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Triple Sampling Inference Procedures for the Mean of the Normal Distribution When the Population Coefficient of Variation Is Known

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Abstract: This paper discusses the triple sampling inference procedures for the mean of a symmetric distribution—the normal distribution when the coefficient of variation is known. We use the Searls' estimator as an initial estimate for the unknown population mean rather than the classical sample mean. In statistics literature, the normal distribution under investigation underlines almost all the natural phenomena with applications in many fields. First, we discuss the minimum risk point estimation problem under a squared error loss function with linear sampling cost. We obtained all asymptotic results that enhanced finding the second-order asymptotic risk and regret. Second, we construct a fixed-width confidence interval for the mean that satisfies at least a predetermined nominal value and find the second-order asymptotic coverage probability. Both estimation problems are performed under a unified optimal framework. The theoretical results reveal that the performance of the triple sampling procedure depends on the numerical value of the coefficient of variation—the smaller the coefficient of variation, the better the performance of the procedure.

Keywords: confidence interval; minimum risk point estimation; Searls' estimator; triple sampling procedure

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1. Introduction

Let $X_1,\ X_2,\ldots$ be a sequence of independent and identically distributed random variables from a symmetric distribution–normal distribution $N\left(\mu, \mu^2\eta^2\right)$, $(\mu \neq 0)$ where $\sqrt{\eta^2}$ is a known coefficient of variation. In usual cases, the normal distribution $N\left(\mu, \sigma^2\right)$ where σ^2 does not depend on μ, the sample mean $\overline{X}_n = n^{-1}\sum_{i=1}^{n} X_i$, $n \geq 1$ is known to be the uniformly minimum variance UMV unbiased estimator of μ, but in the present case of a known coefficient of variation, the sample mean no longer achieves this property. Searls [1] suggested an improved estimator for μ in the form $\hat{\mu}_n = n\left(n+\eta^2\right)^{-1}\overline{X}_n$, $n \geq 1$ and proved that the mean squared error of the Searls' estimator $MSE(\hat{\mu}_n) = \left(n+\eta^2\right)^{-1}\sigma^2$ is smaller than $MSE\left(\overline{X}_n\right)$ with the relative efficiency of $\hat{\mu}_n$ to $\overline{X}_n$ is $\left(\eta^2/n\right)+1$; see Sen [2].

Several authors have intensively studied the point estimation of μ. For example, Arnholt and Hebert [3] considered a wider class of estimators for μ when η is known and showed that Searls' estimator still has minimum mean squared errors among other estimators.

Sinha [4] discussed the Bayesian estimation of the mean of the symmetric distribution–normal distribution when the coefficient of variation is known, while Gleser and Healy [5] considered a class of Bay's estimators against inverted Gamma priors; see also Guo and Pal [6], Anis [7], Srisodaphol and Tongmol [8], and Hinkley [9]. Recently, fuzzy relational inference systems for estimation have been shown in [10,11].

The assumption of a known coefficient of variation is involved in many biological, physical, and engineering applications. For example, for applications in agricultural studies, see Bhat and Rao [12]; for applications in biological and medical experiments, see Brazauskas and Ghorai [13]; see also Hald [14], and Davis and Goldsmith [15].

Despite the extensive literature on point estimation for μ, few literary works are available for interval estimation. Niwitpong [16] proposed two confidence intervals for the mean of the symmetric distribution–normal distribution based on Searls' work. Fu, Wang, and Wong [17] extended Bhat and Rao's [12] approach and proposed the modified signed log-likelihood ratio test for the normal mean.

From a theoretical point of view, the standard inferential methods cannot be used directly to find the inference of the normal mean since the family $N(\mu, \eta^2\mu^2)$, $(\mu \neq 0)$ belongs to the curved exponential family model with a two-dimensional minimal sufficient statistic $(\overline{X}_n, S_n^2)$ where $S_n^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 n \geq 2$; see Efron [18].

Sequential Sampling Procedures

Sequential sampling procedures were mainly developed for statistical inference during and after World War II. Stein [19,20] and Cox [21] presented the two-stage procedure for constructing a fixed-width confidence interval for the population normal mean when the population variance is finite and unknown. It was shown from the literature that the two-stage sampling procedure attains exact consistency and asymptotic consistency but has asymptotic inefficiency (oversampling), especially when the pilot sample size is much smaller than the optimal sample size. To overcome such deficiency, Anscombe [22], Ray [23], and Chow and Robbins [24] proposed a purely sequential procedure. The procedure attains asymptotic consistency and efficiency but lacks exact consistency and time consumption. See Mukhopadhyay and de Silva [25].

