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On the Krýlov–Bogoliúbov-Mitropólsky and Multiple Scales Methods for Analyzing a Time Delay Duffing–Helmholtz Oscillator

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Abstract: This study is divided into two important axes; for the first one, a new symmetric analytical (approximate) solution to the Duffing–Helmholtz oscillatory equation in terms of elementary functions is derived. The obtained solution is compared with the numerical solution using 4th Range–Kutta (RK4) approach and with the exact analytical solution that is obtained using elliptic functions. As for the second axis, we consider the time-delayed version for the same oscillator taking the impact of both forcing and damping terms into consideration. Some analytical approximations for the time delayed Duffing–Helmholtz oscillator are derived using two different perturbation techniques, known as Krylov–Bogoliubov–Mitropolsky method (KBMM) and the multiple scales method (MSM). Moreover, these perturbed approximations are analyzed numerically and compared with the RK4 approximations.

Keywords: time delay Duffing–Helmholtz oscillator; elliptic and elementary functions; Krýlov–Bogoliúbov–Mitropólsky method; multiple scales method; analytical approximations

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1. Introduction

In many scientific domains, especially solid-state physics, plasma waves, fluid mechanics, biology, and robotics motion, nonlinear evolution equations (NLEEs) are frequently employed as models to represent engineering and physical processes. Therefore, it is crucial to find the exact (if possible), analytical and numerical solutions to these NLEEs. Generally, the NLEEs are difficult to solve exactly, and only in a small number of exceptional circumstances, can their solutions be unambiguously recorded in writing. However, during the past few decades, significant progress has been made, and numerous potent and successful methods and techniques for getting the exact and approximate solutions of NLEEs have been proposed in the literature. Some important methods and techniques found in the literature that have been used for solving and analyzing different types of NLEEs and oscillatory equations include the family of the homotopy perturbation technique (HPT) [1–7], Krylov–Bogoliubov–Mitropolsky method (KBMM) [7–15], multiple scales method (MSM) [14–19], He's MSM [20], the equivalent linearized method [21], the non-perturbative approach [22], Galerkin method and ansatz method [23], energy balance method (BM) [24,25], harmonic BM [26–29], Hamiltonian technique [30,31], and many other methods.

In linear oscillators, the presence of friction and an external periodic force can only produce a periodic response, while in a nonlinear oscillator, the response may become chaotic for a certain range of values of some characteristic parameter that we modify. Accordingly, in this investigation and for the first goal, we present the solution to the

Duffing–Helmholtz oscillator (D-HO) for a given arbitrary initial conditions using both elliptic (exact solution) and trigonometric functions (approximate solution). The D-HO, sometimes called mixed-parity Duffing oscillator, which is given by the following form

$$\begin{cases} \ddot{q} + \alpha + \beta q + \gamma q^2 + \delta q^3 = 0, \\ q(0) = q_0 \text{ and } \dot{q}(0) = \dot{q}_0 \,, \end{cases} \tag{1}$$

where $q \equiv q(t)$ denotes the displacement of the system, β is the natural frequency, γ and δ are nonlinear parameters and α is a system parameter independent of the time while q_0 and $\dot{q}_0$ denote the initial position and velocity of the oscillator. Equation (1) has a diversity of applications [32–35]. The standard D-HO (1) can be interpreted as a particle moving in a quadratic potential field, and it has also been studied in a nonlinear circuit theory as well as is related to many nonlinear phenomena that arise in plasma physics. Moreover, the D-HO (1) is a completely integrable equation that contains an abundance of significant properties that occurs in many physical and engineering areas. This equation and some of its extensions have been solved using many different approaches. For instance, the standard D-HO (1) has been solved using the improved harmonic BM [32]. Moreover, some new periodic analytical approximations for the standard D-HO (1) has been derived using the quadrature rules [33]. Some periodic approximations for the D-HO (1) have been obtained via He's Energy BM and He's Frequency Amplitude Formulation [34]. Moreover, the exact solutions to the D-HO (1) with frictional force has been obtained in the framework of Jacobi elliptic functions (JEFs) [35]. It is known that evaluating JEFs is a hard task. Thus, the advantage of solving the standard D-HO using elementary functions allows one to evaluate the obtained solutions using only trigonometric functions, which may be performed using a simple calculator. Moreover, the period of the solution is approximated with high accuracy using the period of a trigonometric function. On the other hand, there are perturbative approaches for solving Duffing and many other weakly nonlinear oscillators. However, the obtained solutions are quite complicated. However, our obtained formulas are short and highly accurate.

Moreover, and for the second goal of this investigation, the following time delayed D-HO is considered

$$\begin{cases} \ddot{x} + r_0 + r_1 x + r_2 x^2 + r_3 x^3 = \varepsilon(Q x(t-\tau) + f(t) - \dot{x}), \\ x(0) = x_0 \text{ and } \dot{x}(0) = \dot{x}_0 \,, \end{cases} \tag{2}$$

where $x \equiv x(t)$ and $(r_0, r_1, r_2, r_3, \varepsilon, Q, \tau)$ are non-zero parameters.

Some limited studies have addressed some different Duffing oscillators using several techniques with delayed feedbacks [36–38]. For example, the chaotic behavior of both the classical Duffing oscillator system without time delay $\left(\ddot{x} + c\dot{x} - kx + \gamma x^3 = f\cos(\lambda t)\right)$ and the Duffing oscillator system with time delay $\left(\ddot{x} + c\dot{x} - kx + \gamma x^3 = f\cos(\lambda t) + \alpha x(t-\tau)\right)$ have been investigated [36]. Moreover, the chaotic dynamics of a delayed Duffing oscillator having harmonic excitation and with delayed displacement and velocity feedbacks $\left(\ddot{x} + \delta\dot{x} + \omega_0^2 x + \alpha x^3 = f\cos(\omega t) + u x(t-\tau_1) + v\dot{x}(t-\tau_2)\right)$ have been investigated [37]. The periodic solutions to the time delay autonomous Duffing oscillatory equations have been reported [38]. In this context, both KBMM and MMS [14,15] are introduced for analyzing to provide approximate analytical solutions of the delayed D-HO (2). Moreover, this problem will solve numerically using 4th-order Range–Kutta (RK4) scheme. Both analytical and numerical approximations will numerically compare with each other. It should be noted that the proposed methods (KBMM and MSM) succeeded in analyzing many different nonlinear oscillators. For instance, both unforced and forced Duffing Van-der Pol (VdP) oscillators were solved using KBMM [13]. The authors made a comparison between the obtained approximations using both KBMM, HPM, and RK4 approach [13]. Moreover, both KBMM and MSM succeeded in solving and analyzing a generalized complex Duffing oscillator in the presence of fractional force, and some high-accurate approximations have been derived and compared with the numerical solutions [15]. Both unforced and forced jerk VdP oscillators were analyzed using both KBMM and MSM. Using the KBMM, both first-order approximation and second-order approximation for (un)forced jerk VdP oscillators have been derived [16]. On the contrary, the authors contented themselves with deriving the first-order approximation using MSM, and the obtained results were compared with the numerical approximation and the second-order approximation of KBMM. Furthermore, the KBMM was implemented in analyzing two different formulas for damped forced complex Duffing oscillators [39]. Motivated by these investigations, thus, we will apply these perturbative methods for analyzing the delayed D-HO (2).

