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Effects of Dipole-Dipole Interaction and Time-Dependent Coupling on the Evolution of Entanglement and Quantum Coherence for Superconducting Qubits in a Nonlinear Field System

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Abstract: We examine the temporal comportment of formation entanglement and quantum coherence in a quantum system made up of two superconducting charge qubits (SC-Qs), in the case of two different classes of nonlinear field. The results discussed the impact role of time-dependent coupling (T-DC) and dipole-dipole interaction (D-DI) on the temporal comportment of quantum coherence and entanglement in the ordinary and nonlinear field. In addition, we show that the main parameters of the quantum model affect the entanglement of formation and the coherence of the system in a similar way.

Keywords: time-dependent coupling; superconducting qubit; dipole–dipole effect; nonlinear field; quantum coherence

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1. Introduction

Superconducting circuits represent a realistic path for the practical implementation of SC-Q in a solid-state Josephson junction system, and these circuits are capable of completing the tasks required for the advancement of quantum technology [1–9]. The SC-Q acts as a system with two charge states [10], but artificial atoms behave in the same manner as natural atoms. A few of the quantum properties of SC-Qs have been identified as a result of experimental findings where the oscillations between the generation of a superposition of charge states are consistent [11,12]. These findings led to the detection of a few of the SC-Qs' quantum properties. A solid-phase circuits and photon interactions have been the subject of recent investigation and it has been shown that SC-Q systems may be integrated with microwave photons [13]. When the SC-Qs are linked with a quantized field, quantum properties such as Rabi oscillations and coherent control become apparent. In general, SC-Qs have short durations of coherence, which reduces the amount of time necessary to carry out gate operations when an external field is present [14,15]. The difficulties of quantum optics are related with the existence of a suitable quantum system, which needs to be taken into consideration.

As a result, a crucial step toward understanding what is the required regime for such quantum systems has been taken. The growth of quantum information processing (QIP) has accelerated recently thanks to increasing knowledge of entanglement phenomena and additional articles published in the trend of QIP. Due to this, optimal execution in quantum metrology protocols and a number of QIP tasks depends on entangled states have been attained [16–21]. Entangled states have been successfully considered with deep understanding of composite quantum systems [22,23]. In this context, the subsystem states are identified by the connections with each other, so non-separable states are essential for achieving this goal. In addition, quantum entanglement can be considered as one of the most important phenomena for the application of quantum processes in contemporary quantum technology [24–26]. The measurement outcomes of quantum entanglement between quantum states have made it feasible to comprehend phenomena and find answers to several issues in the branches of physical sciences [21–29].

The type of quantum system used has a direct bearing on the new possibilities for applying quantum technology. As a result, having a deep understanding of the interactions that occur between nonlinear domains and SC-Qs aids to development of quantum technologies and their applications. For instance, the Tavis-Cummings model is considered as a useful quantum model representing two SC-Qs that are connected to a single field in the context of rotating-wave approximation (RWA) [30]. As a result of this, several generalizations have been postulated for this model in a variety of different directions [31,32]. These advancements include the use of T-DC or the generalization of the sort of constant coupling, both of which are essential for a variety of phenomena in quantum optics and quantum information [33]. The execution of several QIP tasks depends on the finding of actual systems, such as the circuit QED, which are required for completing these tasks. Also, the physical systems have many different applications [34–36].

In light of the information presented above, we continue our investigation into the physical processes in the framework of SC-Qs. We determine the conditions under which it is possible to successfully implement quantum technologies using SC-Qs. We provide a scheme of two SC-Qs that are linked to a nonlinear field by exploring the influence of D-DI when t–dc is ignored and considered. The findings show, by taking into account quantum coherence (QC) and entanglement of formation (EoF), the field nonlinearity, the D–DI, and the t–dc all have an effect on the variation of the two information quantifiers. The remaining parts of the manuscript are structured as follows. In Section 2, the dynamics of the physical system, together with its core notions and measurements of quantumness, are presented. Furthermore, it includes some related terminology. The numerical findings that were obtained are presented in Section 3. In the last part, we provide a summary.

2. Proposed Two SC-Qs Scheme

The proposed system comprises a pair of identical SC-Qs coupled to nonlinear field (NF) with D-DI in the case of constant and T-DC. Over the past several years, solid electronic states have been used to realize a number of quantum device designs [37]. This study provides a straightforward circuit that may be created from a pair of 2-Cooper boxes linked by (Josephson energy E_J and capacitance C_J). We have considered each SC-Q in interaction with a NF.