As a compromising procedure, Hall [26] introduced the triple sampling procedure to achieve two primary objectives, the operational savings made possible by sampling in batches as in a two-stage procedure and the asymptotic efficiency attained by purely sequential sampling. The procedure is based on three stages, as we describe later. The procedure combines the efficiency of Anscombe, Chow, and Robbins' one-by-one purely sequential procedure and the operational saving made possible by sampling in bulk using Stein's group sampling techniques. It is an excellent trade-off between a purely sequential procedure and a two-stage procedure with ease of implementation. The triple sampling procedure was mainly developed to construct a fixed-width confidence interval for the normal mean that satisfies a predetermined width and coverage probability when the population variance is unknown. The procedure attains all customary measures except exact consistency. The following lines describe the above measures as follows:

If N is the final random sample size generated by a multistage sequential procedure and n^* is the optimal sample size needed to estimate the parameter μ, then the procedure is said to be:

(i) first-order asymptotically efficient if $\lim_{n^* \to \infty} E(N/n^*) = 1$ and

(ii) second-order asymptotically efficient if $\lim_{n^* \to \infty} E(N - n^*) < \infty$; see Ghosh and Mukhopadhyay [27].

Moreover, if I_N is the fixed-width confidence interval constructed via a multistage sampling procedure, then the procedure is called (i) consistent or exactly consistent if $P(\mu \in I_N) \geq 1 - \alpha$ while it is asymptotically (first-order) consistent if $\lim_{n^* \to \infty} P(\mu \in I_N) \to 1 - \alpha$; α is the desired nominal value in the sense of Stein [19], Mukhopadhyay [28], and Chow and Robbins [24], respectively. Moreover, if R_N is the multistage sampling risk encountered in estimating the mean μ by the corresponding sample measure, and if R_{n^*} is the optimal fixed-sample-size risk had σ been known, then the procedure is (i) first-order asymptotically risk efficient if $\lim_{n^* \to \infty} R_N/R_{n^*} = 1$ and (ii) second-order asymptotic regret if $\lim_{n^* \to \infty} (R_N - R_{n^*})$ remains bounded in the sense of Ghosh and Mukhopadhyay [27]. For more details, see Mukhopadhyay and de Silva [25], Ghosh, Mukhopadhyay, and Sen [29].

Mukhopadhyay [30] further developed a unified framework for the triple sampling procedure by focusing on higher-order moments of the final stopping variable N. Mukhopadhyay et al. [31] discussed the triple sampling sequential estimation for the normal mean. Hamdy [32] extended Hall's results and proposed a triple sampling procedure to tackle the normal mean minimum risk point estimation problem and fixed-width confidence interval estimation problem. Meanwhile, Liu [33] extended Hall's results to tackle hypothesis-testing problems for the normal mean. Yousef [34] discussed the sensitivity of the normal-based triple sampling sequential point estimation to the normality assumption, considering a class of an absolutely continuous distribution whose absolute first six moments are assumed finite but unknown. After that, he generalized the study to find the second-order asymptotic coverage probability and the second-order characteristic operating function for the mean. He studied the capability of the constructed confidence interval to detect possible shifts in the actual population mean occurring outside the confidence boundaries; see Yousef [35]. Son et al. [36] proposed the triple sampling procedure that tackled a fixed-width confidence interval and a hypothesis testing for the normal mean while controlling Type II error probability. Yousef [37] discussed the performance of the triple sampling procedure to a broader class of underlying continuous distributions applying the second-order Edgeworth series. Both Son et al. [36] and Yousef [35,38] provided second-order approximations of the characteristic operating function of the inference. Yousef [39,40] tackled estimation of the normal inverse coefficient of variation using Monte Carlo simulation. For other underlying distributions, see Yousef et al. [41,42]. For triple sampling minimum risk point estimation for a function of a normal mean under weighted power absolute error loss plus cost, see Banerjee and Mukhopadhyay [43].

Chaturvedi and Tomer [44] discussed the minimum risk and bounded risk point estimation problem for the normal mean when the coefficient of variation is known. They used two sequential procedures: the triple sampling procedure of Hall [26] and the accelerated sequential scheme Hall [45], using the Searls [1] estimator as an estimate for the normal mean. Although we consider the same problem addressed by Chaturvedi and Tomer [44], our approach differs in several ways. First, we combine point and confidence interval estimation in a unified optimal decision framework. This technique utilizes all the available data to construct quality control charts. The point estimation is for determining the center line of the quality control chart (quality mean), and the confidence interval estimation is for establishing the upper and the lower quality limits with a predetermined required specification $(2d)$, $d(>0)$. Second, our theorems and proofs are provided as second-ordered approximations. Third and last, we provide more details regarding the asymptotic distribution characteristics of the final stopping time N, the estimate of the parameter μ, and its higher-ordered moments.

2. Problem Setting

Assume a sample of size n, say, $(x_1, x_2, \ldots, x_n)$, is available from the normal distribution with mean μ and variance $\sigma^2 = \eta^2\mu^2$, $\mu \neq 0$. We use $\hat{\mu}_n = n\left(n+\eta^2\right)^{-1}\overline{X}_n$, $n \geq 1$ as an initial estimate for the population mean μ. The aim is to discuss the minimum risk point estimation problem for the normal population mean and construct a confidence interval for the mean with a predetermined width and coverage probability. It has been shown by Dantziq [46] that there is no fixed sample size n that can solve the problem except sequentially. Therefore, we use the triple sampling procedure of Hall [26] to solve this problem in the presence of a known coefficient of variation.

