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Numerical Investigation of Time-Fractional Phi-Four Equation via Novel Transform

Nidhish Kumar Mishra [1], Mashael M. AlBaidani [2], Adnan Khan [3,*] and Abdul Hamid Ganie [1,*]

[1] Basic Science Department, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia
[2] Department of Mathematics, College of Science and Humanities, Prince Sattam bin Abdulaziz University, Al Kharj 11942, Saudi Arabia
[3] Department of Mathematics, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan
* Correspondence: adnanmummand@gmail.com (A.K.); a.ganie@seu.edu.sa (A.H.G.)

Abstract: This paper examines two methods for solving the nonlinear fractional Phi-four problem with variable coefficients. One of the distinct states of the Klein–Gordon model yields the Phi-four equation. It is also used to simulate the kink and anti-kink solitary wave connections that have recently emerged in biological systems and nuclear particle physics. The approaches that are being suggested consist of the Yang transform, the homotopy perturbation approach, the decomposition approach, and the fractional derivative as stated by Caputo. The advantages of the proposed techniques are their capability of combining two dominant approaches for attaining precise and approximate solutions of nonlinear equations. It is important to keep in mind that the suggested methods can perform better in general as they need less computational effort than the alternative methods, while keeping a high level of numerical precision. The actual and estimated outcomes are demonstrated in graphs and tables to be quite similar, demonstrating the usefulness of the proposed approaches. Additionally, several simulations are used to show the physical behaviors of the found solutions with regard to fractional order. The article's results possess complimentary properties that relate to the symmetry of partial differential equations.

Keywords: nonlinear fractional Phi-four equation; Yang transform; analytical techniques; Caputo operator

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1. Introduction

The noninteger order calculus dates back to the 17th century because it was developed soon after the integer order calculus. Leibniz introduced the notation $\frac{\partial^{\wp}\mathbb{P}(\mathbf{y})}{\partial\varsigma^{\wp}}$ in 1675 to denote the $\wp th$ derivative of a $\mathbb{P}(\mathbf{y})$ function, assuming integer values of $\wp$. L'Hospital posed the following basic problem in 1695: "What if $\wp$ is $\frac{1}{2}$?" Fractional derivative (FD) symbols were later utilised by Leibniz in his studies. As a result, the idea of an FD first came into being practically simultaneously with the integer order derivative. The fractional calculus (FC) was developed in the 18th and 19th centuries by a number of well-known scientists, including Euler, Laplace, Fourier, and many others. The well-known issue with the tautochrone curve was the first issue to be modelled in terms of FC. Based on the Riemann–Liouville formulation of the fractional integration, Abel discovered a solution to an integral problem in 1823. Recent research has established the fundamental features of numerous novel fractional-order operator types and these have been presented in the literature [1–4]. Many features of phenomena are explained using fractional theory, such as optics [5], nanotechnology [6], chaos theory [7], human diseases [8], and others [9–12]. Effective research into several key characteristics of dengue illness can be found in [13]. The analytical and numerical solutions to the equations demonstrating the aforementioned events play a significant part in defining how nonlinear models behave [14–18]. Symmetry is a transformation for the differential equation that maintains the invariance of its family of

solutions, and its exploration can be used to study and demonstrate a variety of differential equation classes. Many scholars from other fields have recently been drawn to the study of fractional calculus in relation to symmetry in order to convey their ideas and examine practical issues.

In the subject of nonlinear science, which has been utilised to determine processes in various fields, nonlinear partial differential equations (PDEs) have grown to be a "hot topic" in various fields, including quantum mechanics, ecology, image processing and economic systems. PDEs are widely used in a wide range of physical applications, including wave dispersion and propagation, magnetohydrodynamic movement via pipes, and supersonic and turbulence flow phenomena [19]. PDEs are employed in population modelling, medical imaging, the electrical signalling of neurons, the correct oxygenation of healing tissues, and other applications [20]. A fairly accurate projection of the number of COVID cases has justified the popularity of PDEs [21,22]. One can create a model of the COVID-19 [23] structure using a PDE. Nonetheless, for a number of complex problems in these areas, fractional partial differential equations (FPDEs) are more precise than integer-order PDEs. Consequently, creating numerical solutions for fractional PDEs is necessary. Resolving a group of related ordinary differential equations may yield symmetries. PDEs with fractional orders have symmetry conditions that are divided into segments through integer and fractional order. The linear structure of fractional PDEs continuously shows a widespread dimensional trivial solution. Numerous famous researchers have made contributions to this topic and some powerful numerical methods have been developed as a result of the importance of solving FPDEs numerically in science and engineering. These include finite volume methods [24], the homotopy perturbation transform method [25], the generalized Kudryashov method [26], the Elzaki transform decomposition method [27], the variational iteration transform method [28,29], and the natural transform decomposition method [30,31], and some other advanced numerical methods as well.

To study the behaviour of relativistic electrons, the physicists Klein and Gordon developed the Klein–Gordon (KG) equation. The Phi-four equation, which is constructed to show wave relations [32] and is a particular instance of the KG equation, is called the fractional Phi-four (FPF) equation since it is thought to be of fractional order.

$$\frac{\partial^{\wp}\mathbb{P}(\mathbf{y},\varsigma)}{\partial\varsigma^{\wp}} = \frac{\partial^2}{\partial\mathbf{y}^2}(\mathbb{P}(\mathbf{y},\varsigma)) + \chi_1\mathbb{P}(\mathbf{y},\varsigma) + \chi_2\mathbb{P}^3(\mathbf{y},\varsigma), \quad 1 < \wp \leq 2 \tag{1}$$

having initial values

$$\mathbb{P}(\mathbf{y},0) = g_1(\mathbf{y}), \frac{\partial}{\partial\varsigma}\mathbb{P}(\mathbf{y},0) = g_2(\mathbf{y}).$$

Here, the constants χ_1 and χ_2 have real values. In this case, the normalized propagation of distance and retraction time ς is represented by the expression $\mathbb{P}(\mathbf{y},\varsigma)$.

The so-called nonlinear problem has recently attracted scientists from a variety of disciplines. Due to the many uses of the suggested model and its significant role in explaining diverse nonlinear processes, multiple researchers have identified and examined the solution numerically as well as analytically. For example, in [33], the authors determined a novel precise solution for the fractional case of the nonlinear problem under consideration, while the authors of [34] applied the spectral collocation approach to find the Phi-four equation's solution. Researchers have been interested in the nonlinear Phi-four equation and have used a variety of ways to present their findings [35–37], which have provided some fascinating arguments.

This study aims to solve the nonlinear fractional Phi-four equation by combining two effective methods: the Yang transform decomposition method (YTDM), which combines the Yang transform and the Adomian decomposition approach, and the homotopy perturbation transform method (HPTM), which is formed from the combination of the Yang transform and the homotopy perturbation approach. The Yang transform (YT), first suggested by Xiao-Jun Yang, was utilised to deal with a variety of differential equations. In 1980, Adomian presented the Adomian decomposition method, a practical method for

finding a numerical as well as an analytical solution of differential equations describing physical problems. We convert differential equations into algebraic equations with the use of the Yang transform, and the nonlinear terms are then broken down using Adomian polynomials. Techniques such as perturbation methods (PM) are widely employed. However, perturbation approaches have their own set of limitations, just as other nonlinear analytical techniques do. The concept that the equation must have a small parameter is the foundation of nearly all perturbation techniques. This so-called small parameter assumption [38] greatly restricts the use of perturbation techniques. The homotopy perturbation method (HPM) was proposed by Ji Huan He [38,39]. Many scholars have used the HPM in recent years to solve various kinds of differential equations, both linear and non-linear. The estimation of the series terms is made simpler by the presented approaches, which do not use the calculation of the fractional derivative or fractional integrals in the recursive mechanism as the traditional Adomian process and perturbation process do. The proposed approaches do not require linearization, perturbation or discretization and avoid any round-off error.

