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# Systems of Sequential $\psi_1$-Hilfer and $\psi_2$-Caputo Fractional Differential Equations with Fractional Integro-Differential Nonlocal Boundary Conditions

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**Abstract:** In this paper, we introduce and study a new class of coupled and uncoupled systems, consisting of mixed-type $\psi_1$-Hilfer and $\psi_2$-Caputo fractional differential equations supplemented with asymmetric and symmetric integro-differential nonlocal boundary conditions (systems (2) and (13), respectively). As far as we know, this combination of $\psi_1$-Hilfer and $\psi_2$-Caputo fractional derivatives in coupled systems is new in the literature. The uniqueness result is achieved via the Banach contraction mapping principle, while the existence result is established by applying the Leray–Schauder alternative. Numerical examples illustrating the obtained results are also presented.

**Keywords:** $\psi$-Hilfer fractional derivative; $\psi$-Caputo fractional derivative; boundary value problems; nonlocal boundary conditions; existence; uniqueness; fixed point



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## 1. Introduction

The topic of coupled fractional-order systems, complemented with different kinds of boundary conditions, constitute an interesting area of research, because such systems appear in mathematical models of real-world problems, such as ecology [1], chaos and fractional dynamics [2], financial economics [3], bio-engineering [4], etc. Nonlocal boundary conditions are found to be more plausible and practical in contrast to the classical boundary conditions in view of their applicability to describe the changes happening within the given domain. In the literature, there are many fractional derivative operators, such as Riemann–Liouville, Caputo, Hadamard, Hilfer, Katugampola, etc., see the monographs [5–10]. For a variety of results on nonlocal single-valued and multi-valued boundary value problems involving different types of fractional-order derivative operators, we refer to the monograph [11].

A generalization of both Riemann–Liouville and Caputo fractional derivatives was given by R. Hilfer in [12]. This derivative can be reduced to the Riemann–Liouville and Caputo fractional derivatives for special cases of the parameters involved in its definition. For detailed advantages of the Hilfer derivative, see [13] and some recent applications in calcium diffusion in [14–16]. The Hilfer fractional derivative with another function, known as $\psi$-Hilfer fractional derivative, has been introduced in [17]. For some recent results on existence and uniqueness of initial and boundary value problems including the $\psi$-Hilfer fractional derivative, see [18–24] and references therein.

Recently, in [25], we introduced and studied a new class of boundary value problems, consisting of mixed-type $\psi_1$-Hilfer and $\psi_2$-Caputo fractional differential equations supplemented with integro-differential nonlocal boundary conditions of the form:

$$\begin{cases} {}^H D^{\alpha,\beta;\psi_1}({}^C D^{\gamma;\psi_2}\pi)(s) = \Upsilon_1(s,\pi(s)), \quad 0<\alpha,\beta,\gamma<1,\ s\in[0,A], \\ {}^C D^{\gamma;\psi_2}\pi(0)=0, \quad \pi(A)=\sum_{i=1}^{m}\lambda_i\, {}^C D^{\gamma;\psi_2}\pi(\eta_i)+\sum_{j=1}^{n}\delta_j\, I^{\mu_j;\psi_2}\pi(\xi_j), \end{cases} \tag{1}$$

where ${}^H D^{\alpha,\beta;\psi_1}$ and ${}^C D^{\gamma;\psi_2}$ are the $\psi_1$-Hilfer and $\psi_2$-Caputo fractional derivatives with respect to functions $\psi_1$ and $\psi_2$, respectively, where $\psi_1'(s),\psi_2'(s)>0$ for all $t\in[0,A]$, $\lambda_i,\delta_j\in\mathbb{R}$, $\eta_i,\xi_j\in(0,A)$, $I^{\mu_j;\psi_2}$ is the Riemann–Liouville fractional integral of order $\mu_j>0$, with respect to a function $\psi_2$, for $i=1,\cdots,m$, $j=1,\cdots,n$ and $f:[0,A]\times\mathbb{R}\to\mathbb{R}$ is a nonlinear continuous function. Existence and uniqueness were established via Banach's fixed point theorem and the Leray–Schauder nonlinear alternative.

The novelty of this study lies in the fact that we introduced a new class of boundary value problems in which we combined $\psi_1$-Hilfer and $\psi_2$-Caputo fractional derivatives and, as far as we know, this combination is new in the literature.

In the present paper, we continue the above investigation, by considering the following system of sequential $\psi_1$-Hilfer and $\psi_2$-Caputo fractional differential equations with fractional integro-differential nonlocal conditions of the form:

$$\begin{cases} {}^H D^{\alpha,\beta;\psi_1}({}^C D^{\gamma;\psi_2}\pi)(s) = \Upsilon_1(s,\pi(s),\rho(s)), \quad s\in[0,A], \\ {}^H D^{\hat\alpha,\hat\beta;\psi_1}({}^C D^{\hat\gamma;\psi_2}\rho)(s) = \Upsilon_2(s,\pi(s),\rho(s)), \quad s\in[0,A], \\ {}^C D^{\gamma;\psi_2}\pi(0)=0, \quad \pi(A)=\lambda_1\, {}^C D^{\hat\gamma;\psi_2}\rho(\xi_1)+\lambda_2\, I^{\hat\mu;\psi_2}\rho(\xi_2), \\ {}^C D^{\hat\gamma;\psi_2}\rho(0)=0, \quad \rho(A)=\delta_1\, {}^C D^{\gamma;\psi_2}\pi(\eta_1)+\delta_2\, I^{\mu;\psi_2}\pi(\eta_2), \end{cases} \tag{2}$$

where the differential operators ${}^H D^{\alpha,\beta;\psi_1}, {}^H D^{\hat\alpha,\hat\beta;\psi_1}$ are the $\psi_1$-Hilfer fractional derivative of orders $0<\alpha,\hat\alpha<1$ with Hilfer parameters $0<\beta,\hat\beta<1$, ${}^C D^{\gamma;\psi_2}, {}^C D^{\hat\gamma;\psi_2}$ are the $\psi_2$-Caputo fractional derivatives of orders $0<\gamma,\hat\gamma<1$, $\lambda_1,\lambda_2,\delta_1,\delta_2\in\mathbb{R}$ are given constants, $\eta_1,\eta_2,\xi_1,\xi_2\in[0,A]$, and $\Upsilon_1,\Upsilon_2:[0,A]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions.

We obtain existence and uniqueness results by applying the classical fixed point theorems. Thus, the uniqueness result is established via Banach's contraction mapping principle, while the basic tool for the existence result is the Leray–Schauder alternative.

The rest of the paper is arranged as follows. In Section 2, we recall some definitions and lemmas from fractional calculus needed in our study and also we present an auxiliary lemma which is used to transform the given nonlinear problem into a fixed-point problem. Section 3 contains the main results, while in Section 4, we indicate the uncoupled fractional integro-differential boundary conditions. Finally, illustrative examples are constructed in Section 5.

## 2. Preliminaries

Now, some notations, definitions, and known results of fractional calculus are reminded [6].

Let $\psi\in C^1([0,A],\mathbb{R})$ with $\psi'(s)>0$ for all $s\in[0,A]$.

**Definition 1** ([6]). *Let $\alpha>0$ and $f\in L^1([0,A],\mathbb{R})$. The $\psi$-Riemann–Liouville fractional integral of order $\alpha$ to a function $f$ with respect to $\psi$ is defined by*

$$I^{\alpha;\psi}f(s)=\frac{1}{\Gamma(\alpha)}\int_0^s\psi'(\tau)(\psi(s)-\psi(\tau))^{\alpha-1}f(\tau)d\tau.$$

**Definition 2** ([17]). *Let $n-1<\alpha<n$, $n\in\mathbb{N}$ and $f,\psi\in C^n([0,A],\mathbb{R})$ such that $\psi'(s)>0$ for all $s\in[0,A]$. The $\psi$-Hilfer fractional derivative ${}^HD^{\alpha,\beta;\psi}(\cdot)$ of order $\alpha$ to a function $f$ and type $0\le\beta\le 1$, is defined by*

$$ {}^HD^{\alpha,\beta;\psi}f(s)=I^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^n I^{(1-\beta)(n-\alpha);\psi}f(s). $$

**Definition 3** ([26]). *Let $n-1<\alpha<n$, $n\in\mathbb{N}$ and $f,\psi\in C^n([0,A],\mathbb{R})$ such that $\psi'(s)>0$ for all $s\in[0,A]$. The $\psi$-Caputo fractional derivative ${}^CD^{\alpha;\psi}(\cdot)$ of order $\alpha$ to a function $f$ is defined by*

$$ {}^CD^{\alpha;\psi}f(s)=I^{n-\alpha;\psi}\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^n f(s). $$

**Lemma 1** ([17]). *The semigroup property and integration of power function formula. Let $\alpha,\chi>0$ and $\delta>1$ be constants. Then, we have*

*(i)* $I^{\alpha;\psi}I^{\chi;\psi}h(s)=I^{\alpha+\chi;\psi}h(s)$;

*(ii)* $I^{\alpha;\psi}(\psi(s)-\psi(a))^{\delta-1}=\dfrac{\Gamma(\delta)}{\Gamma(\alpha+\delta)}(\psi(s)-\psi(a))^{\alpha+\delta-1}$.

