L}{4}\int_0^1(1-i)idi+\tfrac{L}{2}\left|a-\tfrac{a+3b}{4}\right|+\tfrac{7L}{6}\left|\tfrac{a+b}{2}-\tfrac{3a+b}{4}\right|\Bigg) \\
&=\tfrac{13(\kappa-\vartheta)^2}{192}L.
\end{aligned}
$$

The proof is completed. □

4. Applications

Dual Simpson's quadrature formula

Let Λ be the partition of the points $\vartheta=e_0<e_1<\ldots<e_n=\kappa$ of the interval $[\vartheta,\kappa]$, and consider the quadrature formula

$$\int_\vartheta^\kappa \aleph(j)dj=\lambda(\aleph,\Lambda)+R(\aleph,\Lambda),$$

where

$$\lambda(\aleph,\Lambda)=\sum_{i=0}^{n-1}\tfrac{e_{i+1}-e_i}{3}\left(2\aleph\left(\tfrac{3e_i+e_{i+1}}{4}\right)-\aleph\left(\tfrac{e_i+e_{i+1}}{2}\right)+2\aleph\left(\tfrac{e_i+3e_{i+1}}{4}\right)\right),$$

and $R(\aleph,\Lambda)$ denotes the associated approximation error.

Proposition 1. *Let $n\in\mathbb{N}$ and $\aleph:[\vartheta,\kappa]\to\mathbb{R}$ be a differentiable function on (ϑ,κ) with $0\leq\vartheta<\kappa$ and $\aleph'\in L^1[\vartheta,\kappa]$. If $|\aleph'|$ is s-convex function with $s\in(0,1]$, we have*

$$
\begin{aligned}
|R(\aleph,\Upsilon)|\leq&\sum_{i=0}^{n-1}\tfrac{(e_{i+1}-e_i)^2}{16}\left(\tfrac{1}{(s+1)(s+2)}\left(|\aleph'(e_i)|+|\aleph'(e_{i+1})|\right)+\tfrac{4s+14}{3(s+1)(s+2)}\left|\aleph'\left(\tfrac{e_i+e_{i+1}}{2}\right)\right|\right. \\
&\left.+\tfrac{8s+10}{3(s+1)(s+2)}\left(\left|\aleph'\left(\tfrac{3e_i+e_{i+1}}{4}\right)\right|+\left|\aleph'\left(\tfrac{e_i+3e_{i+1}}{4}\right)\right|\right)\right).
\end{aligned}
$$

Proof. Using Theorem 1 on $[e_i,e_{i+1}]$ $(i=0,1,\ldots,n-1)$, we obtain

$$
\begin{aligned}
&\left|\tfrac{1}{3}\left(2\aleph\left(\tfrac{3e_i+e_{i+1}}{4}\right)-\aleph\left(\tfrac{e_i+e_{i+1}}{2}\right)+2\aleph\left(\tfrac{e_i+3e_{i+1}}{4}\right)\right)-\tfrac{1}{e_{i+1}-e_i}\int_{e_i}^{e_{i+1}}\aleph(j)dj\right| \\
&\leq\tfrac{e_{i+1}-e_i}{16}\left(\tfrac{1}{(s+1)(s+2)}\left(|\aleph'(e_i)|+|\aleph'(e_{i+1})|\right)+\tfrac{4s+14}{3(s+1)(s+2)}\left|\aleph'\left(\tfrac{e_i+e_{i+1}}{2}\right)\right|\right. \\
&\quad\left.+\tfrac{8s+10}{3(s+1)(s+2)}\left(\left|\aleph'\left(\tfrac{3e_i+e_{i+1}}{4}\right)\right|+\left|\aleph'\left(\tfrac{e_i+3e_{i+1}}{4}\right)\right|\right)\right).
\end{aligned}
\tag{22}
$$

Multiplying both sides of (22) by $(e_{i+1}-e_i)$, summing the obtained inequalities for all $i=0,1,\ldots,n-1$ and using the triangular inequality, we obtain the result. □

Application to special means

For arbitrary real numbers $e, e_1, e_2, ..., e_n, f$ we have:
The Arithmetic mean: $A(e_1, e_2, ..., e_n) = \frac{e_1+e_2+...+e_n}{n}$.
The p-Logarithmic mean: $L_p(e, f) = \left(\frac{f^{p+1}-e^{p+1}}{(p+1)(f-e)}\right)^{\frac{1}{p}}, e, f > 0, e \neq f$ and $p \in \mathbb{R} \setminus \{-1, 0\}$.

Proposition 2. *Let $e, f \in \mathbb{R}$ with $0 < e < f$, then we have*

$$\left|2A^{\frac{3}{2}}(e,e,e,f) - A^{\frac{3}{2}}(e,f) + 2A^{\frac{3}{2}}(e,f,f,f) - 3L^{\frac{3}{2}}_{\frac{3}{2}}(e,f)\right| \leq \frac{15(f-e)\left(\sqrt{f}-\sqrt{e}\right)}{32}.$$

Proof. Applying Theorem 4 to the function $\aleph(j) = j^{\frac{3}{2}}$ on $[e, f]$. $\square$

5. Conclusions

Many practical studies and engineering problems often lead to calculations of integrals, most of which cannot be solved directly, requiring us to evaluate them by different quadrature rules, hence the need to estimate the error made to better circumvent and manage the problem. Thus, in this work, we have considered the dual Simpson quadrature rule. We have firstly established a novel identity. Based on this identity, we have derived some new dual Simpson type integral inequalities for functions whose first derivatives are -convex. We have also discussed the above-mentioned inequality when the first derivatives lie in the classes of bounded and Lipschitzian functions. We have provided at the end some applications to quadrature formulas and special means. We hope that the obtained results stimulate further research, as well as generalizations in various other types of calculus in this interesting field.

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