In this study, we attempt to represent the physical behaviours in terms of numerical solutions with different arbitrary order. We also offer numerical simulations to demonstrate accuracy and efficiency. We try to clarify the varied model to illustrate a variety of occurrences appearing in daily life with the aid of these observations. Our paper is presented using the following format: Section 2 provides the fundamental definitions of FC, the Yang transform, and its properties. Section 3 introduces the concept of HPTM, while Section 4 introduces the concept of YTDM. In Section 5, how the techniques can be used with nonlinear FPF equations is illustrated. In Section 6, we present the conclusions.

2. Preliminaries

In this section, we highlight some of the basic definitions related to our present work.

Definition 1. *The operator in terms of Caputo is stated as [1]*

$$D_{\varsigma}^{\wp}\mathbb{P}(\boldsymbol{y},\varsigma)=\frac{1}{\Gamma(k-\wp)}\int_{0}^{\varsigma}(\varsigma-\wp)^{k-\wp-1}\mathbb{P}^{(k)}(\boldsymbol{y},\varphi)d\varphi,\quad k-1<\wp\leq k,\quad k\in\mathbf{N}. \tag{2}$$

Definition 2. *The definition of YT of a function $\mathbb{P}(\varsigma)$ is stated as [40]*

$$Y\{\mathbb{P}(\varsigma)\}=M(u)=\int_{0}^{\infty}e^{\frac{-\varsigma}{u}}\mathbb{P}(\varsigma)d\varsigma,\ \ \varsigma>0,\ \ u\in(-\varsigma_1,\varsigma_2), \tag{3}$$

defining inverse YT as

$$Y^{-1}\{M(u)\}=\mathbb{P}(\varsigma). \tag{4}$$

Definition 3. *The definition of the YT of a function having a derivative of nth order as [40]*

$$Y\{\mathbb{P}^{n}(\varsigma)\}=\frac{M(u)}{u^{n}}-\sum_{k=0}^{n-1}\frac{\mathbb{P}^{k}(0)}{u^{n-k-1}},\ \ \forall\ n=1,2,3,\cdots \tag{5}$$

Definition 4. *The definition of the YT of a function having a fractional derivative as [40]*

$$Y\{\mathbb{P}^{\wp}(\varsigma)\}=\frac{M(u)}{u^{\wp}}-\sum_{k=0}^{n-1}\frac{\mathbb{P}^{k}(0)}{u^{\wp-(k+1)}},\ \ n-1<\wp\leq n. \tag{6}$$

3. Solution Procedure of HPTM

In this section, we take into account the nonlinear FPDE to illustrate the fundamental principle of the projected method as provided in [41,42]

$$D_{\varsigma}^{\wp}\mathbb{P}(\mathbf{y},\varsigma)=\mathcal{F}_1[\mathbf{y}]\mathbb{P}(\mathbf{y},\varsigma)+\mathcal{G}_1[\mathbf{y}]\mathbb{P}(\mathbf{y},\varsigma),\ \ 1<\wp\leq 2, \tag{7}$$

and

$$\mathbb{P}(\mathbf{y},0)=\xi(\mathbf{y}),\ \ \frac{\partial}{\partial\varsigma}\mathbb{P}(\mathbf{y},0)=\zeta(\mathbf{y}).$$

where $D_{\varsigma}^{\wp}=\frac{\partial^{\wp}}{\partial\varsigma^{\wp}}$ signifies the fractional Caputo operator, and $\mathcal{F}_1[\mathbf{y}]$, $\mathcal{G}_1[\mathbf{y}]$ are, respectively, linear and nonlinear operators.

On taking the YT and then by means of the definition (4), we obtain

$$Y[D_{\varsigma}^{\wp}\mathbb{P}(\mathbf{y},\varsigma)]=Y[\mathcal{F}_1[\mathbf{y}]\mathbb{P}(\mathbf{y},\varsigma)+\mathcal{G}_1[\mathbf{y}]\mathbb{P}(\mathbf{y},\varsigma)], \tag{8}$$

$$\frac{1}{u^{\wp}}\{M(u)-u\mathbb{P}(0)-u^2\mathbb{P}^{'}(0)\}=Y[\mathcal{F}_1[\mathbf{y}]\mathbb{P}(\mathbf{y},\varsigma)+\mathcal{G}_1[\mathbf{y}]\mathbb{P}(\mathbf{y},\varsigma)]. \tag{9}$$

After, we have

$$M(\mathbb{P})=u\mathbb{P}(0)+u^2\mathbb{P}^{'}(0)+u^{\wp}Y[\mathcal{F}_1[\mathbf{y}]\mathbb{P}(\mathbf{y},\varsigma)+\mathcal{G}_1[\mathbf{y}]\mathbb{P}(\mathbf{y},\varsigma)]. \tag{10}$$

Now, by employing inverse YT, we get

$$\mathbb{P}(\mathbf{y},\varsigma)=\mathbb{P}(0)+\mathbb{P}^{'}(0)+Y^{-1}[u^{\wp}Y[\mathcal{F}_1[\mathbf{y}]\mathbb{P}(\mathbf{y},\varsigma)+\mathcal{G}_1[\mathbf{y}]\mathbb{P}(\mathbf{y},\varsigma)]]. \tag{11}$$

By HPM

$$\mathbb{P}(\mathbf{y},\varsigma)=\sum_{k=0}^{\infty}\epsilon^k\mathbb{P}_k(\mathbf{y},\varsigma). \tag{12}$$

having homotopy parameter $\epsilon\in[0,1]$.

The nonlinear terms are discarded as

$$\mathcal{G}_1[\mathbf{y}]\mathbb{P}(\mathbf{y},\varsigma)=\sum_{k=0}^{\infty}\epsilon^k H_n(\mathbb{P}), \tag{13}$$

with $H_k(\mathbb{P})$ representing the He's polynomials

$$H_n(\mathbb{P}_0,\mathbb{P}_1,...,\mathbb{P}_n)=\frac{1}{\Gamma(n+1)}D_{\epsilon}^{k}\left[\mathcal{G}_1\left(\sum_{k=0}^{\infty}\epsilon^i\mathbb{P}_i\right)\right]_{\epsilon=0}, \tag{14}$$

with $D_{\epsilon}^{k}=\frac{\partial^k}{\partial\epsilon^k}$.

Using (12) and (13) in (11), we obtain

$$\sum_{k=0}^{\infty}\epsilon^k\mathbb{P}_k(\mathbf{y},\varsigma)=\mathbb{P}(0)+\mathbb{P}^{'}(0)+\epsilon\times\left(Y^{-1}\left[u^{\wp}Y\{\mathcal{F}_1\sum_{k=0}^{\infty}\epsilon^k\mathbb{P}_k(\mathbf{y},\varsigma)+\sum_{k=0}^{\infty}\epsilon^k H_k(\mathbb{P})\}\right]\right). \tag{15}$$

Comparing the ϵ coefficients, we obtain

$$\begin{aligned}
&\epsilon^0:\mathbb{P}_0(\mathbf{y},\varsigma)=\mathbb{P}(0)+\mathbb{P}^{'}(0),\\
&\epsilon^1:\mathbb{P}_1(\mathbf{y},\varsigma)=Y^{-1}[u^{\wp}Y(\mathcal{F}_1[\mathbf{y}]\mathbb{P}_0(\mathbf{y},\varsigma)+H_0(\mathbb{P}))],\\
&\epsilon^2:\mathbb{P}_2(\mathbf{y},\varsigma)=Y^{-1}[u^{\wp}Y(\mathcal{F}_1[\mathbf{y}]\mathbb{P}_1(\mathbf{y},\varsigma)+H_1(\mathbb{P}))],\\
&.\\
&.\\
&.\\
&\epsilon^k:\mathbb{P}_k(\mathbf{y},\varsigma)=Y^{-1}[u^{\wp}Y(\mathcal{F}_1[\mathbf{y}]\mathbb{P}_{k-1}(\mathbf{y},\varsigma)+H_{k-1}(\mathbb{P}))],
\end{aligned} \tag{16}$$

$k>0, k\in\mathbf{N}$.