In the context of a nonlinear oscillator, the Hamiltonian of a field is given by

$$\hat{H}_{NF} = \omega_r \left(\frac{\hat{A}\hat{A}^{\dagger} + \hat{A}^{\dagger}\hat{A}}{2} \right), \ \hbar = 1 \tag{1}$$

where the frequency of the field is designed by ω_r. Also, we consider the frequency of the LC-resonance or the NF at the resonance $\omega_r = \frac{1}{\sqrt{LC}}$. The annihilation (creation) operator of the NF $\hat{A}^{\dagger}$ $(\hat{A})$ is given in terms of the usual operators $\hat{a}^{\dagger}$ and $\hat{a}$ by

$$\begin{aligned} \hat{A} &= \hat{a} f(\hat{n}) = f(\hat{n}+1)\hat{a}, \\ \hat{A}^{\dagger} &= f(\hat{n})\hat{a}^{\dagger} = \hat{a}^{\dagger} f(\hat{n}+1), \end{aligned} \tag{2}$$

with $\hat{n} = \hat{a}^\dagger \hat{a}$ denotes the number operator. The function f describes the field's nonlinearity with respect to the commutation of $\hat{A}^\dagger$ and $\hat{A}$

$$[\hat{A}, \hat{n}] = \hat{A}, \ \left[\hat{A}^\dagger, \hat{n}\right] = -\hat{A}^\dagger$$

and

$$\left[A^\dagger, \hat{A}\right] = \hat{n}f^2(\hat{n}) - (\hat{n}+1)f^2(\hat{n}+1).$$

If $f(\hat{n}) = 1$, the form of usual Heisenberg algebra is obtained and we have

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1, \ \ [\hat{n}, \hat{a}] = -\hat{a}, \text{ and } \left[\hat{n}, \hat{a}^\dagger\right] = \hat{a}^\dagger$$

The free Hamiltonian of the two SC-Qs in terms of the Pauli operator in z-spin component $\hat{\sigma}_z^1$ and $\hat{\sigma}_z^2$

$$\hat{H}_{SC-Qs} = E_J\left(\frac{\hat{\sigma}_z^1 + \hat{\sigma}_z^2}{2}\right) \tag{3}$$

Now, the Hamiltonian of the system can be provided as [38].

$$\hat{H} = \hat{H}_{NF} + \hat{H}_{SC-Qs} + \xi\left(\hat{A}^\dagger + \hat{A}\right)\left(\hat{\sigma}_x^1 + \hat{\sigma}_x^2\right), \ \hbar = 1 \tag{4}$$

where the coupling parameter takes $\xi = \left(\frac{eC_g}{C_g+C_J}\right)\sqrt{\frac{\hbar\omega_r}{LC}}$ **and** $C_g + C_J$ represents the total capacitance, where C_g is the gate capacitance, $\hat{\sigma}_x^j$, $j = 1,2$ denotes to the Pauli matrix of z- spin component. A lot of applications were obtained in numerous domains of physics since the achievement of quantum algebra using deformed bosonic operators. A lot of attention has been given to deformed algebra in the framework of different mathematical and physical problems, including non-linear finite W-symmetries [39] and geometry of symmetrized states [40]. Applying the RWA, the Hamiltonian in the interaction picture is formulated as

$$\hat{H}_I = p(t)\sum_{j=1,2}\left(\hat{A}\left|0\right\rangle_{jj}\left\langle 1\right| + \hat{A}^\dagger\left|1\right\rangle_{jj}\left\langle 0\right|\right), \tag{5}$$

where $p(t) = \xi$ in the case of constant coupling between the NF and two SC-Qs while $p(t) = \xi\sin^2(t)$ **with respect to** t–dc. Here, the jth SC-Q corresponds to the case of the excited (ground) state $|0\rangle_j$ $(|1\rangle_j)$. Moreover, we consider the influence of D–DI via

$$H_{dd} = \hbar\lambda_{dd}\left(\hat{\sigma}_-^{(2)}\hat{\sigma}_+^{(1)} + \hat{\sigma}_+^{(1)}\hat{\sigma}_-^{(2)}\right), \tag{6}$$

where λ_{dd} is the D-DI, and the SC-Qs operators is defined as $\hat{\sigma}_+(\hat{\sigma}_-) = |0\rangle\langle 1|$ $(|1\rangle\langle 0|)$. A deformed coherent state is described in detail [41,42] and the preparation of NF states highlights the necessity for a trapping system [43]. Depending on the value of the function f, we assume the initial states of the quantized field is a coherent and a nonlinear coherent state with the initial density operator

$$\rho_{NF}(0) = |z,f\rangle\langle z,f|, \tag{7}$$

and a maximally entangled state for the pair of SC-Qs, $|E\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ with initial density matrix $\rho_{qq}(0) = |E\rangle\langle E|$. The product states then provide the overall system state $\rho(0) = \rho_{qq}(0) \otimes \rho_{NF}(0)$.

