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Strong Differential Subordination and Superordination Results for Extended q-Analogue of Multiplier Transformation

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Abstract: The results obtained by the authors in the present article refer to quantum calculus applications regarding the theories of strong differential subordination and superordination. The q-analogue of the multiplier transformation is extended, in order to be applied on the specific classes of functions involved in strong differential subordination and superordination theories. Using this extended q-analogue of the multiplier transformation, a new class of analytic normalized functions is introduced and investigated. The convexity of the set of functions belonging to this class is proven and the symmetry properties derive from this characteristic of the class. Additionally, due to the convexity of the functions contained in this class, interesting strong differential subordination results are proven using the extended q-analogue of the multiplier transformation. Furthermore, strong differential superordination theory is applied to the extended q-analogue of the multiplier transformation for obtaining strong differential superordinations for which the best subordinants are provided.

Keywords: convex function; differential operator; extended q-analogue operator; strong differential subordination; strong differential superordination; best dominant; best subordinant

MSC: 30C45; 30A20; 34A40; 33D05

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1. Introduction

Jackson initiated the use of the q-calculus by defining the q-derivative [1] and q-integral [2]. Ismail et al. provided the first examples of q-calculus applications in geometric function theory in a paper published in 1990 [3], where an extension of the set of starlike functions was introduced and studied related to q-calculus aspects. Many applications of quantum calculus in geometric function theory have appeared in recent years, following Srivastava's establishment of the broad background for such study in a book chapter released in 1989 [4]. In addition to the numerous q-operators generated by utilizing well-known differential and integral operators specific to geometric function theory, some aspects of the application of quantum calculus in geometric function theory are highlighted in recent papers [5,6], respectively.

Several studies focused on the q-analogues of the Ruscheweyh differential operator described in paper [7] and the q-analogues of the Sălăgean differential operator established in [8]. In [9], for instance, differential subordinations were investigated using a specific q-Ruscheweyh-type derivative operator; in [10], a new class of analytic functions was defined, and its coefficient estimates were analyzed; and in [11], classes of analytic univalent functions were introduced and investigated using both Ruscheweyh and Sălăgean q-analogue operators. In [12,13], a generalization of the Sălăgean q-differential operator was used to investigate certain differential subordinations. Subordination outcomes using the q-analogue of the Sălăgean differential operator were achieved in [12,13]. The q-Bernardi integral operator was introduced in [14], and the multiplier transformation and Srivastava–Attiya operator was studied, involving the quantum calculus in [15].

The concept of strong differential subordination was first used by Antonino and Romaguera [16] for the investigation of Briot–Bouquet's strong differential subordination. It was intended to be an extension of the classical notion of differential subordination, due to Miller and Mocanu [17,18]. The concept was developed, setting the basis for the theory of strong differential subordination in 2009 [19], where the authors extended the concepts familiar to the established theory of differential subordination [20]. The introduction of the dual notion of strong differential superordination followed in 2009 [21], based on the pattern set for classical differential superordination theory [22]. Both theories developed nicely during the next years. Means for obtaining the best subordinant of a strong differential superordination were provided in [23], and special cases of strong differential subordinations and superordinations were considered for the studies [24]. Strong differential subordinations began to be obtained by associating different operators to the studies, such as the Sălăgean differential operator [25], Liu–Srivastava operator [26], Ruscheweyh operator [27], combinations of Sălăgean and Ruscheweyh operators [28], multiplier transformation [29,30], Komatu integral operator [31,32], Mittag-Leffler-confluent hypergeometric functions [33–35], or general differential operators [36,37]. The topic remains of interest at the present, as it was proved by citing recently published works [38–41].

Using q-analogue of the multiplier transformation, we have defined and studied new subclasses of harmonic univalent functions in [42] and have obtained fuzzy differential subordinations in [43].

We first remind of the notions and results used in this study.

Denote by $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$, where $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$.

In [44], the authors introduced some special subclasses of $\mathcal{H}(U \times \overline{U})$ that were used only in relation to the theories of strong differential subordination and its dual strong differential superordination:

$$\mathcal{A}^*_{n\zeta} = \{f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \cdots \in \mathcal{H}(U \times \overline{U})\},$$

with $\mathcal{A}^*_{1\zeta} = \mathcal{A}^*_{\zeta}$ and $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n+1$, $n \in \mathbb{N}^*$, and

$$\mathcal{H}^*[a,n,\zeta] = \{f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \cdots \in \mathcal{H}(U \times \overline{U})\},$$

with $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n$, $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$.

The next definitions concern the concept of strong differential subordination, as it was used in [16] and further developed in [19,44].

Definition 1 ([19]). *The analytic function $f(z,\zeta)$ is strongly subordinate to the analytic function $H(z,\zeta)$ if there exists an analytic function w in U, such that $w(0) = 0$, $|w(z)| < 1$ and $f(z,\zeta) = H(w(z),\zeta)$ for all $\zeta \in \overline{U}$. It is denoted $f(z,\zeta) \prec\prec H(z,\zeta)$, $(z,\zeta) \in U \times \overline{U}$.*

Remark 1 ([19]). *(i) In the particular case when the function $f(z,\zeta)$ is univalent in U, for all $\zeta \in \overline{U}$, the conditions from Definition 1 can be written as $f(U \times \overline{U}) \subset H(U \times \overline{U})$ and $f(0,\zeta) = H(0,\zeta)$, for all $\zeta \in \overline{U}$.*

(ii) In the particular case when $H(z,\zeta) = H(z)$ and $f(z,\zeta) = f(z)$, the strong differential subordination is reduced to differential subordination.

The next definitions are connected to strong differential superordination theory.