Hence, the analytical solution is as

$$\mathbb{P}(\mathbf{y},\varsigma)=\lim_{M\to\infty}\sum_{k=1}^{M}\mathbb{P}_k(\mathbf{y},\varsigma). \tag{17}$$

4. Solution Procedure of YTDM

In this section, we take into account the nonlinear FPDE to illustrate the fundamental principle of the projected method as provided in [41,42]

$$D_{\varsigma}^{\wp}\mathbb{P}(\mathbf{y},\varsigma)=\mathcal{F}_1(\mathbf{y},\varsigma)+\mathcal{G}_1(\mathbf{y},\varsigma),\ \ 1<\wp\leq 2, \tag{18}$$

and

$$\mathbb{P}(\mathbf{y},0)=\xi(\mathbf{y}),\ \ \frac{\partial}{\partial\varsigma}\mathbb{P}(\mathbf{y},0)=\zeta(\mathbf{y}).$$

where $D_{\varsigma}^{\wp}=\frac{\partial^{\wp}}{\partial\varsigma^{\wp}}$ signifies the fractional Caputo operator, and $\mathcal{F}_1$, $\mathcal{G}_1$ are, respectively, linear and nonlinear operators.

On taking the YT and then, by means of the definition (4), we obtain

$$\begin{aligned}&Y[D_{\varsigma}^{\wp}\mathbb{P}(\mathbf{y},\varsigma)]=Y[\mathcal{F}_1(\mathbf{y},\varsigma)+\mathcal{G}_1(\mathbf{y},\varsigma)],\\&\frac{1}{u^{\wp}}\{M(u)-u\mathbb{P}(0)-u^2\mathbb{P}^{'}(0)\}=Y[\mathcal{F}_1(\mathbf{y},\varsigma)+\mathcal{G}_1(\mathbf{y},\varsigma)].\end{aligned} \tag{19}$$

After, we have

$$M(\mathbb{P})=u\mathbb{P}(0)+u^2\mathbb{P}^{'}(0)+u^{\wp}Y[\mathcal{F}_1(\mathbf{y},\varsigma)+\mathcal{G}_1(\mathbf{y},\varsigma)]. \tag{20}$$

Now, by employing inverse YT, we get

$$\mathbb{P}(\mathbf{y},\varsigma)=\mathbb{P}(0)+\mathbb{P}^{'}(0)+Y^{-1}[u^{\wp}Y[\mathcal{F}_1(\mathbf{y},\varsigma)+\mathcal{G}_1(\mathbf{y},\varsigma)]. \tag{21}$$

Now, the solution is as

$$\mathbb{P}(\mathbf{y},\varsigma)=\sum_{m=0}^{\infty}\mathbb{P}_m(\mathbf{y},\varsigma). \tag{22}$$

The nonlinear terms $\mathcal{G}_1$ are discarded as

$$\begin{aligned}&\mathcal{G}_1(\mathbf{y},\varsigma)=\sum_{m=0}^{\infty}\mathcal{A}_m,\\&\mathcal{A}_m=\frac{1}{m!}\left[\frac{\partial^m}{\partial\ell^m}\left\{\mathcal{G}_1\left(\sum_{k=0}^{\infty}\ell^k\mathbf{y}_k,\sum_{k=0}^{\infty}\ell^k\varsigma_k\right)\right\}\right]_{\ell=0}.\end{aligned} \tag{23}$$

Using Equations (22) and (23) into (21), we have

$$\sum_{m=0}^{\infty}\mathbb{P}_m(\mathbf{y},\varsigma)=\mathbb{P}(0)+\mathbb{P}^{'}(0)+Y^{-1}u^{\wp}\left[Y\left\{\mathcal{F}_1(\sum_{m=0}^{\infty}\mathbf{y}_m,\sum_{m=0}^{\infty}\varsigma_m)+\sum_{m=0}^{\infty}\mathcal{A}_m\right\}\right]. \tag{24}$$

The approximation can be easily obtained by comparing both sides; we have

$$\begin{aligned}&\mathbb{P}_0(\mathbf{y},\varsigma)=\mathbb{P}(0)+\varsigma\mathbb{P}^{'}(0),\\&\mathbb{P}_1(\mathbf{y},\varsigma)=Y^{-1}[u^{\wp}Y\{\mathcal{F}_1(\mathbf{y}_0,\varsigma_0)+\mathcal{A}_0\}],\end{aligned} \tag{25}$$

The general recursive relation can be obtained as,

$$\mathbb{P}_{m+1}(\mathbf{y},\varsigma)=Y^{-1}[u^{\wp}Y\{\mathcal{F}_1(\mathbf{y}_m,\varsigma_m)+\mathcal{A}_m\}].$$

5. Applications of the Proposed Techniques

Example 1. *Let us assume the fractional Phi-four equation:*

$$\frac{\partial^{\wp}\mathbb{P}(\boldsymbol{y},\varsigma)}{\partial\varsigma^{\wp}} = \frac{\partial^2}{\partial \boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) - \chi_1\mathbb{P}(\boldsymbol{y},\varsigma) - \chi_2\mathbb{P}^3(\boldsymbol{y},\varsigma) \quad 1 < \wp \leq 2 \tag{26}$$

with

$$\mathbb{P}(\boldsymbol{y},0) = \sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right), \frac{\partial}{\partial\varsigma}\mathbb{P}(\boldsymbol{y},0) = -\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\,\mathrm{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right),$$

On taking the YT

$$Y\left(\frac{\partial^{\wp}\mathbb{P}}{\partial\varsigma^{\wp}}\right) = Y\left[\frac{\partial^2}{\partial \boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) - \chi_1\mathbb{P}(\boldsymbol{y},\varsigma) - \chi_2\mathbb{P}^3(\boldsymbol{y},\varsigma)\right], \tag{27}$$

and then by means of definition (4), we have

$$\frac{1}{u^{\wp}}\{M(u) - u\mathbb{P}(0) - u^2\mathbb{P}^{'}(0)\} = Y\left[\frac{\partial^2}{\partial \boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) - \chi_1\mathbb{P}(\boldsymbol{y},\varsigma) - \chi_2\mathbb{P}^3(\boldsymbol{y},\varsigma)\right], \tag{28}$$

$$M(u) = u\mathbb{P}(0) + u^2\mathbb{P}^{'}(0) + u^{\wp}Y\left[\frac{\partial^2}{\partial \boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) - \chi_1\mathbb{P}(\boldsymbol{y},\varsigma) - \chi_2\mathbb{P}^3(\boldsymbol{y},\varsigma)\right]. \tag{29}$$

Now, by employing inverse YT, we get

$$\begin{aligned}
&\mathbb{P}(\boldsymbol{y},\varsigma) = Y^{-1}[u\mathbb{P}(0) + u^2\mathbb{P}^{'}(0)] + Y^{-1}\left[u^{\wp}\left\{Y\left[\frac{\partial^2}{\partial \boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) - \chi_1\mathbb{P}(\boldsymbol{y},\varsigma) - \chi_2\mathbb{P}^3(\boldsymbol{y},\varsigma)\right]\right\}\right],\\
&\mathbb{P}(\boldsymbol{y},\varsigma) = \left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right) - \varsigma\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\,\mathrm{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right)\right) +\\
&Y^{-1}\left[u^{\wp}\left\{Y\left[\frac{\partial^2}{\partial \boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) - \chi_1\mathbb{P}(\boldsymbol{y},\varsigma) - \chi_2\mathbb{P}^3(\boldsymbol{y},\varsigma)\right]\right\}\right].
\end{aligned} \tag{30}$$

In terms of HPM, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}\epsilon^k\mathbb{P}_k(\boldsymbol{y},\varsigma) = \left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right) - \varsigma\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\,\mathrm{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right)\right) +\\
&\epsilon\left(Y^{-1}\left[u^{\wp}Y\left[\left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{P}_k(\boldsymbol{y},\varsigma)\right)_{\boldsymbol{yy}} - \chi_1\left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{P}_k(\boldsymbol{y},\varsigma)\right) - \chi_2\left(\sum_{k=0}^{\infty}\epsilon^k H_k(\mathbb{P})\right) + \right]\right]\right).
\end{aligned} \tag{31}$$