NF coherent states were defined as eigenvectors of the annihilation operator $\hat{A}$ as $\hat{A}|z,f\rangle = z|z,f\rangle$ with z is the eigenvalue of $\hat{A}$. The form of this state is given by [44]

$$|z,f\rangle = \left(\exp_f\left[|z|^2\right]\right)^{-\frac{1}{2}}\sum_{m=0}^{\infty}\sqrt{\frac{z^{2m}}{[m]_f!}}|m\rangle, \tag{8}$$

where

$$\mathbf{exp}_f[x] = \sum_{m=0}^{\infty} \frac{x^m}{[m]_f!},$$

$$\text{and } [m]_f! = \left[mf^2(m)\right] \times \left[(m-1)f^2(m-1)\right] \cdots \left[f^2(1)\right].$$

is the deformed exponential function.

We note that although $\mathbf{exp}_f[x]\,\mathbf{exp}_f[y] \neq \mathbf{exp}_f[x+y]$, it may be used to investigate a number of physical issues. Consider two forms of deformation provided by the functions

$$f_1(n) = \frac{1-q_1^{2n}}{1-q_1^2}, \tag{9}$$

$$f_2(n) = \frac{q_2(q_2^{-n} - q_2^n)}{1-q_2^2}\,. \tag{10}$$

The wavefunction at instant t under the above initial condition can be written as

$$|S(t)\rangle = \sum_{m=0}^{\infty} (V_1|m,00\rangle + (V_2|01\rangle + V_3|10\rangle)|m+1\rangle + V_4|m+2,11\rangle). \tag{11}$$

By using the Schrödinger equation, $i\hbar\frac{\partial}{\partial t}|S(t)\rangle = \hat{H}|S(t)\rangle$, the coefficients V_l $(l = 1, \cdots, 4)$ are obtained through the solve of the following differential equation

$$\frac{dV}{dt} = -iLV, \tag{12}$$

where

$$V = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}, L = \begin{pmatrix} 0 & p(t)\sqrt{m+1} & p(t)\sqrt{m+1} & 0 \\ p(t)\sqrt{m+1} & 0 & \lambda_{qq} & p(t)\sqrt{m+1} \\ p(t)\sqrt{m+1} & \lambda_{qq} & 0 & p(t)\sqrt{m+1} \\ 0 & p(t)\sqrt{m+1} & p(t)\sqrt{m+1} & 0 \end{pmatrix} \tag{13}$$

By taking the trace over the NF, the SC-Qs matrix operator may be found as

$$\hat{\rho}_{qq}(t) = Tr_{NF}|\,S(t)\rangle\langle S(t)\,|, \tag{14}$$

and $1st(2nd)$ SC-Q as the density matrix

$$\hat{\rho}_{1st(2nd)\ \mathbf{q}}(t) = Tr_{2nd(1st)\ \mathbf{q}}(\hat{\rho}_{qq}(t)). \tag{15}$$

Based on the above formulation, the effect of field nonlinearity on the measures of quantumness can be examined by the value of the parameter $q_1(q_2)$ in the box function $f_1(n)$ $(f_2(n))$.

The entanglement of formation (EoF) can be used to detect the entanglement for SC-Qs state, $\hat{\rho}_{qq}$, and its expression is introduced as [45]

$$EoF(t) = H\left\{\frac{1}{2}\left(\sqrt{1 - C^2(\hat{\rho}_{qq}(t))} + 1\right)\right\}, \tag{16}$$

where H stands for the information entropy defined by

$$H(\vartheta) = -\vartheta log_2\vartheta - (1-\vartheta)log_2(1-\vartheta), \tag{17}$$

and the concurrence $C(\hat{\rho}_{qq}(t))$ is formulated in terms of the eigenvalues η_j of the matrix $\sqrt{\sqrt{\hat{\rho}_{qq}(t)}\widetilde{\rho}_{qq}(t)\sqrt{\hat{\rho}_{qq}(t)}}$ in the decreasing order as

$$C(\hat{\rho}_{qq}(t)) = max\left\{0, \sqrt{\eta_1(t)} - \sqrt{\eta_2(t)} - \sqrt{\eta_3(t)} - \sqrt{\eta_4(t)}\right\}, \tag{18}$$

with $\widetilde{\rho}_{qq}(t)$ is determined from

$$\widetilde{\rho}_{qq}(t) = (\hat{\sigma}_y \otimes \hat{\sigma}_y)\rho^*_{qq}(t)(\hat{\sigma}_y \otimes \hat{\sigma}_y), \tag{19}$$

and $\rho^*_{qq}(t)$ denotes the conjugate of $\hat{\rho}_{qq}(t)$ with $\hat{\sigma}_y$ designs the Pauli y-operator.