This paper is organized as follows: In Section 2, the exact symmetric solutions to the D-HO (1) are derived and discussed based on the concept of a discriminant of the Duffing equation. In Section 3, a symmetric approximation to the D-HO (1) in the framework of a trigonometric function is derived and compared with Jacobi elliptic exact solution and numerical approximation. In Section 4, the delayed D-HO (2) is solved and analyzed using both KBMM and MSM. Note that in the delayed D-HO (2), the impact of time delay, frictional force, and excited periodic force is considered. Thus, this equation is not completely integrable and does not support an exact solution. Consequently, the obtained approximations using both KBMM and MSM will be compared with RK numerical approximations.

2. Exact Solution for D-HO

In the beginning, we should mention that the case for $\alpha = \gamma = 0$, was discussed in Ref. [40] but in our case, $|\alpha| + |\gamma| \neq 0$ is considered. Accordingly, the solution of the problem (1) can be assumed in the ansatz form

$$q \equiv q(t) = A + \frac{B}{1+v}, \tag{3}$$

where the function $v \equiv v(t)$ indicates a solution of the following normal Duffing equation

$$\begin{cases} \ddot{v} + av + bv^3 = 0, \\ v(0) = v_0 := \frac{q_0 - A - B}{A - q_0} \text{ and } \dot{v}(0) = \dot{v}_0 := -\frac{B\dot{q}_0}{(A-q_0)^2}. \end{cases} \tag{4}$$

Moreover, we get

$$\begin{cases} \dot{v}^2 = D - av^2 - \frac{b}{2}v^4, \\ \mathbb{R}_1 \equiv \frac{1}{2}\dot{q}^2 + \alpha q + \frac{1}{2}\beta q^2 + \frac{1}{3}\gamma q^3 + \frac{1}{4}\delta q^4 - C = 0, \end{cases} \tag{5}$$

with

$$\begin{cases} D = \dot{v}_0^2 + av_0^2 + \frac{b}{2}v_0^4, \\ C = \frac{1}{2}\dot{q}_0^2 + \alpha q_0 + \frac{1}{2}\beta q_0^2 + \frac{1}{3}\gamma q_0^3 + \frac{1}{4}\delta q, \end{cases} \tag{6}$$

where (a, b, A, B) are undetermined variables.

Inserting ansatz (3) into the second equation of system (5) and taking into account the first equation in the system (5), we get

$$12(1+v^4)\mathbb{R}_1 = Y_0 + \sum_{i=1}^{4} Y_i v^i, \tag{7}$$

with

$$
\begin{array}{l}
Y_0 = 12AB^3\delta + 12AB^2\gamma + 12AB\beta + 12A\alpha + 3B^4\delta \\
+4B^3\gamma + 6B^2D + 6B^2\beta + +12B\alpha - 12C + 3A^4\delta \\
+12A^3B\delta + 4A^3\gamma + 18A^2B^2\delta + 12A^2B\gamma + 6A^2\beta, \\
Y_1 = -4\left(\begin{array}{c} -3A^4\delta - 9A^3B\delta - 4A^3\gamma - 9A^2B^2\delta - 9A^2B\gamma \\ -6A^2\beta - 3AB^3\delta - 6AB^2\gamma - 9AB\beta \\ -12A\alpha - B^3\gamma - 3B^2\beta - 9B\alpha + 12C \end{array}\right), \\
Y_2 = -6\left(\begin{array}{c} aB^2 - 3A^4\delta - 6A^3B\delta - 4A^3\gamma - 3A^2B^2\delta - 6A^2B\gamma \\ -6A^2\beta - 2AB^2\gamma - 6AB\beta - 12A\alpha - B^2\beta - 6B\alpha + 12C \end{array}\right), \\
Y_3 = -4\left(\begin{array}{c} -3A^4\delta - 3A^3B\delta - 4A^3\gamma - 3A^2B\gamma - 6A^2\beta \\ -3AB\beta - 12A\alpha - 3B\alpha + 12C \end{array}\right), \\
Y_4 = -\left(-3A^4\delta - 4A^3\gamma - 6A^2\beta - 12A\alpha + 3bB^2 + 12C\right).
\end{array}
$$

By solving the algebraic system $Y_i = 0$ $(i, 0, 1, 2, 3, 4)$, we obtain

$$
\begin{array}{l}
a = \frac{1}{2}\left(-3A^2\delta - 3AB\delta - 2A\gamma - B\gamma - \beta\right), \\
b = \frac{1}{6}\left(9A^2\delta + 3AB\delta + 6A\gamma + B\gamma + 3\beta\right), \\
B = \frac{-3A^4\delta - 4A^3\gamma - 6A^2\beta - 12A\alpha + 12C}{3(A^3\delta + A^2\gamma + A\beta + \alpha)}, \\
C = q_0\alpha + \frac{q_0^2\beta}{2} + \frac{q_0^3\gamma}{3} + \frac{q_0^4\delta}{4} + \frac{\dot{q}_0^2}{2},
\end{array} \tag{8}
$$

whereas the number A must obey the following equation

$$
\begin{array}{l}
\left(9\beta\gamma\delta - 27\alpha\delta^2 - 2\gamma^3\right)A^6 + 3\left(36C\delta^2 - 6\alpha\gamma\delta + 9\beta^2\delta - 2\beta\gamma^2\right)A^5 \\
-15\left(-12C\gamma\delta - 9\alpha\beta\delta + 2\alpha\gamma^2\right)A^4 + 30\left(4C\gamma^2 + 9\alpha^2\delta\right)A^3 \\
+90\left(-6C\alpha\delta + 2C\beta\gamma + \alpha^2\gamma\right)A^2 + 18\left(24C^2\delta - 4C\alpha\gamma + 6C\beta^2 + 3\alpha^2\beta\right)A \\
+18\left(8C^2\gamma + 6C\alpha\beta + 3\alpha^3\right) = 0.
\end{array}
$$

$$
3\delta A^4 + 4\gamma A^3 + 6\beta A^2 + 12\alpha A - \left(12q_0\alpha + 6q_0^2\beta + 4q_0^3\gamma + 3q_0^4\delta + 6\dot{q}_0^2\right) = 0. \tag{9}
$$

Thus, the solution to the normal Duffing Equation (4) can be constructed based on the following discriminant of Equation (4)

$$
\Delta = (a + bv_0^2)^2 + 2b\dot{v}_0^2.
$$

Based on this discriminant Δ, three cases can be discussed and investigated as illustrated below.