The nonlinear terms are discarded in terms of He's polynomial $H_k(\mathbb{P})$ as

$$\sum_{k=0}^{\infty}\epsilon^k H_k(\mathbb{P}) = \mathbb{P}^3 \tag{32}$$

Some terms are calculated as

$$\begin{aligned}
H_0(\mathbb{P}) &= \mathbb{P}_0^3,\\
H_1(\mathbb{P}) &= 3\mathbb{P}_0^2\mathbb{P}_1,
\end{aligned}$$

Comparing the ϵ coefficients, we obtain

$$\epsilon^0 : \mathbb{P}_0(\boldsymbol{y},\varsigma) = \left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right) - \varsigma\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right)\right),$$

$$\epsilon^1 : \mathbb{P}_1(\boldsymbol{y},\varsigma) = \frac{\chi_1^2\rho^2\varsigma^{\wp}}{8(-1+\rho^2)\Gamma(\wp+1)}\operatorname{sech}^6\left(\frac{\chi_1\boldsymbol{y}\sqrt{\frac{1}{-1+\rho^2}}}{\sqrt{2}}\right)\left(-3\sqrt{2}\chi_1\rho\varsigma\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}} + 2\sqrt{2}\chi_1^3\rho\varsigma^3\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}} + \right.$$

$$2\sqrt{2}\chi_1\rho\varsigma\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(\sqrt{2}\chi_1\varsigma\sqrt{\frac{1}{-1+\rho^2}}\right) + \sqrt{2}\chi_1\rho\varsigma\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(2\sqrt{2}\chi_1\varsigma\sqrt{\frac{1}{-1+\rho^2}}\right)$$

$$2\sqrt{\frac{-\chi_1^2}{\chi_2}}\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\boldsymbol{y}\right) + 6\varsigma^2\left(-\frac{\chi_1}{\chi_2}\right)^{\frac{3}{2}}\chi_2\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\boldsymbol{y}\right)$$

$$\left.+\sqrt{-\frac{\chi_1^2}{\chi_2}}\sinh\left(2\sqrt{2}\chi_1\sqrt{\frac{1}{(-1+\rho^2)}}\boldsymbol{y}\right)\right)$$

$$\vdots$$

Hence, the analytical solution is as

$$\mathbb{P}(\boldsymbol{y},\varsigma) = \mathbb{P}_0(\boldsymbol{y},\varsigma) + \mathbb{P}_1(\boldsymbol{y},\varsigma) + \cdots$$

$$\mathbb{P}(\boldsymbol{y},\varsigma) = \left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right) - \varsigma\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right)\right) +$$

$$\frac{\chi_1^2\rho^2\varsigma^{\wp}}{8(-1+\rho^2)\Gamma(\wp+1)}\operatorname{sech}^6\left(\frac{\chi_1\boldsymbol{y}\sqrt{\frac{1}{-1+\rho^2}}}{\sqrt{2}}\right)\left(-3\sqrt{2}\chi_1\rho\varsigma\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}} + 2\sqrt{2}\chi_1^3\rho\varsigma^3\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}} + \right.$$

$$2\sqrt{2}\chi_1\rho\varsigma\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(\sqrt{2}\chi_1\varsigma\sqrt{\frac{1}{-1+\rho^2}}\right) + \sqrt{2}\chi_1\rho\varsigma\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(2\sqrt{2}\chi_1\varsigma\sqrt{\frac{1}{-1+\rho^2}}\right)$$

$$2\sqrt{\frac{-\chi_1^2}{\chi_2}}\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\boldsymbol{y}\right) + 6\varsigma^2\left(-\frac{\chi_1}{\chi_2}\right)^{\frac{3}{2}}\chi_2\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\boldsymbol{y}\right)$$

$$\left.+\sqrt{-\frac{\chi_1^2}{\chi_2}}\sinh\left(2\sqrt{2}\chi_1\sqrt{\frac{1}{(-1+\rho^2)}}\boldsymbol{y}\right)\right) + \cdots$$

Implementation of the YTDM

On taking the YT

$$Y\left\{\frac{\partial^{\wp}\mathbb{P}}{\partial\varsigma^{\wp}}\right\} = Y\left[\frac{\partial^2}{\partial\boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) - \chi_1\mathbb{P}(\boldsymbol{y},\varsigma) - \chi_2\mathbb{P}^3(\boldsymbol{y},\varsigma)\right], \tag{33}$$

and then, by means of definition (4), we have

$$\frac{1}{u^{\wp}}\{M(u) - u\mathbb{P}(0) - u^2\mathbb{P}'(0)\} = Y\left[\frac{\partial^2}{\partial\boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) - \chi_1\mathbb{P}(\boldsymbol{y},\varsigma) - \chi_2\mathbb{P}^3(\boldsymbol{y},\varsigma)\right], \tag{34}$$

$$M(u) = u\mathbb{P}(0) + u^2\mathbb{P}'(0) + u^{\wp}Y\left[\frac{\partial^2}{\partial\boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) - \chi_1\mathbb{P}(\boldsymbol{y},\varsigma) - \chi_2\mathbb{P}^3(\boldsymbol{y},\varsigma)\right]. \tag{35}$$

Now, by employing inverse YT, we get

$$\mathbb{P}(\boldsymbol{y},\boldsymbol{z},\varsigma) = \Upsilon^{-1}[u\mathbb{P}(0)+u^2\mathbb{P}^{'}(0)] + \Upsilon^{-1}\left[u^{\wp}\left\{\Upsilon\left[\frac{\partial^2}{\partial \boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) - \chi_1\mathbb{P}(\boldsymbol{y},\varsigma) - \chi_2\mathbb{P}^3(\boldsymbol{y},\varsigma)\right]\right\}\right],$$
$$\mathbb{P}(\boldsymbol{y},\boldsymbol{z},\varsigma) = \left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right) - \varsigma\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right)\right) \tag{36}$$
$$+\Upsilon^{-1}\left[u^{\wp}\left\{\Upsilon\left[\frac{\partial^2}{\partial \boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) - \chi_1\mathbb{P}(\boldsymbol{y},\varsigma) - \chi_2\mathbb{P}^3(\boldsymbol{y},\varsigma)\right]\right\}\right].$$

Now, the solution is as

$$\mathbb{P}(\boldsymbol{y},\boldsymbol{z},\varsigma) = \sum_{m=0}^{\infty}\mathbb{P}_m(\boldsymbol{y},\boldsymbol{z},\varsigma) \tag{37}$$

The nonlinear term $\mathbb{P}^3 = \sum_{m=0}^{\infty}\mathcal{A}_m$*, by means of the Adomian polynomial, are discarded as*

$$\sum_{m=0}^{\infty}\mathbb{P}_m(\boldsymbol{y},\boldsymbol{z},\varsigma) = \mathbb{P}(\boldsymbol{y},\boldsymbol{z},0) + \Upsilon^{-1}\left[u^{\wp}\left\{\Upsilon\left[\frac{\partial^2}{\partial \boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) - \chi_1\mathbb{P}(\boldsymbol{y},\varsigma) - \sum_{m=0}^{\infty}\mathcal{A}_m\right]\right\}\right],$$
$$\sum_{m=0}^{\infty}\mathbb{P}_m(\boldsymbol{y},\boldsymbol{z},\varsigma) = \left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right) - \varsigma\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right)\right) \tag{38}$$
$$+\Upsilon^{-1}\left[u^{\wp}\left\{\Upsilon\left[\frac{\partial^2}{\partial \boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) - \chi_1\mathbb{P}(\boldsymbol{y},\varsigma) - \sum_{m=0}^{\infty}\mathcal{A}_m\right]\right\}\right],$$

Some terms are calculated as

$$\mathcal{A}_0 = \mathbb{P}_0^3,$$
$$\mathcal{A}_1 = 3\mathbb{P}_0^2\mathbb{P}_1,$$

The approximation can be easily obtained by comparing both sides; we have

$$\mathbb{P}_0(\boldsymbol{y},\varsigma) = \left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right) - \varsigma\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right)\right),$$