The diagonal elements of the density operator of a quantum system are responsible for the majority of the properties that define the coherence. The quantum coherence Q_C based on l_1 norm is defined in terms of the off-diagonal elements as [46]

$$Q_C = \min_{\delta \in M} \parallel \rho - \delta \parallel_{l_1} = \sum\nolimits_{k \neq j} |\rho_{kj}|, \tag{20}$$

where $\parallel \rho - \delta \parallel_{l_1}$ represents the distance of the state ρ to a set of incoherent states δ. The function Q_C verifies the property of monotonicity for different quantum states. For a d-dimensional system, it satisfies $0 \leq Q_C \leq d - 1$.

3. Numerical Findings and Discussion

Here, we analyze the QC and EoF with respect to the nonlinearity in the quantized field, D-DI, as well as T-DC to investigates the temporal behavior of the two suggested measures of quantumness.

Figure 1 shows how the deformation parameter and T-DC affect the SC-Qs entanglement's temporal fluctuation. Generally, we can see that the SC-Qs-NF coupling and deformation parameter values have a big impact on the evolution entanglement. From the figure, it can be seen that the EoF exhibits the phenomena of sudden death and sudden birth of quantum entanglement for diverse values of the parameter q [47]. To confirm these phenomena, we present the results of different values of the deformed parameter in Figure 2. For $q = 1$ and $p(t) = \xi$, the function EoF displays a behavior with rapid oscillations accompanied by amplitudes with maximal value that is less than in the initial instant. For $q \neq 1$ and $p(t) = \xi$, the deformation effect leads to increase in the value of the function EoF and then reinforce the amount of entanglement during the dynamics. Moreover, the presence of T-DC, $p(t) = \xi \sin^2(t)$, results a reduction in the oscillations of the function EoF and its comportment becomes more regular. Furthermore, we can note that the deformation effect on the quantum entanglement is smaller in the presence of T-DC. These results show that the protection of the SC-Qs entanglement during the evolution can be made by a considerable control of the field nonlinearity and the T-DC. Figure 3 displays the time evolution of EoF by taking into consideration the D-DI effect. From the figure, it can be seen that the D-DI affects the comportment of the entanglement in the presence of deformation and T-DC effect. In this context, the existence of D-DI can enhance the amount of the entanglement with the quasi-periodic behavior of the EoF during the evolution. To confirm this result, we present the time variation of EoF for several values of λ_{dd} in Figure 4.