2.1. First Case: For Positive Discriminant ($\Delta > 0$)

In this case, the solution of problem (4) reads

$$
v = \frac{v_0 \mathrm{cn}\left(\sqrt{\omega}t|m\right) + \frac{\dot{v}_0}{\sqrt{\omega}}\mathrm{sn}\left(\sqrt{\omega}t|m\right)\mathrm{dn}\left(\sqrt{\omega}t|m\right)}{1 + b\,\mathrm{sn}\left(\sqrt{\omega}t|m\right)^2}, \tag{10}
$$

with

$$
\omega = \sqrt{\Delta},\ m = \frac{1}{2}\left(1 - \frac{a}{\sqrt{\Delta}}\right),\ b = \frac{1}{2}\left(\frac{a + bv_0^2}{\sqrt{\Delta}} - 1\right). \tag{11}
$$

This solution is periodic for $m \neq 1$ and its main period reads

$$
T = \frac{4K(m)}{\sqrt{\omega}} \text{ for } 0 \leq m < 1. \tag{12}
$$

2.2. Second Case: For Negative Discriminant ($\Delta < 0$)

To discuss this case, let us define the following new quantity

$$
\delta = 2av_0^2 + bv_0^4 + 2\dot{v}_0^2. \tag{13}
$$

Since $\Delta < 0$, necessarily $b < 0$, thus, we have

$$\delta = \frac{\Delta - a^2}{b}, \tag{14}$$

which mean that $\delta > 0$. Accordingly, we try to find a solution to the i.v.p. (4) in the following form

$$v = A - \frac{2A}{1+u}, \tag{15}$$

where $u \equiv u(t)$ denotes the solution to the following Duffing equation

$$\begin{cases} \ddot{u} + cu(t) + du^3 = 0, \\ u(0) = u_0 := \frac{A+v_0}{A-v_0} \text{ and } \dot{u}_0 := \frac{2A\dot{v}_0}{(A-v_0)^2}. \end{cases} \tag{16}$$

By substituting the ansatz (15) into the main problem

$$c = \frac{1}{2}\left(3\sqrt{-b\delta} - a\right),\ d = \frac{1}{2}\left(a + \sqrt{-b\delta}\right),$$
$$A = \sqrt[4]{-\frac{\delta}{b}},\ \delta = 2av_0^2 + bv_0^4 + 2\dot{v}_0^2 > 0.$$

Thus, the Duffing Equation (16) has a positive discriminant $\Delta > 0$. The solution to the i.v.p. (4) is then given by

$$u = \sqrt[4]{\frac{\delta}{-b}} - \frac{2\sqrt[4]{\frac{\delta}{-b}}}{1+u}. \tag{17}$$

The period of $v(t)$ is that of $u(t)$.

2.3. Third Case: For Vanishing the Discriminant $(\Delta = 0)$

For vanishing the discriminant $(\Delta = 0)$ then $b < 0$ and the only solution to the i.v.p. (1) with $\dot{v}(0)^2 = \dot{v}_0^2$ reads

$$v(t) = -\sqrt{\frac{a}{-b}}\tanh\left(\sqrt{\frac{a}{2}}t - \tanh^{-1}\left(\sqrt{\frac{-b}{a}}v_0\right)\right), \tag{18}$$

which may be verified by direct computation.

Remark 1. *For $\delta = 0$, we obtain the solution to the following Helmholtz oscillator*

$$\begin{cases} \ddot{q} + \alpha + \beta q + \gamma q^2 = 0, \\ q(0) = q_0 \textit{ and } \dot{q}(0) = \dot{q}_0. \end{cases} \tag{19}$$

Remark 2. *The solution of the i.v.p. (1) may also be expressed in terms of the following Weierstrass elliptic function*

$$q(t) = A + \frac{B}{1 + C\wp(t + t_0; g_2, g_3)}, \tag{20}$$

with

$$\begin{aligned} &B = -\frac{6\left(A^3\delta + A^2\gamma + A\beta + \alpha\right)}{3A^2\delta + 2A\gamma + \beta}, \\ &C = \frac{12}{3A^2\delta + 2A\gamma + \beta}, \\ &t_0 = \pm\wp^{-1}\left(\frac{q_0 - A - B}{C(A - q_0)}; g_2, g_3\right), \\ &g_2 = \frac{1}{12}\left(-3A^4\delta^2 - 4A^3\gamma\delta - 6A^2\beta\delta - 12A\alpha\delta - 4\alpha\gamma + \beta^2\right), \\ &g_3 = \frac{1}{216}\left(-3A^4\delta\left(\gamma^2 - 3\beta\delta\right) - 4A^3\gamma\left(\gamma^2 - 3\beta\delta\right) + 6A^2\beta\left(3\beta\delta - \gamma^2\right)\right. \\ &\left. -12A\alpha\left(\gamma^2 - 3\beta\delta\right) + 27\alpha^2\delta - 6\alpha\beta\gamma + \beta^3\right), \end{aligned} \tag{21}$$

and the value of the coefficient A can be estimated by solving the following algebraic equation

$$3\delta A^4 + 4\gamma A^3 + 6\beta A^2 + 12A\alpha - (12q_0\alpha + 6q_0^2\beta + 4q_0^3\gamma + 3q_0^4\delta + 6\dot{q}_0^2) = 0.$$

Remember that solution (20) is valid even if $\alpha = \gamma = 0$*, i.e., for the standard Duffing oscillator*

$$\begin{cases} \ddot{q} + \beta q + +\delta q^3 = 0, \\ q(0) = q_0 \ \text{ and } \ \dot{q}(0) = \dot{q}_0 \,. \end{cases} \tag{22}$$

3. Approximate Solution for D-HO in Terms of Elementary Functions

Let us consider the following D-HO

$$\begin{cases} \ddot{q} + \alpha + \beta q + \gamma q^2 + \delta q^3 = 0, \\ q(0) = q_0 \ \text{ and } \ \dot{q}(0) = 0. \end{cases} \tag{23}$$

The exact solution to this problem in the form of JEFs is given by

$$q(t) = A + \frac{B}{1 + v_0 \operatorname{cn}\left(\sqrt{a + b v_0^2}\, t, \frac{b v_0^2}{2(a + b v_0^2)}\right)}, \tag{24}$$

where (A, B, a, b, v_0) are undermined parameters. By inserting solution (24) into the problem (23) and after several simple and successive arithmetic operations, we can obtain

$$\begin{aligned} a &= \frac{\begin{array}{c} A^4\left(-\left(\gamma^2 - 3\beta\delta\right)\right) - 2A^3(\beta\gamma - 9\alpha\delta) \\ -3A^2\left(12q_0\alpha\delta + 6q_0^2\beta\delta + 4q_0^3\gamma\delta + 3q_0^4\delta^2 - 2\alpha\gamma + \beta^2\right) \\ -2A\left(12q_0\alpha\gamma + 6q_0^2\beta\gamma + 4q_0^3\gamma^2 + 3q_0^4\gamma\delta + 3\alpha\beta\right) \\ -12q_0\alpha\beta - 6q_0^2\beta^2 - 4q_0^3\beta\gamma - 3q_0^4\beta\delta - 9\alpha^2 \end{array}}{(A-q_0)\left(3A^3\delta + A^2(3q_0\delta + 4\gamma) + A\left(4q_0\gamma + 3q_0^2\delta + 6\beta\right) + 6q_0\beta + 4q_0^2\gamma + 3q_0^3\delta + 12\alpha\right)}. \\ b &= -\frac{3\left(A^3\delta + A^2\gamma + A\beta + \alpha\right)^2}{(A-q_0)\left(3A^3\delta + A^2(3q_0\delta + 4\gamma) + A\left(4q_0\gamma + 3q_0^2\delta + 6\beta\right) + 6q_0\beta + 4q_0^2\gamma + 3q_0^3\delta + 12\alpha\right)}. \\ B &= \frac{\left(A^3\delta + A^2\gamma + A\beta + \alpha\right)}{b}. \\ v_0 &= \frac{q_0 - A - B}{A - q_0}, \end{aligned} \tag{25}$$