The following lemmas contain the compositional property of the Riemann–Liouville fractional integral operator with the $\psi$-Hilfer fractional derivative and $\psi$-Caputo fractional derivative.

**Lemma 2** ([17]). *Let $f\in L(0,A)$, $n-1<\alpha\le n$, $n\in\mathbb{N}$, $0\le\beta\le 1$, $\gamma^*=\alpha+n\beta-\alpha\beta$, $\left(I^{(n-\alpha)(1-\beta)}f\right)\in AC^k[0,A]$. Then,*

$$ \left(I^{\alpha;\psi}\,{}^HD^{\alpha,\beta;\psi}f\right)(s)=f(s)-\sum_{k=1}^{n}\frac{(\psi(s)-\psi(0))^{\gamma^*-k}}{\Gamma(\gamma^*-k+1)}\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^{n-k}\left(I^{(1-\beta)(n-\alpha);\psi}f\right)(0). $$

**Lemma 3** ([26]). *Let $f\in L(0,A)$ and $\alpha>0$, we have*

$$ \left(I^{\alpha;\psi}\,{}^CD^{\alpha;\psi}f\right)(s)=f(s)-\sum_{k=0}^{n-1}\frac{\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^k f(0)}{k!}(\psi(s)-\psi(0))^k. $$

Our first task is to transform the boundary value problem (2) into an integral equation.

**Lemma 4.** *Let $h,\hat{h}\in C([0,A],\mathbb{R})$ be given functions and $\Omega\neq 0$. Then, the unique solution of the following linear system*

$$ \begin{cases} {}^HD^{\alpha,\beta;\psi_1}({}^CD^{\gamma;\psi_2}\pi)(s)=h(s), \\ {}^HD^{\hat{\alpha},\hat{\beta};\psi_1}({}^CD^{\hat{\gamma};\psi_2}\rho)(s)=\hat{h}(s), \\ {}^CD^{\gamma;\psi_2}\pi(0)=0, \quad \pi(A)=\lambda_1\,{}^CD^{\hat{\gamma};\psi_2}\rho(\xi_1)+\lambda_2\,I^{\hat{\mu};\psi_2}\rho(\xi_2), \\ {}^CD^{\hat{\gamma};\psi_2}\rho(0)=0, \quad \rho(A)=\delta_1\,{}^CD^{\gamma;\psi_2}\pi(\eta_1)+\delta_2\,I^{\mu;\psi_2}\pi(\eta_2), \end{cases} \tag{3} $$

*is given by*

$$ \begin{aligned} \pi(s) \;=\; & \frac{1}{\Omega}\Big[\lambda_1 I^{\hat{\alpha};\psi_1}\hat{h}(\xi_1)-I^{\gamma;\psi_2}I^{\alpha;\psi_1}h(A)+\lambda_2 I^{\hat{\mu}+\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\hat{h}(\xi_2) \\ & +\Omega_2\Big\{\delta_1 I^{\alpha;\psi_1}h(\eta_1)-I^{\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\hat{h}(A)+\delta_2 I^{\mu+\gamma;\psi_2}I^{\alpha;\psi_1}h(\eta_2)\Big\}\Big] \\ & +I^{\gamma;\psi_2}I^{\alpha;\psi_1}h(s), \end{aligned} \tag{4} $$

*and*

$$\rho(s) = \frac{1}{\Omega}\Bigg[\Big(\delta_1 I^{\alpha;\psi_1}h(\eta_1) - I^{\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\hat{h}(A) + \delta_2 I^{\mu+\gamma;\psi_2}I^{\alpha;\psi_1}h(\eta_2)\Big) + \Omega_1\Big\{\lambda_1 I^{\hat{\alpha};\psi_1}\hat{h}(\xi_1) - I^{\gamma;\psi_2}I^{\alpha;\psi_1}h(A) + \lambda_2 I^{\hat{\mu}+\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\hat{h}(\xi_2)\Big\}\Bigg] + I^{\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\hat{h}(s), \tag{5}$$

*where*

$$\Omega_1 = \delta_2\frac{[\psi_2(\eta_2)-\psi_2(0)]^{\mu}}{\Gamma(\mu+1)},\ \ \Omega_2 = \lambda_2\frac{[\psi_2(\xi_2)-\psi_2(0)]^{\hat{\mu}}}{\Gamma(\hat{\mu}+1)},\ \ \Omega = 1-\Omega_1\Omega_2.$$

**Proof.** Assume that $x, y$ are solutions of the nonlocal system (3) on $[0, A]$. Taking the fractional integrals $I^{\alpha;\psi_1}$, $I^{\hat{\alpha};\psi_1}$ on both sides of the first and second equations in (3), respectively, and using Lemma 2, we obtain for $s \in [0, A]$,

$${}^{C}D^{\gamma;\psi_2}\pi(s) = c_0\frac{[\psi_1(s)-\psi_1(0)]^{\alpha^*-1}}{\Gamma(\alpha^*)} + I^{\alpha;\psi_1}h(s),$$

$${}^{C}D^{\hat{\gamma};\psi_2}\rho(s) = d_0\frac{[\psi_1(s)-\psi_1(0)]^{\hat{\alpha}^*-1}}{\Gamma(\hat{\alpha}^*)} + I^{\hat{\alpha};\psi_1}\hat{h}(s),$$

where $\alpha^* = \alpha + (1-\alpha)\beta$ and $\hat{\alpha}^* = \hat{\alpha} + (1-\hat{\alpha})\hat{\beta}$, $c_0, d_0 \in \mathbb{R}$. Since $\alpha^* \in (\alpha, 1)$ and $\hat{\alpha}^* \in (\hat{\alpha}, 1)$, and from conditions ${}^{C}D^{\gamma;\psi_2}\pi(0) = 0$, ${}^{C}D^{\hat{\gamma};\psi_2}\rho(0) = 0$, we obtain $c_0 = 0$ and $d_0 = 0$. Hence, we have

$$\begin{cases} {}^{C}D^{\gamma;\psi_2}\pi(s) = I^{\alpha;\psi_1}h(s), \\ {}^{C}D^{\hat{\gamma};\psi_2}\rho(s) = I^{\hat{\alpha};\psi_1}\hat{h}(s). \end{cases} \tag{6}$$

The fractional integration of the above two equations of orders $\gamma$ and $\hat{\gamma}$, respectively, leads to

$$\begin{cases} \pi(s) = c_1 + I^{\gamma;\psi_2}I^{\alpha;\psi_1}h(s), \\ \rho(s) = d_1 + I^{\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\hat{h}(s), \quad c_1, d_1 \in \mathbb{R}. \end{cases} \tag{7}$$

From (6), we have

$${}^{C}D^{\gamma;\psi_2}\pi(\eta_1) = I^{\alpha;\psi_1}h(\eta_1) \quad \text{and} \quad {}^{C}D^{\hat{\gamma};\psi_2}\rho(\xi_1) = I^{\hat{\alpha};\psi_1}\hat{h}(\xi_1). \tag{8}$$

In addition, the Riemann–Liouville fractional integral with respect to a function $\psi_2$ of orders $\mu$ and $\hat{\mu}$ is applied in (7) to the points $\eta_2$ and $\xi_2$, respectively, then,

$$I^{\mu;\psi_2}\pi(\eta_2) = c_1\frac{[\psi_2(\eta_2)-\psi_2(0)]^{\mu}}{\Gamma(\mu+1)} + I^{\mu+\gamma;\psi_2}I^{\alpha;\psi_1}h(\eta_2), \tag{9}$$

and

$$I^{\hat{\mu};\psi_2}\rho(\xi_2) = d_1\frac{[\psi_2(\xi_2)-\psi_2(0)]^{\hat{\mu}}}{\Gamma(\hat{\mu}+1)} + I^{\hat{\mu}+\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\hat{h}(\xi_2). \tag{10}$$

Substituting $s = A$ in (7) and using (8)–(10) in boundary conditions, $c_1$ and $d_1$ can be expressed as

$$c_1 = \frac{1}{\Omega}\Bigg[\lambda_1 I^{\hat{\alpha};\psi_1}\hat{h}(\xi_1) - I^{\gamma;\psi_2}I^{\alpha;\psi_1}h(A) + \lambda_2 I^{\hat{\mu}+\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\hat{h}(\xi_2) + \Omega_2\Big\{\delta_1 I^{\alpha;\psi_1}h(\eta_1) - I^{\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\hat{h}(A) + \delta_2 I^{\mu+\gamma;\psi_2}I^{\alpha;\psi_1}h(\eta_2)\Big\}\Bigg],$$

$$d_1 = \frac{1}{\Omega}\Bigg[\Big(\delta_1 I^{\alpha;\psi_1}h(\eta_1) - I^{\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\hat{h}(A) + \delta_2 I^{\mu+\gamma;\psi_2}I^{\alpha;\psi_1}h(\eta_2)\Big) + \Omega_1\Big\{\lambda_1 I^{\hat{\alpha};\psi_1}\hat{h}(\xi_1) - I^{\gamma;\psi_2}I^{\alpha;\psi_1}h(A) + \lambda_2 I^{\hat{\mu}+\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\hat{h}(\xi_2)\Big\}\Bigg].$$

Substituting the constants into (7), we obtain (4) and (5).

