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# The Geometry of the Inextensible Flows of Timelike Curves according to the Quasi-Frame in Minkowski Space $\mathbb{R}^{2,1}$

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**Abstract:** The study of the flows of curves is one of the most fascinating research areas in differential geometry. In this paper, we investigate the geometry of the flows of timelike curves according to the quasi-frame in Minkowski space $\mathbb{R}^{2,1}$ (In this paper, we refer to these curves as "quasi-timelike curves"). We investigate the evolution of quasi-timelike curves using the velocity functions and obtain the necessary and sufficient conditions for inextensibility. Additionally, we obtain the explicit forms of the time evolution equations for the quasi-orthonormal frames (tangent, quasi-normal, and quasi-binormal vectors) of the quasi-timelike curve as well as the time evolution equations of their quasi-curvatures. We present a new application for motion with velocities equal to the quasi-curvatures of the quasi-timelike curve. In this application, the time evolution equations of the quasi-curvatures arise as a system of partial differential equations with the form of the heat equation, and by solving this system, we visualize the evolution of quasi-curvatures and the evolution of the quasi-timelike curve. In addition, the acceleration functions are used to investigate the flows of inextensible quasi-timelike curves, and an application for accelerations equal to the quasi-curvatures is given. Through this application, the position vector of the quasi-timelike curve satisfies the one-dimensional wave equation, and the time evolution equations of the quasi-curvatures arise as a system of transport equations. We obtain the solutions and graph them using Wolfram Mathematica 12.

**Keywords:** motion of curves; inextensible flows; evolution equations; timelike curves; quasi-frame

**MSC:** 35C08; 51B20; 53A04; 53A17; 53A35; 53B20; 53C50; 53E10; 53Z0

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## 1. Introduction

The theories of curves and surfaces are essential topics in differential geometry. They have diverse applications in many fields of science, such as physics, engineering, and image processing. Researchers have studied the motion and integrability of curves widely. Geometric characterizations of the integrable curves and the evolution of inelastic plane curves have been extensively studied in [1–5].

The study of the flows of curves in Euclidean space has attracted the interest of many researchers. Hasimoto [6] studied the motion of a vortex filament (smoke ring) and obtained the equations of the time evolution of the curves. In [7], the inextensible flows of curves (IFC) that move according to the type−2 Bishop frame in $\mathbb{R}^3$ were studied. In [8], a general formulation for IFC on an oriented surface in $\mathbb{R}^3$ was derived. In [9], the motion of curves in $\mathbb{R}^n$ was studied, and the time evolution equations for the given orthonormal Frenet frame and for the higher curvatures were derived. In [10], the surfaces were generated by the motion of IFC in $\mathbb{R}^3$ according to the Frenet frame. The geometric properties of these surfaces were described and visualized.

In [11], the dynamics of inextensible flows for adjoint curves in three-dimensional Euclidean space were investigated. A new method for the inextensible flows of adjoint curves using the Frenet frame was constructed, and the ferromagnetic and antiferromagnetic chain equations were given as an application. Gaber [12] constructed new models of the normal motions of inextensible curves that are moving according to the type$-1$ Bishop frame in $\mathbb{R}^3$. In [13], an analysis was presented for the acceleration and jerk vectors of particles that are moving along a space curve in three-dimensional Euclidean space. Based on the quasi-frame, the alternative resolutions of the accelerations and jerks were derived by applying this resolution and Siacci's theorem. An acceleration vector was resolved in the osculating plane by summing its tangential and radial components. In the rectifying and osculating planes, the jerk vector was also resolved along the tangential direction and two special radial directions. According to the jerk vector formula, the maximum permissible speed on a space curve was determined at all trajectory points.

The flows of curves in Minkowski space and De Sitter space have been investigated by many authors. In [14], the soliton solutions to the KdV equation were studied, and a motion of spacelike curves in $\mathbb{R}^{2,1}$ was described. In [15], the IFC of nonnull curves was investigated in $n$-dimensional pseudo-Euclidean space according to the Frenet frame. Gaber [16] studied and described the binormal motions of curves in De Sitter space $\mathbb{S}^{2,1}$ and constructed Hasimoto surfaces. In [17], a new type of Bishop frame called "Type$-2$ Bishop Frame" was introduced by using the binormal vector field of a regular curve. A new spherical image called type$-2$ Bishop spherical images was introduced by translating type$-2$ Bishop frame vectors into the center of the unit sphere of three-dimensional Euclidean space.

In [18], a new method was constructed for inextensible flows of timelike curves in a conformally flat, quasi-conformally flat, and conformally symmetric 4-dimensional LP-Sasakian manifold. The necessary and sufficient condition for the timelike curve to be inextensible was derived. Some characterizations were given for the curvatures of the timelike curve in a conformally flat, quasi-conformally flat, and conformally symmetric 4-dimensional LP-Sasakian manifold. Moreover, the flows of some associated curves of timelike curves were obtained. In [19], some conditions for nonnull curve flows to be inextensible in the 6-dimensional Lorentzian space $L_6$ were studied. The family of inextensible nonnull curves was characterized by partial differential equations.