On $m = 0$

$$\mathbb{P}_1(\boldsymbol{y},\varsigma) = \frac{\chi_1^2\rho^2\varsigma^{\wp}}{8(-1+\rho^2)\Gamma(\wp+1)}\operatorname{sech}^6\left(\frac{\chi_1\boldsymbol{y}\sqrt{\frac{1}{-1+\rho^2}}}{\sqrt{2}}\right)\left(-3\sqrt{2}\chi_1\rho\varsigma\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}} + 2\sqrt{2}\chi_1^3\rho\varsigma^3\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}} + \right.$$
$$2\sqrt{2}\chi_1\rho\varsigma\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(\sqrt{2}\chi_1\varsigma\sqrt{\frac{1}{-1+\rho^2}}\right) + \sqrt{2}\chi_1\rho\varsigma\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(2\sqrt{2}\chi_1\varsigma\sqrt{\frac{1}{-1+\rho^2}}\right)$$
$$2\sqrt{\frac{-\chi_1^2}{\chi_2}}\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\boldsymbol{y}\right) + 6\varsigma^2\left(-\frac{\chi_1}{\chi_2}\right)^{\frac{3}{2}}\chi_2\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\boldsymbol{y}\right)$$
$$\left.+\sqrt{-\frac{\chi_1^2}{\chi_2}}\sinh\left(2\sqrt{2}\chi_1\sqrt{\frac{1}{(-1+\rho^2)}}\boldsymbol{y}\right)\right)$$
$$\vdots$$

The series form YTDM solution is as follows:

$$\mathbb{P}(\boldsymbol{y},\varsigma)=\sum_{m=0}^{\infty}\mathbb{P}_m(\boldsymbol{y},\varsigma)=\mathbb{P}_0(\boldsymbol{y},\varsigma)+\mathbb{P}_1(\boldsymbol{y},\varsigma)+\cdots$$

$$\mathbb{P}(\boldsymbol{y},\varsigma)=\left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right)-\varsigma\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\boldsymbol{y}\right)\right)+$$
$$\frac{\chi_1^2\rho^2\varsigma^{\wp}}{8(-1+\rho^2)\Gamma(\wp+1)}\operatorname{sech}^6\left(\frac{\chi_1\boldsymbol{y}\sqrt{\frac{1}{-1+\rho^2}}}{\sqrt{2}}\right)\left(-3\sqrt{2}\chi_1\rho\varsigma\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}+2\sqrt{2}\chi_1^3\rho\varsigma^3\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}+\right.$$
$$2\sqrt{2}\chi_1\rho\varsigma\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(\sqrt{2}\chi_1\varsigma\sqrt{\frac{1}{-1+\rho^2}}\right)+\sqrt{2}\chi_1\rho\varsigma\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(2\sqrt{2}\chi_1\varsigma\sqrt{\frac{1}{-1+\rho^2}}\right)$$
$$2\sqrt{\frac{-\chi_1^2}{\chi_2}}\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\boldsymbol{y}\right)+6\varsigma^2\left(-\frac{\chi_1}{\chi_2}\right)^{\frac{3}{2}}\chi_2\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\boldsymbol{y}\right)$$
$$\left.+\sqrt{-\frac{\chi_1^2}{\chi_2}}\sinh\left(2\sqrt{2}\chi_1\sqrt{\frac{1}{(-1+\rho^2)}}\boldsymbol{y}\right)\right)+\cdots$$

If we take $\wp=2$, we get an exact solution as

$$\mathbb{P}(\boldsymbol{y},\varsigma)=\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}(\boldsymbol{y}-\rho\varsigma)\right). \tag{39}$$

Example 2. *Let us assume the fractional Phi-four equation when $\chi_1=-1$ and $\chi_2=1$*

$$\frac{\partial^{\wp}\mathbb{P}(\boldsymbol{y},\varsigma)}{\partial\varsigma^{\wp}}=\frac{\partial^2}{\partial\boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma))+\mathbb{P}(\boldsymbol{y},\varsigma)-\mathbb{P}^3(\boldsymbol{y},\varsigma)\quad 1<\wp\leq 2 \tag{40}$$

with

$$\mathbb{P}(\boldsymbol{y},0)=\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right),\frac{\partial}{\partial\varsigma}\mathbb{P}(\boldsymbol{y},0)=\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right),$$

On taking the YT

$$Y\left(\frac{\partial^{\wp}\mathbb{P}}{\partial\varsigma^{\wp}}\right)=Y\left[\frac{\partial^2}{\partial\boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma))+\mathbb{P}(\boldsymbol{y},\varsigma)-\mathbb{P}^3(\boldsymbol{y},\varsigma)\right], \tag{41}$$

and then, by means of definition (4), we have

$$\frac{1}{u^{\wp}}\{M(u)-u\mathbb{P}(0)-u^2\mathbb{P}'(0)\}=Y\left[\frac{\partial^2}{\partial\boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma))+\mathbb{P}(\boldsymbol{y},\varsigma)-\mathbb{P}^3(\boldsymbol{y},\varsigma)\right], \tag{42}$$

$$M(u)=u\mathbb{P}(0)+u^2\mathbb{P}'(0)+u^{\wp}Y\left[\frac{\partial^2}{\partial\boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma))+\mathbb{P}(\boldsymbol{y},\varsigma)-\mathbb{P}^3(\boldsymbol{y},\varsigma)\right]. \tag{43}$$

Now, by employing inverse YT, we get

$$\begin{aligned}&\mathbb{P}(\boldsymbol{y},\varsigma)=Y^{-1}[u\mathbb{P}(0)+u^2\mathbb{P}'(0)]+Y^{-1}\left[u^{\wp}\left\{Y\left[\frac{\partial^2}{\partial\boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma))+\mathbb{P}(\boldsymbol{y},\varsigma)-\mathbb{P}^3(\boldsymbol{y},\varsigma)\right]\right\}\right],\\&\mathbb{P}(\boldsymbol{y},\varsigma)=(1+\varsigma)\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)+Y^{-1}\left[u^{\wp}\left\{Y\left[\frac{\partial^2}{\partial\boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma))+\mathbb{P}(\boldsymbol{y},\varsigma)-\mathbb{P}^3(\boldsymbol{y},\varsigma)\right]\right\}\right].\end{aligned} \tag{44}$$

In terms of HPM, we have

$$\sum_{k=0}^{\infty}\epsilon^k\mathbb{P}_k(\boldsymbol{y},\varsigma)=(1+\varsigma)\left(\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)\right)+ \\ \epsilon\left(\Upsilon^{-1}\left[u^{\wp}\Upsilon\left[\left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{P}_k(\boldsymbol{y},\varsigma)\right)_{\boldsymbol{yy}}+\left(\sum_{k=0}^{\infty}\epsilon^k\mathbb{P}_k(\boldsymbol{y},\varsigma)\right)-\left(\sum_{k=0}^{\infty}\epsilon^kH_k(\mathbb{P})\right)+\right]\right]\right). \tag{45}$$

The nonlinear terms are discarded in terms of the He's polynomial $H_k(\mathbb{P})$ as

$$\sum_{k=0}^{\infty}\epsilon^kH_k(\mathbb{P})=\mathbb{P}^3 \tag{46}$$

Some terms are calculated as

$$H_0(\mathbb{P})=\mathbb{P}_0^3,$$
$$H_1(\mathbb{P})=3\mathbb{P}_0^2\mathbb{P}_1,$$