Definition 2 ([21]). *The analytic function $f(z,\zeta)$ is strongly superordinate to the analytic function $H(z,\zeta)$ if there exists an analytic function w in U, such that $w(0) = 0$, $|w(z)| < 1$, $z \in U$, and $H(z,\zeta) = f(w(z),\zeta)$, for all $\zeta \in \overline{U}$. It is denoted $H(z,\zeta) \prec\prec f(z,\zeta)$, $(z,\zeta) \in U \times \overline{U}$.*

Remark 2 ([21]). *(i) In the particular case when the function $f(z,\zeta)$ is univalent in U, for all $\zeta \in \overline{U}$, the conditions from Definition 2 can be written as $H(U \times \overline{U}) \subset f(U \times \overline{U})$ and $H(0,\zeta) = f(0,\zeta)$, for all $\zeta \in \overline{U}$.*

(ii) In the particular case when $H(z,\zeta) = H(z)$ and $f(z,\zeta) = f(z)$, the strong differential superordination is reduced to the differential superordination.

Definition 3 ([45]). *Q^* represents the set of analytic and injective functions on $\overline{U} \times \overline{U} \backslash E(f,\zeta)$, with property $f'_z(y,\zeta) \neq 0$ for $y \in \partial U \times \overline{U} \backslash E(f,\zeta)$, where $E(f,\zeta) = \{y \in \partial U : \lim_{z \to y} f(z,\zeta) = \infty\}$. $Q^*(a)$ represents the subclass of Q^* with $f(0,\zeta) = a$.*

The following lemmas are useful to prove the new results exposed in the next sections.

Lemma 1 ([46]). *Consider $\gamma \in \mathbb{C}^*$ a complex number such that $Re\gamma \geq 0$ and a convex function $h(z,\zeta)$ with the property $h(0,\zeta) = a$ for every $\zeta \in \overline{U}$. If $p \in \mathcal{H}^*[a,n,\zeta]$ and*

$$\frac{1}{\gamma} z p'_z(z,\zeta) + p(z,\zeta) \prec\prec h(z,\zeta),$$

then

$$p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta),$$

where $g(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$ is convex and it represents the best dominant.

Lemma 2 ([46]). *Consider a convex function $g(z,\zeta)$ in $U \times \overline{U}$ and let*

$$h(z,\zeta) = g(z,\zeta) + n\alpha z g'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

with $\alpha > 0$. If

$$p(z,\zeta) = g(0,\zeta) + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \ldots, \quad z \in U,\ \zeta \in \overline{U},$$

is holomorphic in $U \times \overline{U}$ and

$$p(z,\zeta) + \alpha z p'_z(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

then

$$p(z,\zeta) \prec\prec g(z,\zeta),$$

this result is sharp.

Lemma 3 ([47]). *Consider $\gamma \in \mathbb{C}^*$, such that $Re\ \gamma \geq 0$ and a convex function $h(z,\zeta)$ with the property $h(0,\zeta) = a$. If $p \in Q^* \cap \mathcal{H}^*[a,n,\zeta]$, $\frac{1}{\gamma} z p'_z(z,\zeta) + p(z,\zeta)$ is univalent in $U \times \overline{U}$ and*

$$h(z,\zeta) \prec\prec \frac{1}{\gamma} z p'_z(z,\zeta) + p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$, $z \in U$, $\zeta \in \overline{U}$. The convex function q represents the best subordinant.

Lemma 4 ([47]). *Consider a convex function $q(z,\zeta)$ in $U \times \overline{U}$ and let $h(z,\zeta) = \frac{1}{\gamma} z q'_z(z,\zeta) + q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, with $Re\ \gamma \geq 0$.*

If $p \in Q^* \cap \mathcal{H}^*[a,n,\zeta]$, $\frac{1}{\gamma}zp'_z(z,\zeta)+p(z,\zeta)$ is univalent in $U \times \overline{U}$ and

$$\frac{1}{\gamma}zq'_z(z,\zeta)+q(z,\zeta) \prec\prec \frac{1}{\gamma}zp'_z(z,\zeta)+p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta)=\frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z h(t,\zeta)t^{\frac{\gamma}{n}-1}dt$, $z \in U$, $\zeta \in \overline{U}$, represents the best subordinant.

The notations and notions from q-calculus theory are presented below.
For $n \in \mathbb{N}$ and $0<q<1$, we denote

$$[n]_q = \frac{1-q^n}{1-q},$$

and

$$[n]_q! = \begin{cases} \prod_{k=1}^{n} [k]_q, & n \in \mathbb{N}^*, \\ 1, & n=0. \end{cases}$$

The q-derivative operator $\mathcal{D}_q$ applied to a function $f \in \mathcal{A}$ is defined by [2]

$$\mathcal{D}_q(f(z)) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z=0. \end{cases}$$

When f is a differentiable function, we can see that

$$\lim_{q\to 1}\mathcal{D}_q(f(z)) = \lim_{q\to 1}\frac{f(z)-f(qz)}{(1-q)z} = f'(z).$$

For the special case when $f(z)=z^k$, we have $\mathcal{D}_q(f(z)) = \mathcal{D}_q\left(z^k\right) = \frac{1-q^k}{1-q}z^{k-1} = [k]_q z^{k-1}$.

In Section 2 of the paper, the q-analogue of the multiplier transformation is extended and defined on the class $\mathcal{A}^*_\zeta$. Next, a new class of analytic normalized functions $\mathcal{S}^q_{m,l,\zeta}(\alpha)$ is introduced using the extended q-analogue of the multiplier transformation. The convexity of the class $\mathcal{S}^q_{m,l,\zeta}(\alpha)$ is shown, and strong differential subordination theorems are proved involving the extended q-analogue of the multiplier transformation and the convex functions from class $\mathcal{S}^q_{m,l,\zeta}(\alpha)$. In Section 3, the dual theory of strong differential superordinations is employed in connection to the extended q-analogue of the multiplier transformation, in order to establish strong differential superordination results, for which the best subordinants are also obtained.

2. Strong Differential Subordination Results

We extend the q-analogue of the multiplier transformation to the new class of analytic functions $\mathcal{A}^*_\zeta$.