The flows of curves in Galilean space have been discussed by many researchers. In [20], the flows of curves were investigated in the pseudo-Galilean three-dimensional space and its equiform geometry without any constraints, and the motions were described in terms of the inviscid and viscous Burgers' equations. In [21], the inextensible flows of curves in three-dimensional pseudo-Galilean space were studied. The necessary and sufficient conditions for the inextensible flows of curves were derived according to equiform geometry in pseudo-Galilean space. In [22], the inextensible flows of curves in three-dimensional Galilean space were investigated by using the Sabban frame, and the necessary and sufficient conditions for inextensible flows were derived. In [23], Fermi–Walker derivatives for inextensible curve flows were investigated in three-dimensional Galilean space $\mathbb{G}_3$. A novel approach to these flows was expressed using Frenet and Darboux frames with the help of Fermi–Walker derivatives. Sorour [24] studied the inextensible flows of focal curves associated with tube-like surfaces in Galilean three-dimensional space $\mathbb{G}_3$ and gave some characterizations for the curvatures of the focal curves associated with tube-like surfaces.

The main purpose of the current work is to investigate the geometric flows of timelike curves by using the quasi-frame in Minkowski space $\mathbb{R}^{2,1}$. Through this work, we derive the evolution of quasi-timelike curves by velocity and acceleration functions. In addition, we give some new applications with certain types of velocity and acceleration functions. In these applications, we choose the velocity and acceleration functions equal to the quasi-curvatures of the quasi-timelike curve. We obtain and visualize the evolution of the quasi-timelike curve and the evolution of the quasi-curvatures.

The present work is organized as follows: In Section 2, some geometric concepts of timelike curves in Minkowski space are given according to the Frenet frame and the quasi frame. In Section 3, the main results and discussions for the time evolution equations (TEEs) for quasi-timelike curves (QTIC) in $\mathbb{R}^{2,1}$ are obtained. In Section 4, an application on the motion of QTIC by velocity fields is given. In Section 6, the inextensible flows of the QTIC via acceleration fields are presented, and an application of the motion of the QTIC by acceleration fields is provided. Finally, we give our conclusions.

## 2. Basic Geometric Concepts of Timelike Curves in Minkowski Space $\mathbb{R}^{2,1}$

**Definition 1.** *The three dimensional Minkowski space $\mathbb{R}^{2,1}$ is the real vector space $\mathbb{R}^3$ provided with the Lorentzian inner product given by $-dx_0^2 + dx_1^2 + dx_2^2$ with $\{X = (x_0, x_1, x_2) \mid x_0, x_1, x_2 \in \mathbb{R}\}$ [25]. Consider the vectors $X, Y \in \mathbb{R}^{2,1}$, where $X = (x_0, x_1, x_2), Y = (y_0, y_1, y_2)$, the inner product is defined by $\langle X, Y \rangle = -x_0y_0 + x_1y_1 + x_2y_2$, and the vector product is defined by $X \times Y = (x_2y_1 - x_1y_2, x_2y_0 - x_0y_2, x_0y_1 - x_1y_0)$. The vector $u \in \mathbb{R}^{2,1}$ is spacelike if $\langle u, u \rangle > 0$, timelike if $\langle u, u \rangle < 0$, and null (lightlike) if $\langle u, u \rangle = 0$. The signature of the vector $u$ is 1 if $u$ is spacelike, $-1$ if $u$ is timelike, and 0 if $u$ is lightlike.*

**Definition 2.** *Let $\alpha = \alpha(v)$, $\alpha : I \to \mathbb{R}^{2,1}$ be a regular parameterized curve in $\mathbb{R}^{2,1}$. Then, $\alpha$ is [25]:*

$$\begin{cases} Spacelike\ \ curve & , \quad if\ \langle \dot{\alpha}, \dot{\alpha} \rangle > 0, \\ Timelike\ \ curve & , \quad if\ \langle \dot{\alpha}, \dot{\alpha} \rangle < 0, \\ Null\ \ curve & , \quad \langle \dot{\alpha}, \dot{\alpha} \rangle = 0. \end{cases}$$

**Definition 3.** *The angle between any two nonnull vectors $X$ and $Y$ is defined in [26] according to the classification of vectors in $\mathbb{R}^{2,1}$ as follows:*

- *Let $X$ and $Y$ be spacelike vectors $\mathbb{R}^{2,1}$; if $X$ and $Y$ span a timelike vector subspace, then $|\langle X, Y \rangle| > \|X\| \|Y\|$, and there is a unique positive real number $\theta$ such that*

  $$|\langle X, Y \rangle| = \|X\| \|Y\| \cosh \theta,$$

  *where the real number $\theta$ is called the Lorentz timelike angle between $X$ and $Y$.*