3. Estimation of the Population Mean

3.1. Minimum Risk Point Estimation

Let $L_n(A)$ be the loss function incurred by estimating the population mean μ by Searls' [1] estimator $\hat{\mu}_n$ for all $n \geq 1$. That is,

$$L_n(A) = A|\hat{\mu}_n - \mu|^2 + cn, \tag{1}$$

where c is the cost per unit sample and assumed to be known to the experimenter, cn is the cost of sampling, while the constant $A(>0)$ will be described after subsequent lines.

The risk associated with (1) is defined by

$$R_n(A) = AE|\hat{\mu}_n - \mu|^2 + cn.$$

However,

$$E(\hat{\mu}_n - \mu)^2 = \left(n + \eta^2\right)^{-2} n^2 E\left(\overline{X}_n - \mu\right)^2 + \left(n + \eta^2\right)^{-2} \mu^2 \eta^4,$$

substituting $\sigma = \eta\mu$, follows $E(\hat{\mu}_n - \mu)^2 = \sigma^2 \left(n + \eta^2\right)^{-1}$.

Hence,

$$R_n(A) = A\sigma^2 \left(n + \eta^2\right)^{-1} + cn. \tag{2}$$

By treating n as a continuous random variable, the minimum value for n is

$$n \geq \sqrt{A/c}\sigma - \eta^2 = \lambda\sigma - \eta^2 = n^*, \text{ (Say)} \tag{3}$$

where $\lambda = \sqrt{A/c}$. As $c \to 0$, $\lambda \to \infty$. Since σ is unknown, then n^* is unknown. It was shown by Dantzig [46], Stein [19,20], and Seelbinder [47] that no fixed sample size procedure exists that minimizes (2) uniformly over σ. Therefore, we propose the triple sampling procedure of Hall [26] to estimate n^* through estimation of σ.

The optimal risk, had σ been known, is

$$R_{n^*}(A) = 2cn^* + c\eta^2. \tag{4}$$

To obtain further insight into the nature of A, write (3) as $A = c\sigma^{-2}\left(n^* + \eta^2\right)^2$ from which we obtain the following representation of A:

$$A = c\left(n^* + \eta^2\right) I\left(n^*, \sigma^2\right). \tag{5}$$

From (5), A is partially known (knowable) since it depends on the unknown n^*. If we assume that A is known as mentioned in the literature on sequential estimation (see Chaturvedi and Tomer [44], Hamdy [32], and Mukhopadhyay et al. [31]), then this will impose restrictions on the parameter space of the population mean μ, as it can be seen from (5) $A \propto \frac{n^* + \eta^2}{\sigma^2}$. Now $c\left(n^* + \eta^2\right)$ is the cost of optimal sampling, and $I(n^*, \theta) = \sigma^{-2}\left(n^* + \eta^2\right)$ is the optimal Fisher information. Hence, we can define A as the cost of optimal sampling information.

3.2. Fixed-Width Confidence Interval Estimation

Assume we need to establish a fixed-width confidence interval for the mean of the normal distribution with a prescribed width of $2d$, $d(>0)$, and coverage probability of at least $(1 - \alpha)$, $0 < \alpha < 1$. That is, we need to find a solution to the inequality

$$P(|\hat{\mu}_n - \mu| \leq d) \geq 1 - \alpha. \tag{6}$$

Since $\frac{n}{n+\eta^2} \to 1$ as $n \to \infty$, in probability, then it follows from Slutsky's Theorem as $n \to \infty$, $\hat{\mu}_n = \frac{n}{n+\eta^2}\overline{X}_n \to N\left(\mu, \frac{\sigma^2}{n}\right)$ in distribution. This leads to

$$P\left(\frac{\sqrt{n}}{\sigma} \mid \hat{\mu}_n - \mu \mid \leq d\frac{\sqrt{n}}{\sigma}\right) \geq 1 - \alpha = 2\Phi(a) - 1 \Rightarrow 2\Phi\left(d\frac{\sqrt{n}}{\sigma}\right) - 1 \geq 2\Phi(a) - 1,$$

where $\Phi(u) = \int_{-\infty}^{u}\left(\sqrt{2\pi}\right)^{-1} e^{-y^2/2}dy$ and $a = \Phi^{-1}(1 - \alpha/2)$.

It follows immediately that

$$n \geq (a/d)^2\, \sigma^2 = n^0. \text{ (say)} \tag{7}$$

If σ is known, then n^0 is the optimal fixed-sample size required to solve (6) uniformly over $\sigma > 0$. Consequently, the desired fixed-width confidence interval for μ is $I_n = (\hat{\mu}_{n^0} - d,\ \hat{\mu}_{n^0} + d\ )$. Similarly, as in the previous section, we propose the triple sampling procedure of Hall [26] to estimate n^0 through estimation of σ^2.

4. Triple Sampling Procedure and Asymptotic Results

The following lines describe the triple sampling procedure based on (3).

Stage 1. Fix m, η, and the design factor δ, $0 < \delta < 1$ and generate a pilot sample of size m (≥ 2) from the normal distribution and compute $\hat{\mu}_m = m\left(m + \eta^2\right)^{-1}\overline{X}_m$ and S_m^2 as initial estimates of μ and σ^2, respectively.