Definition 4. *The extended q-analogue of multiplier transformation has the the following form*

$$\mathcal{I}_q^{m,l}f(z,\zeta) = z + \sum_{k=2}^{\infty}\left(\frac{[k+l]_q}{[1+l]_q}\right)^m a_k(\zeta)z^k, \ z \in U,\ \zeta \in \overline{U},$$

with $l>-1$, $q \in (0,1)$, m a real number and $f(z,\zeta)=z+\sum_{k=2}^{\infty}a_k(\zeta)z^k \in \mathcal{A}^*_\zeta$.

Applying the properties of q-calculus, we obtain

$$z\mathcal{D}_q\left(\mathcal{I}_q^{m,l}f(z,\zeta)\right) = \left(1+\frac{[l]_q}{q^l}\right)\mathcal{I}_q^{m+1,l}f(z,\zeta) - \frac{[l]_q}{q^l}\mathcal{I}_q^{m,l}f(z,\zeta).$$

We define a new class of normalized analytic functions using the extended q-analogue of the multiplier transformation $\mathcal{I}_q^{m,l}$ introduced in Definition 4.

Definition 5. *The class $\mathcal{S}_{m,l,\zeta}^{q}(\alpha)$ consists of the functions $f \in \mathcal{A}_\zeta^*$ with the property*

$$Re\left(\mathcal{I}_q^{m,l}f(z,\zeta)\right)_z' > \alpha, \quad z \in U,\ \zeta \in \overline{U}, \tag{1}$$

for $\alpha \in [0,1)$.

The convexity of the class $\mathcal{S}_{m,l,\zeta}^{q}(\alpha)$ is established by the first result.

Theorem 1. *$\mathcal{S}_{m,l,\zeta}^{q}(\alpha)$ is a convex set.*

Proof. Taking the functions

$$f_j(z,\zeta) = z + \sum_{k=2}^{\infty} a_{jk}(\zeta)z^k,\ \ z \in U,\ \zeta \in \overline{U},\ j = 1,2,$$

from the class $\mathcal{S}_{m,l,\zeta}^{q}(\alpha)$, it is enough to prove that the function

$$f(z,\zeta) = \lambda_1 f_1(z,\zeta) + \lambda_2 f_2(z,\zeta)$$

belongs to the class $\mathcal{S}_{m,l,\zeta}^{q}(\alpha)$, when λ_1 and λ_2 are positive real numbers with the property $\lambda_1 + \lambda_2 = 1$.

The function f can be written by the following relation

$$f(z,\zeta) = z + \sum_{k=2}^{\infty}(\lambda_1 a_{1k}(\zeta) + \lambda_2 a_{2k}(\zeta))z^k, \quad z \in U,\ \zeta \in \overline{U},$$

and

$$\mathcal{I}_q^{m,l}f(z,\zeta) = z + \sum_{k=2}^{\infty}\left(\frac{[k+l]_q}{[1+l]_q}\right)^m(\lambda_1 a_{1k}(\zeta) + \lambda_2 a_{2k}(\zeta))z^k,\ \ z \in U, \zeta \in \overline{U}. \tag{2}$$

Differentiating relation (2), with respect to z, we obtain

$$\left(\mathcal{I}_q^{m,l}f(z,\zeta)\right)_z' = 1 + \sum_{k=2}^{\infty}\left(\frac{[k+l]_q}{[1+l]_q}\right)^m(\lambda_1 a_{1k}(\zeta) + \lambda_2 a_{2k}(\zeta))kz^{k-1}, \quad z \in U, \zeta \in \overline{U}.$$

Hence

$$\begin{aligned} Re\left(\mathcal{I}_q^{m,l}f(z,\zeta)\right)_z' \quad &= \quad 1 + Re\left(\lambda_1 \sum_{k=2}^{\infty} k\left(\frac{[k+l]_q}{[1+l]_q}\right)^m a_{1k}(\zeta)z^{k-1}\right) \\ &\quad + Re\left(\lambda_2 \sum_{k=2}^{\infty} k\left(\frac{[k+l]_q}{[1+l]_q}\right)^m a_{2k}(\zeta)z^{k-1}\right). \end{aligned} \tag{3}$$

Taking account that the functions $f_1, f_2 \in \mathcal{S}^q_{m,l,\zeta}(\alpha)$, we can write

$$Re\left(\lambda_j \sum_{k=2}^{\infty} k\left(\frac{[k+l]_q}{[1+l]_q}\right)^m a_{jk}(\zeta) z^{k-1}\right) > \lambda_k(\alpha-1), \quad j=1,2. \tag{4}$$

Using relation (4), we obtain, from (3),

$$Re\left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z > 1+\lambda_1(\alpha-1)+\lambda_2(\alpha-1)=\alpha,$$

which showed that $\mathcal{S}^q_{m,l,\zeta}(\alpha)$ is a convex set. $\square$

We next expose a series of strong differential subordinations using the convex functions from the class $\mathcal{S}^q_{m,l,\zeta}(\alpha)$ and the extended q-analogue of the multiplier transformation $\mathcal{I}_q^{m,l}$.

Theorem 2. *Taking $g(z,\zeta)$ a convex function, we consider the function*

$$h(z,\zeta) = \frac{z g'_z(z,\zeta)}{a+2} + g(z,\zeta), \;\; a>0,\; z\in U,\, \zeta \in \overline{U}. \tag{5}$$

For $f \in \mathcal{S}^q_{m,l,\zeta}(\alpha)$ set

$$F(z,\zeta) = \frac{a+2}{z^{a+1}} \int_0^z t^a f(t,\zeta)dt, \quad z\in U,\ \zeta\in\overline{U}, \tag{6}$$

then the strong differential subordination

$$\left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z \prec\prec h(z,\zeta), \;\; z\in U,\ \zeta\in\overline{U}, \tag{7}$$

implies the sharp strong differential subordination

$$\left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)'_z \prec\prec g(z,\zeta), \;\; z\in U,\ \zeta\in\overline{U},$$

with the function g as best dominant.