- *Let $X$ and $Y$ be spacelike vectors in $\mathbb{R}^{2,1}$ that span a spacelike vector subspace; then, there is a unique real number $\theta \in [0, \frac{\pi}{2}]$, such that*

  $$|\langle X, Y \rangle| = \|X\| \|Y\| \cos \theta,$$

  *where the real number $\theta$ is called the Lorentz spacelike angle between $X$ and $Y$.*
- *Let $X$ and $Y$ be future pointing (past pointing) timelike vectors in $\mathbb{R}^{2,1}$; then, there is a unique nonnegative real number $\theta$, such that*

  $$|\langle X, Y \rangle| = \|X\| \|Y\| \cosh \theta,$$

  *where the real number $\theta$ is called the Lorentz timelike angle between $X$ and $Y$.*
- *Let $X$ be a spacelike vector and $Y$ a future pointing timelike vector in $\mathbb{R}^{2,1}$; then, there is a unique nonnegative real number $\theta \geq 0$, such that*

  $$|\langle X, Y \rangle| = \|X\| \|Y\| \sinh \theta,$$

  *where the real number $\theta$ is called the Lorentz timelike angle between $X$ and $Y$.*

### *2.1. Frenet Frame for Timelike Curves in $\mathbb{R}^{2,1}$*

The Frenet frame is the most famous adapted frame for curves. It plays a significant role in investigating curves and tube surfaces in the classical differential geometry of curves.

**Definition 4.** *Let $v \in I$ be the parameter of the timelike curve $\alpha$ and define $s(v)$ as the arc length of the regular parameterized timelike curve $\alpha$ by [25]*

$$s(v) = \int_0^v \| \frac{d\alpha(\sigma)}{d\sigma} \| d\sigma, \qquad (\dot{\ }) = \frac{d}{dv}. \tag{1}$$

*For a regular timelike curve $\alpha$, we define $g > 0$ by $\frac{ds}{dv} = \| \frac{d\alpha(\sigma)}{d\sigma} \| = \sqrt{g}$. If $\| \frac{d\alpha(\sigma)}{d\sigma} \| = 1$ for all $v \in I$, then the timelike curve called an arc-length parameterized curve ($\alpha = \alpha(s)$ ).*

**Definition 5.** *Let $\alpha$ be an arc-length parameterized timelike curve. Assume $\langle \alpha''(s), \alpha''(s) \rangle \neq 0$, where $(') = \frac{d}{ds}$. Define the unit tangent vector $T$, the unit principal normal vector $N$, and the unit binormal vector $B$ to the timelike curve $\alpha$, and let $\{T, N, B\}$ be the Frenet–Serret frame for the timelike curve $\alpha$. The Frenet–Serret frame $\{T, N, B\}$ in $\mathbb{R}^{2,1}$ satisfies the following properties [27]:*

- $Sign(T) = -1, \quad Sign(T') = 1, \quad \alpha' = T,$ *and* $B = T \times N.$
- $\langle T, N \rangle = 0, \ \langle N, B \rangle = 0, \ \langle T, B \rangle = 0, \langle T, T \rangle = -1,$ *and* $\langle N, N \rangle = \langle B, B \rangle = 1.$
- $N \times B = -T,$ *and* $B \times T = N.$
- $N = \frac{1}{k} T', \quad k = \|T'\|, \quad \tau = \langle N', B \rangle,$ *where $k$ and $\tau$ are the curvature and torsion for the timelike curve, respectively.*

**Lemma 1.** *Let $\alpha(s)$ be the timelike curve in $\mathbb{R}^{2,1}$, then the Frenet–Serret equations are given by [27]:*

$$\frac{dF}{ds} = A \cdot F, \tag{2}$$

*where*

$$F = \begin{pmatrix} T \\ N \\ B \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & k & 0 \\ k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}.$$

### 2.2. *Quasi-Frame for Timelike Curves in $\mathbb{R}^{2,1}$*

The Frenet–Serret frame is inadequate to study space curves when the curvatures have discrete zero points. To solve this problem, we use a quasi-frame and a formula that corresponds to the Frenet–Serret equations. We study the timelike curves according to a quasi-frame (q-frame) and we call them the quasi-timelike curve (QTIC). The q-frame has some advantages, such as that it can be defined even along the tangent line equal to zero. Moreover, whether or not the space curve has unit speed does not affect the construction of the q-frame. In addition, the q-frame can be easily calculated.

**Definition 6.** *Let $v \in I$ be the parameter of the timelike curve $\alpha$. The q-frame consists of three orthonormal vectors $\{T, N_q, B_q\}$, where $T$ is the unit tangent vector, $N_q$ is the quasi-normal vector, and $B_q$ is the quasi-binormal vector. The q-frame $\{T, N_q, B_q, u\}$ is defined by [28]:*

$$\begin{aligned} T &= \frac{\alpha'(v)}{\|\alpha'(v)\|} \quad , \\ N_q &= \frac{T \wedge u}{\|T \wedge u\|} \quad , \\ B_q &= T \wedge N_q \quad , \end{aligned} \tag{3}$$

*where $u$ is the projection vector, and for simplicity, we can choose it as the unit vector $u = (1, 0, 0)$.*

**Definition 7.** *Let $\{T, N_q, B_q\}$ be the q-frame of the timelike curve $\alpha(s)$ on a point p, and let $\{T, N, B\}$ be the Frenet frame at the same point p on the same timelike curve. The relation between these frames is defined in [28] by:*

$$\begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{4}$$

*or*

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N_q \\ B_q \end{bmatrix}. \tag{5}$$