Comparing the ϵ coefficients, we obtain

$$\epsilon^0:\mathbb{P}_0(\boldsymbol{y},\varsigma)=(1+\varsigma)\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right),$$

$$\epsilon^1:\mathbb{P}_1(\boldsymbol{y},\varsigma)=-\frac{\varsigma^{\wp}}{\Gamma(\wp+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)\left(-1-\varsigma-\frac{(1+\varsigma)\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)}{\kappa^2-1}\right)+(1+\varsigma)^3$$
$$\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)$$
$$\epsilon^2:\mathbb{P}_2(\boldsymbol{y},\varsigma)=\frac{\varsigma^{2\wp}}{\Gamma(2\wp+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)(1+\varsigma+\frac{1+\varsigma}{-1+\kappa^2}\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)$$
$$\frac{2(1+\varsigma)(\kappa^2(1+\varsigma)^2-\varsigma(2+\varsigma))}{(-1+\kappa^2)^2}\times\operatorname{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)-$$
$$\frac{(1+\varsigma)(\kappa^2(1+\varsigma)^2-\varsigma(2+\varsigma))\left(-2+\cosh\left(\sqrt{2}\sqrt{\frac{1}{1-\kappa^2}}\right)\right)\boldsymbol{y}}{(-1+\kappa^2)^2}$$
$$\operatorname{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)-(1+\varsigma)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)+\frac{2\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)}{2-2\kappa^2}$$
$$(-1-\varsigma-\frac{(1+\varsigma)\operatorname{sech}\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)^2}{-1+\kappa^2}+(1+\varsigma)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right))+3(1+\varsigma)^2\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)$$
$$(-1-\varsigma-\frac{(1+\varsigma)\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)}{-1+\kappa^2}+(1+\varsigma)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right))$$

$\vdots$

Hence, the analytical solution is as

$$\mathbb{P}(\boldsymbol{y},\varsigma) = \mathbb{P}_0(\boldsymbol{y},\varsigma) + \mathbb{P}_1(\boldsymbol{y},\varsigma) + \mathbb{P}_2(\boldsymbol{y},\varsigma) + \cdots$$

$$\mathbb{P}(\boldsymbol{y},\varsigma) = (1+\varsigma)\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right) - \frac{\varsigma^{\wp}}{\Gamma(\wp+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)$$
$$\left(-1-\varsigma-\frac{(1+\varsigma)\,\mathrm{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)}{\kappa^2-1}\right) + (1+\varsigma)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)$$
$$-\frac{\varsigma^{2\wp}}{\Gamma(2\wp+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)(1+\varsigma+\frac{1+\varsigma}{-1+\kappa^2}\,\mathrm{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)$$
$$\frac{2(1+\varsigma)(\kappa^2(1+\varsigma)^2-\varsigma(2+\varsigma))}{(-1+\kappa^2)^2}\times\mathrm{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right) -$$
$$\frac{(1+\varsigma)(\kappa^2(1+\varsigma)^2-\varsigma(2+\varsigma))\left(-2+\cosh\left(\sqrt{2}\sqrt{\frac{1}{1-\kappa^2}}\right)\right)\boldsymbol{y}}{(-1+\kappa^2)^2}$$
$$\mathrm{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right) - (1+\varsigma)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right) + \frac{2\,\mathrm{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)}{2-2\kappa^2}$$
$$(-1-\varsigma-\frac{(1+\varsigma)\,\mathrm{sech}\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)^2}{-1+\kappa^2} + (1+\varsigma)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)) + 3(1+\varsigma)^2\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)$$
$$(-1-\varsigma-\frac{(1+\varsigma)\,\mathrm{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)}{-1+\kappa^2} + (1+\varsigma)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)))$$

Implementation of the YTDM

On taking the YT

$$Y\left\{\frac{\partial^{\wp}\mathbb{P}}{\partial\varsigma^{\wp}}\right\} = Y\left[\frac{\partial^2}{\partial\boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) + \mathbb{P}(\boldsymbol{y},\varsigma) - \mathbb{P}^3(\boldsymbol{y},\varsigma)\right], \tag{47}$$

and then, by means of definition (4), we have

$$\frac{1}{u^{\wp}}\{M(u) - u\mathbb{P}(0) - u^2\mathbb{P}'(0)\} = Y\left[\frac{\partial^2}{\partial\boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) + \mathbb{P}(\boldsymbol{y},\varsigma) - \mathbb{P}^3(\boldsymbol{y},\varsigma)\right], \tag{48}$$

$$M(u) = u\mathbb{P}(0) + u^2\mathbb{P}'(0) + u^{\wp}Y\left[\frac{\partial^2}{\partial\boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) + \mathbb{P}(\boldsymbol{y},\varsigma) - \mathbb{P}^3(\boldsymbol{y},\varsigma)\right]. \tag{49}$$

Now, by employing inverse YT, we get

$$\begin{aligned}
&\mathbb{P}(\boldsymbol{y},\varsigma) = Y^{-1}[u\mathbb{P}(0) + u^2\mathbb{P}'(0)] + Y^{-1}\left[u^{\wp}\left\{Y\left[\frac{\partial^2}{\partial\boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) + \mathbb{P}(\boldsymbol{y},\varsigma) - \mathbb{P}^3(\boldsymbol{y},\varsigma)\right]\right\}\right],\\
&\mathbb{P}(\boldsymbol{y},\varsigma) = (1+\varsigma)\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right) + Y^{-1}\left[u^{\wp}\left\{Y\left[\frac{\partial^2}{\partial\boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma)) + \mathbb{P}(\boldsymbol{y},\varsigma) - \mathbb{P}^3(\boldsymbol{y},\varsigma)\right]\right\}\right].
\end{aligned} \tag{50}$$

Now, the solution is as

$$\mathbb{P}(\boldsymbol{y},\varsigma)=\sum_{m=0}^{\infty}\mathbb{P}_m(\boldsymbol{y},\varsigma) \tag{51}$$

The nonlinear term $\mathbb{P}^3=\sum_{m=0}^{\infty}\mathcal{A}_m$*, by means of the Adomian polynomial, is discarded as*

$$\begin{aligned}
&\sum_{m=0}^{\infty}\mathbb{P}_m(\boldsymbol{y},\varsigma)=\mathbb{P}(\boldsymbol{y},0)+\Upsilon^{-1}\left[u^{\wp}\left\{\Upsilon\left[\frac{\partial^2}{\partial \boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma))-\chi_1\mathbb{P}(\boldsymbol{y},\varsigma)-\sum_{m=0}^{\infty}\mathcal{A}_m\right]\right\}\right],\\
&\sum_{m=0}^{\infty}\mathbb{P}_m(\boldsymbol{y},\varsigma)=(1+\varsigma)\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)\\
&+\Upsilon^{-1}\left[u^{\wp}\left\{\Upsilon\left[\frac{\partial^2}{\partial \boldsymbol{y}^2}(\mathbb{P}(\boldsymbol{y},\varsigma))-\chi_1\mathbb{P}(\boldsymbol{y},\varsigma)-\sum_{m=0}^{\infty}\mathcal{A}_m\right]\right\}\right],
\end{aligned} \tag{52}$$

Some terms are calculated as

$$\begin{aligned}
\mathcal{A}_0&=\mathbb{P}_0^3,\\
\mathcal{A}_1&=3\mathbb{P}_0^2\mathbb{P}_1,
\end{aligned}$$

The approximation can be easily obtained by comparing both sides; we have

$$\mathbb{P}_0(\boldsymbol{y},\varsigma)=(1+\varsigma)\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right),$$

On $m=0$

$$\mathbb{P}_1(\boldsymbol{y},\varsigma)=-\frac{\varsigma^{\wp}}{\Gamma(\wp+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)\left(-1-\varsigma-\frac{(1+\varsigma)\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)}{\kappa^2-1}\right)+(1+\varsigma)^3$$
$$\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right),$$

On $m=1$

$$\begin{aligned}
&\mathbb{P}_2(\boldsymbol{y},\varsigma)=\frac{\varsigma^{2\wp}}{\Gamma(2\wp+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)(1+\varsigma+\frac{1+\varsigma}{-1+\kappa^2}\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)\\
&\frac{2(1+\varsigma)(\kappa^2(1+\varsigma)^2-\varsigma(2+\varsigma))}{(-1+\kappa^2)^2}\times\operatorname{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)-\\
&\frac{(1+\varsigma)(\kappa^2(1+\varsigma)^2-\varsigma(2+\varsigma))\left(-2+\cosh\left(\sqrt{2}\sqrt{\frac{1}{1-\kappa^2}}\right)\right)\boldsymbol{y}}{(-1+\kappa^2)^2}\\
&\operatorname{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)-(1+\varsigma)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)+\frac{2\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)}{2-2\kappa^2}\\
&(-1-\varsigma-\frac{(1+\varsigma)\operatorname{sech}\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)^2}{-1+\kappa^2}+(1+\varsigma)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right))+3(1+\varsigma)^2\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)\\
&(-1-\varsigma-\frac{(1+\varsigma)\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)}{-1+\kappa^2}+(1+\varsigma)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)),
\end{aligned}$$