and the value of the coefficient A can be calculated by finding the root of the following equation

$$3\delta A^4 + 4\gamma A^3 + 6\beta A^2 + 12\alpha A - (12q_0\alpha + 6q_0^2\beta + 4q_0^3\gamma + 3q_0^4\delta) = 0.$$

To give approximate analytic solution to the i.v.p. (23) in terms of elementary functions, we will approximate the Jacobian function "cn" using the trigonometric cosine function as follows:

First let us assume that [40]

$$\begin{aligned} x(t) &= \frac{\sqrt{1 + \lambda + \mu}\cos(\sqrt{w}t)}{\sqrt{1 + \lambda\cos^2(\sqrt{w}t) + \mu\cos^4(\sqrt{w}t)}}, \\ y(t) &= \operatorname{cn}(t, m), \end{aligned} \tag{26}$$

where (λ, μ, w) are undermined parameters. Now, by applying an approximation to $x(t)$ and $y(t)$ until t^4, we have

$$\begin{aligned} x(t) - y(t) = &\frac{(\lambda + \mu + \mu w - w + 1)}{2(\lambda + \mu + 1)}t^2 \\ &+ \left(\frac{w^2 - 4\lambda\mu w^2 - 8\lambda w^2 + 5\mu^2 w^2 - 30\mu w^2}{24(\lambda + \mu + 1)^2} - \frac{1}{24}(4m + 1)\right)t^4 + O\left(t^5\right). \end{aligned}$$

with

$$w=\frac{1+\lambda+\mu}{1-\mu}\text{ and }\lambda=\frac{\mu(\mu-7)-m(\mu-1)^2}{\mu+2}. \tag{27}$$

On the other hand, if the residual error $R(z(t))=\dot{z}(t)^2+mz(t)^4+(1-2m)z(t)^2+(m-1)$, it is easy to see that $R(y(t))\equiv 0$. We will choose the value of μ to get a small value for the residual $R(x(t))$. Thus, we have

$$2m\Big(\lambda\cos^2\big(t\sqrt{w}\big)+\mu\cos^4\big(t\sqrt{w}\big)+1\Big)^3R(x(t))=A_0+\sum_{k=1}^{6}A_k\cos(2k\sqrt{w}t), \tag{28}$$

with

$$\begin{aligned}A_0=&\tfrac{7(m+4w+3)}{1024}\mu^3+\tfrac{1}{256}(14\lambda+3\lambda m+7\lambda w-25w+13)\mu^2\\&+\tfrac{1}{128}\big(5\lambda^2+6\lambda-6\lambda m-13m-16\lambda w+48w\big)\mu\\&+\tfrac{1}{16}\big(-\lambda^2-4\lambda+\lambda m+6m+8\lambda w+8w-8\big).\end{aligned}$$

We will choose the value of μ to fulfill $A_0=0$ and by using Equation (27), the following quintic algebraic equation is obtained

$$\begin{aligned}&320m^2-40\big(128-128m+27m^2\big)\mu+4\big(1240-1240m+309m^2\big)\mu^2\\&-4\big(513-513m+128m^2\big)\mu^3+4\big(32-32m+9m^2\big)\mu^4+5(1-m)\mu^5=0.\end{aligned} \tag{29}$$

Using the Padé-approximate technique, then the approximate value of μ reads

$$\mu=\frac{40m^2\big(27m^2-128m+128\big)}{2409m^4-29,600m^3+111,520m^2-163,840m+81,920}. \tag{30}$$

Accordingly, we get

$$\mathrm{cn}(t,m)\approx\cos_m(t):=\frac{\sqrt{1+\lambda+\mu}\cos(\sqrt{w}t)}{\sqrt{1+\lambda\cos^2(\sqrt{w}t)+\mu\cos^4(\sqrt{w}t)}}, \tag{31}$$

where the values of (w,λ,μ) are evaluated using the Formulas (27)–(30). For more information about the relation between the JEFs and elementary functions, the reader can see Ref. [40]. In the end, the approximate trigonometric solution to the i.v.p. (1) can be written in the following trigonometric form

$$q(t)=A+\frac{B}{1+\frac{\cos(\sqrt{w\omega}t+\bar{t}_0)}{\sqrt{1+\lambda\cos^2(\sqrt{w\omega}t+t_0)+\mu\cos^4(\sqrt{w\omega}t+\bar{t}_0)}}}. \tag{32}$$

4. Approximate Solutions for the Time Delayed D-HO

Let us consider the i.v.p.

$$\begin{cases}\ddot{x}+r_0+r_1x+r_2x^2+r_3x^3=\varepsilon(Qx(t-\tau)+f(t)-\dot{x}),\\ x(0)=x_0\ \text{ and }\ \dot{x}(0)=\dot{x}_0\,,\end{cases} \tag{33}$$

For anatomy this problem, we first consider $z\equiv z(t)$ as a solution to the i.v.p. (33) for $f(t)=0$ and assuming that

$$\lim_{t\to\infty}z(t)=d, \tag{34}$$

which leads to

$$d^3r_3+d^2r_2+d(r_1-\varepsilon Q)+r_0=0. \tag{35}$$

After that, the solution of the original problem (33) is considered in the following form

$$x(t)=d+u(t), \tag{36}$$

where $u \equiv u(t)$ needs to determine. Inserting solution (36) into the i.v.p. (33) yields

$$\left\{ \begin{array}{c} \ddot{u} + \omega_0^2 u + \alpha u^2 + \beta u^3 + \varepsilon[\dot{u} - Qx(t-\tau) - f(t)] = 0, \\ u(0) = d - x_0 \text{ and } \dot{u}(0) = \dot{x}_0, \end{array} \right. \tag{37}$$

with

$$\omega_0^2 = \left(r_1 + 2dr_2 + 3d^2 r_3\right),\ \alpha = (3dr_3 + r_2) \text{ and } \beta = r_3. \tag{38}$$

Since we will use one of the perturbation methods, the following construction (the homotopy) is considered

$$H_p(t) = \ddot{u} + \omega_0^2 u + p\left[\alpha u^2 + \beta u^3 + \varepsilon \dot{u} - \varepsilon Q x(t-\tau) - \varepsilon f(t)\right], \tag{39}$$

with the periodically excited force

$$f(t) = \gamma \cos(\omega t), \tag{40}$$

where p indicates the perturbation parameter.