**Definition 8.** *The q-frame $\{T, N_q, B_q\}$ has the following properties [28]:*

$$\begin{gathered} \langle T, T \rangle = -1, \langle N_q, N_q \rangle = \langle B_q, B_q \rangle = 1, \\ \langle T, N_q \rangle = \langle T, B_q \rangle = \langle N_q, B_q \rangle = 0, \\ \langle N, N_q \rangle = \cos\theta\,, \langle B, B_q \rangle = \cos\theta. \end{gathered} \tag{6}$$

**Definition 9.** *The relation between the curvature k and torsion $\tau$ of the timelike curve specified by the Frenet frame and the curvatures $k_1, k_2$, and $k_3$ specified by the q-frame are defined in [28] by:*

$$k_1 = k\cos\theta, \;\; k_2 = -k\sin\theta, \;\; k_3 = d\theta + \tau. \tag{7}$$

*In this paper, we call $k_1, k_2$, and $k_3$ quasi-curvatures, and they are defined by:*

$$\begin{aligned} k_1 &= \langle T', N_q \rangle, \\ k_2 &= \langle T', B_q \rangle, \\ k_3 &= \langle N', B_q \rangle. \end{aligned} \tag{8}$$

## 3. Main Results and Discussion

*Time Evolution Equations for the Quasi-Timelike Curves (QTIC) in $\mathbb{R}^{2,1}$*

**Theorem 1.** *Let $\alpha(s)$ be the QTIC with a spacelike quasi-normal vector $N_q$ and a spacelike quasi-binormal vector $B_q$; then,*

$$\alpha_s = T. \tag{9}$$

$$\begin{bmatrix} T'(s) \\ N_q'(s) \\ B_q'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) \\ k_1(s) & 0 & k_3(s) \\ k_2(s) & -k_3(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N_q(s) \\ B_q(s) \end{bmatrix}, \quad (') = \frac{d}{ds}. \tag{10}$$

**Proof.** The tangent vector $T$ is the same tangent vector in both frames (the Frenet frame and the quasi-frame), where

$$T' = k\,N. \tag{11}$$

Substituting (5) and (7) into (11),

$$T' = k_1 N_q + k_2 B_q. \tag{12}$$

From (4), we have

$$N_q = \cos\theta N + \sin\theta B. \tag{13}$$

Taking the $s-$derivative of (13) and using (2), (4), and (7), we obtain:

$$N_q' = k_1 T + k_3 B_q. \tag{14}$$

From (4), we have

$$B_q = -\sin\theta N + \cos\theta B. \tag{15}$$

Taking the $s$-derivative of (15), and using (2), (4), and (7),

$$B_q' = k_2 T - k_3 N_q. \tag{16}$$

From (12), (14), and (16), the theorem holds. □

**Definition 10.** *Let $\alpha(s,0) : I = [0,l] \longrightarrow \mathbb{R}^{2,1}$ be the initial QTIC in $\mathbb{R}^{2,1}$, where s is the arc length and it is defined by (1). Assume that the curve moves with the time parameter t. The family of curves $\alpha(s,t)$ represents the flows of the QTIC at different time values, and it can be defined as $\alpha(s,t) : I \times [0,\infty) \longrightarrow \mathbb{R}^{2,1}$. The time evolution equation that describes the motion of the QTIC according to the q-frame in $\mathbb{R}^{2,1}$ is expressed by the velocity functions $W_1, W_2$, and $W_3$ as follows:*

$$\frac{\partial\alpha(s,t)}{\partial t} = W_1 T + W_2 N_q + W_3 B_q, \tag{17}$$

*where $W_1, W_2$, and $W_3$ are the velocity functions in the direction of the tangent vector $T$, the quasi-normal vector $N_q$, and the quasi binormal vector $B_q$. In this study, we call $W_1$ the q-tangential velocity, $W_2$ the q-normal velocity, and $W_3$ the q-binormal velocity.*

**Theorem 2.** *The TEEs for the q-frame in $\mathbb{R}^{2,1}$ are given by:*

$$\begin{aligned} T_t &= \psi_1 N_q + \psi_2 B_q, \\ N_{q,t} &= \psi_1 T, \\ B_{q,t} &= \psi_2 T, \end{aligned} \tag{18}$$

*and the TEEs for the quasi-curvatures $k_1, k_2$, and $k_3$ are:*

$$\begin{aligned} \kappa_{1,t} &= -\frac{g_t}{2g} k_1 + \psi_{1,s} - k_3\psi_2, \\ \kappa_{2,t} &= -\frac{g_t}{2g} k_2 + \psi_{2,s} + k_3\psi_1, \\ \kappa_{3,t} &= -\frac{g_t}{2g} k_3 - k_1\psi_2 + k_2\psi_1, \end{aligned} \tag{19}$$

*where*

$$\begin{aligned} \psi_1 &= W_{2,s} + k_1 W_1 - k_3 W_3\,, \\ \psi_2 &= W_{3,s} + k_2 W_1 + k_3 W_2. \end{aligned} \tag{20}$$