Proof. Relation (6) can take the following form

$$z^{a+1}F(z,\zeta) = (a+2)\int_0^z t^a f(t,\zeta)dt, \tag{8}$$

and after differentiating it, with respect to z, we obtain

$$zF'_z(z,\zeta) + (a+1)F(z,\zeta) = (a+2)f(z,\zeta) \tag{9}$$

and

$$z\left(\mathcal{I}_q^{m,l}F(z,\zeta)\right)'_z + (a+1)\mathcal{I}_q^{m,l}F(z,\zeta) = (a+2)\mathcal{I}_q^{m,l}f(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}. \tag{10}$$

Differentiating again the last relation with respect to z, we obtain

$$\frac{z\left(\mathcal{I}_q^{m,l}F(z,\zeta)\right)''_{z^2}}{a+2} + \left(\mathcal{I}_q^{m,l}F(z,\zeta)\right)'_z = \left(\mathcal{I}_q^{m,l}f(z,\zeta)\right)'_{z}, \quad z\in U,\ \zeta\in\overline{U}, \tag{11}$$

and the strong differential subordination (7) can be written in the form

$$\frac{z\left(\mathcal{I}_q^{m,l}F(z,\zeta)\right)''_{z^2}}{a+2}+\left(\mathcal{I}_q^{m,l}F(z,\zeta)\right)'_z\prec\prec\frac{zg'_z(z,\zeta)}{a+2}+g(z,\zeta). \tag{12}$$

Denoting

$$p(z,\zeta)=\left(\mathcal{I}_q^{m,l}F(z,\zeta)\right)'_z,\ z\in U,\ \zeta\in\overline{U}, \tag{13}$$

strong differential subordination (12) can be written as

$$\frac{zp'_z(z,\zeta)}{a+2}+p(z,\zeta)\prec\prec\frac{zg'_z(z,\zeta)}{a+2}+g(z,\zeta),\quad z\in U,\zeta\in\overline{U}.$$

Applying Lemma 2, we obtain

$$p(z,\zeta)\prec\prec g(z,\zeta),$$

equivalent with

$$\left(\mathcal{I}_q^{m,l}F(z,\zeta)\right)'_z\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

and the sharpness of this result is given by the best dominant g. □

Theorem 3. *Let* $h(z,\zeta)=\frac{\zeta+(2\delta-\zeta)z}{1+z}$, $z\in U,\zeta\in\overline{U}$. *Denoting*

$$I_a(f)(z,\zeta)=\frac{a+2}{z^{a+1}}\int_0^z t^a f(t,\zeta)dt,\ \ a>0, \tag{14}$$

then

$$I_a\left[\mathcal{S}^q_{m,l,\zeta}(\alpha)\right]\subset\mathcal{S}^q_{m,l,\zeta}(\alpha^*), \tag{15}$$

where

$$\alpha^*=2\alpha-\zeta+2(a+2)(\zeta-\alpha)\int_0^1\frac{t^{a+1}}{t+1}dt. \tag{16}$$

Proof. Following the steps used in the proof of Theorem 2, taking account the hypothesis of Theorem 3 and taking the convex function $h(z,\zeta)=\frac{\zeta+(2\delta-\zeta)z}{1+z}$, $z\in U,\zeta\in\overline{U}$, we obtain the strong differential subordination

$$\frac{zp'_z(z,\zeta)}{a+2}+p(z,\zeta)\prec\prec h(z,\zeta),$$

where p is given by relation (13).

Applying Lemma 1, we obtain the strong differential subordinations

$$p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),$$

written in the following form

$$\left(\mathcal{I}_q^{m,l}F(z,\zeta)\right)'_z\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),$$

where

$$\begin{aligned}g(z,\zeta)&=\frac{a+2}{z^{a+2}}\int_0^z t^{a+1}\frac{\zeta+(2\alpha-\zeta)t}{1+t}dt\\&=2\alpha-\zeta+\frac{2(a+2)(\zeta-\alpha)}{z^{a+2}}\int_0^z\frac{t^{a+1}}{t+1}dt.\end{aligned}$$

Taking account that g is a convex function with $g(U\times\overline{U})$ symmetric to the real axis, we obtain

$$Re\left(\mathcal{I}_q^{m,l}F(z,\zeta)\right)'_z\geq\min_{|z|=1}Reg(z,\zeta)=Reg(1,\zeta)=\alpha^*$$

$$= 2\alpha - \zeta + 2(a+2)(\zeta - \alpha) \int_0^z \frac{t^{a+1}}{t+1}.$$

□

Theorem 4. *Taking the convex function $g(z,\zeta)$ with the property $g(0,\zeta) = 1$, we consider the function*

$$h(z,\zeta) = zg'_z(z,\zeta) + g(z,\zeta), \quad z \in U, \zeta \in \overline{U}.$$

If $f \in \mathcal{A}^*_\zeta$ satisfies the strong differential subordination

$$\left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z \prec\prec h(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}, \tag{17}$$

then the sharp strong differential subordination

$$\frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z} \prec\prec g(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}$$

holds, with the function g as best dominant.