Now, problem (37) can be analyzed using many perturbation techniques such as the KBMM and MSM.

4.1. KBMM for Analyzing the Time Delayed D-HO

Based on the KBMM, the solution of the i.v.p. (39) is assumed in the ansatz form

$$u = a\cos\psi + \sum_{n=1}^{N} p^n \Upsilon_n(a, \psi), \tag{41}$$

with

$$\dot{a} = \sum_{n=1}^{N} p^n A_n(a) \text{ and } \dot{\psi} = \omega_0 + \sum_{n=1}^{N} p^n \Phi_n(a), \tag{42}$$

Approximation (41) is called the N-th order approximations. For the second-order approximation, i.e., $N = 2$, we get

$$u(t) = a\cos(\psi) + p\Upsilon_1(a,\psi) + p^2\Upsilon_2(a,\psi) + O\left(p^3\right), \tag{43}$$

with

$$\left\{ \begin{array}{c} \dot{a} = pA_1(a) + p^2 A_2(a) + O(p^3), \\ \dot{\psi} = \omega_0 + p\Phi_1(a) + p^2\Phi_2(a) + O(p^3). \end{array} \right. \tag{44}$$

where the functions $a \equiv a(t)$, $\psi \equiv \psi(t)$, $\Upsilon_1 \equiv \Upsilon_1(a,\psi)$, and $\Upsilon_2 \equiv \Upsilon_2(a,\psi)$ will be set later.

Inserting both Equations (43) with (44) into Equation (39) yields

$$a\varepsilon\omega_0 + a\varepsilon Q\sin(\tau\omega_0) - 2A_1\omega_0 = 0, \tag{45}$$

$$\frac{1}{4}a\left(-8\omega_0\Phi_1(a) + 3a^2\beta + 4\varepsilon Q\cos(\tau\omega_0)\right) = 0, \tag{46}$$

$$-2A_2\omega_0 + a\varepsilon\Phi_1(a) - 2A_1\Phi_1(a) - aA_1\dot{\Phi}_1(a) = 0, \tag{47}$$

$$\frac{1}{4}\left(4\omega_0^2\Upsilon_1 + 4\omega_0^2\Upsilon_1^{(0,2)} + 2\alpha a^2\cos(2\psi) + 2\alpha a^2 + a^3\beta\cos(3\psi) - 4\varepsilon f(t)\right) = 0, \tag{48}$$

$$-A_1\varepsilon + 2a\alpha\Upsilon_1 - a\Phi_1(a)^2 - 2a\omega_0\Phi_2(a) + A_1\dot{A}_1 = 0. \tag{49}$$

$$\frac{1}{2}\left(\begin{array}{c} 3a^2\beta\Upsilon_1 + 2\varepsilon Q\Upsilon_1 + 3a^2\beta\cos(2\psi)\Upsilon_1 + 2\omega_0^2\Upsilon_2 - 2\varepsilon\omega_0\Upsilon_1^{(0,1)} \\ +4\omega_0\Phi_1(a)\Upsilon_1^{(0,2)} + 2\omega_0^2\Upsilon_2^{(0,2)} + 4A_1\omega_0\Upsilon_1^{(1,1)} \end{array} \right) = 0, \tag{50}$$

Now, by solving Equations (45) and (46), we obtain

$$\begin{cases} A_1(a) = \frac{a\varepsilon}{2\omega_0}(\omega_0 + Q\sin(\omega_0\tau)), \\ \Phi_1(a) = \frac{\varepsilon Q\cos(\omega_0\tau)}{2\omega_0} + \frac{3a^2\beta}{8\omega_0}, \end{cases} \tag{51}$$

Solving Equations (47)–(50) with the help of values given in Equation (51), we have

$$\begin{aligned} A_2(a) &= -\frac{\varepsilon a}{16\omega_0^3}\Big(3a^2\beta\omega_0 + 6a^2\beta Q\sin(\omega_0\tau) + 2\varepsilon Q^2\sin(2\omega_0\tau)\Big), \\ \Phi_2(a) &= \frac{1}{384\omega_0^3}\Big(-27a^4\beta^2 - 192a^2\alpha^2 - 48\varepsilon^2\omega_0^2 + 384\varepsilon\alpha f(t)\Big), \\ \Upsilon_1(a,\psi) &= \frac{1}{96\omega_0^2}\Big[3a^3\beta\cos(3\psi) + 16\alpha a^2(\cos(2\psi) - 3) + 96\varepsilon f(t)\Big], \end{aligned} \tag{52}$$

and

$$\begin{aligned} \Upsilon_2(a,\psi) = &-\frac{a^4\alpha\beta\cos(2\psi)}{3\omega_0^4} + \frac{a^4\alpha\beta\cos(4\psi)}{120\omega_0^4} + \frac{5\alpha a^4\beta}{8\omega_0^4} - \frac{a^2\alpha\varepsilon\sin(2\psi)}{9\omega_0^3} - \frac{21a^5\beta^2\cos(3\psi)}{1024\omega_0^4} \\ &+ \frac{a^5\beta^2\cos(5\psi)}{1024\omega_0^4} - \frac{3a^3\beta\varepsilon\sin(3\psi)}{128\omega_0^3} + \frac{a^2\beta\varepsilon f(t)\cos(2\psi)}{2\omega_0^4} - \frac{2a^2\alpha\varepsilon Q\cos(2\psi - \tau\omega_0)}{9\omega_0^4} \\ &+ \frac{a^2\alpha\varepsilon Q\cos(2\psi)}{18\omega_0^4} + \frac{a^2\varepsilon\alpha Q}{2\omega_0^4} - \frac{9a^3\beta\varepsilon Q\cos(3\psi - \tau\omega_0)}{256\omega_0^4} + \frac{a^3\beta\varepsilon Q\cos(3\psi)}{256\omega_0^4} - \frac{\varepsilon^2 Q f(t)}{\omega_0^4} - \frac{3a^2\beta\varepsilon f(t)}{2\omega_0^4}. \end{aligned} \tag{53}$$

Inserting Equations (51) and (51) into system (44), and for $p \to 1$, we get

$$\dot{a} = -S_1 a + S_2 a^3 \tag{54}$$

and

$$\dot{\psi} = W_0 + W_1 a^2 + W_2 a^4 + \frac{384\varepsilon\alpha}{384\omega_0^3} f(t) \tag{55}$$

with

$$\begin{aligned} S_1 &= -\frac{\varepsilon}{2\omega_0}(\omega_0 + Q\sin(\omega_0\tau)) + \frac{2a\varepsilon^2 Q^2}{16\omega_0^3}\sin(2\omega_0\tau), \\ S_2 &= -\frac{\varepsilon a}{16\omega_0^3}\Big(3a^2\beta\omega_0 + 6a^2\beta Q\sin(\omega_0\tau)\Big), \\ W_0 &= \omega_0 + \frac{\varepsilon Q\cos(\omega_0\tau)}{2\omega_0} - \frac{48\varepsilon^2\omega_0^2}{384\omega_0^3}, \\ W_1 &= \frac{3\beta}{8\omega_0} - \frac{192\alpha^2}{384\omega_0^3}, \\ W_2 &= \frac{-27\beta^2}{384\omega_0^3}. \end{aligned}$$