**Proof.** Since $\alpha = \alpha(v,t)$, by differentiating this equation with respect to $v$ we obtain:

$$\frac{\partial\alpha}{\partial v} = \sqrt{g} T, \tag{21}$$

Taking the derivative of (21) with respect to $t$, we have:

$$\frac{\partial^2\alpha}{\partial t\partial v} = \sqrt{g}\Big(\frac{g_t}{2g}\, T + T_t\Big). \tag{22}$$

Differentiating (17) with respect to $v$, we have:

$$\frac{\partial\alpha}{\partial v\partial t} = \sqrt{g}\big((W_{1,s} + k_1 W_2 + k_2 W_3)T + (W_{2,s} + k_1 W_1 - k_3 W_3)N_q + (W_{3,s} + k_2 W_1 + k_3 W_2)B_q\big). \tag{23}$$

For simplicity, we choose:

$$\begin{aligned} W_{2,s} + k_1 W_1 - k_3 W_3 &= \psi_1, \\ W_{3,s} + k_2 W_1 + k_3 W_2 &= \psi_2. \end{aligned} \tag{24}$$

Then,

$$\frac{\partial \alpha}{\partial v \partial t} = \sqrt{g}\big((W_{1,s} + k_1 W_2 + k_2 W_3)T + \psi_1 N_q + \psi_2 B_q\big). \tag{25}$$

Applying the compatibility condition ($\frac{\partial \alpha}{\partial t \partial v} = \frac{\partial \alpha}{\partial v \partial t}$) for Equations (22) and (25), we obtain:

$$g_t = 2g\big(W_{1,s} + k_1 W_2 + k_2 W_3\big), \tag{26}$$

and

$$T_t = \psi_1 N_q + \psi_2 B_q. \tag{27}$$

Taking the derivative of (27) with respect to $v$ and by a straightforward computation, we obtain:

$$T_{tv} = \sqrt{g}\big((k_1\psi_1 + k_2\psi_2)T + (\psi_{1,s} - k_3\psi_2)N_q + (\psi_{2,s} + k_3\psi_1)B_q\big). \tag{28}$$

From (10), we have $T_v = \sqrt{g}(k_1 N_q + k_2 B_q)$; by taking the $t-$derivative of this equation, we obtain:

$$T_{vt} = \sqrt{g}\bigg((k_{1,t} + \frac{g_t}{2g}k_1)N_q + (k_{2,t} + \frac{g_t}{2g}k_2)B_q + k_1\frac{\partial N_q}{\partial t} + k_2\frac{\partial B_q}{\partial t}\bigg). \tag{29}$$

Substituting (28) and (29) into the compatibility condition ($T_{tv} = T_{vt}$), we obtain:

$$\begin{aligned} k_{1,t} &= -\frac{g_t}{2g}k_1 + \psi_{1,s} - k_3\psi_2 \ , \\ k_{2,t} &= -\frac{g_t}{2g}k_2 + \psi_{2,s} + k_3\psi_1 \ , \end{aligned} \tag{30}$$

and

$$\kappa_1\frac{\partial N_q}{\partial t} + k_2\frac{\partial B_q}{\partial t} = (k_1\psi_1 + k_2\psi_2)T. \tag{31}$$

To determine $\frac{\partial N_q}{\partial t}$ and $\frac{\partial B_q}{\partial t}$ explicitly, we assume that:

$$\begin{aligned} \frac{\partial N_q}{\partial t} &= a_{11}T + a_{12}N_q + a_{13}B_q \ , \\ \frac{\partial B_q}{\partial t} &= a_{21}T + a_{22}N_q + a_{23}B_q \ . \end{aligned} \tag{32}$$

By using the properties of the q-frame that are given in **Definition 8**, we obtain

$$\begin{aligned} &a_{12} = a_{23} = 0, \\ &a_{11} = \psi_1, \quad a_{21} = \psi_2, \quad a_{13} = -a_{22}. \end{aligned} \tag{33}$$

Substituting (33) into (32), we have:

$$\begin{aligned} \frac{\partial N_q}{\partial t} &= \psi_1 T + a_{13}B_q \ , \\ \frac{\partial B_q}{\partial t} &= \psi_2 T - a_{13}N_q \ . \end{aligned} \tag{34}$$

Substituting (34) into (31), we obtain:

$$(k_1 B_q - k_2 N_q)a_{13} = 0. \tag{35}$$

Assuming that $k_1 \neq 0$ and $k_2 \neq 0$; then $a_{13} = 0$. Hence, we obtain

$$\frac{\partial N_q}{\partial t} = \psi_1 T \tag{36}$$

$$\frac{\partial B_q}{\partial t} = \psi_2 T \, . \tag{37}$$

Since $\frac{\partial N_q}{\partial v} = \sqrt{g}\frac{\partial N_q}{\partial s}$, we have:

$$\frac{\partial N_q}{\partial v} = \sqrt{g}(k_1 T + k_3 B_q). \tag{38}$$

By taking the $t$-derivative of (38) and then using (27) and (37), we obtain:

$$\frac{\partial^2 N_q}{\partial t \partial v} = \sqrt{g}((\frac{g_t}{2g}k_1 + k_{1,t} + k_3\psi_2)T + k_1\psi_1 N_q + (\frac{g_t}{2g}k_3 + k_1\psi_2 + k_{3,t})B_q). \tag{39}$$