Proof. Considering

$$p(z,\zeta) = \frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z} = \frac{z + \sum_{k=2}^{\infty} \left(\frac{[k+l]_q}{[1+l]_q}\right)^m a_k(\zeta) z^k}{z} = 1 + p_1(\zeta)z + p_2(\zeta)z^2 + ...,$$

$z \in U, \zeta \in \overline{U}$, so we can write

$$zp(z,\zeta) = \mathcal{I}_q^{m,l} f(z,\zeta), \quad z \in U, \ \zeta \in \overline{U},$$

and differentiating it, with respect to z, we obtain

$$\left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z = zp'_z(z,\zeta) + p(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}.$$

The strong differential subordination (17) takes the form

$$zp'_z(z,\zeta) + p(z,\zeta) \prec\prec h(z,\zeta) = zg'_z(z,\zeta) + g(z,\zeta), \quad z \in U, \ \zeta \in \overline{U},$$

and applying Lemma 2, we have

$$p(z,\zeta) \prec\prec g(z,\zeta), \quad z \in U, \ \zeta \in \overline{U},$$

that means

$$\frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z} \prec\prec g(z,\zeta), \quad z \in U, \ \zeta \in \overline{U},$$

and the sharpness of this result is given by the best dominant g. □

Theorem 5. *Taking the convex function $h(z,\zeta)$ with the property $h(0,\zeta) = 1, \zeta \in \overline{U}$, for $f \in \mathcal{A}^*_\zeta$, such that the strong subordination*

$$\left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z \prec\prec h(z,\zeta), \quad z \in U, \ \zeta \in \overline{U} \tag{18}$$

holds, we obtain the strong differential subordination

$$\frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z} \prec\prec g(z,\zeta), \quad z \in U, \ \zeta \in \overline{U},$$

for the convex function $g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$*, considered as the best dominant.*

Proof. Let

$$p(z,\zeta) = \frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z} = 1 + \sum_{k=2}^{\infty} \left(\frac{[k+l]_q}{[1+l]_q} \right)^m a_k(\zeta) z^{k-1}, \ \ z \in U, \ \zeta \in \overline{U}.$$

Differentiating, with respect to z this relation, we obtain

$$\left(\mathcal{I}_q^{m,l} f(z,\zeta) \right)'_z = z p'_z(z,\zeta) + p(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}$$

and strong differential subordination (18) can be written as

$$z p'_z(z,\zeta) + p(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}.$$

After applying Lemma 1, we have

$$p(z,\zeta) \prec\prec g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt, \quad z \in U, \ \zeta \in \overline{U},$$

equivalent with

$$\frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z} \prec\prec g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt, \ \ z \in U, \ \zeta \in \overline{U}$$

with g being the best dominant. □

Theorem 6. *Taking a convex function* $g(z,\zeta)$ *with the property* $g(0,\zeta) = 1$*, we consider the function* $h(z,\zeta) = z g'_z(z,\zeta) + g(z,\zeta)$*,* $z \in U, \zeta \in \overline{U}$*. If* $f \in \mathcal{A}_\zeta^*$ *and the strong subordination*

$$\left(\frac{z \mathcal{I}_q^{m+1,l} f(z,\zeta)}{\mathcal{I}_q^{m,l} f(z,\zeta)} \right)'_z \prec\prec h(z,\zeta), \ \ z \in U, \ \zeta \in \overline{U} \tag{19}$$

holds, then we obtain the sharp strong differential subordination

$$\frac{\mathcal{I}_q^{m+1,l} f(z,\zeta)}{\mathcal{I}_q^{m,l} f(z,\zeta)} \prec\prec g(z,\zeta), \ \ z \in U, \ \zeta \in \overline{U},$$

with the function g *as best dominant.*

Proof. Let

$$p(z,\zeta) = \frac{\mathcal{I}_q^{m+1,l} f(z,\zeta)}{\mathcal{I}_q^{m,l} f(z,\zeta)} = \frac{z + \sum_{k=2}^{\infty} \left(\frac{[k+l]_q}{[1+l]_q} \right)^{m+1} a_k(\zeta) z^k}{z + \sum_{k=2}^{\infty} \left(\frac{[k+l]_q}{[1+l]_q} \right)^{m} a_k(\zeta) z^k}.$$

Differentiating this relation, with respect to z, we obtain $p'_z(z,\zeta) = \frac{\left(\mathcal{I}_q^{m+1,l} f(z,\zeta) \right)'_z}{\mathcal{I}_q^{m,l} f(z,\zeta)} - p(z,\zeta) \frac{\left(\mathcal{I}_q^{m,l} f(z,\zeta) \right)'_z}{\mathcal{I}_q^{m,l} f(z,\zeta)}$, written as $z p'_z(z,\zeta) + p(z,\zeta) = \left(\frac{z \mathcal{I}_q^{m+1,l} f(z,\zeta)}{\mathcal{I}_q^{m,l} f(z,\zeta)} \right)'_z$.

Strong differential subordination (19) can be written for $z \in U, \zeta \in \overline{U}$ as

$$z p'_z(z,\zeta) + p(z,\zeta) \prec\prec h(z,\zeta) = z g'_z(z,\zeta) + g(z,\zeta),$$

and applying Lemma 2, we have the strong differential subordination, for $z \in U, \zeta \in \overline{U}$,

$$p(z,\zeta) \prec\prec g(z,\zeta),$$

equivalent with

$$\frac{\mathcal{I}_q^{m+1,l} f(z,\zeta)}{\mathcal{I}_q^{m,l} f(z,\zeta)} \prec\prec g(z,\zeta),$$

and the sharpness of this result is given by the best dominant g. □

3. Strong Differential Superordination Results

In this section, strong differential superordinations are studied, regarding the extended q-analogue of the multiplier transformation $\mathcal{I}_q^{m,l}$. The best subordinant is established for each of the studied strong differential superordinations.

Theorem 7. *Taking $f \in \mathcal{A}_\zeta^*$ and a convex function $h(z,\zeta)$ in $U \times \overline{U}$ with the property $h(0,\zeta) = 1$, consider $F(z,\zeta) = \frac{a+2}{z^{a+1}} \int_0^z t^a f(t,\zeta)dt$, $z \in U$, $\zeta \in \overline{U}$, Re $a > -2$, and suppose that $\left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)_z'$ is a univalent function in $U \times \overline{U}$, $\left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)_z' \in Q^* \cap \mathcal{H}^*[1,1,\zeta]$. If the strong differential superordination*

$$h(z,\zeta) \prec\prec \left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)_z', \quad z \in U, \zeta \in \overline{U}, \tag{20}$$

states, then we obtain the strong differential superordination

$$g(z,\zeta) \prec\prec \left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)_z', \quad z \in U, \zeta \in \overline{U},$$

with the convex function $g(z,\zeta) = \frac{a+2}{z^{a+2}} \int_0^z h(t,\zeta) t^{a+1} dt$ the best subordinant.