Stage 2. Let $S^* = \left[\delta\left(\lambda S_m - \eta^2\right)\right] + 1$, where $[x]$ is the largest integer less than x. Calculate

$$N_1 = max\{m, S^*\ \}. \tag{8}$$

If $m \geq S^*$, then stop sampling; consequently, the experiment terminates. Otherwise, sample extra observations $(S^* - m\ )$ and augment them with the previous observations. The resultant sample is of size N_1^*.

Stage 3. Let $T^* = \left[\left(\lambda S_{N_1^*} - \eta^2\right)\right] + 1$. Calculate

$$N = max\{N_1,\ T^*\}. \tag{9}$$

If $N_1 \geq T^*$, then no further observations are needed; otherwise, sample an extra observation $(T^* - N_1)$ and augment them with the previous sample. As a result, we propose $\hat{\mu}_N = N\left(N + \eta^2\right)^{-1}\overline{X}_N$, and $\hat{\sigma} = S_N$ are, respectively, the sequential point estimates for μ and σ.

To proceed further, the following assumption is necessary to setup all the upcoming theorems. This assumption was setup by Hall [26].

Assumption A. The triple sampling procedure is carried out under the choice of m such that as $m \to \infty$, $n^* = O(\lambda^r)$, $r \geq 1$,, and $lim\ sup(m/n^*) < \delta$.

4.1. Minimum Risk Point Estimation

Theorem 1. Under assumption (A), for the triple sampling procedure (8) and (9) as $\lambda \to \infty$.

(i) $E\left(\overline{X}_{N_1}\right) = \mu - \frac{\sigma\eta}{\delta n^*} + o\left(\lambda^{-1}\right)$

(ii) $E\left(\overline{X}_{N_1}^2\right) = \mu^2 - \frac{\sigma^2}{\delta n^*} + o\left(\lambda^{-1}\right)$

(iii) $Var\left(\overline{X}_{N_1}\right) = \frac{\sigma^2}{\delta n^*} + \text{o}\left(\lambda^{-1}\right)$

(iv) $E\left(S_{N_1}\right) = \sigma - \frac{\sigma\eta^2}{\delta n^*} + o\left(\lambda^{-1}\right)$

(v) $E\left(S_{N_1}^2\right) = \sigma^2 - \frac{\sigma^2\eta^2}{\delta n^*} + o\left(\lambda^{-1}\right)$

(vi) $Var\left(S_{N_1}\right) = \frac{\sigma^2\eta^2}{\delta n^*} + o\left(\lambda^{-1}\right)$

Proof. To prove (i), we condition on the $\sigma - field$ generated by the pilot study phase by $X_1, X_2, \ldots, X_m,$ and write

$$\begin{aligned} E\left(\overline{X}_{N_1}\right) &= E\left\{N_1^{-1}E\left\{\left(\sum_{i=1}^{N_1}(X_i-\mu+\mu)\right)|X_1,X_2,\ldots X_m\right\}\right\} \\ &= \mu + E\left\{N_1^{-1}\sigma E\left(\sum_{i=1}^{m}Z_i+\sum_{i=m+1}^{N_1}Z_i|Z_1,Z_2,\ldots,Z_M\right)\right\}, \end{aligned} \tag{10}$$

where $Z_i = \frac{X_i-\mu}{\sigma}$ $i = 1,\ 2,\ \ldots, m$, are *i.i.d* random variables distributed as $\mathcal{N}(0,1)$.

Provided $Z_1, Z_2, \ldots, Z_m$, the summation $\sum_{i=1}^{m} Z_i$ is non-random, so is N_1. Therefore,

$$E\sum_{i=m+1}^{N_1} Z_i = 0. \tag{11}$$

$$\text{and } E\left(\overline{X}_{N_1}\right) = \mu + \sigma E\left\{N_1^{-1}\ \sum_{i=1}^{m} Z_i\right\} \tag{12}$$

Then we expand N_1^{-1} in Taylor series around δn^* as

$$N_1^{-1} = (\delta n^*)^{-1} - \left(N_1-\delta n^*\right)(\delta n^*)^{-2} + \left(N_1-\delta n^*\right)^2(\delta n^*)^{-3} + R_{1N_1},$$

where R_{1N_1} is the remainder term. Recall the second term in (12); we obtain

$$\begin{aligned} \sigma E\left\{N_1^{-1}\ \sum_{i=1}^{m} Z_i\right\} &= \frac{-\sigma^2\lambda\delta\eta\ E(\sum_{i=1}^{m} Z_i)^2}{m(\delta n^*)^2} + \frac{-\sigma\lambda^2\delta^2\eta^2\ E(\sum_{i=1}^{m} Z_i)^3}{m(\delta n^*)^3} + E(R_{1N_1}) \\ &= \frac{-\sigma\eta}{\delta n^*} + E(R_{1N_1}) \end{aligned}$$