Taking the $v$-derivative of (36),

$$\frac{\partial^2 N_q}{\partial v \partial t} = \sqrt{g}(\psi_{1,s}T + \psi_1 k_1 N_q + \psi_1 k_2 B_q) \, . \tag{40}$$

We apply the compatibility condition ($\frac{\partial^2 N_q}{\partial t \partial v} = \frac{\partial^2 N_q}{\partial v \partial t}$) for the Equations (39) and (40). By using the properties of the q-frame in **Definition** 8,

$$\langle \frac{\partial^2 N_q}{\partial t \partial v}, B_q \rangle = \langle \frac{\partial^2 N_q}{\partial v \partial t}, B_q \rangle.$$

Hence,

$$k_{3,t} = -\frac{g_t}{2g}k_3 - k_1\psi_2 + k_2\psi_1.$$

Based on the above, the theorem holds. □

**Lemma 2.** *Let $\alpha(v,t)$ be the QTIC; if $\alpha(v,t)$ is inextensible (the arc length of the curve is preserved along the motion; so, $g_t = 0$); then, from (26), we obtain the following condition:*

$$W_{1,s} = -k_1 W_2 - k_2 W_3 \, , \tag{41}$$

*and the quasi-curvatures that are given by (19) take the form:*

$$\begin{aligned} k_{1,t} &= \psi_{1,s} - k_3\psi_2 \, , \\ k_{2,t} &= \psi_{2,s} + k_3\psi_1 \, , \\ k_{3,t} &= -k_1\psi_2 + k_2\psi_1 \, , \end{aligned} \tag{42}$$

*or*

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}_{,t} = \begin{pmatrix} 0 & 0 & -\psi_2 \\ 0 & 0 & \psi_1 \\ -\psi_2 & \psi_1 & 0 \end{pmatrix} . \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} + \begin{pmatrix} \psi_{1,s} \\ \psi_{2,s} \\ 0 \end{pmatrix}, \tag{43}$$

*where*

$$\begin{aligned} \psi_1 &= W_{2,s} + k_1 W_1 - k_3 W_3 \, , \\ \psi_2 &= W_{3,s} + k_2 W_1 + k_3 W_2 \, . \end{aligned} \tag{44}$$

## 4. An Application for the Motion of the QTIC by Velocity Fields

Consider the inextensible flows of the QTIC, and assume that $k_3 = 0$. Suppose that the QTIC evolves with q-tangential velocity $W_1 = 0$; then from (41), we have $W_3 = -\frac{k_1}{k_2} W_2$. Substituting in (42), we obtain the TEEs for the quasi-curvatures:

$$\begin{aligned} k_{1,t} &= \psi_{1,s} \ , \\ k_{2,t} &= \psi_{2,s} \ , \\ \psi_2 k_1 &= k_2 \psi_1 \ . \end{aligned} \tag{45}$$

From (44), we have

$$\begin{aligned} \psi_1 &= W_{2,s} \ , \\ \psi_2 &= W_{3,s} \ . \end{aligned} \tag{46}$$

Substituting (46) into (45),

$$\begin{aligned} k_{1,t} &= W_{2,ss} \ , \\ k_{2,t} &= W_{3,ss} \ , \\ k_1 W_{3,s} &= k_2 W_{2,s} \ . \end{aligned} \tag{47}$$

Now, we study the case when $W_2 = k_2$ and $W_3 = -k_1$. Then, the TEEs (47) for the quasi-curvatures $k_1$ and $k_2$ take the form of heat equations:

$$\begin{aligned} k_{1,t} &= k_{2,ss} \ , \\ k_{2,t} &= -k_{1,ss} \ , \\ -k_1 k_{1,s} &= k_2 k_{2,s} \ . \end{aligned} \tag{48}$$

One solution of the system (48) takes the following form:

$$\begin{aligned} k_1(s,t) &= A_0 \cos(c_1 s + c_1^2 t + c_2) \ , \\ k_2(s,t) &= A_0 \sin(c_1 s + c_1^2 t + c_2) \ . \end{aligned} \tag{49}$$

Hence, the curve is defined by their quasi-curvatures $k_1$, $k_2$, and $k_3 = 0$; these curvatures are illustrated in Figures 1 and 2.