Proof. Using the relation $z^{a+1}F(z,\zeta) = (a+2)\int_0^z t^a f(t,\zeta)dt$ from Theorem 2 and differentiating it, with respect to z, we can write $zF_z'(z,\zeta) + (a+1)F(z,\zeta) = (a+2)f(z,\zeta)$ in the following form $z\left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)_z' + (a+1)\mathcal{I}_q^{m,l} F(z,\zeta) = (a+2)\mathcal{I}_q^{m,l} f(z,\zeta)$, $z \in U, \zeta \in \overline{U}$, which, after differentiating it again, with respect to z, has the form

$$\frac{z\left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)_{z^2}''}{a+2} + \left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)_z' = \left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)_z', \quad z \in U, \zeta \in \overline{U}.$$

Using the last relation, the strong superordination (20) has the following form

$$h(z,\zeta) \prec\prec \frac{z\left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)_{z^2}''}{a+2} + \left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)_z'. \tag{21}$$

Define

$$p(z,\zeta) = \left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)_z', \quad z \in U, \zeta \in \overline{U}, \tag{22}$$

and replacing (22) in (21), we have $h(z,\zeta) \prec\prec \frac{zp_z'(z,\zeta)}{a+2} + p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. Applying Lemma 3, considering $n = 1$ and $\gamma = a+2$, it yields $g(z,\zeta) \prec\prec p(z,\zeta)$, equivalently with $g(z,\zeta) \prec\prec \left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)_z'$, $z \in U$, $\zeta \in \overline{U}$, with the best subordinant $g(z,\zeta) = \frac{a+2}{z^{a+2}} \int_0^z h(t,\zeta) t^{a+1} dt$ convex function. □

Theorem 8. *Taking a convex function $g(z,\zeta)$, we consider the function $h(z,\zeta) = \frac{zg_z'(z,\zeta)}{a+2} + g(z,\zeta)$, with Re $a > -2$, $z \in U$, $\zeta \in \overline{U}$. For $f \in \mathcal{A}_\zeta^*$, set $F(z,\zeta) = \frac{a+2}{z^{a+1}} \int_0^z t^a f(t,\zeta)dt$,*

$z \in U$, $\zeta \in \overline{U}$ and suppose that $\left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z$ is univalent in $U \times \overline{U}$ and $\left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)'_z \in Q^* \cap \mathcal{H}^*[1,1,\zeta]$. When the strong differential superordination

$$h(z,\zeta) \prec\prec \left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{23}$$

states, then we obtain the strong differential superordination

$$g(z,\zeta) \prec\prec \left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U},$$

for $g(z,\zeta) = \frac{a+2}{z^{a+2}} \int_0^z h(t,\zeta) t^{a+1} dt$ the best subordinant.

Proof. Considering $p(z,\zeta) = \left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)'_z$, $z \in U, \zeta \in \overline{U}$, following the proof of Theorem 7, we can write the strong differential superordination (23) in the following form

$$h(z,\zeta) = \frac{z g'_z(z,\zeta)}{a+2} + g(z,\zeta) \prec\prec \frac{z p'_z(z,\zeta)}{a+2} + p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Applying Lemma 4 for $\gamma = a+2$ and $n = 1$, we obtain the strong differential superordination $g(z,\zeta) \prec\prec p(z,\zeta) = \left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)'_z$, $z \in U$, $\zeta \in \overline{U}$, having $g(z,\zeta) = \frac{a+2}{z^{a+2}} \int_0^z h(t,\zeta) t^{a+1} dt$ the best subordinant. □

Theorem 9. *For $f \in \mathcal{A}^*_\zeta$, set $F(z,\zeta) = \frac{a+2}{z^{a+1}} \int_0^z t^a f(t,\zeta) dt$, $z \in U$, $\zeta \in \overline{U}$, and $h(z,\zeta) = \frac{\zeta + (2\alpha - \zeta) z}{1+z}$, where Re $a > -2$, $\alpha \in [0,1)$. Assume that $\left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z$ is univalent in $U \times \overline{U}$, $\left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)'_z \in Q^* \cap \mathcal{H}^*[1,1,\zeta]$ and the strong differential superordination*

$$h(z,\zeta) \prec\prec \left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z, \ z \in U,\ \zeta \in \overline{U}, \tag{24}$$

is satisfied, then the strong differential superordination

$$g(z,\zeta) \prec\prec \left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U},$$

is satisfied for the convex function $g(z,\zeta) = 2\alpha - \zeta + \frac{2(a+2)(\zeta-\alpha)}{z^{a+2}} \int_0^z \frac{t^{a+1}}{t+1} dt$, $z \in U, \zeta \in \overline{U}$ as the best subordinant.