Now, the remainder term $E(R_{1N_1}) = \sigma\lambda^3\delta^3\eta^3 E\{(\sum_{i=1}^{m} Z_i)^3\ (\nu)^{-3}\}$, where v is a random variable lying in between N_1 and δn^*. If $N_1 \leq \nu \leq \delta n^*$ and since $m \leq N_1$, we obtain

$$E(R_{1N_1}) = \sigma\lambda^3\delta^3\eta^3 E\{(\sum_{i=1}^{m} Z_i)^4\ (\nu)^{-4}\} \leq \frac{\sigma^4\lambda^3\delta^3\eta^3}{m^5}E\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{m}}\right)^4 = 3\frac{\eta^3}{m} = o(\lambda^{-2})$$

as $m \to \infty$. Similarly, when $\delta n^* \leq \nu \leq N_1$. Thus, we have

$$E(R_{1N_1}) = \sigma^4\lambda^3\delta^3\eta^3 E\{(\sum_{i=1}^{m} Z_i)^4\ (\nu)^{-4}\} \leq \frac{\sigma^4\lambda^3\delta^3\eta^3}{m^2(\delta n^*)^3}E\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{m}}\right)^4 = \mathrm{o}\left(\lambda^{-2}\right)$$

as $m \to \infty$, where we have used the assumption that $m/n^* \approx \delta$ as $m \to \infty$. Finally, we obtain

$$E\left(\overline{X}_{N_1}\right) = \mu - \frac{\sigma\eta}{\delta n^*} + o(\lambda^{-1}\Big),$$

which proves (i) of Theorem 1.

Hence, (iv) of Theorem 1 is straightforward if we write $S_{N_1} = \eta\overline{X}_{N_1}$.

To prove (ii), we condition on the $\sigma - field$ generated by $X_1, X_2, \ldots, X_m$ and write

$$\begin{aligned} E\left(\overline{X}_{N_1}^2\right) &= E\left\{N_1^{-2}E\left(\sum_{i=1}^{N_1} X_i-\mu+\mu\right)^2|X_1,\ X_2,\ldots,\ X_m\right\} \\ &= E\ \left\{N_1^{-2}E\{(\sum_{i=1}^{N_1}(X_i-\mu))^2 + 2\mu N_1\sum_{i=1}^{N_1}(X_i-\mu) + N_1^2\mu^2\}|X_1,X_{2,\ldots}X_m\right\} \\ &= \mu^2 + E\{\ N_1^{-2}E\{(\sum_{i=1}^{m}(X_i-\mu)+\sum_{i=m+1}^{N_1}(X_i-\mu))^2 + 2\mu N_1\sum_{i=1}^{m}(X_i-\mu)\}\}|X_1,\ X_2,\ldots,\ X_m \\ &= \mu^2 + I + II. \end{aligned}$$

Thus,

$$I = E\{N_1^{-2}E(\sum_{i=1}^{m}(X_i-\mu))^2\} + E\{N_1^{-2}\sigma^2(N_1-m)\}|X_1,X_2,\ldots,X_m. \tag{13}$$

The second term in (13)

$$E\Big\{N_1^{-2}\sigma^2(N_1-m)\,\}\le\sigma^2(E\Big(N^{-1}\Big)=o(\lambda^{-1}\Big).$$

Therefore, the first term in (13)

$$I=E\{\overset{-2}{\underset{1}{N}}\{E(\sum_{i=1}^{m}(X_i-\mu))^2+2\sum_{i=1}^{m}(X_i-\mu)E\sum_{i=m+1}^{N_1}(X_i-\mu)$$
$$+E(\sum_{i=m+1}^{N_1}(X_i-\mu))^2\}\}|X_1,\ X_2,\ldots,\ X_m.$$

Provided the $\sigma-field$ generated $by\ X_1,\ X_2,\ldots\ \ X_m$, the summation $\sum_{i=1}^{m}(X_i-\mu)$ is non-random, so is N_1. Thus, similar to the arguments used above and the fact that

$$E\sum_{i=m+1}^{N_1}(X_i-\mu)=0,$$

and

$$EN^{-2}E(\sum_{i=m+1}^{N_1}(X_i-\mu))^2\}=E\{N_1^{-2}\sigma^2(N_1-m)\}\le\sigma^2E(N^{-1})=o(\lambda^{-1})$$

provides

$$I=E\{N_1^{-2}(\sum_{i=1}^{m}(X_i-\mu))^2\}+o(\lambda^{-1}).$$

Consequently, we expand N_1^{-2} in Taylor series around δn^* and substitute $N_1=\eta\overline{X}_{N_1}$, to obtain

$$I=\frac{E(\sum_{i=1}^{m}(X_i-\mu))^2}{(\delta n^*)^2}-\frac{2\mu\delta\lambda E(\sum_{i=1}^{m}(X_i-\mu))^3}{(\delta n^*)^2m}+E(R_{2N_1}),$$

for the normal distribution, $E(\sum_{i=1}^{m}(X_i-\mu))^3=0$, and $E(\sum_{i=1}^{m}(X_i-\mu))^2=\sigma^2m$ and $I=\frac{\sigma^2m}{(\delta n^*)^2}+E(R_{2N_1})$ by assumption A, $\frac{m}{n^*}\approx\delta$. Furthermore, $E(R_{2N_1})=o(\lambda^{-2})$ by arguments similar to those used to evaluate $E(R_{1N_1})=o(\lambda^{-1})$, and finally,

$$I=\frac{\sigma^2}{\delta n^*}+\mathrm{o}\Big(\lambda^{-1}\Big).$$

Now, recall II:

$$II=E\{2\mu N_1^{-1}\{\sum_{i=1}^{m}(X_i-\mu)\}\},$$

and expand N_1^{-1} in Taylor series expansion; while we substitute N_1= $\eta\overline{X}_{N_1}$, we obtain

$$II=-\frac{2\mu\sigma\eta}{\delta n^*}+o\Big(\lambda^{-1}\Big)=-\frac{2\sigma^2}{\delta n^*}+o\Big(\lambda^{-1}\Big).$$