Solving Equation (54) for $a(0) = c_0$, we obtain

$$a(t) = \frac{\sqrt{S_1}}{\sqrt{e^{2tS_1}\left(\frac{S_1}{c_0^2} - S_2\right) + S_2}}, \tag{56}$$

Solving both system (55) for $f(t) = \gamma\cos(\omega t)$ and $\psi(0) = c_1$, the value of ψ is obtained

$$\begin{aligned}\psi(t) = c_1 + \tfrac{c_0^2 W_2}{2S_2}\left(\tfrac{S_1}{c_0^2 S_2 + (S_1 - c_0^2 S_2)e^{2S_1 t}} - 1\right) + t\left(W_0 + \tfrac{S_1}{S_2^2}(S_2 W_1 + S_1 W_2)\right)\\ + \tfrac{\gamma\varepsilon S_2 \sin(t\omega)}{\omega\omega_0^3} + \tfrac{(S_2 W_1 + S_1 W_2)}{2S_2^2}\left(\log(-S_1) - \log\left((c_0^2 S_2 - S_1)e^{2S_1 t} - c_0^2 S_2\right)\right).\end{aligned} \tag{57}$$

The constants c_0 and c_1 are obtained from the initial conditions.

Inserting the value of $u(t)$ given in (43) into solution (36), we finally obtain the second-order approximation to the time-delayed D-HO (33)

$$\begin{aligned}x(t) = {} & d + a\cos(\psi) + \frac{1}{96\omega_0^2}\left[3a^3\beta\cos(3\psi) + 16\alpha a^2(\cos(2\psi) - 3) + 96\varepsilon f(t)\right]\\ & - \frac{a^4\alpha\beta\cos(2\psi)}{3\omega_0^4} + \frac{a^4\alpha\beta\cos(4\psi)}{120\omega_0^4} + \frac{5\alpha a^4\beta}{8\omega_0^4} - \frac{a^2\alpha\varepsilon\sin(2\psi)}{9\omega_0^3} - \frac{21a^5\beta^2\cos(3\psi)}{1024\omega_0^4}\\ & + \frac{a^5\beta^2\cos(5\psi)}{1024\omega_0^4} - \frac{3a^3\beta\varepsilon\sin(3\psi)}{128\omega_0^3} + \frac{a^2\beta\varepsilon f(t)\cos(2\psi)}{2\omega_0^4} - \frac{2a^2\alpha\varepsilon Q\cos(2\psi - \tau\omega_0)}{9\omega_0^4}\\ & + \frac{a^2\alpha\varepsilon Q\cos(2\psi)}{18\omega_0^4} + \frac{a^2\varepsilon\alpha Q}{2\omega_0^4} - \frac{9a^3\beta\varepsilon Q\cos(3\psi - \tau\omega_0)}{256\omega_0^4}\\ & + \frac{a^3\beta\varepsilon Q\cos(3\psi)}{256\omega_0^4} - \frac{\varepsilon^2 Q f(t)}{\omega_0^4} - \frac{3a^2\beta\varepsilon f(t)}{2\omega_0^4},\end{aligned} \tag{58}$$

where a, ψ, and $f(t)$ are defined above. Note that Q is not too large and r_1 is large as compared with Q.

4.2. MSM for Analyzing the Time Delayed D-HO

We will assume that the delay is not too large, say $0 < \tau \le 2$, accordingly, Taylor series for $x(t-\tau)$ until the second-order reads

$$x(t-\tau) \approx x(t) - \tau\dot{x}(t) + \frac{1}{2}\tau^2\ddot{x}(t). \tag{59}$$

Inserting this approximation in the i.v.p. (2) yields

$$\begin{cases} \ddot{x} + r_0 + r_1 x + r_2 x^2 + r_3 x^3 = \varepsilon\left[Q\left(x(t) - \tau\dot{x}(t) + \frac{1}{2}\tau^2\ddot{x}(t)\right) + f(t) - \dot{x}\right],\\ x(0) = x_0 \text{ and } \dot{x}(0) = \dot{x}_0, \end{cases} \tag{60}$$

Inserting solution (36) $(x(t) = d + u(t))$ into problem (60) and rearrange the obtained results, we finally get

$$\begin{cases} \ddot{u} + w_1^2 u + w_2 u^2 + w_3 u^3 + 2\delta\dot{u} + F(t) = 0,\\ u(0) = d - x_0 \text{ and } \dot{u}(0) = \dot{x}_0, \end{cases} \tag{61}$$

with

$$\begin{aligned}\delta = \frac{\varepsilon(1+Q\tau)}{2-\varepsilon Q\tau^2}, w_1 = \sqrt{\frac{2(r_1 + 2dr_2 + 3d^2 r_3 - \varepsilon Q)}{2-\varepsilon Q\tau^2}},\\ w_2 = \frac{2(r_2 + 3dr_3)}{2-\varepsilon Q\tau^2}, w_3 = \frac{2r_3}{2-\varepsilon Q\tau^2}, F(t) = \frac{2\varepsilon f(t)}{2-\varepsilon Q\tau^2}\end{aligned}$$

and the value of d can be calculated by finding a suitable root to the following equation

$$r_0 + r_1 d + r_2 d^2 + r_3 d^3 - \varepsilon Q d = 0.$$

Applying the MSM, then the solution to the i.v.p. (61) is assumed in the ansatz form

$$u(t) = v_0(T_0, T_1) + pv_1(T_0, T_1) + O(p^2), \tag{62}$$

where $T_0 = t$, $T_1 = pt$, and p denotes the perturbation parameter.

Inserting the perturbed solution (62) into problem (61), we have

$$\ddot{u} + w_1^2 u + w_2 u^2 + w_3 u^3 + 2\delta\dot{u} + F(t) = p^0 K_0 + pK_1 + O(p^2), \tag{63}$$

with

$$\begin{aligned}
K_0 &= v_0{}^{(2,0)}(T_0, T_1) + w_1^2 v_0(T_0, T_1), \\
K_1 &= F(t) + w_1^2 v_1(T_0, T_1) + w_2 v_0^2(T_0, T_1) + w_3 v_0^3(T_0, T_1) \\
&\quad + 2\delta v_0{}^{(1,0)}(T_0, T_1) + 2v_0{}^{(1,1)}(T_0, T_1) + v_1{}^{(2,0)}(T_0, T_1)
\end{aligned}$$

Equating the coefficient K_0 to zero and solved the obtained results, we get

$$v_0(T_0, T_1) = a(T_1)\cos(\theta), \tag{64}$$

then inserting the solution (64) into the coefficient $K_1 = 0$ and rearrange the obtained results, we get

$$\begin{array}{l}
-2w_1\left[a^{(1,0)}(T_1) + \delta a(T_1)\right]\sin(\theta) \\
+\frac{1}{4}a(T_1)\left[3w_3 a^2(T_1) - 8w_1\varphi^{(1,0)}(T_1)\right]\cos(\theta) \\
+\frac{1}{2}w_2 a^2(T_1) + \frac{1}{2}w_2 a^2(T_1)\cos(2\theta) + \frac{1}{4}w_3 a^3(T_1)\cos(3\theta) \\
+F(t) + w_1^2 v_1(T_0, T_1) + v_1{}^{(2,0)}(T_0, T_1) = 0,
\end{array} \tag{65}$$

where $\theta = \omega_0 T_0 + \varphi(T_1)$.