On the other hand, taking the $\psi_2$-Caputo fractional derivative of orders $\gamma$ and $\hat{\gamma}$, to (4) and (5), respectively, we obtain (6) which satisfies the first condition at lines 3 and 4 of (3) when $s = 0$. Applying the $\psi_1$-Hilfer fractional derivative of orders $\alpha$ and $\hat{\alpha}$ to the first and second equations in (6), respectively, leads to the first two equations in (3). Using the fractional integration $\psi_2$-Riemann–Liouville of orders $\mu$ and $\hat{\mu}$ in (4) and (5) with points $s = \eta_2$ and $s = \xi_2$, respectively, and from (6) at the points $s = \eta_1$ and $s = \eta_2$, we can show by direct computation that the second condition at lines 3 and 4 of (3) holds. Therefore, this lemma is proved. □

## 3. Main Results

From Lemma 4, we define an operator $\mathbb{M} : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ by

$$\mathbb{M}(\pi,\rho)(s) = \begin{pmatrix} \mathbb{M}_1(\pi,\rho)(s) \\ \mathbb{M}_2(\pi,\rho)(s) \end{pmatrix},$$

where

$$\begin{aligned} &\mathbb{M}_1(\pi,\rho)(s) \\ = &\frac{1}{\Omega}\Bigg[\lambda_1 I^{\hat{\alpha};\psi_1}\Upsilon_2(\xi_1,\pi(\xi_1),\rho(\xi_1)) - I^{\gamma;\psi_2}I^{\alpha;\psi_1}\Upsilon_1(A,\pi(A),\rho(A)) \\ &+ \lambda_2 I^{\hat{\mu}+\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\Upsilon_2(\xi_2,\pi(\xi_2),\rho(\xi_2)) + \Omega_2\Big\{\delta_1 I^{\alpha;\psi_1}\Upsilon_1(\eta_1,\pi(\eta_1),\rho(\eta_1)) \\ &- I^{\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\Upsilon_2(A,\pi(A),\rho(A)) + \delta_2 I^{\mu+\gamma;\psi_2}I^{\alpha;\psi_1}\Upsilon_1(\eta_2,\pi(\eta_2),\rho(\eta_2))\Big\}\Bigg] \\ &+ I^{\gamma;\psi_2}I^{\alpha;\psi_1}\Upsilon_1(s,\pi(s),\rho(s)), \end{aligned}$$

and

$$\begin{aligned} &\mathbb{M}_2(\pi,\rho)(s) \\ = &\frac{1}{\Omega}\Bigg[\delta_1 I^{\alpha;\psi_1}\Upsilon_1(\eta_1,\pi(\eta_1),\rho(\eta_1)) - I^{\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\Upsilon_2(A,\pi(A),\rho(A)) \\ &+ \delta_2 I^{\mu+\gamma;\psi_2}I^{\alpha;\psi_1}\Upsilon_1(\eta_2,\pi(\eta_2),\rho(\eta_2)) + \Omega_1\Big\{\lambda_1 I^{\hat{\alpha};\psi_1}\Upsilon_2(\xi_1,\pi(\xi_1),\rho(\xi_1)) \\ &- I^{\gamma;\psi_2}I^{\alpha;\psi_1}\Upsilon_1(A,\pi(A),\rho(A)) + \lambda_2 I^{\hat{\mu}+\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\Upsilon_2(\xi_2,\pi(\xi_2),\rho(\xi_2))\Big\}\Bigg] \\ &+ I^{\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\Upsilon_2(s,\pi(s),\rho(s)), \end{aligned}$$

and $\mathfrak{X} = C([0,A],\mathbb{R})$ is the Banach space of all continuous functions $\pi$ from $[0,A]$ to $\mathbb{R}$ endowed with the norm $\|\pi\| = \max\{|\pi(s)|, s \in [0,A]\}$. The product space $(\mathfrak{X} \times \mathfrak{X}, \|(\pi,\rho)\|)$ is also a Banach space with norm $\|(\pi,\rho)\| = \|\pi\| + \|\rho\|$.

For simplicity in computation, we put:

$$\Phi^{\alpha,\varphi}_{\psi_1,\psi_2}(b) := I^{\varphi;\psi_2}I^{\alpha;\psi_1}(1)(b)$$

$$= \frac{1}{\Gamma(\alpha+1)\Gamma(\varphi)}\int_0^b \psi_2'(u)(\psi_1(u)-\psi_1(0))^{\alpha}(\psi_2(b)-\psi_2(u))^{\varphi-1}du,$$

and

$$\hat{\Phi}_{\psi}^{\varphi}(b) := I^{\varphi;\psi}(1)(b) = \frac{1}{\Gamma(\varphi)}\int_0^b \psi'(s)(\psi(b)-\psi(s))^{\varphi-1}ds,$$

and some constants as

$$\begin{aligned}
Q_1 &= \frac{1}{|\Omega|}\Big[|\Omega_2|\Big(|\delta_1|\tilde{\Phi}_{\psi_1}^{\alpha}(\eta_1)+|\delta_2|\Phi_{\psi_1,\psi_2}^{\alpha,\mu+\gamma}(\eta_2)\Big)+(1+|\Omega_2|)\Phi_{\psi_1,\psi_2}^{\alpha,\gamma}(A)\Big],\\
Q_2 &= \frac{1}{|\Omega|}\Big[|\lambda_1|\tilde{\Phi}_{\psi_1}^{\hat{\alpha}}(\xi_1)+|\lambda_2|\Phi_{\psi_1,\psi_2}^{\hat{\alpha},\hat{\mu}+\hat{\gamma}}(\xi_2)+|\Omega_2|\Phi_{\psi_1,\psi_2}^{\hat{\alpha},\hat{\gamma}}(A)\Big],\\
Q_3 &= \frac{1}{|\Omega|}\Big[|\Omega_1|\Big(|\lambda_1|\tilde{\Phi}_{\psi_1}^{\hat{\alpha}}(\xi_1)+|\lambda_2|\Phi_{\psi_1,\psi_2}^{\hat{\alpha},\hat{\mu}+\hat{\gamma}}(\xi_2)\Big)+(1+|\Omega|)\Phi_{\psi_1,\psi_2}^{\hat{\alpha},\hat{\gamma}}(A)\Big],\\
Q_4 &= \frac{1}{|\Omega|}\Big[|\delta_1|\tilde{\Phi}_{\psi_1}^{\alpha}(\eta_1)+|\delta_2|\Phi_{\psi_1,\psi_2}^{\alpha,\mu+\gamma}(\eta_2)+|\Omega_1|\Phi_{\psi_1,\psi_2}^{\alpha,\gamma}(A)\Big].
\end{aligned}$$

Now, the existence of a unique solution to the coupled system of sequential $\psi_1$-Hilfer and $\psi_2$-Caputo fractional differential equations with fractional integro-differential nonlocal conditions (2) is presented by applying Banach's contraction mapping principle.

**Theorem 1.** *Assume that $\Omega \neq 0$ and $\Upsilon_1, \Upsilon_2 : [0, A] \times \mathbb{R}^2 \to \mathbb{R}$ are two functions for which there exist constants $m_i, n_i, i = 1, 2$ such that, for all $s \in [0, A]$ and $\pi_i, \rho_i \in \mathbb{R}, i = 1, 2,$*

$$|\Upsilon_1(s,\pi_1,\rho_1)-\Upsilon_1(s,\pi_2,\rho_2)| \le m_1|\pi_1-\pi_2|+m_2|\rho_1-\rho_2|$$

*and*

$$|\Upsilon_2(s,\pi_1,\rho_1)-\Upsilon_2(s,\pi_2,\rho_2)| \le n_1|\pi_1-\pi_2|+n_2|\rho_1-\rho_2|.$$

*If*

$$(Q_1+Q_4)(m_1+m_2)+(Q_2+Q_3)(n_1+n_2) < 1,$$

*then the coupled system of sequential $\psi_1$-Hilfer and $\psi_2$-Caputo fractional differential equations with fractional integro-differential nonlocal conditions (2) has a unique solution $(\pi, \rho)$ on $[0, A]$.*