The series form YTDM solution is as follows:

$$\mathbb{P}(\boldsymbol{y},\varsigma)=\sum_{m=0}^{\infty}\mathbb{P}_m(\boldsymbol{y},\varsigma)=\mathbb{P}_0(\boldsymbol{y},\varsigma)+\mathbb{P}_1(\boldsymbol{y},\varsigma)+\mathbb{P}_2(\boldsymbol{y},\varsigma)+\cdots$$

$$\mathbb{P}(\boldsymbol{y},\varsigma)=(1+\varsigma)\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)-\frac{\varsigma^{\wp}}{\Gamma(\wp+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)$$
$$\left(-1-\varsigma-\frac{(1+\varsigma)\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)}{\kappa^2-1}\right)+(1+\varsigma)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)$$
$$-\frac{\varsigma^{2\wp}}{\Gamma(2\wp+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)(1+\varsigma+\frac{1+\varsigma}{-1+\kappa^2}\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)$$
$$\frac{2(1+\varsigma)(\kappa^2(1+\varsigma)^2-\varsigma(2+\varsigma))}{(-1+\kappa^2)^2}\times\operatorname{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)-$$
$$\frac{(1+\varsigma)(\kappa^2(1+\varsigma)^2-\varsigma(2+\varsigma))\left(-2+\cosh\left(\sqrt{2}\sqrt{\frac{1}{1-\kappa^2}}\right)\right)\boldsymbol{y}}{(-1+\kappa^2)^2}$$
$$\operatorname{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)-(1+\varsigma)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)+\frac{2\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)}{2-2\kappa^2}$$
$$(-1-\varsigma-\frac{(1+\varsigma)\operatorname{sech}\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)^2}{-1+\kappa^2}+(1+\varsigma)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right))+3(1+\varsigma)^2\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)$$
$$(-1-\varsigma-\frac{(1+\varsigma)\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)}{-1+\kappa^2}+(1+\varsigma)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\boldsymbol{y}\right)))$$

If we take $\wp=2$*, we get an exact solution as*

$$\mathbb{P}(\boldsymbol{y},\varsigma)=\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}(\boldsymbol{y}-\kappa\varsigma)\right). \tag{53}$$

Numerical Simulation Studies

Here, we present the numerical simulations of the Phi-four equation of fractional-order by applying the proposed methodologies. We use graphs and tables to show how the obtained solution behaves. Maple is used to complete all of the computational work for the problems stated. The behaviour of the exact and suggested approaches' solutions at $\wp=2$ is depicted by the graphs in Figure 1a,b. Figure 1c displays the results of the proposed methodologies at various fractional orders of $\wp=2,1.9,1.8,1.7$, and $-5\leq\mathbf{y}\leq5$ for example 1 and Figure 1d, respectively, at $\varsigma=0.1$ and $0\leq\varsigma\leq0.1$. The absolute error comparison of the suggested approaches with q-HATM [43] is shown in Table 1. The behaviour of the exact and suggested approaches' solutions at $\wp=2$ is depicted by the graphs in Figure 2a,b. Figure 2c displays the results of the proposed methodologies at various fractional orders of $\wp=2,1.9,1.8,1.7$, and $-3\leq\mathbf{y}\leq3$, for example 2 and Figure 2d, respectively, at $\varsigma=0.1$ and $0\leq\varsigma\leq0.1$. The absolute error comparison of the suggested approaches with q-HATM [43] is shown in Table 2. The demonstrated plots help us to better understand the nature of the fractional Phi-four equation when temporal-spatial variables vary in comparison with the arbitrary order. The comparison of absolute errors demonstrates that our methods converge more quickly than other methods. Additionally,

the graphical depiction demonstrates good agreement between the exact solution and the suggested approaches' solution.

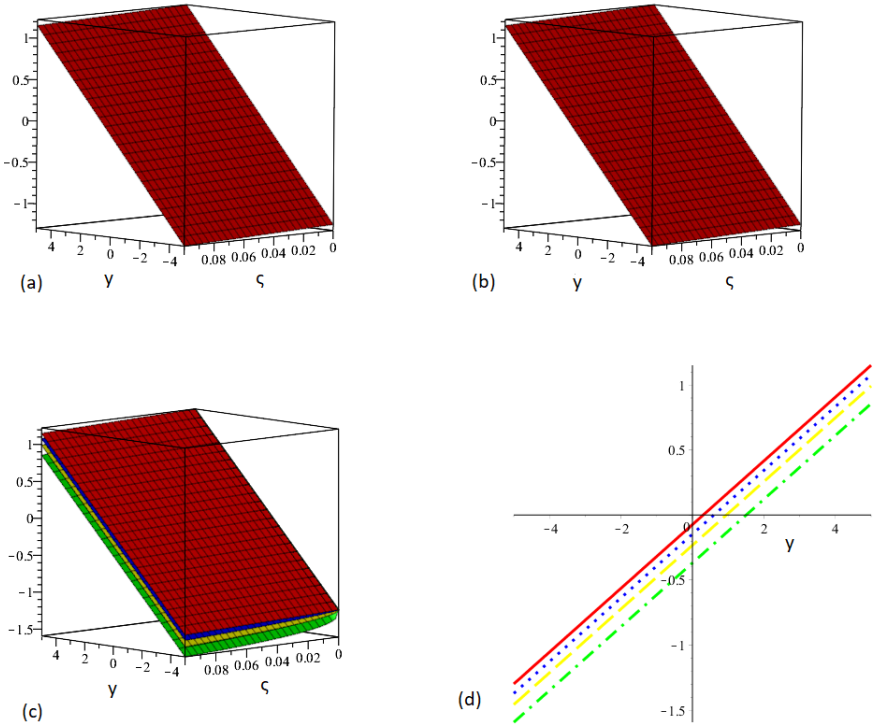


Figure 1. The graphical behaviour of $\mathbb{P}(\mathbf{y}, \varsigma)$ of example 1 for various orders of $\wp$.

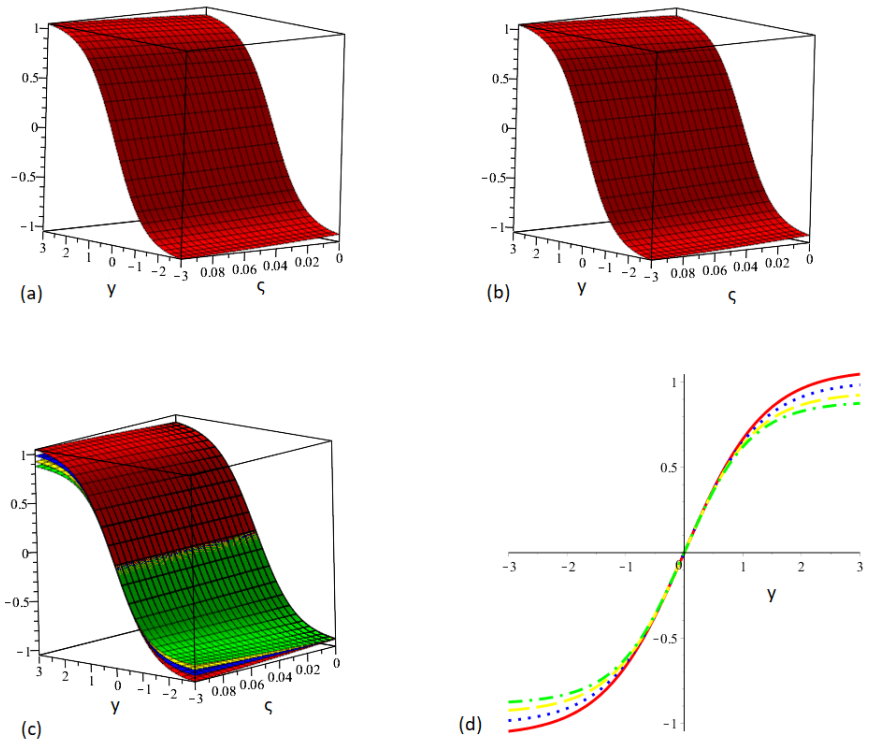


Figure 2. The graphical behaviour of $\mathbb{P}(\mathbf{y}, \varsigma)$ of example 2 for different values of $\wp$.