To avoid the secular terms, the coefficients of $\sin(\theta)$ and $\cos(\theta)$ must be vanished which lead to

$$\begin{cases} a^{(1,0)}(T_1) + \delta a(T_1, T_2) = 0, \\ 3w_3 a^2(T_1) - 8w_1\varphi^{(1,0)}(T_1) = 0. \end{cases} \tag{66}$$

Solving these two differential equations, we obtain

$$\begin{aligned}
a(T_1) &= c_0 e^{-t\delta}, \\
\varphi(T_1) &= c_1 + \frac{3c_0^2 w_3}{8w_1} t e^{-2t\delta}.
\end{aligned} \tag{67}$$

and by solving the following other part in Equation (65) with $f(t) = \gamma\cos(\omega t)$ and $F(t) = 2\varepsilon f(t)/\left(2 - \varepsilon Q\tau^2\right)$

$$\begin{aligned}
&\frac{1}{2}w_2 a^2(T_1) + \frac{1}{2}w_2 a^2(T_1)\cos(2\theta) + \frac{1}{4}w_3 a^3(T_1)\cos(3\theta) \\
&+F(t) + w_1^2 v_1(T_0, T_1) + v_1{}^{(2,0)}(T_0, T_1) = 0,
\end{aligned} \tag{68}$$

we get

$$v_1(T_0, T_1) = \frac{2\gamma\varepsilon\cos(\omega T_0)}{(w_1^2 - \omega^2)(\varepsilon Q\tau^2 - 2)} + \frac{a(T_1)^2}{96w_1^2}\left[\begin{array}{c} 3w_3 a(T_1)\cos(3\theta) \\ +16w_2(\cos(2\theta) - 3) \end{array}\right]. \tag{69}$$

For $p \to 1$, $v_0(t,t) \equiv U_0(t)$, and $v_1(t,t) \equiv U_1(t)$, we finally obtain the MSM first-order approximation to the time delayed D-HO (33)

$$\begin{aligned} x(t) &= d + u(t) = d + U_0(t) + U_1(t) + O(\varepsilon^2) \\ &= d + c_0 e^{-\frac{1}{2}t\delta}\cos(\theta) + \frac{2\gamma\varepsilon\cos(\omega t)}{(w_1^2-\omega^2)(\varepsilon Q\tau^2-2)} + \frac{a(t)^2}{96w_1^2}\left[\begin{array}{c} 3w_3 a(t)\cos(3\theta) \\ +16w_2(\cos(2\theta)-3) \end{array}\right] + O(\varepsilon^2), \end{aligned} \tag{70}$$

with

$$\theta = (\omega_0 t + c_1) + \frac{3c_0^2 w_3}{8w_1} t e^{-2t\delta}.$$

The constants c_0 and c_1 are obtained from the initial conditions.

5. Numerical Example and Discussion

Let us consider the following numerical example of the time-delayed D-HO

$$\begin{cases} \ddot{x} + 1 + 2x + x^2 + x^3 = \varepsilon\left(\frac{1}{3}x(t-1) + \frac{1}{10}\cos\left(\frac{t}{4}\right) - \dot{x}\right), \\ x(0) = 0 \text{ and } \dot{x}(0) = 0. \end{cases} \tag{71}$$

For $\varepsilon = 0.1$, the numerical value of the second-order perturbed approximation using the KBMM is obtained as follows

$$\begin{aligned} x_{\text{KBM}} &= c_0 e^{-0.1t}\cos\left(c_0^2\left(1.37959 - 1.37959e^{-0.2t}\right) + c_0^4\left(0.070019e^{-0.4t} - 0.070019\right) + c_1 + 1.3591t\right) \\ &+ 0.0000782538\sin\left(\frac{t}{4}\right) - 0.000220198e^{-0.05t}\sin(1.35912t) + 0.00559423\cos\left(\frac{t}{4}\right) \\ &- 0.00559423e^{-0.05t}\cos(1.35912t) - 0.580342, \end{aligned} \tag{72}$$

where $c_0 = 0.581722$ and $c_1 = -0.0688913$, whereas the numerical value to the first-order perturbed approximation using the MSM reads

$$\begin{aligned} x_{\text{MSM}} = &- 0.580342 + e^{-0.135593t}\Big(0.0603533 - 0.0201178\cos\Big(\Big(-0.166039e^{-0.135593t} - 2.7182\Big)t + 0.121715\Big)\Big) \\ &- 0.00569828\cos\left(\frac{t}{4}\right) + 0.00276884e^{-0.20339t}\cos\Big(\Big(-0.249059e^{-0.135593t} - 4.07731\Big)t + 0.182572\Big) \\ &+ 0.54394e^{-0.0677966t}\cos\Big(\Big(-0.0830196e^{-0.135593t} - 1.3591\Big)t + 0.0608574\Big). \end{aligned} \tag{73}$$

For $\varepsilon = 0.2$, the numerical value of the second-order perturbed approximation using the KBMM is given by

$$\begin{aligned} x_{\text{KBM}} =& c_0 e^{-0.2t}\cos\left(c_0^2\left(0.687142 - 0.687142e^{-0.4t}\right) + c_0^4\left(0.0346073e^{-0.8t} - 0.0346073\right) + c_1 + 1.36435t\right) \\ &+ 0.000307188\sin\left(\frac{t}{4}\right) - 0.000869703e^{-0.1t}\sin(1.36237t) + 0.0110806\cos\left(\frac{t}{4}\right) \\ &- 0.0110806e^{-0.1t}\cos(1.36237t) - 0.591142, \end{aligned} \tag{74}$$

where $c_0 = 0.596673$ and $c_1 = -0.136259$, whereas the numerical value to the first-order perturbed approximation using the MSM reads

$$\begin{aligned} x_{\text{MSM}} = & e^{-0.275862t}\Big(0.0671009 - 0.022367\cos\Big(\Big(-0.177552e^{-0.275862t} - 2.7287\Big)t + 0.250504\Big)\Big) \\ &- 0.011501\cos\left(\frac{t}{4}\right) + 0.00302986e^{-0.413793t}\cos\Big(\Big(-0.266328e^{-0.275862t} - 4.09304\Big)t + 0.375756\Big) \\ &+ 0.55877e^{-0.137931t}\cos\Big(\Big(-0.088776e^{-0.275862t} - 1.36435\Big)t + 0.125252\Big) - 0.591142. \end{aligned} \tag{75}$$