Proof. Let $p(z,\zeta) = \left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)'_z$, and following the proof of Theorem 7, the strong superordination (24) can be written as $h(z,\zeta) = \frac{\zeta + (2\alpha-\zeta)z}{1+z} \prec\prec \frac{z p'_z(z,\zeta)}{a+2} + p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Applying Lemma 3, we obtain the strong differential superordination $g(z,\zeta) \prec\prec p(z,\zeta)$, where $g(z,\zeta) = \frac{a+2}{z^{a+2}} \int_0^z \frac{\zeta + (2\alpha-\zeta)t}{1+t} t^{a+1} dt = 2\alpha - \zeta + \frac{2(a+2)(\zeta-\alpha)}{z^{a+2}} \int_0^z \frac{t^{a+1}}{t+1} dt \prec\prec \left(\mathcal{I}_q^{m,l} F(z,\zeta)\right)'_z$, $z \in U, \zeta \in \overline{U}$, and g is the best subordinant and it is convex. □

Theorem 10. *Consider $f \in \mathcal{A}^*_\zeta$ and $h(z,\zeta)$ a convex function with the property $h(0,\zeta) = 1$. Assume that $\left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z$ is univalent and $\frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z} \in Q^* \cap \mathcal{H}^*[1,1,\zeta]$. When the strong superordination*

$$h(z,\zeta) \prec\prec \left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{25}$$

states, then the following strong differential superordination

$$g(z,\zeta) \prec\prec \frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

is satisfied, for the convex function $g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$ *the best subordinant.*

Proof. Let $p(z,\zeta) = \frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z} = \frac{z+\sum_{k=2}^{\infty}\left(\frac{[k+l]_q}{[1+l]_q}\right)^m a_k(\zeta)z^k}{z} \in \mathcal{H}^*[1,1,\zeta]$, $z \in U$, $\zeta \in \overline{U}$. With this notation we can write $\mathcal{I}_q^{m,l} f(z,\zeta) = zp(z,\zeta)$, and differentiating it, with respect to z, we obtain $\left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z = zp'_z(z,\zeta) + p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Using this notation, the strong differential superordination (25) becomes $h(z,\zeta) \prec\prec zp'_z(z,\zeta) + p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and applying Lemma 3, we obtain $g(z,\zeta) \prec\prec p(z,\zeta) = \frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z}$, $z \in U$, $\zeta \in \overline{U}$, for $g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$ is the best subordinant and convex. □

Theorem 11. *Taking a convex function* $g(z,\zeta)$ *in* $U \times \overline{U}$, *we consider the function* $h(z,\zeta) = zg'_z(z,\zeta) + g(z,\zeta)$. *Suppose* $\left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z$ *is univalent,* $\frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z} \in Q^* \cap \mathcal{H}^*[1,1,\zeta]$ *for* $f \in \mathcal{A}_\zeta^*$ *and the strong superordination*

$$h(z,\zeta) = zg'_z(z,\zeta) + g(z,\zeta) \prec\prec \left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{26}$$

is satisfied, then the strong differential superordination

$$g(z,\zeta) \prec\prec \frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

is satisfied for $g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$ *the best subordinant.*

Proof. Taking account the proof of Theorem 10 for $p(z,\zeta) = \frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z}$, the strong superordination (26), can be written in the following form $zg'_z(z,\zeta) + g(z,\zeta) \prec\prec zp'_z(z,\zeta) + p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Applying Lemma 4, we obtain the strong differential superordination $g(z,\zeta) \prec\prec p(z,\zeta)$, equivalently with $g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt \prec\prec \frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z}$, $z \in U$, $\zeta \in \overline{U}$, for g the best subordinant. □

Theorem 12. *Let* $h(z,\zeta) = \frac{\zeta+(2\alpha-\zeta)z}{1+z}$ *with* $0 \le \alpha < 1$, $z \in U$, $\zeta \in \overline{U}$. *For* $f \in \mathcal{A}_\zeta^*$, *assume that* $\left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)'_z$ *is univalent and* $\frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z} \in Q^* \cap \mathcal{H}^*[1,1,\zeta]$. *If the strong differential superordination*

$$h(z,\zeta) \prec\prec \left(\mathcal{I}_q^{m,l} f(z,\zeta)\right)', \quad z \in U,\ \zeta \in \overline{U}, \tag{27}$$

holds, then we have the following strong differential superordination

$$g(z,\zeta) \prec\prec \frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

and the best subordinant is the convex function $g(z,\zeta) = 2\alpha - \zeta + 2(\zeta-\alpha)\frac{\ln(1+z)}{z}$, $z \in U$, $\zeta \in \overline{U}$.

Proof. Following the proof of Theorem 10 for $p(z,\zeta) = \frac{\mathcal{I}_q^{m,l} f(z,\zeta)}{z}$, the strong superordination (27) takes the form $h(z,\zeta) = \frac{\zeta+(2\alpha-\zeta)z}{1+z} \prec\prec zp'_z(z,\zeta) + p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Applying Lemma 3, we obtain the following strong differential superordination $g(z,\zeta) \prec\prec p(z,\zeta)$, equivalent with $g(z,\zeta) = \frac{1}{z}\int_0^z \frac{\zeta+(2\alpha-\zeta)t}{1+t}dt = 2\alpha - \zeta + \frac{2(\zeta-\alpha)}{z}\ln(z+1) \prec\prec \frac{\mathcal{I}_q^{m,l}f(z,\zeta)}{z}$, $z \in U$, $\zeta \in \overline{U}$. The convex function g is the best subordinant. □

Theorem 13. *Taking a convex function $h(z,\zeta)$ with the property $h(0,\zeta) = 1$, for $f \in \mathcal{A}_\zeta^*$, assume that $\left(\frac{z\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}\right)_z'$ is univalent and $\frac{\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)} \in Q^* \cap \mathcal{H}^*[1,1,\zeta]$. If the strong differential superordination*

$$h(z,\zeta) \prec\prec \left(\frac{z\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}\right)_z', \quad z \in U,\ \zeta \in \overline{U}, \tag{28}$$

holds, then we obtain the following strong differential superordination

$$g(z,\zeta) \prec\prec \frac{\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

and the best subordinant is the convex function $g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$.

Proof. Let $p(z,\zeta) = \frac{\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}$, after differentiating this relation, with respect to z, we obtain $p_z'(z,\zeta) = \frac{\left(\mathcal{I}_q^{m+1,l}f(z,\zeta)\right)_z'}{\mathcal{I}_q^{m,l}f(z,\zeta)} - p(z,\zeta)\frac{\left(\mathcal{I}_q^{m,l}f(z,\zeta)\right)_z'}{\mathcal{I}_q^{m,l}f(z,\zeta)}$, written in the following form $zp_z'(z,\zeta) + p(z,\zeta) = \left(\frac{z\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}\right)_z'$.