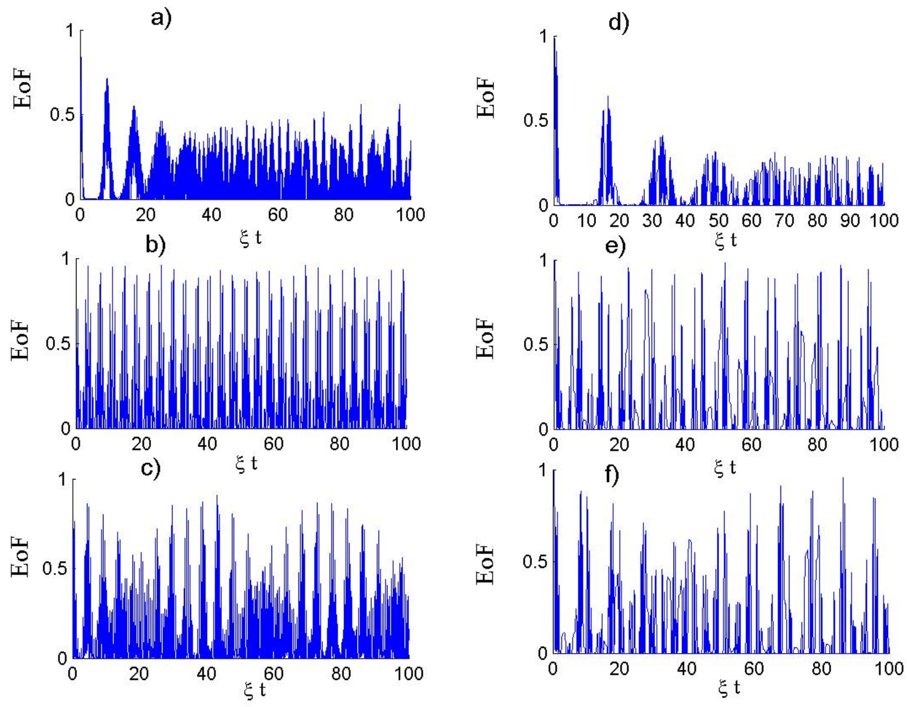

Figure 1. Evolution of the EoF of the pair state of SC-Qs versus the scaled time $T = \xi t$ in the absence of D-DI effect (i.e., $\lambda_{dd} = 0$). For the field with $f(n) = 1$ and in the coherent state for $z = \mathbf{5}$, panels (**a**,**d**) are displayed. For the field in a nonlinear coherent state for $f_1(n)$ with $q_1 = 1.5$ panels (**b**,**e**) and $f_2(n)$ with $q_2 = 1.5$ panels (**c**,**f**), respectively. Panels (**a**–**c**) are for constant coupling $p(t) = \xi$ and (**d**–**f**) in the presence of the T-DC with $p(t) = \xi \sin^2(t)$.

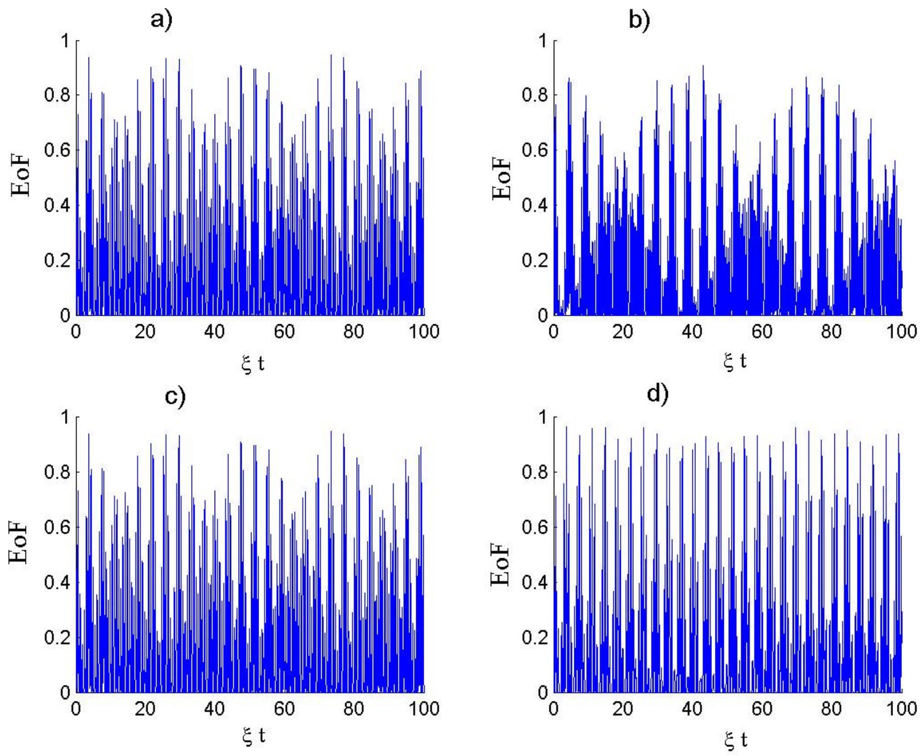

Figure 2. Evolution of the EoF of the pair state of SC-Qs against the scaled time $T = \xi t$ with the absence of D-DI effect for $z = \mathbf{5}$ and several values of q. Panel (**a**) is defined for $q_2 = 0.5$, panel (**b**) corresponds to $q_2 = 1.5$, panel (**c**) is defined for $q_2 = 2$, and panel (**d**) corresponds to $q_2 = 2.5$.