Combine terms until we finally obtain

$$E\Big(\overline{X}_{N_1}^2\Big)=\mu^2-\frac{\sigma^2}{\delta n^*}+o\Big(\lambda^{-2}\Big),$$

which proves (ii) Theorem 1. Parts (iii), (v), and (vi) follow immediately; we omit further details for brevity.

The following Theorem 2 provides a second-order approximation of the expectation of a real-valued function $g(>0)of\ S_{N_1}$. □

Theorem 2. Let $g(>0)$ be a real-valued continuously differentiable and bounded function in a neighborhood around σ, such that $\underset{n\ge m}{Sup}\,g(n)=o|g'''(n^*)|$; then, as $\lambda\to\infty$, we obtain

$$Eg(S_{N_1})=g(\sigma)-\frac{\sigma\eta^2}{\delta n^*}g'(\sigma)+\frac{\sigma^2\eta^2}{2(\delta n^*)}g''(\sigma)+o(g'''(\lambda))$$

Proof. By expanding $g(S_{N_1})$ around σ using the Taylor series and taking the expectation over the terms we obtain

$$Eg(S_{N_1}) = g(\sigma) + g'(\sigma)E(S_{N_1} - \sigma) + \frac{1}{2}g''(\sigma)E(S_{N_1} - \sigma)^2 + o(g'''(\lambda))$$

utilizing parts (iv) and (v) of Theorem 1, and the assumption that g''' is a bounded function; the proof is complete. □

Theorem 3. Under assumption (A), for the triple sampling procedure (8) and (9), for all fixed μ and σ^2, with L_n provided by (1), we obtain, as $\lambda \to \infty$,

(i) $E(N) = n^* - \frac{\eta^2(1+\delta)}{\delta} + \frac{1}{2} + o(1)$
(ii) $E(N - n^*)^2 = \frac{n^*\eta^2}{\delta} + o(\lambda)$
(iii) $E|N - n^*|^3 = o(\lambda^2)$

Proof. Part (i): We noticed $N = T$ except possibly on a set of measures zero. That is,

$$\psi = \{S < m\} \cup \left\{\lambda S_{N_1} - \eta^2 < \delta\left(\lambda S_m - \eta^2\right) + 1\right\},$$

where $\int_\psi N^m dP = o(\lambda^{m-1})$. Therefore,

$$N^m = \left(\left(\lambda S_{N_1} - \eta^2\right) + \beta_{N_1}\right)^m + o\left(\lambda^{m-1}\right),$$

where $\beta_{N_1} = 1 - \{(\lambda S_{N_1} - \eta^2) - [(\lambda S_{N_1} - \eta^2)]\}$. From Hall [26], as $\lambda \to \infty$, $\beta_{N_1} \xrightarrow{D} U(0,1)$.
By setting $m = 1$, we obtain

$$E(N) = E\left(\lambda S_{N_1} - \eta^2\right) + E(\beta_{N_1}) + o(1).$$

Part (iv) of Theorem 1 justifies (i) of Theorem 3. Next,

$$\mathrm{E}(\mathrm{N} - \mathrm{n}^*)^2 = \lambda^2 \mathrm{E}(\mathrm{S}_{\mathrm{N}_1} - \sigma)^2$$

Substitute (vi) of Theorem 1, and the proof of part (ii) is immediate. Part (iii) of Theorem 3 is straightforward if we use Theorem 2 with $\mathrm{g}(\mathrm{S}_{\mathrm{N}_1}) = \mathrm{S}^3_{\mathrm{N}_1}$; we obtain

$$E|S_{N_1} - \sigma|^3 = o\left(\lambda^{-1}\right).$$

Part (i) of Theorem 1 shows that if $\eta^2 > \frac{\delta}{2(1+\delta)}$ then we attain early stopping. □

Lemma 1 provides a second-order approximation of a real-valued continuously differentiable function $\langle(> 0)$ of the final stage stopping time N.