**Proof.** Define $\sup_{s\in[0,A]} \Upsilon_1(A, 0, 0) = M < \infty$ and $\sup_{s\in[0,A]} \Upsilon_2(A, 0, 0) = N < \infty$ and choose

$$r \ge \frac{(Q_1+Q_4)M+(Q_2+Q_3)N}{1-[(Q_1+Q_4)(m_1+m_2)+(Q_2+Q_3)(n_1+n_2)]},$$

where $r$ is a radius of the ball $B_r = \{(\pi,\rho) \in \mathfrak{X}\times\mathfrak{X} : \|(\pi,\rho)\| \le r\}$. Next, we show that $(\mathbb{M}B_r) \subset B_r$. For each $(\pi, \rho) \in B_r$, we have

$$\begin{aligned}
&|\mathbb{M}_1(\pi,\rho)(s)|\\
\le\ & \frac{1}{|\Omega|}\Bigg[\Bigg(|\lambda_1|I^{\hat{\alpha};\psi_1}\Big[|\Upsilon_2(\xi_1,\pi(\xi_1),\rho(\xi_1))-\Upsilon_2(\xi_1,0,0)|+|\Upsilon_2(\xi_1,0,0)|\Big]\\
&+I^{\gamma;\psi_2}I^{\alpha;\psi_1}\Big[|\Upsilon_1(A,\pi(A),\rho(A))-\Upsilon_1(A,0,0)|+|\Upsilon_1(A,0,0)|\Big]\\
&+|\lambda_2|I^{\hat{\mu}+\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\Big[|\Upsilon_2(\xi_2,\pi(\xi_2),\rho(\xi_2)-\Upsilon_2(\xi_2,0,0))|+|\Upsilon_2(\xi_2,0,0)|\Big]\Bigg)\\
&+|\Omega_2|\Bigg\{|\delta_1|I^{\alpha;\psi_1}\Big[|\Upsilon_1(\eta_1,\pi(\eta_1),\rho(\eta_1))-\Upsilon_1(\eta_1,0,0)|+|\Upsilon_1(\eta_1,0,0)|\Big]\\
&+I^{\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\Big[|\Upsilon_2(A,\pi(A),\rho(A))-\Upsilon_2(A,0,0)|+|\Upsilon_2(A,0,0)|\Big]\\
&+|\delta_2|I^{\mu+\gamma;\psi_2}I^{\alpha;\psi_1}\Big[|\Upsilon_1(\eta_2,\pi(\eta_2),\rho(\eta_2))-\Upsilon_1(\eta_2,0,0)|+|\Upsilon_1(\eta_2,0,0)|\Big]\Bigg\}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
& + I^{\gamma;\psi_2} I^{\alpha;\psi_1}\Big[|\Upsilon_1(A,\pi(A),\rho(A)) - \Upsilon_1(A,0,0)| + |\Upsilon_1(A,0,0)|\Big] \\
\leq\ & \frac{1}{|\Omega|}\Bigg[\bigg(|\lambda_1|[n_1\|\pi\| + n_2\|\rho\| + N] I^{\hat{\alpha};\psi_1}(1)(\xi_1) \\
& + [m_1\|\pi\| + m_2\|\rho\| + M] I^{\gamma;\psi_2} I^{\alpha;\psi_1}(1)(A) \\
& + |\lambda_2|[n_1\|\pi\| + n_2\|\rho\| + N] I^{\hat{\mu}+\hat{\gamma};\psi_2} I^{\hat{\alpha};\psi_1}(1)(\xi_2)\bigg) \\
& + |\Omega_2|\bigg\{|\delta_1|[m_1\|\pi\| + m_2\|\rho\| + M] I^{\alpha;\psi_1}(1)(\eta_1) \\
& + [n_1\|\pi\| + n_2\|\rho\| + N] I^{\hat{\gamma};\psi_2} I^{\hat{\alpha};\psi_1}(1)(A) \\
& + |\delta_2|[m_1\|\pi\| + m_2\|\rho\| + M] I^{\mu+\gamma;\psi_2} I^{\alpha;\psi_1}(1)(\eta_2)\bigg\}\Bigg] \\
& + [m_1\|\pi\| + m_2\|\rho\| + M] I^{\gamma;\psi_2} I^{\alpha;\psi_1}(1)(s),
\end{aligned}
$$

by using the following relations $|\Upsilon_1(s,\pi,\rho)| \leq |\Upsilon_1(s,\pi,\rho) - \Upsilon_1(s,0,0)| + |\Upsilon_1(s,0,0)| \leq m_1|x| + m_2|y| + M$ and $|\Upsilon_2(s,\pi,\rho)| \leq |\Upsilon_2(s,\pi,\rho) - \Upsilon_2(s,0,0)| + |\Upsilon_2(s,0,0)| \leq n_1|x| + n_2|y| + N$. Then, we have

$$
\begin{aligned}
& |\mathbb{M}_1(\pi,\rho)(s)| \\
\leq\ & \frac{1}{|\Omega|}\Big[|\Omega_2|\Big(|\delta_1|\tilde{\Phi}^{\alpha}_{\psi_1}(\eta_1) + |\delta_2|\Phi^{\alpha,\mu+\gamma}_{\psi_1,\psi_2}(\eta_2)\Big) + (1+|\Omega_2|)\Phi^{\alpha,\gamma}_{\psi_1,\psi_2}(A)\Big][m_1\|\pi\| + m_2\|\rho\| + M] \\
& + \frac{1}{|\Omega|}\Big[|\lambda_1|\tilde{\Phi}^{\hat{\alpha}}_{\psi_1}(\xi_1) + |\lambda_2|\Phi^{\hat{\alpha},\hat{\mu}+\hat{\gamma}}_{\psi_1,\psi_2}(\xi_2) + |\Omega_2|\Phi^{\hat{\alpha},\hat{\gamma}}_{\psi_1,\psi_2}(A)\Big][n_1\|\pi\| + n_2\|\rho\| + N] \\
=\ & Q_1[m_1\|\pi\| + m_2\|\rho\| + M] + Q_2[n_1\|\pi\| + n_2\|\rho\| + N] \\
=\ & (Q_1 m_1 + Q_2 n_1)\|\pi\| + (Q_1 m_2 + Q_2 n_2)\|\rho\| + Q_1 M + Q_2 N \\
\leq\ & (Q_1 m_1 + Q_2 n_1 + Q_1 m_2 + Q_2 n_2)r + Q_1 M + Q_2 N.
\end{aligned}
$$

Next, we consider boundedness of the operator $\mathbb{M}_2$ as

$$
\begin{aligned}
\mathbb{M}_2(\pi,\rho)(s) \leq\ & \frac{1}{|\Omega|}\Bigg[|\delta_1|[m_1\|\pi\| + m_2\|\rho\| + M] I^{\alpha;\psi_1}(1)(\eta_1) \\
& + [n_1\|\pi\| + n_2\|\rho\| + N] I^{\hat{\gamma};\psi_2} I^{\hat{\alpha};\psi_1}(1)(A) \\
& + |\delta_2|[m_1\|\pi\| + m_2\|\rho\| + M] I^{\mu+\gamma;\psi_2} I^{\alpha;\psi_1}(1)(\eta_2) \\
& + |\Omega_1|\Big\{|\lambda_1|[n_1\|\pi\| + n_2\|\rho\| + N] I^{\hat{\alpha};\psi_1}(1)(\xi_1) \\
& + [m_1\|\pi\| + m_2\|\rho\| + M] I^{\gamma;\psi_2} I^{\alpha;\psi_1}(1)(A) \\
& + |\lambda_2|[n_1\|\pi\| + n_2\|\rho\| + N] I^{\hat{\mu}+\hat{\gamma};\psi_2} I^{\hat{\alpha};\psi_1}(1)(\xi_2)\Big\}\Bigg] \\
& + [n_1\|\pi\| + n_2\|\rho\| + N] I^{\hat{\gamma};\psi_2} I^{\hat{\alpha};\psi_1}(1)(A) \\
=\ & Q_3[n_1\|\pi\| + n_2\|\rho\| + N] + Q_4[m_1\|\pi\| + m_2\|\rho\| + M] \\
=\ & (Q_4 m_1 + Q_3 n_1)\|\pi\| + (Q_4 m_2 + Q_3 n_2)\|\rho\| + Q_4 M + Q_3 N \\
\leq\ & (Q_4 m_1 + Q_3 n_1 + Q_4 m_2 + Q_3 n_2)r + Q_4 M + Q_3 N.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\|\mathbb{M}(\pi,\rho)\| =\ & \|\mathbb{M}_1(\pi,\rho)\| + \|\mathbb{M}_2(\pi,\rho)\| \\
\leq\ & (Q_1 m_1 + Q_2 n_1 + Q_1 m_2 + Q_2 n_2)r + Q_1 M + Q_2 N \\
& +(Q_4 m_1 + Q_3 n_1 + Q_4 m_2 + Q_3 n_2)r + Q_4 M + Q_3 N \\
=\ & [(Q_1 + Q_4)(m_1 + m_2) + (Q_2 + Q_3)(n_1 + n_2)]r
\end{aligned}
$$

$$+(Q_1+Q_4)M+(Q_2+Q_3)N \leq r,$$

which implies the fact that $(\mathbb{M}B_r) \subset B_r$.