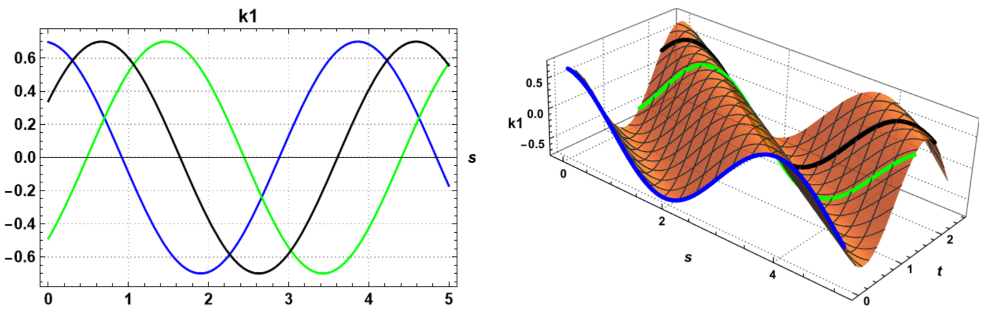
Substituting (49) into (9), (10), (17), and (18) and solving the PDE systems numerically, we can determine and plot the surface $\alpha(s,t) = (\alpha_1, \alpha_2, \alpha_3)$ that is generated by the flows of the QTIC. It is important to use the properties of the $q$-frame defined by **Definition 8**, which can be rewritten explicitly as:

$$\begin{aligned} -\alpha_{1,s}^2 + \alpha_{2,s}^2 + \alpha_{3,s}^2 &= -1 \ , \\ -\alpha_{1,ss}^2 + \alpha_{2,ss}^2 + \alpha_{3,ss}^2 &= k_1^2 + k_2^2 \ . \end{aligned} \tag{50}$$

The conditions (50) are necessary to validate the general solution. The surface generated by the motion of the QTIC $\alpha(s,t)$ is illustrated in Figure 3.

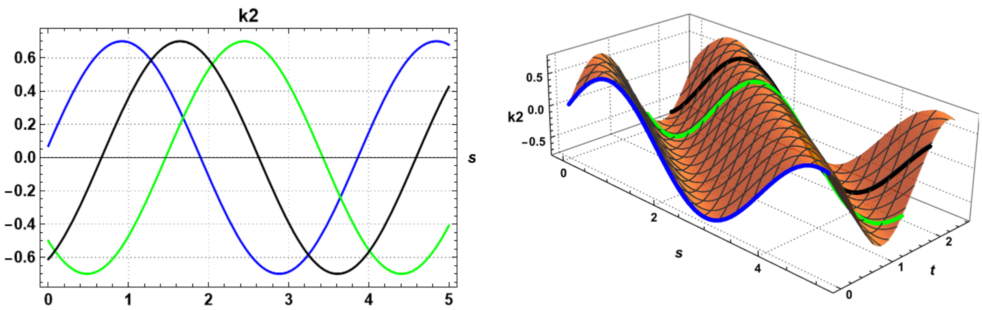
Figure 1a represents the evolution of the first quasi-curvature $k_1$ of the QTIC for different time values at $t = 0, 1.5, 2$ in two dimensions. Figure 1b represents the three-dimensional graph of the first quasi-curvature $k_1$ of the QTIC for $s \in [0,5]$ and $t = [0, 2.5]$, $A_0 = 0.7, c_1 = 1.6$, and $c_2 = 0.1$. The blue, green, and black curves represent the flows of the QTIC at time $t = 0, 1.5, 2$, respectively. It is obvious that the flows of the first quasi-curvature $k_1$ had a forward shift with increasing time, but the shape did not change.

Figure 2a represents the evolution of the second quasi-curvature $k_2$ of the QTIC for some different time values at $t = 0, 1.5, 2$ in two dimensions. Figure 2b represents the three-dimensional graph of the second quasi-curvature $k_2$ of the QTIC for $s \in [0,5]$ and $t = [0, 2.5]$, $A_0 = 0.7, c_1 = 1.6$, and $c_2 = 0.1$. The blue, green, and black curves represent the flows of the QTIC at time $t = 0, 1.5, 2$, respectively. We note that the flows of the second quasi-curvature $k_2$ had a forward shift with increasing time, but the shape did not change.



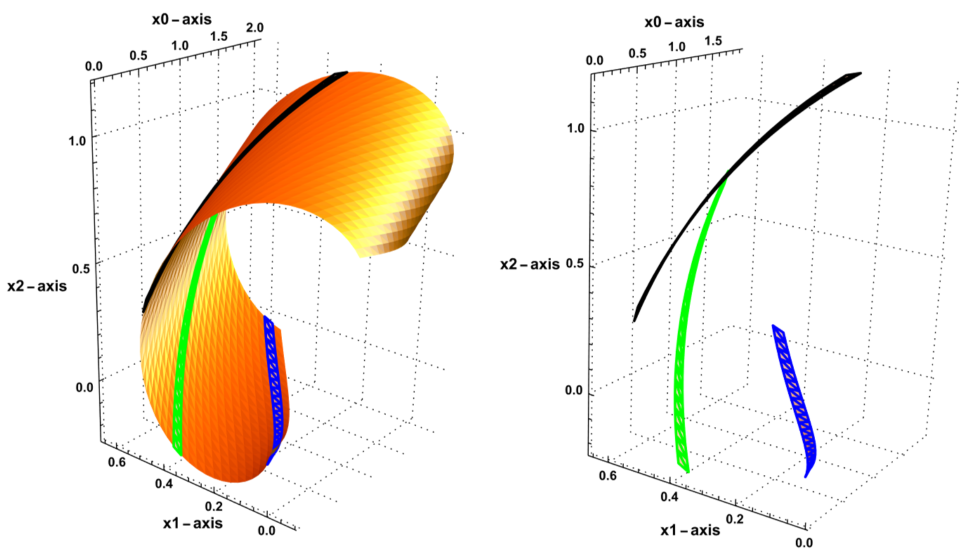

(**a**) The quasi-curvature $k_1(s,t)$ in 2-D. (**b**) The quasi-curvature $k_1(s,t)$ in 3-D.