The two perturbed approximations (72) and (73) for $\varepsilon = 0.1$ and (74) and (75) for $\varepsilon = 0.2$, are compared with the RK4 numerical approximations as illustrated in Figures 1 and 2, respectively. Moreover, the global maximum error for the two perturbed approximations as compared to the RK4 numerical approximations are estimated

$$\begin{cases} L_{KBM}|_{\varepsilon=0.1} = \max_{0<t\leq 40}|\text{RK4} - \text{KBM-approx.}| = 0.029302, \\ L_{MSM}|_{\varepsilon=0.1} = \max_{0<t\leq 40}|\text{RK4} - \text{MSM-approx.}| = 0.0522202, \end{cases} \tag{76}$$

and

$$\begin{cases} L_{KBM}|_{\varepsilon=0.2} = \max_{0<t\leq 40}|\text{RK4} - \text{KBM-approx.}| = 0.0545619, \\ L_{MSM}|_{\varepsilon=0.2} = \max_{0<t\leq 40}|\text{RK4} - \text{MSM-approx.}| = 0.0293156. \end{cases} \tag{77}$$

The comparison results are shown that both KBMM and MSM give good results compared the RK4 numerical approximations as illustrated from Figures 1 and 2 as well as from the estimated error.

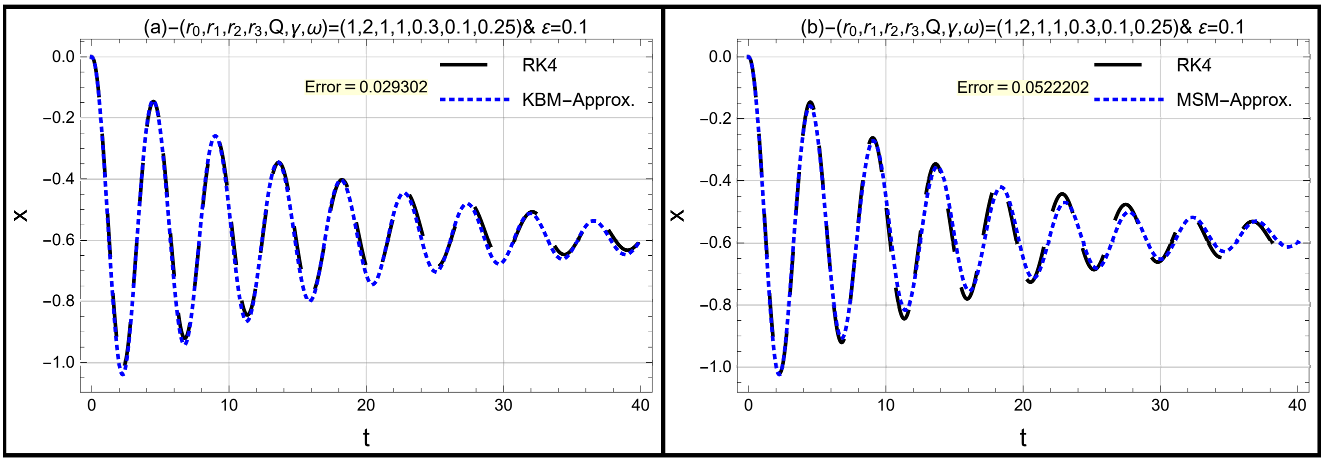

Figure 1. Both (**a**) second-order perturbed approximation (72) using the KBMM and (**b**) first-order approximation (73) using MSM to the i.v.p. (71) are compared with the RK4 numerical approximations for $\varepsilon = 0.1$.

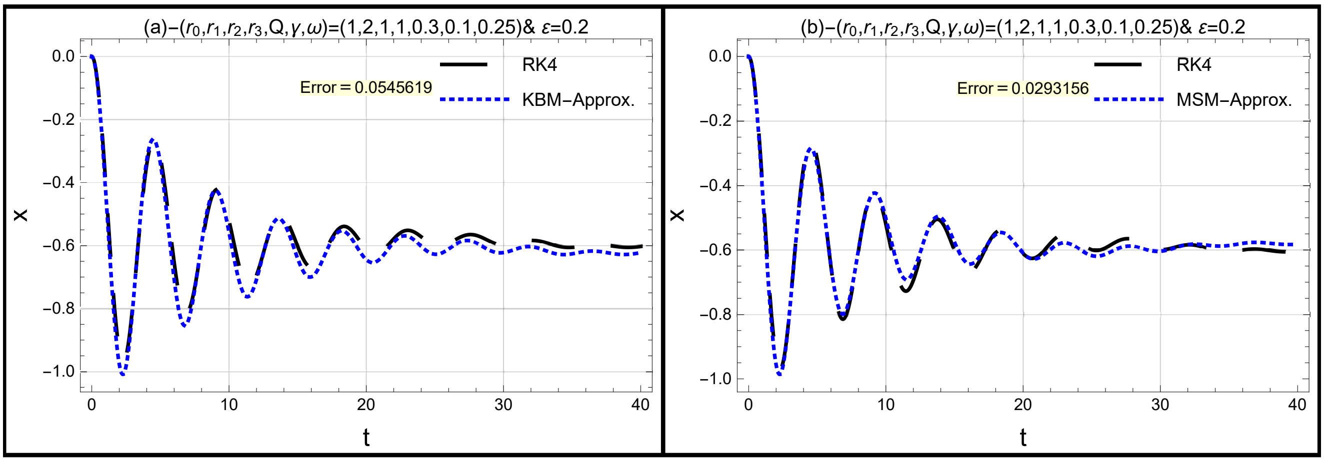

Figure 2. Both (**a**) second-order perturbed approximation (74) using the KBMM and (**b**) first-order approximation (75) using MSM to the i.v.p. (71) are compared with the RK4 numerical approximations for $\varepsilon = 0.2$.

6. Conclusions

In summary, using MATHEMATICA, the exact symmetric solutions of the standard/undamped Duffing–Helmholtz oscillator (D-HO) have been derived and discussed. Moreover, an approximation to this oscillator in the framework of elementary functions was derived. The trigonometric approximation was compared with both exact solutions and the 4^{th} Range–Kutta (RK4) numerical approximation. It has been noted that the trigonometric approximation is characterized by high accuracy and more stability for long time intervals. On the other hand, both Krylov–Bogoliubov–Mitropolsky method (KBMM) and the multiple scales method (MSM) were applied to derive some approximate solutions to the time-delayed forced damped D-HO. Due to KBMM being less complicated than the other methods, thus, the approximate perturbed solution was derived up to the second order. In addition, it is possible to derive the solutions up to the upper degree in the same way. For the MSM, the first perturbed approximation was derived, but the higher-order approximations cost a lot of time; also in the same way, the higher-order approximations can be obtained. All perturbed approximations were analyzed numerically using suitable numerical values to the related parameters and compared with RK4 numerical approximations. It has been noticed that KBMM gives more accurate perturbed approximations to the current problem than the MSM.

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