Now, we show that the operator $\mathbb{M}$ is a contraction. For each $(\pi_2,\rho_2),(\pi_1,\rho_1) \in \mathfrak{X}\times\mathfrak{X}$, and for any $t \in [0,A]$, we obtain:

$$\begin{aligned}
& |\mathbb{M}_1(\pi_2,\rho_2)(s)-\mathbb{M}_1(\pi_1,\rho_1)(s)| \\
\leq\ & \frac{1}{|\Omega|}\Bigg[\Bigg(|\lambda_1|I^{\hat{\alpha};\psi_1}|\Upsilon_2(\xi_1,\pi_2(\xi_1),\rho_2(\xi_1))-\Upsilon_2(\xi_1,\pi_1(\xi_1),\rho_1(\xi_1))| \\
& +I^{\gamma;\psi_2}I^{\alpha;\psi_1}|\Upsilon_1(A,\pi_2(A),\rho_2(A))-\Upsilon_1(A,\pi_1(A),\rho_1(A))| \\
& +|\lambda_2|I^{\hat{\mu}+\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}|\Upsilon_2(\xi_2,\pi_2(\xi_2),\rho_2(\xi_2))-\Upsilon_2(\xi_2,\pi_1(\xi_2),\rho_1(\xi_2))|\Bigg) \\
& +|\Omega_2|\Bigg\{|\delta_1|I^{\alpha;\psi_1}|\Upsilon_1(\eta_1,\pi_2(\eta_1),\rho_2(\eta_1))-\Upsilon_1(\eta_1,\pi_1(\eta_1),\rho_1(\eta_1))| \\
& +I^{\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}|\Upsilon_2(A,\pi_2(A),\rho_2(A))-\Upsilon_2(A,\pi_1(A),\rho_1(A))| \\
& +|\delta_2|I^{\mu+\gamma;\psi_2}I^{\alpha;\psi_1}|\Upsilon_1(\eta_2,\pi_2(\eta_2),\rho_2(\eta_2))-\Upsilon_1(\eta_2,\pi_1(\eta_2),\rho_1(\eta_2))|\Bigg\}\Bigg] \\
& +I^{\gamma;\psi_2}I^{\alpha;\psi_1}|\Upsilon_1(A,\pi_2(A),\rho_2(A))-\Upsilon_1(A,\pi_2(A),\rho_2(A))| \\
\leq\ & [m_1\|\pi_2-\pi_1\|+m_2\|\rho_2-\rho_1\|]\frac{1}{|\Omega|}\Big[|\Omega_2|\Big(|\delta_1|\tilde{\Phi}^{\alpha}_{\psi_1}(\eta_1)+|\delta_2|\Phi^{\alpha,\mu+\gamma}_{\psi_1,\psi_2}(\eta_2)\Big) \\
& +(1+|\Omega|)\Phi^{\alpha,\gamma}_{\psi_1,\psi_2}(A)\Big]+[n_1\|\pi_2-\pi_1\|+n_2\|\rho_2-\rho_1\|]\frac{1}{|\Omega|} \\
& \times\Big[|\lambda_1|\tilde{\Phi}^{\hat{\alpha}}_{\psi_1}(\xi_1)+|\lambda_2|\Phi^{\hat{\alpha},\hat{\mu}+\hat{\gamma}}_{\psi_1,\psi_2}(\xi_2)+|\Omega_2|\Phi^{\hat{\alpha},\hat{\gamma}}_{\psi_1,\psi_2}(A)\Big] \\
=\ & [m_1\|\pi_2-\pi_1\|+m_2\|\rho_2-\rho_1\|]Q_1+[n_1\|\pi_2-\pi_1\|+n_2\|\rho_2-\rho_1\|]Q_2, \\
\leq\ & (m_1Q_1+n_1Q_2+m_2Q_1+n_2Q_2)[\|\pi_2-\pi_1\|+\|\rho_2-\rho_1\|]. \qquad (11)
\end{aligned}$$

By the same way of computation, we have

$$\begin{aligned}
& |\mathbb{M}_2(\pi_2,\rho_2)(s)-\mathbb{M}_2(\pi_1,\rho_1)(s)| \\
\leq\ & (m_1Q_4+n_1Q_3+m_2Q_4+n_2Q_3)[\|\pi_2-\pi_1\|+\|\rho_2-\rho_1\|]. \qquad (12)
\end{aligned}$$

From the two inequalities (11) and (12) above, we can conclude that

$$\begin{aligned}
& \|\mathbb{M}(\pi_2,\rho_2)-\mathbb{M}(\pi_1,\rho_1)\| \\
\leq\ & [(Q_1+Q_4)(m_1+m_2)+(Q_2+Q_3)(n_1+n_2)][\|\pi_2-\pi_1\|+\|\rho_2-\rho_1\|].
\end{aligned}$$

From the assumption that $[(Q_1+Q_4)(m_1+m_2)+(Q_2+Q_3)(n_1+n_2)] < 1$, $\mathbb{M}$ is a contraction operator. Applying Banach's contraction mapping principle, a unique solution of the operator $\mathbb{M}$ exists on the interval $[0,A]$. □

Next, the Leray–Schauder alternative is used to prove an existence result [27].

**Theorem 2.** *Assume that* $\Omega \neq 0$ *and* $\Upsilon_1,\Upsilon_2 : [0,A]\times\mathbb{R}^2 \to \mathbb{R}$ *are continuous functions such that*

$$|\Upsilon_1(s,\pi,\rho)| \leq F_0+F_1|\pi|+F_2|\rho| \quad \textit{and} \quad |\Upsilon_2(s,\pi,\rho)| \leq G_0+G_1|\pi|+G_2|\rho|,$$

*for all* $\pi,\rho \in \mathbb{R}$, *where constants* $F_i,G_i \geq 0$ $(i=1,2)$ *and* $F_0>0, G_0>0$. *In addition, it is assumed that*

$$(Q_1+Q_4)F_1+(Q_2+Q_3)G_1<1 \ \textit{and}\ (Q_1+Q_4)F_2+(Q_2+Q_3)G_2<1.$$

*Then, there exists at least one solution to the coupled system of sequential $\psi_1$-Hilfer and $\psi_2$-Caputo fractional differential equations with fractional integro-differential nonlocal conditions* (2) *on* $[0, A]$.

**Proof.** In view of the continuity of functions $\Upsilon_1$ and $\Upsilon_2$, the operator $\mathbb{M}$ is continuous. Next, we show that the operator $\mathbb{M}$ is completely continuous. Let $\mathbb{K}_\zeta \subset \mathfrak{X} \times \mathfrak{X}$ be a bounded set defined by

$$\mathbb{K}_\zeta = \{(\pi, \rho) \in \mathfrak{X} \times \mathfrak{X} : \|(\pi, \rho)\| \leq \zeta\}.$$

Then, there exist $L_1, L_2 > 0$ such that

$$|\Upsilon_1(s, \pi(s), \rho(s)| \leq F_0 + (F_1 + F_2)\zeta := L_1,$$

and

$$|\Upsilon_2(s, \pi(s), \rho(s)| \leq G_0 + (G_1 + G_2)\zeta := L_2, \;\; \forall (\pi, \rho) \in \mathbb{K}_\zeta.$$

Then, for any $(\pi, \rho) \in \mathbb{K}_\zeta$, we have

$$\begin{aligned} |\mathbb{M}_1(\pi,\rho)(s)| \;\; &\leq \;\; \frac{1}{|\Omega|}\Big[|\Omega_2|\Big(|\delta_1|\tilde{\Phi}^{\alpha}_{\psi_1}(\eta_1) + |\delta_2|\Phi^{\alpha,\mu+\gamma}_{\psi_1,\psi_2}(\eta_2)\Big) + (1+|\Omega|)\Phi^{\alpha,\gamma}_{\psi_1,\psi_2}(A)\Big]L_1 \\ &\quad + \frac{1}{|\Omega|}\Big[|\lambda_1|\tilde{\Phi}^{\hat{\alpha}}_{\psi_1}(\xi_1) + |\lambda_2|\Phi^{\hat{\alpha},\hat{\mu}+\hat{\gamma}}_{\psi_1,\psi_2}(\xi_2) + |\Omega_2|\Phi^{\hat{\alpha},\hat{\gamma}}_{\psi_1,\psi_2}(A)\Big]L_2, \end{aligned}$$

which leads to

$$\|\mathbb{M}_1(\pi,\rho)\| \;\; \leq \;\; Q_1 L_1 + Q_2 L_2.$$

In the same way, we have

$$\|\mathbb{M}_2(\pi,\rho)\| \leq Q_4 L_1 + Q_3 L_2.$$

Hence,

$$\|\mathbb{M}(\pi,\rho)\| = \|\mathbb{M}_1(\pi,\rho)\| + \|\mathbb{M}_2(\pi,\rho)\| \leq (Q_1 + Q_4)L_1 + (Q_2 + Q_3)L_2,$$

which implies the uniformly bounded property of the operator $\mathbb{M}$.