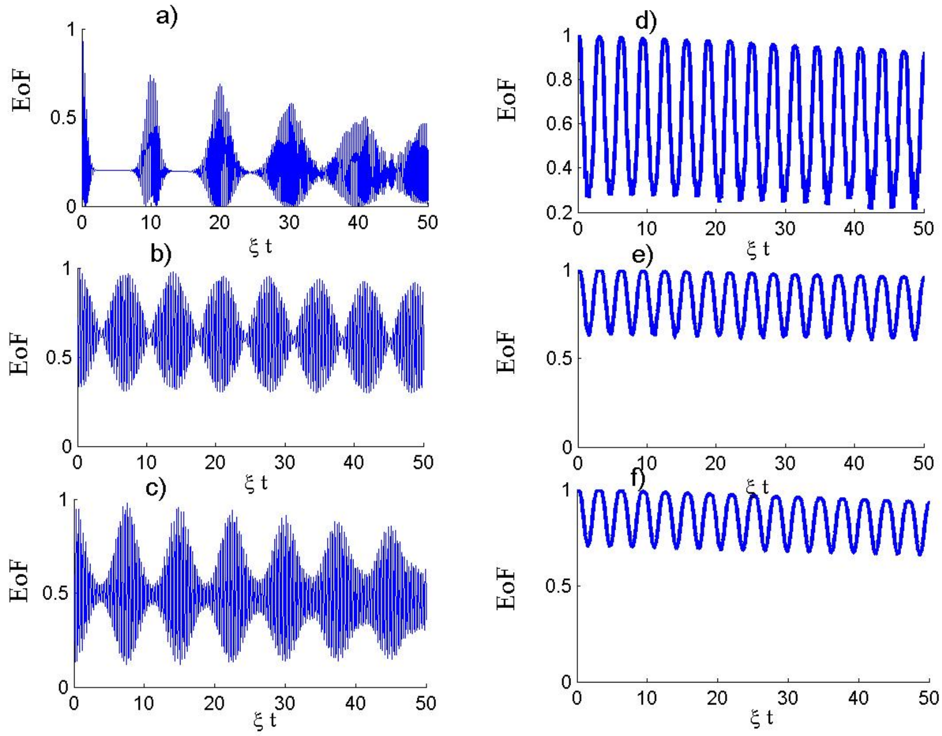

Figure 3. Effect of D-DI with $\lambda_{dd} = 15$ on the evolution of *EoF* for the field with $f(n) = 1$ and in the coherent state for $z = \mathbf{5}$, panels (**a**,**d**) are displayed. For the field in a nonlinear coherent state for $f_1(n)$ with $q_1 = 1.5$ panels (**b**,**e**) and $f_2(n)$ with $q_2 = 1.5$ panels (**c**,**f**), respectively. Panels (**a**–**c**) are for constant coupling $p(t) = \xi$ and (**d**–**f**) in the presence of the T-DC with $p(t) = \xi \sin^2(t)$.

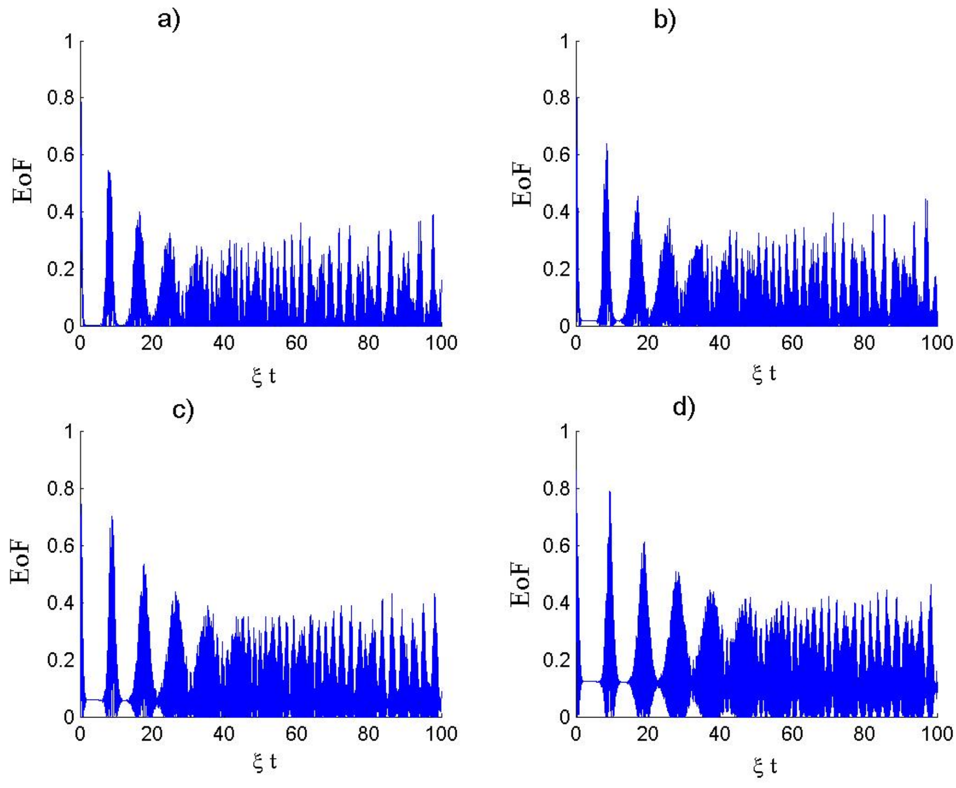

Figure 4. Evolution of the *EoF* of the pair state of SC-Qs against the scaled time $T = \xi t$, $z = \mathbf{5}$ and several values of λ_{dd} with $q = 1$. Panel (**a**) is defined for $\lambda_{dd} = 3$, panel (**b**) corresponds to $\lambda_{dd} = 6$, panel (**c**) is defined for $\lambda_{dd} = 9$, and panel (**d**) corresponds to $\lambda_{dd} = 12$.