Lemma 1. Let $\langle(>0)$ be a real-valued, continuously differentiable, and bounded function around n^*, such that $\sup_{n>m}|\langle h'''(n)| = o|\langle'''(n^*)|$; as $\lambda \to \infty$, then we obtain

$$E(\langle(N)) = \langle(n^*) + \langle'(n^*)\left\{-\frac{\eta^2(1+\delta)}{\delta} + \frac{1}{2}\right\} + \frac{n^*\eta^2\langle''(n^*)}{2\delta} + o\left(\lambda^2|\langle'''(n^*)|\right).$$

Proof. The proof follows immediately by expanding $\langle(N)$ around n^* and taking the expectation all over the terms; we obtain

$$E(\langle(N)) = \langle(n^*) + \langle'(n^*)E(N - n^*) + \frac{\langle''(n^*)}{2}E(N - n^*)^2 + \frac{1}{6}E\left\{\langle'''(\rho)(N - n^*)^3\right\}$$

By utilizing Theorem 3, parts (i), (ii), and (iii), the proof is complete. □

Theorem 3. For the triple sampling procedure (8) and (9) and (1), the asymptotic risk and regret, as $\lambda \to \infty$, is

(i) $R_N(A) = 2cn^* + c\eta^2 + \frac{c}{\delta}\eta^2 + o(\lambda)$

(ii) $\omega(A) = \frac{c}{\delta}\eta^2 + o(1)$

Proof. (i) It is seen that

$$ER_N(A) = AE(\hat{\mu}_N - \mu)^2 + cE(N),$$
$$E(\hat{\mu}_n - \mu)^2 = \sum_{n=m}^{\infty} E\Big((\hat{\mu}_N - \mu)^2 | N = n\Big)\ P(N = n).$$

Since the events $\{N = n\}$ and $\hat{\mu}_N$ are stochastically independent for all $n = m$, $m + 1, \ldots$ then,

$$E(\hat{\mu}_n - \mu)^2 = \sum_{n=m}^{\infty} E(\hat{\mu}_n - \mu)^2\ P(N = n) = \sum_{n=m}^{\infty} \frac{\sigma^2}{n + \eta^2}\ P(N = n) = E\left(\frac{\sigma^2}{N + \eta^2}\right).$$

It follows that

$$R_N(A) = c\Big(n^* + \eta^2\Big)^2 E\left(\frac{1}{N + \eta^2}\right) + cE(N).$$

By using Lemma 1 and (i) of Theorem 3, we obtain

$$E\left(\frac{1}{N + \eta^2}\right) = \mathrm{c}\Big(n^* + \eta^2) + \frac{c\eta^2(1 + \delta)}{\delta} - \frac{c}{2} + \frac{c\eta^2 n^*}{\delta(n^* + \eta^2)} + \mathrm{o}\Big(\lambda^{-2}\Big).$$

Hence,

$$R_N(A) = \mathrm{c}\Big(n^* + \eta^2\Big) + \frac{c\eta^2(1 + \delta)}{\delta} - \frac{c}{2} + \frac{c\eta^2 n^*}{\delta(n^* + \eta^2)} + o\Big(\lambda^{-2}\Big).$$

The proof of part (i) is complete.
(ii) It is known that

$$\omega(A) = R_N(A) - R_{n^*}(A) = \frac{c\eta^2 n^*}{\delta(n^* + \eta^2)} + \mathrm{o}\Big(\lambda^{-1}\Big),$$

$$\omega(A) = \frac{c\eta^2}{\delta}\left(1 - \frac{\eta^2}{n^* + \eta^2}\right) + o\Big(\lambda^{-1}\Big) \text{ as } \lambda \to \infty.$$

The proof of part (ii) is complete. □

Part (ii) shows that the amount of regret incurred by estimating the population mean by the Searls' estimator is $\frac{c\eta^2}{\delta}\left(1 - \frac{\eta^2}{n^* + \eta^2}\right)$. Thus, the smaller the coefficient of variation, the smaller the regret.

4.2. Triple Sampling Fixed-Width Confidence Interval

Recall that the triple-sampling confidence interval $I_N = (\hat{\mu}_N - d,\ \hat{\mu}_N + d,)$. Then, the asymptotic coverage probability is

$$P(\mu \epsilon\ I_N) = \sum_{n=m}^{\infty} (P|\ \hat{\mu}_N - \mu| \le d,\ N = n) = \sum_{n=m}^{\infty} (P|\hat{\mu}_N - \mu| \le d | N = n)\ P(N = n)$$

The results of Anscombe [48] provide the asymptotic distribution of $\hat{\mu}_N$ as standard normal, $\frac{\sqrt{N}(\hat{\mu}_N-\mu)}{\sigma} \to \mathcal{N}(0,1)$, as $m \to \infty$, independent of the random variable $N = m,\ m+1,\ m+2,\ \ldots$.

Thus,

$$P_\eta(\mu \in I_N) = \sum_{n=m}^{\infty} \left(P \left| \frac{\sqrt{n}}{\sigma}(\hat{\mu}_n - \mu) \right| \leq \frac{d\sqrt{n}}{\sigma} \right) P(N = n) = E\left\{ 2\Phi\left(\frac{d}{\sigma}\sqrt{N} \right) - 1 \right\}, \quad (14)$$

where $\Phi(a) = \int\limits_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-t^2/2\sigma^2} dt$.

By using Lemma 1 with $h(N) = \Phi(N)$ we acquire as $d \to 0$, we obtain

$$P_\eta(\mu \in I_N) = (1-\alpha) - \frac{a\phi(a)}{4\delta n^*} \left\{ \eta^2 \left(a^2 + 5 + 4\delta \right) - 2\delta \right\} + o\left(d^2\right), \quad (15)$$

where ϕ is the probability density function of $N(0,\ 1)$.