For the equicontinuity of $\mathbb{M}$, we set $s_1, s_2 \in [0, A]$ such that $s_1 < s_2$. Then, by putting $(\Upsilon_1)_{\pi\rho}(s) = \Upsilon_1(s, \pi(s), \rho(s))$ and $(\Upsilon_2)_{\pi\rho}(s) = \Upsilon_2(s, \pi(s), \rho(s))$, we obtain:

$$\begin{aligned} & |\mathbb{M}_1(\pi,\rho)(s_2) - \mathbb{M}_1(\pi,\rho)(s_1)| \\ = \;\; & |I^{\gamma;\psi_2} I^{\alpha;\psi_1}(\Upsilon_1)_{\pi\rho}(s_2) - I^{\gamma;\psi_2} I^{\alpha;\psi_1}(\Upsilon_1)_{\pi\rho}(s_1)| \\ = \;\; & \Big|\frac{1}{\Gamma(\alpha+1)\Gamma(\gamma)}\int_0^{s_2}\psi_2'(u)(\psi_1(u)-\psi_1(0))^{\alpha}(\psi_2(s_2)-\psi_2(u))^{\gamma-1}(\Upsilon_1)_{\pi\rho}(u)du \\ & -\frac{1}{\Gamma(\alpha+1)\Gamma(\gamma)}\int_0^{s_1}\psi_2'(u)(\psi_1(u)-\psi_1(0))^{\alpha}(\psi_2(s_1)-\psi_2(u))^{\gamma-1}(\Upsilon_1)_{\pi\rho}(u)du\Big| \\ \leq \;\; & L_1\Big|\frac{1}{\Gamma(\alpha+1)\Gamma(\gamma)}\int_0^{s_1}\psi_2'(u)(\psi_1(u)-\psi_1(0))^{\alpha}\Big\{(\psi_2(s_2)-\psi_2(u))^{\gamma-1} \\ & -(\psi_2(s_1)-\psi_2(u))^{\gamma-1}\Big\}du \\ & +\frac{1}{\Gamma(\alpha+1)\Gamma(\gamma)}\int_{s_1}^{s_2}\psi_2'(u)(\psi_1(u)-\psi_1(0))^{\alpha}(\psi_2(s_2)-\psi_2(u))^{\gamma-1}du\Big| \end{aligned}$$

which is independent of $(\pi, \rho)$ and tends to zero as $s_2 - s_1 \to 0$. Analogously, we can obtain $|\mathbb{M}_2(\pi,\rho)(s_2) - \mathbb{M}_2(\pi,\rho)(s_1)| \to 0$ as $s_1 \to s_2$.

Consequently, the set $(\mathbb{M}\mathbb{K}_\zeta)$ is equicontinuous. By the Arzelá–Ascoli theorem, the operator $\mathbb{M}(\pi, \rho)$ is completely continuous.

This final step shows the boundedness of the set $\mathcal{E} = \{(\pi,\rho) \in \mathfrak{X} \times \mathfrak{X} : (\pi,\rho) = \lambda\mathbb{M}(\pi,\rho), 0 \le \lambda \le 1\}$. Suppose that $(\pi,\rho) \in \mathcal{E}$, then we obtain $(\pi,\rho) = \lambda\mathbb{M}(\pi,\rho)$. For any $s \in [0,A]$, we have

$$\pi(s) = \lambda\mathbb{M}_1(\pi,\rho)(s), \quad \rho(s) = \lambda\mathbb{M}_2(\pi,\rho)(s).$$

Then, we can compute that

$$\begin{aligned} |\pi(s)| \le\ & \frac{1}{|\Omega|}\Big[|\Omega_2|\Big(|\delta_1|\tilde{\Phi}^{\alpha}_{\psi_1}(\eta_1) + |\delta_2|\Phi^{\alpha,\mu+\gamma}_{\psi_1,\psi_2}(\eta_2)\Big) + (1+|\Omega|)\Phi^{\alpha,\gamma}_{\psi_1,\psi_2}(A)\Big] \\ & \times(F_0 + F_1|\pi| + F_2|\rho|) \\ & + \frac{1}{|\Omega|}\Big[|\lambda_1|\tilde{\Phi}^{\hat{\alpha}}_{\psi_1}(\xi_1) + |\lambda_2|\Phi^{\hat{\alpha},\hat{\mu}+\hat{\gamma}}_{\psi_1,\psi_2}(\xi_2) + |\Omega_2|\Phi^{\hat{\alpha},\hat{\gamma}}_{\psi_1,\psi_2}(s)\Big](G_0 + G_1|\pi| + G_2|\rho|), \end{aligned}$$

and

$$\begin{aligned} |\rho(s)| \le\ & \frac{1}{|\Omega|}\Big[|\Omega_1|\Big(|\lambda_1|\tilde{\Phi}^{\hat{\alpha}}_{\psi_1}(\xi_1) + |\lambda_2|\Phi^{\hat{\alpha},\hat{\mu}+\hat{\gamma}}_{\psi_1,\psi_2}(\xi_2)\Big) + (1+|\Omega|)\Phi^{\hat{\alpha},\hat{\gamma}}_{\psi_1,\psi_2}(A)\Big] \\ & \times(G_0 + G_1|\pi| + G_2|\rho|) \\ & + \frac{1}{|\Omega|}\Big[|\delta_1|\tilde{\Phi}^{\alpha}_{\psi_1}(\eta_1) + |\delta_2|\Phi^{\alpha,\mu+\gamma}_{\psi_1,\psi_2}(\eta_2) + |\Omega_1|\Phi^{\alpha,\gamma}_{\psi_1,\psi_2}(s)\Big](F_0 + F_1|\pi| + F_2|\rho|). \end{aligned}$$

Therefore, we obtain:

$$\|\pi\| \le Q_1(F_0 + F_1\|\pi\| + F_2\|\rho\|) + Q_2(G_0 + G_1\|\pi\| + G_2\|\rho\|)$$

and

$$\|\rho\| \le Q_3(G_0 + G_1\|\pi\| + G_2\|\rho\|) + Q_4(F_0 + F_1\|\pi\| + F_2\|\rho\|),$$

which yield

$$\begin{aligned} \|\pi\| + \|\rho\| \le\ & (Q_1+Q_4)F_0 + (Q_2+Q_3)G_0 + [(Q_1+Q_4)F_1 + (Q_2+Q_3)G_1]\|\pi\| \\ & + [(Q_1+Q_4)F_2 + (Q_2+Q_3)G_2]\|\rho\|. \end{aligned}$$

Then, we have

$$\begin{aligned} M_0(\|\pi\| + \|\rho\|) \le\ & (1 - [(Q_1+Q_4)F_1 + (Q_2+Q_3)G_1])\|\pi\| \\ & + (1 - [(Q_1+Q_4)F_2 + (Q_2+Q_3)G_2])\|\rho\| \\ \le\ & (Q_1+Q_4)F_0 + (Q_2+Q_3)G_0, \end{aligned}$$

which implies that

$$\|(\pi,\rho)\| \le \frac{(Q_1+Q_4)F_0 + (Q_2+Q_3)G_0}{M_0},$$

where $M_0$ is defined as

$$M_0 = \min\{1 - [(Q_1+Q_4)F_1 + (Q_2+Q_3)G_1], 1 - [(Q_1+Q_4)F_2 + (Q_2+Q_3)G_2]\},$$

which shows that $\mathcal{E}$ is bounded. By the Leray–Schauder alternative, we deduce that the operator $\mathbb{M}$ has at least one fixed point, which is a solution of the system (2) on $[0,A]$. The proof is finished. □

**4. Uncoupled Fractional Integro-Differential Boundary Conditions**

In this section, we consider the following system of sequential $\psi_1$-Hilfer and $\psi_2$-Caputo fractional differential equations with uncoupled fractional integro-differential nonlocal conditions:

$$\begin{cases} {}^H D^{\alpha,\beta;\psi_1}({}^C D^{\gamma;\psi_2}\pi)(s) = \Upsilon_1(s,\pi(s),\rho(s)), \quad s \in [0,A], \\ {}^H D^{\hat{\alpha},\hat{\beta};\psi_1}({}^C D^{\hat{\gamma};\psi_2}\rho)(s) = \Upsilon_2(s,\pi(s),\rho(s)), \quad s \in [0,A], \\ {}^C D^{\gamma;\psi_2}\pi(0) = 0, \quad \pi(s) = \lambda_1\, {}^C D^{\gamma;\psi_2}\pi(\eta_1) + \lambda_2\, I^{\mu;\psi_2}\pi(\eta_2), \\ {}^C D^{\hat{\gamma};\psi_2}\rho(0) = 0, \quad \rho(s) = \delta_1\, {}^C D^{\hat{\gamma};\psi_2}\rho(\xi_1) + \delta_2\, I^{\hat{\mu};\psi_2}\rho(\xi_2), \end{cases} \tag{13}$$

where all constants and notations are as in the problem (2). The following lemma is not difficult to derive and, therefore, we omit the proof.