Figure 5 shows the influence of deformation parameter and t–dc on the time variation of the SC-Qs coherence. Generally, we can note that the amount of QC is very sensitive to the values of the deformation parameter and the SC–Q-NF coupling. We can observe that the coherence experiences a dynamical behavior with rapid oscillations and amplitudes that depend on the values of $\boldsymbol{q}$. For the classical limit $\boldsymbol{q} \to \mathbf{1}$ or the absence of the deformation effect, the results have some similarities with the results of ref. [48]**.** The presence of the T-DC leads to reduce the oscillations in the coherence measure during the time evolution. On the other hand, the presence of D-DI results an enhancement in the amount of SC-Qs coherence with more regularity in the dynamical behavior, as shown in Figure 6 for $\boldsymbol{p(t)} = \boldsymbol{\xi}$. Whereas for $\boldsymbol{p(t)} = \boldsymbol{\xi}\mathbf{sin}^{\mathbf{2}}\boldsymbol{(t)}$, the behavior of the QC becomes quasi periodic with the appearance of the trapping phenomenon of coherence. These results show that the enhancement and protection of the SC-Qs coherence during the evolution can be made through the suitable control of the nonlinearity of the field, T-DC and D-DI. We can note that the behavior of coherence is directly related to the correlation between the SC-Qs. In fact, the presence of D-DI results an energy exchange between the SC-Qs, leading to the enhancement of correlation, which will be accompanied by an increase in the amount of coherence. On the other hand, when the deformed parameter of the field gets further from one, the measure of correlation (entanglement) presents rapid oscillations with large amplitudes, which can be explained by the change of emitted energy and thus results an enhancement in the coherence value.

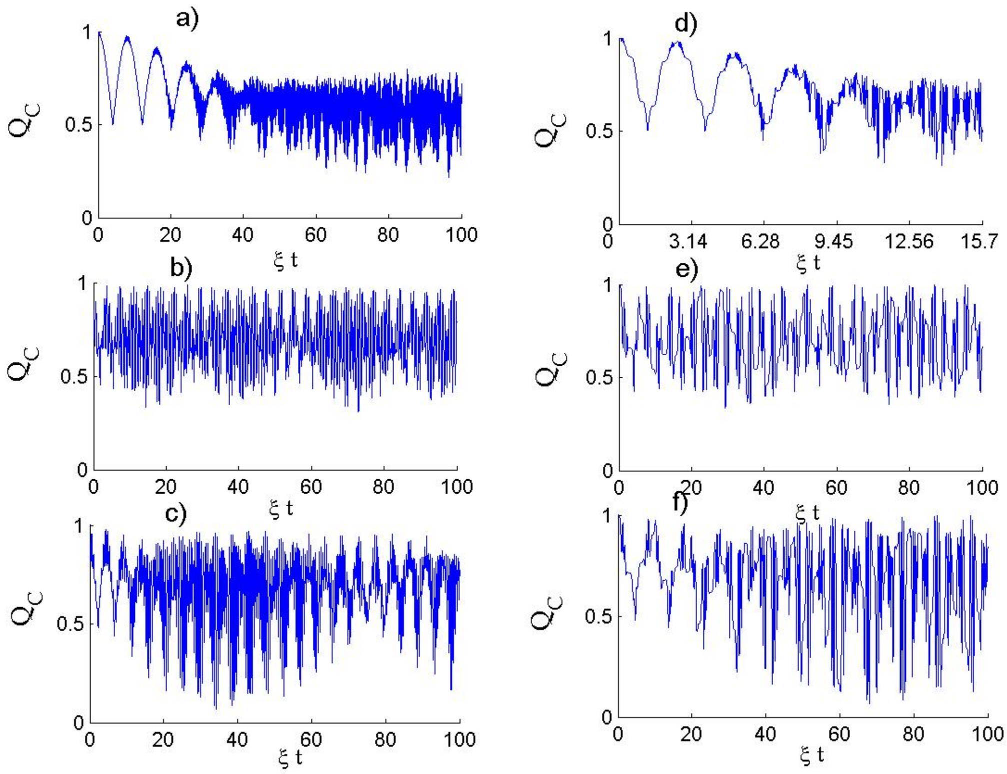

Figure 5. Evolution of the quantum coherence of SC-Qs state versus the scaled time $T = \xi t$ in the absence of d–d interaction effect (i.e., $\lambda_{qq} = 0$). For the field with $f(n) = 1$ and in the coherent state, panels (**a**,**d**) are displayed. The field in a nonlinear coherent state for $f_1(n)$ with $q_1 = 1.5$ panels (**b**,**e**) and $f_2(n)$ with $q_2 = 1.5$, panels (**c**,**f**) respectively. Panels (**a**–**c**) are for constant coupling $p(t) = \xi$ and (**d**–**f**) in the presence of the t–d coupling with $p(t) = \xi \sin^2(t)$.