It is evident from (15) that the performance of the asymptotic coverage probability depends on the value of η^2. That is, if $\eta^2 > \frac{2\delta}{a^2+5+4\delta}$, then the asymptotic coverage probability is always less than the desired nominal value, while if $\eta^2 < \frac{2\delta}{a^2+5+4\delta}$, the procedure exceeds the desired nominal value. This shows that the value of the coefficient of variation controls the procedure. For example, at $\delta = 0.5$ and for $1-\alpha = 0.9,\ 0.95$, and 0.99, the respective $\eta^2 = 0.10303,\ 0.09224$, and 0.07323. For example, the calculated coverage probability at $\delta = 0.5$, $n^* = 500$, and for $1-\alpha = 0.95$ at $\eta = 0.01,\ 0.3$, and 0.5 is, respectively, 0.95011, 0.95, and 0.9498. This shows that knowing the coefficient of variation would control the coverage probability.

For the triple sampling coverage probability for μ using the classical sample mean as an estimator of the mean and with unknown η as $d \to 0$, see Hall [26] and Yousef [35,37].

$$P(\mu \in I_n) = (1-\alpha) - \frac{a\phi(a)}{2\delta n^*} \left(a^2 - \delta + 5 \right) + o\left(d^2\right) \quad (16)$$

It is evident from (16) that the asymptotic coverage probability is always less than $(1-\alpha)$ and attains the nominal value only asymptotically.

5. Monte Carlo Simulation

To visualize the asymptotic results obtained in the above theorems, wrote FORTRAN codes and ran them using Microsoft Developer Studio software with IMSL. We generated a pilot sample of size m from the normal distribution with mean μ and variance $\sigma^2 = \eta^2\mu^2$. We took $n^* = 24,\ 43,\ 61,\ 76,\ 96,\ 125,\ 171,\ 246$, and 500; see Hall [26]. Such selected values of the optimal sample size allowed us to explore the procedure's performance as the optimal sample size increased. For brevity, we took $\mu = 2,\ \delta = 0.5,\ m = 15, 1-\alpha = 95\%$, and $\eta = 0.3$. The number of replications was taken at 50,000. For more details about the simulation methodology, see Yousef [37,40].

The estimates were as follows: $\overline{N}$ was the simulated estimate for n^* with standard errors $S(\overline{N})$; $\hat{\mu}$ was the simulated estimate for the population mean with standard errors $S(\hat{\mu})$; $\hat{\sigma}$ was the simulated estimate for the population variance with standard errors $S(\hat{\sigma})$; $\hat{\omega}$ was the simulated estimate for the asymptotic regret; and finally, $1-\hat{\alpha}$ was the simulated estimate for the asymptotic coverage probability. Table 1 shows that, as the optimal sample size increased, $\overline{N}$ was always less than n^* (early stopping), with standard errors decreasing. $\hat{\mu}$ approached the actual value of the mean, with standard errors decreasing. $\hat{\sigma}$ approached the actual value of 0.6, with standard error decreasing. $\hat{\omega}$ was finite and positive (positive regret). The simulated coverage probability $1-\hat{\alpha}$ was always less than the targeted value and attained it only asymptotically. This means the triple sampling procedure attains all the above customary measures except exact consistency and provides good estimates in the presence of η.

Table 1. The simulated estimates of the triple sampling procedure at η **= 0.3**.

n^*	$\overline{N}$	$S(\overline{N})$	$\hat{\mu}$	$S(\hat{\mu})$	$\hat{\sigma}$	$S(\hat{\sigma})$	$\hat{\omega}$	$1-\hat{\alpha}$
24	19.36	0.037	1.9901	0.0006	0.5700	0.0004	4.27	0.9062
43	38.22	0.068	1.9937	0.0005	0.5660	0.0004	13.59	0.9040
61	56.55	0.080	1.9960	0.0004	0.5773	0.0003	12.43	0.9218
76	71.65	0.088	1.9971	0.0003	0.5838	0.0003	11.43	0.9279
96	91.81	0.098	1.9982	0.0003	0.5889	0.0002	9.81	0.9344
125	120.95	0.112	1.9985	0.0003	0.5923	0.0002	8.02	0.9380
171	166.99	0.130	1.9984	0.0002	0.5949	0.0002	4.60	0.9433
246	242.39	0.154	1.9992	0.0002	0.5968	0.0001	7.62	0.9445
500	496.57	0.222	1.9996	0.0001	0.5985	0.0001	11.04	0.9474

6. Conclusions

We discussed the triple sampling estimation for the mean of the symmetric distribution–normal distribution when the coefficient of variation is known. We used the Searls' estimator as an initial estimator for the mean. Such a problem can be used in quality control. We studied the minimum risk point estimation under a squared error loss function with linear sampling cost and found the asymptotic risk and regret. Then, we utilized the asymptotic results to construct a confidence interval with a precise width and coverage probability. We found the region where the asymptotic coverage probability was either less than or exceeded the desired nominal value. The theoretical results show that the procedure is sensitive to the choice of the coefficient of variation. Finally, a series of simulation results were conducted to explore the performance of the estimates as the optimal sample size increased, and these agreed with the theoretical results.

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