**Lemma 5.** *For $h \in C([0,A],\mathbb{R})$ and $\Lambda_1 \neq 0$, the unique solution of the problem*

$$\begin{cases} {}^H D^{\alpha,\beta;\psi_1}({}^C D^{\gamma;\psi_2}\pi)(s) = h(s), \quad s \in [0,A], \\ {}^C D^{\gamma;\psi_2}\pi(0) = 0, \quad \pi(s) = \lambda_1\, {}^C D^{\gamma;\psi_2}\pi(\eta_1) + \lambda_2\, I^{\mu;\psi_2}\pi(\eta_2), \end{cases} \tag{14}$$

*is given by*

$$\begin{aligned} \pi(s) &= \frac{1}{\Lambda_1}\big(\lambda_1 I^{\alpha;\psi_1}h(\eta_1) - I^{\gamma;\psi_2}I^{\alpha;\psi_1}h(A) + \lambda_2 I^{\gamma+\mu;\psi_2}I^{\alpha;\psi_1}h(\eta_2)\big) \\ &\quad + I^{\gamma;\psi_2}I^{\alpha;\psi_1}h(s), \end{aligned} \tag{15}$$

*where*

$$\Lambda_1 = 1 - \lambda_2\frac{(\psi_2(\eta_2)-\psi_2(0))^{\mu}}{\Gamma(\mu+1)}.$$

From the above Lemma, we can define operator $\mathbb{P} : \mathfrak{X}\times\mathfrak{X} \to \mathfrak{X}\times\mathfrak{X}$ by

$$\mathbb{P}(\pi,\rho)(s) = \begin{pmatrix} \mathbb{P}_1(\pi,\rho)(s) \\ \mathbb{P}_2(\pi,\rho)(s) \end{pmatrix},$$

to prove the existence criteria to the system of uncoupled boundary conditions in (13), where

$$\begin{aligned} \mathbb{P}_1(\pi,\rho)(s) &= \frac{1}{\Lambda_1}\big\{\lambda_1 I^{\alpha;\psi_1}\Upsilon_1(\eta_1,\pi(\eta_1),\rho(\eta_1)) - I^{\gamma;\psi_2}I^{\alpha;\psi_1}\Upsilon_1(A,\pi(A),\rho(A)) \\ &\quad + \lambda_2 I^{\gamma+\mu;\psi_2}I^{\alpha;\psi_1}\Upsilon_1(\eta_2,\pi(\eta_2),\rho(\eta_2))\big\} + I^{\gamma;\psi_2}I^{\alpha;\psi_1}\Upsilon_1(s,\pi(s),\rho(s)), \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}_2(\pi,\rho)(s) &= \frac{1}{\Lambda_2}\big\{\delta_1 I^{\hat{\alpha};\psi_1}\Upsilon_2(\xi_1,\pi(\xi_1),\rho(\xi_1)) - I^{\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\Upsilon_2(A,\pi(A),\rho(A)) \\ &\quad + \delta_2 I^{\hat{\gamma}+\hat{\mu};\psi_2}I^{\hat{\alpha};\psi_1}\Upsilon_2(\xi_2,\pi(\xi_2),\rho(\xi_2))\big\} + I^{\hat{\gamma};\psi_2}I^{\hat{\alpha};\psi_1}\Upsilon_2(s,\pi(s),\rho(s)). \end{aligned}$$

The following existence theorems can be presented without proof by using the Banach contraction principle and also the Leray–Schauder alternative technique. In addition, we have to give some constants as

$$\begin{aligned} Q_5 &= \frac{1}{|\Lambda_1|}\Big[|\lambda_1|\tilde{\Phi}^{\alpha}_{\psi_1}(\eta_1) + (1+|\Lambda_1|)\Phi^{\alpha,\gamma}_{\psi_1,\psi_2}(A) + |\lambda_2|\Phi^{\alpha,\mu+\gamma}_{\psi_1,\psi_2}(\eta_2)\Big], \\ Q_6 &= \frac{1}{|\Lambda_2|}\Big[|\delta_1|\tilde{\Phi}^{\hat{\alpha}}_{\psi_1}(\xi_1) + (1+|\Lambda_1|)\Phi^{\hat{\alpha},\hat{\gamma}}_{\psi_1,\psi_2}(A) + |\delta_2|\Phi^{\hat{\alpha},\hat{\mu}+\hat{\gamma}}_{\psi_1,\psi_2}(\xi_2)\Big], \end{aligned}$$

and

$$\Lambda_2 = 1 - \delta_2 \frac{(\psi_2(\xi_2) - \psi_2(0))^{\hat{\mu}}}{\gamma(\hat{\mu}+1)} \neq 0.$$

**Theorem 3.** *Let $f, g$ be two functions satisfy the Lipschitz conditions in Theorem 1. If $(m_1 + m_2)Q_5 + (n_1 + n_2)Q_6 < 1$, then problem (13) has a unique solution on the interval $[0, A]$.*

**Theorem 4.** *Suppose that the continuous functions $f, g$ satisfy the growth conditions as in Theorem 2. If $Q_5F_1 + Q_6G_1 < 1$ and $Q_5F_2 + Q_6G_2 < 1$, then the problem of fractional integro-differential nonlocal conditions (13) has at least one solution on $[0, A]$.*

## 5. Illustrative Examples

**Example 1.** *Let us consider the following coupled system of sequential $\psi_1$-Hilfer and $\psi_2$-Caputo fractional differential equations with fractional integro-differential nonlocal conditions of the form:*

$$\begin{cases} {}^{H}D^{\frac{1}{8},\frac{5}{8};e^{s/12}}({}^{C}D^{\frac{3}{4};s^2+t}\pi)(s) = \Upsilon_1(s, \pi(s), \rho(s)), & s \in [0, 3/2], \\ {}^{H}D^{\frac{7}{8},\frac{3}{8};e^{s/12}}({}^{C}D^{\frac{1}{4};s^2+t}\rho)(s) = \Upsilon_2(s, \pi(s), \rho(s)), & s \in [0, 3/2], \end{cases} \tag{16}$$

*subject to*

$$\begin{cases} {}^{C}D^{\frac{3}{4};s^2+s}\pi(0) = 0, \quad \pi\left(\frac{3}{2}\right) = \frac{2}{55}\,{}^{C}D^{\frac{1}{4};s^2+s}\rho\left(\frac{1}{4}\right) + \frac{4}{77}\,I^{\frac{3}{2};s^2+s}\rho\left(\frac{5}{4}\right), \\ {}^{C}D^{\frac{1}{4};s^2+s}\rho(0) = 0, \quad \rho\left(\frac{3}{2}\right) = \frac{3}{88}\,{}^{C}D^{\frac{3}{4};s^2+s}\pi\left(\frac{1}{2}\right) + \frac{5}{99}\,I^{\frac{11}{8};s^2+s}\pi\left(\frac{3}{4}\right). \end{cases} \tag{17}$$

From the above problem: $\alpha = 1/8$, $\hat{\alpha} = 7/8$, $\beta = 5/8$, $\hat{\beta} = 3/8$, $\gamma = 3/4$, $\hat{\gamma} = 1/4$, $A = 3/2$, $\lambda_1 = 2/55$, $\lambda_2 = 4/77$, $\delta_1 = 3/88$, $\delta_2 = 5/99$, $\xi_1 = 1/4$, $\xi_2 = 5/4$, $\eta_1 = 1/2$, $\eta_2 = 3/4$, $\mu = 11/8$, $\hat{\mu} = 3/2$ and functions $\psi_1(s) = e^{(s/12)}$ and $\psi_2(s) = s^2 + s$. This information leads to constants as $\Omega_1 \approx 0.0600563771$, $\Omega_2 \approx 0.1843197460$, $\Omega \approx 0.9889304238$, $Q_1 \approx 1.276579172$, $Q_2 \approx 0.2900508368$, $Q_3 \approx 2.069536146$ and $Q_4 \approx 0.1538946945$.

$(i)$ Let the functions $\Upsilon_1$ and $\Upsilon_2$ are given on $[0, 3/2]$ as

$$\begin{cases} \Upsilon_1(s, \pi, \rho) = \dfrac{1}{2(s+7)}\left(\dfrac{\pi^2 + 2|\pi|}{1+|\pi|}\right) + \dfrac{1}{3s+8}\sin|\rho| + \dfrac{1}{4}s^2 + 2s + 3, \\ \Upsilon_2(s, \pi, \rho) = \dfrac{1}{s+9}\tan^{-1}|\pi| + \dfrac{1}{3(s+10)}\left(\dfrac{3|\rho| + \rho^2}{1+|\rho|}\right) + \sqrt{s^2+1}. \end{cases} \tag{18}$$

Then, we have

$$|\Upsilon_1(s, \pi_1, \rho_1) - \Upsilon_1(s, \pi_2, \rho_2)| \leq \frac{1}{7}|\pi_1 - \pi_2| + \frac{1}{8}|\rho_1 - \rho_2|,$$

and

$$|\Upsilon_2(s, \pi_1, \rho_1) - \Upsilon_2(s, \pi_2, \rho_2)| \leq \frac{1}{9}|\pi_1 - \pi_2| + \frac{1}{10}|\rho_1 - \rho_2|,$$

$t \in [0, 3/2]$, $(\pi_i, \rho_i) \in \mathbb{R}^2$, $i = 1, 2$ and, hence, $\Upsilon_1$ and $\Upsilon_2$ satisfy the Lipschitz condition with Lipschitz constants $m_1 = 1/7$, $m_2 = 1/8$, $n_1 = 1/9$, and $n_2 = 1/10$. The last condition in Theorem 1 is fulfilled since $(Q_1 + Q_4)(m_1 + m_2) + (Q_2 + Q_3)(n_1 + n_2) \approx 0.8812976724 < 1$. Therefore, the nonlinear coupled system of sequential $\psi_1$-Hilfer and $\psi_2$-Caputo fractional differential equations with fractional integro-differential nonlocal conditions (16) and (17) with $\Upsilon_1$ and $\Upsilon_2$ given by (18) has a unique solution $(\pi, \rho)$ on $[0, 3/2]$.