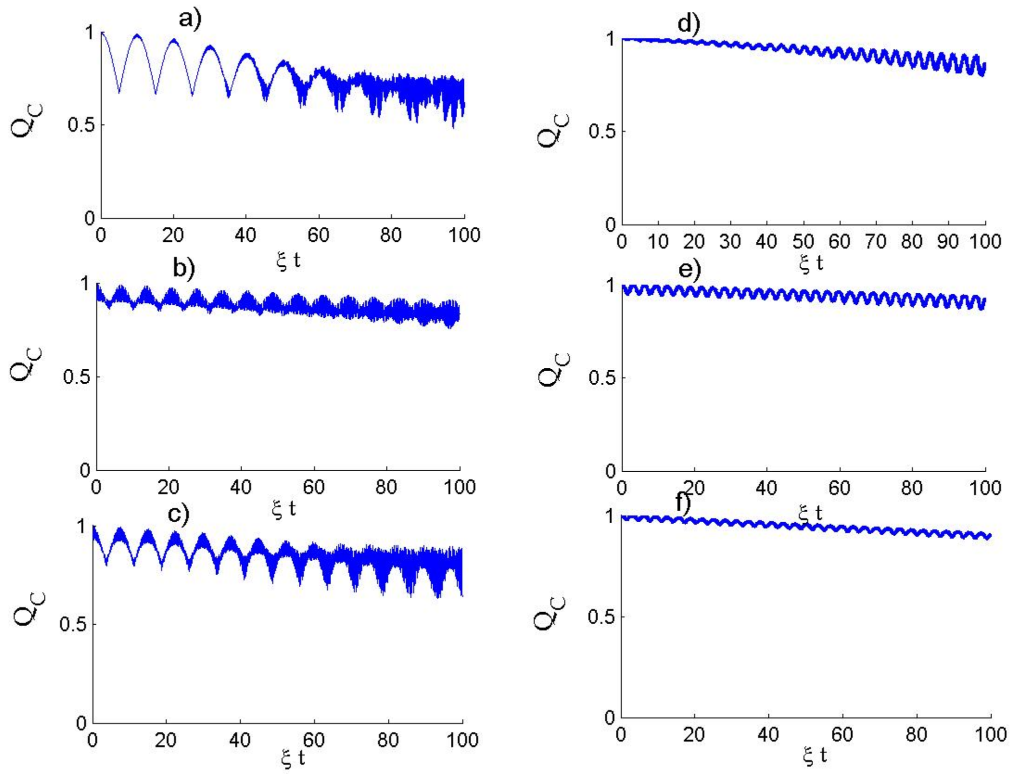

Figure 6. Evolution of quantum coherence under d–dI effect for $\lambda_{dd} = 15$. For the field with $f(n) = 1$ and in the coherent state, panels (**a**,**d**) are displayed. The field in a nonlinear coherent state for $f_1(n)$ with $q_1 = 1.5$ panels (**b**,**e**) and $f_2(n)$ with $q_2 = 1.5$, panels (**c**,**f**) respectively. Panels (**a–c**) are for constant coupling $p(t) = \xi$ and (**d–f**) in the presence of the t–d coupling with $p(t) = \xi \sin^2(t)$.

4. Conclusions

In conclusion, using current QED technologies, the QC and EoF for a pair of SC-Qs interacting with an NF were examined. In the scenario of an ordinary field and an NF, we have taken into account the consequences of the D-DI and T-DC. We have discussed about how the EoF and QC of the SC-Qs state behave dynamically both with and without the influence of D-DI and T-DC. By making an appropriate selection of the class of field nonlinearity and the t–dc, it is shown that the possibility to improve and control the entanglement and QC of SC-Qs state. We have illustrated that the quantifiers are more susceptible to the field's nonlinearity and the t–dc compared with the d–dI. In addition, we have shown that the main physical parameters of the model affect the entanglement of formation and the quantum coherence of the system in a similar way. A substantial consideration for future study the dynamics of the quantumness for open nonlinear schemes in the presence of noise at finite temperatures.

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