Strong differential superordination (28) for $z \in U$, $\zeta \in \overline{U}$, becomes $h(z,\zeta) \prec\prec zp_z'(z,\zeta) + p(z,\zeta)$.

Applying Lemma 3, we obtain the following strong differential superordination $g(z,\zeta) \prec\prec p(z,\zeta) = \frac{\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}$, $z \in U$, $\zeta \in \overline{U}$, for the best subordinant $g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$ convex. □

Theorem 14. *Taking a convex function $g(z,\zeta)$, we consider $h(z,\zeta) = zg_z'(z,\zeta) + g(z,\zeta)$. For $f \in \mathcal{A}_\zeta^*$, assume that $\left(\frac{z\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}\right)_z'$ is univalent and $\frac{\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)} \in Q^* \cap \mathcal{H}^*[1,1,\zeta]$. If the strong differential superordination*

$$h(z,\zeta) = zg_z'(z,\zeta) + g(z,\zeta) \prec\prec \left(\frac{z\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}\right)_z', \quad z \in U,\ \zeta \in \overline{U}, \tag{29}$$

states, then we obtain the strong differential superordination

$$g(z,\zeta) \prec\prec \frac{\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

and the best subordinant is $g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$.

Proof. Following the proof of Theorem 13 for $p(z,\zeta) = \frac{\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}$, the strong superordination (29) has the form $h(z,\zeta) = zg_z'(z,\zeta) + g(z,\zeta) \prec\prec zp_z'(z,\zeta) + p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Applying Lemma 4, it yields $g(z,\zeta) \prec\prec p(z,\zeta)$, equivalently with $g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt \prec\prec \frac{\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}$, $z \in U$, $\zeta \in \overline{U}$, and the best subordinant is g. □

Theorem 15. *Consider* $h(z,\zeta) = \frac{\zeta+(2\alpha-\zeta)z}{1+z}$, *with* $0 \le \alpha < 1$. *For* $f \in \mathcal{A}_{\zeta}^{*}$ *assume that* $\left(\frac{z\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}\right)_z^{\prime}$ *is univalent and* $\frac{\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)} \in Q^* \cap \mathcal{H}^*[1,1,\zeta]$. *If the strong differential superordination*

$$h(z,\zeta) \prec\prec \left(\frac{z\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}\right)_z^{\prime}, \quad z \in U,\ \zeta \in \overline{U}, \tag{30}$$

holds, then the strong differential superordination

$$g(z,\zeta) \prec\prec \frac{\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

states, and the best subordinant is the convex function $g(z) = 2\alpha - \zeta + 2(\zeta-\alpha)\frac{\ln(1+z)}{z}$, $z \in U$, $\zeta \in \overline{U}$.

Proof. Considering the notation $p(z,\zeta) = \frac{\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}$, the strong differential superordination (30) can be written $h(z,\zeta) = \frac{\zeta+(2\alpha-\zeta)z}{1+z} \prec\prec zp_z^{\prime}(z,\zeta) + p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Applying Lemma 3, we have the strong differential superordination $g(z,\zeta) \prec\prec p(z,\zeta)$, equivalently with $g(z,\zeta) = \frac{1}{z}\int_0^z \frac{\zeta+(2\alpha-\zeta)t}{1+t}dt = 2\alpha - \zeta + 2(\zeta-\alpha)\frac{1}{z}\ln(z+1) \prec\prec \frac{\mathcal{I}_q^{m+1,l}f(z,\zeta)}{\mathcal{I}_q^{m,l}f(z,\zeta)}$, $z \in U$, $\zeta \in \overline{U}$.

The best subordinant is the convex function g. □

4. Conclusions

The significant findings in this paper are connected to a new class of mathematically normalized analytic functions in $U \times \overline{U}$, $\mathcal{S}_{m,l,\zeta}^{q}(\alpha)$, defined in Definition 5, using the multiplier transformation shown in Definition 4 as an expanded version of the q-analogue of the $I_q^{m,l}$ expression. The class is presented, and its convexity property is established in Section 2 of the article. Sharp strong differential subordinations are next studied in five theorems using the property of the functions belonging to the class $\mathcal{S}_{m,l,\zeta}^{q}(\alpha)$. The best dominant for the strong differential subordination is similarly given in Theorem 2, and in Theorem 3, a specific inclusion relation for the class $\mathcal{S}_{m,l,\zeta}^{q}(\alpha)$ is established. Strong differential superordinations are established in the nine theorems involving the extended q-analogue of the multiplier transformation $I_q^{m,l}$, its first derivative with regard to z, $\left(I_q^{m,l}f(z,\zeta)\right)_z^{\prime}$, second derivative $\left(I_q^{m,l}f(z,\zeta)\right)_{z^2}^{\prime\prime}$, and the representation $\frac{zI_q^{m+1,l}f(z,\zeta)}{I_q^{m,l}f(z,\zeta)}$ and its derivative, with respect to z, in Section 3 of the article.

Strong subordination and superordination outcomes such as those shown here may serve as an inspiration for future research that substitutes various extended q-operators for the multiplier transformation $I_q^{m,l}f(z,\zeta)$. An additional set of conditions for the univalence of the operator $I_q^{m,l}f(z,\zeta)$ under investigation might be derived because the best dominant of the strong differential subordinations in Theorem 2, and the best subordinants for the strong differential superordinations discussed in Section 3 are both presented. Using the extended q-analogue of the multiplier transformation, $I_q^{m,l}f(z,\zeta)$, and other strong subordination relations, further classes of univalent functions might be created. It will also be possible to look for coefficient estimates for the class $\mathcal{S}_{m,l,\zeta}^{q}(\alpha)$. With the previously established convexity of this class, more research might be performed to demonstrate other symmetry features of this class.

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