$(ii)$ Now, we consider the functions $\Upsilon_1$ and $\Upsilon_2$ defined on $[0, 3/2]$, as

$$\begin{cases} \Upsilon_1(s, \pi, \rho) = \dfrac{2}{s+4} + \dfrac{\pi^{130} e^{-\rho^2}}{(s+3)(1+|\pi|^{129})} + \dfrac{|\rho^5| \cos^2 \pi^4}{(s+5)(1+\rho^4)}, \\ \Upsilon_2(s, \pi, \rho) = \dfrac{1}{6}s + \dfrac{\pi^8 \sin^4 \rho^6}{(s^2+5)(1+|\pi|^7)} + \dfrac{|\rho|^{2023} \tan^{-1} \pi}{2\pi(1+y^{2022})}. \end{cases} \tag{19}$$

Observe that the above two nonlinear functions in (19) are non-Lipschitzian, but we can find the bounded planes as follows:

$$|\Upsilon_1(s, \pi, \rho)| \le \frac{1}{2} + \frac{1}{3}|\pi| + \frac{1}{5}|\rho| \quad \text{and} \quad |\Upsilon_2(s, \pi, \rho)| \le \frac{1}{4} + \frac{1}{5}|\pi| + \frac{1}{4}|\rho|.$$

Hence, we choose the constants $F_0 = 1/2$, $F_1 = 1/3$, $F_2 = 1/5$, $G_0 = 1/4$, $G_1 = 1/5$, and $G_2 = 1/4$. Then, we obtain two inequalities $(Q_1 + Q_4)F_1 + (Q_2 + Q_3)G_1 \approx 0.9487420186 < 1$ and $(Q_1 + Q_4)F_2 + (Q_2 + Q_3)G_2 \approx 0.8759915190 < 1$. Thus, all conditions of Theorem 2 are satisfied. So, the coupled system (16) and (17), with $\Upsilon_1$ and $\Upsilon_2$ given by (19) has at least one solution $(\pi, \rho)$ on $[0, 3/2]$.

**Example 2.** *Assume that the sequential $\psi_1$-Hilfer and $\psi_2$-Caputo fractional differential Equation (16) subject to the following uncoupled fractional integro-differential boundary conditions:*

$$\begin{cases} {}^C D^{\frac{3}{4}; s^2+s} \pi(0) = 0, \quad \pi\left(\dfrac{3}{2}\right) = \dfrac{2}{55}\, {}^C D^{\frac{3}{4}; s^2+s} \pi\left(\dfrac{1}{2}\right) + \dfrac{4}{77} I^{\frac{11}{8}; s^2+s} \pi\left(\dfrac{3}{4}\right), \\ {}^C D^{\frac{1}{4}; s^2+s} \rho(0) = 0, \quad \rho\left(\dfrac{3}{2}\right) = \dfrac{3}{88}\, {}^C D^{\frac{1}{4}; s^2+s} y\left(\dfrac{1}{4}\right) + \dfrac{5}{99} I^{\frac{3}{2}; s^2+s} \rho\left(\dfrac{5}{4}\right). \end{cases} \tag{20}$$

Then, we can find the constants $\Lambda_1 \approx 0.9382277264$, $\Lambda_2 \approx 0.8208002471$, $Q_5 \approx 2.159965388$, and $Q_6 \approx 2.321388442$.

$(I)$ If two nonlinear functions are presented on $[0, 3/2]$ by

$$\begin{cases} \Upsilon_1(s, \pi, \rho) = \dfrac{|\pi|}{(s+8)(1+|\pi|)} + \dfrac{1}{\sqrt{s}+11} \sin|\rho| + \dfrac{1}{3}s + 1, \\ \Upsilon_2(s, \pi, \rho) = \dfrac{\pi^2 + 2|\pi|}{6(s^2+3)(1+|\pi|)} + \dfrac{1}{s+10} \tan^{-1}|\rho| + s^2 + \dfrac{1}{5}, \end{cases} \tag{21}$$

then it is obvious by direct computation that $\Upsilon_1$ and $\Upsilon_2$ satisfy the Lipschitz condition with Lipschitz constants $m_1 = 1/8$, $m_2 = 1/11$, $n_1 = 1/9$, and $n_2 = 1/10$. Then, the relation $(m_1 + m_2)Q_5 + (n_1 + n_2)Q_6 \approx 0.9564270566 < 1$ holds. By Theorem 3, the sequential $\psi_1$-Hilfer and $\psi_2$-Caputo fractional differential Equation (16), subject to uncoupled fractional integro-differential boundary conditions (16)–(20) with $\Upsilon_1$ and $\Upsilon_2$ given by (21), has a unique solution $(\pi(s), \rho(s))$, $s \in [0, 3/2]$.

$(II)$ Let $f$ and $g$ be two nonlinear functions defined by

$$\begin{cases} \Upsilon_1(s, \pi, \rho) = \dfrac{1}{2}s^2 + \dfrac{(1+|\rho|)\pi}{(s+2)^2(2+|\rho|)} + \dfrac{1}{s+5}\left(\dfrac{\rho^4 e^{-\pi^2}}{1+|\rho|^3}\right), \\ \Upsilon_2(s, \pi, \rho) = \dfrac{1}{3}s + \dfrac{1}{2} + \dfrac{\pi e^{-|\rho|}}{2(s+3)} + \dfrac{1}{s+7}\left(\dfrac{2^{-|\pi|}|\rho|^5}{1+\rho^6}\right). \end{cases} \tag{22}$$

It is easy to see that the above two functions are bounded, for $s \in [0, 3/2]$, by

$$|\Upsilon_1(s, \pi, \rho)| \le \frac{9}{8} + \frac{1}{4}|\pi| + \frac{1}{5}|\rho| \quad \text{and} \quad |\Upsilon_2(s, \pi, \rho)| \le 1 + \frac{1}{6}|\pi| + \frac{1}{7}|\rho|.$$

Setting constants $F_0 = 9/8$, $F_1 = 1/4$, $F_2 = 1/5$, $G_0 = 1$, $G_1 = 1/6$, and $G_2 = 1/7$ leads to the relations $Q_5F_1 + Q_6G_1 \approx 0.9268894207 < 1$ and $Q_5F_2 + Q_6G_2 \approx 0.7636199979 < 1$.

By Theorem 4, the uncoupled system (16)–(20), with $\Upsilon_1$ and $\Upsilon_2$ given by (22), has at least one solution $(\pi, \rho)$ on the interval $[0, 3/2]$.

## 6. Conclusions

In the present work, we presented the criteria concerning the existence and uniqueness of solutions for a coupled system of mixed-type $\psi_1$-Hilfer and $\psi_2$-Caputo fractional differential equations subjected to integro-differential nonlocal boundary conditions. After transforming the given nonlinear problem into an equivalent fixed point problem, we applied the Banach contraction mapping principle to establish the existence of a unique solution, while an existence result is proved via the Leray–Schaude alternative. Numerical examples are also constructed for illustrating the obtained results. The results obtained here are new and initiate the study of mixed nonlocal systems of $\psi_1$-Hilfer and $\psi_2$-Caputo fractional differential equations. Hence, our results enrich the existing literature with this new research area of nonlocal fractional coupled systems. In addition, our results yield several new results as special cases by fixing the parameters involved in the problems appropriately. For example, our results correspond to the ones with: (i) coupled system of Hilfer and Caputo fractional differential equations supplemented with integro-differential boundary conditions if $\psi_1(s) = \psi_2(s) = s$; (ii) coupled system of Hilfer and $\psi_2$-Caputo fractional differential equations supplemented with integro-differential boundary conditions if $\psi_1(s) = s$; (iii) coupled system of $\psi_1$-Hilfer and Caputo fractional differential equations supplemented with integro-differential boundary conditions if $\psi_2(s) = s$.

For future work, we plan to study boundary value problems and coupled systems of mixed-type $\psi_1$-Hilfer and $\psi_2$-Caputo fractional differential equations subject to new kinds of boundary conditions.

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