Table 1. Comparison of the proposed methods with q-$HATM$ in terms of absolute error for problem 1.

ς	y	$\wp = 2(q\text{-}HATM)$	$\wp = 2(HPTM)$	$\wp = 2(YTDM)$
0.01	−5	9.18570×10^{-08}	2.65743×10^{-09}	2.65743×10^{-09}
	−3	3.68683×10^{-08}	2.44352×10^{-09}	2.44352×10^{-09}
	−1	2.15263×10^{-07}	2.85464×10^{-09}	2.85464×10^{-09}
	1	2.18306×10^{-07}	2.14212×10^{-09}	2.14212×10^{-09}
	3	3.36893×10^{-08}	1.64233×10^{-09}	1.64233×10^{-09}
	5	9.09190×10^{-08}	1.72362×10^{-09}	1.72362×10^{-09}
0.02	−5	7.38629×10^{-07}	2.74521×10^{-08}	2.74521×10^{-08}
	−3	3.07718×10^{-07}	2.64861×10^{-08}	2.64861×10^{-08}
	−1	1.70951×10^{-06}	2.42235×10^{-08}	2.42235×10^{-08}
	1	1.75819×10^{-06}	2.96853×10^{-07}	2.96853×10^{-07}
	3	2.56849×10^{-07}	1.12425×10^{-08}	1.12425×10^{-08}
	5	7.23617×10^{-07}	1.56423×10^{-08}	1.56423×10^{-08}
0.03	−5	2.50566×10^{-06}	2.53324×10^{-08}	2.53324×10^{-08}
	−3	1.08179×10^{-06}	2.63423×10^{-07}	2.63423×10^{-07}
	−1	5.72612×10^{-06}	2.64673×10^{-07}	2.64673×10^{-07}
	1	5.97254×10^{-06}	1.86552×10^{-08}	2.33212×10^{-08}
	3	8.24228×10^{-07}	1.86552×10^{-08}	1.86552×10^{-08}
	5	2.42963×10^{-06}	1.24312×10^{-07}	1.24312×10^{-07}
0.04	−5	5.96979×10^{-06}	2.64367×10^{-07}	2.64367×10^{-07}
	−3	2.66706×10^{-06}	2.76561×10^{-07}	2.76561×10^{-07}
	−1	1.34678×10^{-05}	2.12122×10^{-07}	2.12122×10^{-07}
	1	1.42464×10^{-05}	2.43563×10^{-06}	2.43563×10^{-06}
	3	1.85290×10^{-06}	1.75440×10^{-07}	1.75440×10^{-07}
	5		1.65373×10^{-07}	1.65373×10^{-07}
0.05	−5	1.17195×10^{-05}	2.47735×10^{-06}	2.47735×10^{-06}
	−3	5.41063×10^{-06}	2.40471×10^{-06}	2.40471×10^{-06}
	−1	2.60945×10^{-05}	2.29902×10^{-06}	2.29902×10^{-06}
	1	2.79946×10^{-05}	2.11023×10^{-06}	2.11023×10^{-06}
	3	3.42245×10^{-06}	1.67250×10^{-06}	1.67250×10^{-06}
	5	1.11322×10^{-05}	1.97820×10^{-06}	1.97820×10^{-06}

Table 2. Comparison of the proposed methods with q-$HATM$ in terms of absolute error for problem 2.

ς	y	$\wp = 2(q\text{-}HATM)$	$\wp = 2(HPTM)$	$\wp = 2(YTDM)$
0.01	−5	1.96030×10^{-05}	2.47795×10^{-06}	2.47795×10^{-06}
	−3	3.17168×10^{-04}	2.40477×10^{-05}	2.40477×10^{-05}
	−1	4.33502×10^{-03}	2.28902×10^{-04}	2.28902×10^{-04}
	1	1.65573×10^{-02}	2.14103×10^{-03}	2.14103×10^{-03}
	3	1.97624×10^{-02}	1.96890×10^{-03}	1.96890×10^{-03}
	5	1.99868×10^{-02}	1.96890×10^{-03}	1.96890×10^{-03}
0.02	−5	9.96170×10^{-03}	2.72234×10^{-04}	2.72234×10^{-04}
	−3	9.36662×10^{-03}	2.19087×10^{-04}	2.19087×10^{-04}
	−1	1.33124×10^{-03}	2.27543×10^{-04}	2.27543×10^{-04}
	1	2.31142×10^{-02}	2.42901×10^{-03}	2.42901×10^{-03}
	3	2.95239×10^{-02}	1.11443×10^{-03}	1.11443×10^{-03}
	5	2.99728×10^{-02}	1.64921×10^{-03}	1.64921×10^{-03}

Table 2. *Cont.*

ς	y	$\wp = 2(q\text{-}HATM)$	$\wp = 2(HPTM)$	$\wp = 2(YTDM)$
0.03	−5	1.99430×10^{-02}	2.87213×10^{-03}	2.87213×10^{-03}
	−3	1.90505×10^{-02}	2.68263×10^{-03}	2.68263×10^{-03}
	−1	6.99789×10^{-03}	2.09897×10^{-03}	2.09897×10^{-03}
	1	2.96714×10^{-02}	2.25643×10^{-03}	2.25643×10^{-03}
	3	3.92856×10^{-02}	1.33221×10^{-03}	1.33221×10^{-03}
	5	3.99588×10^{-02}	1.44231×10^{-03}	1.44231×10^{-03}
0.04	−5	2.99243×10^{-02}	2.33241×10^{-03}	2.33241×10^{-03}
	−3	2.87343×10^{-02}	2.24215×10^{-03}	2.24215×10^{-03}
	−1	1.26649×10^{-02}	2.23415×10^{-03}	2.23415×10^{-03}
	1	3.62290×10^{-02}	2.14253×10^{-03}	2.14253×10^{-03}
	3	4.90472×10^{-02}	1.95362×10^{-03}	1.95362×10^{-03}
	5	4.99447×10^{-02}	1.54820×10^{-03}	1.54820×10^{-03}
0.05	−5	3.99056×10^{-02}	2.47883×10^{-03}	2.47883×10^{-03}
	−3	3.84183×10^{-02}	2.77402×10^{-03}	2.77402×10^{-03}
	−1	1.83324×10^{-02}	2.97842×10^{-03}	2.97842×10^{-03}
	1	4.27869×10^{-02}	2.56482×10^{-03}	2.56482×10^{-03}
	3	5.88089×10^{-02}	1.96730×10^{-03}	1.96730×10^{-03}
	5	5.99307×10^{-02}	1.75689×10^{-03}	1.75689×10^{-03}

6. Conclusions

The fractional Phi-four problem has numerical solutions in the current framework using the Yang transform decomposition method and the homotopy perturbation transform method. We looked at two case studies in order to demonstrate the projected algorithm's effectiveness and applicability. The current work demonstrates that the projected nonlinear issue has extraordinary properties with respect to both the time instant and the time history, and that these properties may be successfully illustrated using the idea of fractional calculus. The proposed model plays a significant role in the analysis of many physical events. Future research can study this model utilising recently proposed and suggested accurate and effective approaches. We can discover further interesting results with the aid of the acquired results. Lastly, it has been shown that the proposed schemes are incredibly methodical, more precise, and highly successful. They can be used to illustrate the numerous kinds of nonlinear models that are present in science. The results show that the methods under consideration are easier to use and more effective for examining the behaviours of fractional differential equations with multiple dimensions that arise in related fields of science and engineering.

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