**Figure 1.** The time evolution of the quasi-curvature $k_1(s,t) = A_0 \cos(c_1 s + c_1^2 t + c_2)$ of the QTIC and its evolution in 2-D and 3-D for $s \in [0,5]$, $t = [0, 2.5]$, $A_0 = 0.7, c_1 = 1.6$, and $c_2 = 0.1$. The blue, green, and black curves represent the flows of the quasi-curvatures of the QTIC at time $t = 0, 1.5, 2$, respectively.




(**a**) The quasi-curvature $k_2(s,t)$ in 2-D. (**b**) The quasi-curvature $k_2(s,t)$ in 3-D.

**Figure 2.** The time evolution of the quasi-curvature $k_2(s,t) = A_0 \sin(c_1 s + c_1^2 t + c_2)$ of the QTIC and its evolution in 2-D and 3-D for $s \in [0,5]$, $t = [0, 2.5]$, $A_0 = 0.7, c_1 = 1.6$, and $c_2 = 0.1$. The blue, green, and black curves represent the flows of the quasi-curvatures of the QTIC at time $t = 0, 1.5, 2$, respectively.




(**a**) (**b**)

**Figure 3.** The evolution of the QTIC via velocity functions ($W_1 = 0, W_2 = k_1(s,t), W_3 = k_2(s,t)$) for $s \in [0,1]$, $t = [0, 2.5]$, $A_0 = 0.7, c_1 = 1.6$, and $c_2 = 0.1$. The blue, green, and black curves represent the flows of the QTIC at time $t = 0.05, 0.7, 1.3$, respectively. (**a**) The evolution of the QTIC for $s \in [0,1]$ and time $t = [0, 2.5]$; (**b**) the evolution of the QTIC for $s \in [0,1]$ at time $t = 0.05, 0.7, 1.3$.

Figure 3a represents the flows of the QTIC via the velocity functions ($W_1 = 0, W_2 = k_1(s,t), W_3 = k_2(s,t)$) for $s \in [0,1]$ and $t = [0,2.5]$, $A_0 = 0.7, c_1 = 1.6$, and $c_2 = 0.1$. The blue, green, and black curves in Figure 3a,b represent the flows of the QTIC at time $t = 0.05, 0.7, 1.3$, respectively.

## 5. The Flows of the QTIC via the Acceleration Fields

Let $\alpha(s,0) : I \longrightarrow \mathbb{R}^{2,1}$ be the initial inextensible QTIC in $\mathbb{R}^{2,1}$. Let $C_t$ be the family of curves, it represents the flows of the QTIC, where $C_t = \alpha(s,t) : I = [0,l] \times [0,\infty) \longrightarrow \mathbb{R}^{2,1}$, and $t$ represents the time parameter. Assume that the inextensible QTIC evolves by the acceleration fields $\varpi_1, \varpi_2$, and $\varpi_3$ in the direction of the q-tangent vector $T$, q-normal vector $N_q$, and q-principal binormal vector $B_q$, where $\varpi_i, i = 1,2,3$ are functions in the quasi-curvatures and their derivatives. The QTIC evolves according to the acceleration fields in the q-frame by the following law:

$$\frac{\partial^2 \alpha}{\partial t^2} = \varpi_1 \, T + \varpi_2 \, N_q + \varpi_3 \, B_q. \tag{51}$$

**Lemma 3.** *The relation between the acceleration fields $\varpi_1, \varpi_2$, and $\varpi_3$ that describe the evolution of the QTIC by (51) and the velocity fields $W_1, W_2$, and $W_3$ that describe the evolution of the QTIC (17) is given by:*

$$\begin{aligned} \varpi_1 &= W_{1,t} + W_2 \psi_1 + W_3 \psi_2 \,, \\ \varpi_2 &= W_{2,t} + W_1 \psi_1 \,, \\ \varpi_3 &= W_{3,t} + W_1 \psi_2 \,, \end{aligned} \tag{52}$$

*or*

$$\begin{aligned} \varpi_{1,s} + \varpi_2 k_1 + \varpi_3 k_2 &= \psi_1^2 + \psi_2^2 \quad , \\ \varpi_{2,s} + \varpi_1 k_1 - \varpi_3 k_3 &= \psi_{1,t} \quad , \\ \varpi_{3,s} + \varpi_1 k_2 + \varpi_2 k_3 &= \psi_{2,t} \quad , \end{aligned} \tag{53}$$

*where $\psi_1$ and $\psi_2$ are given by (20).*

**Proof.** We differentiate (17) with respect to $t$; then, we have:

$$\alpha_{tt} = W_{1,t} \, T + W_1 \, T_t + W_{2,t} \, N_q + W_2 \, N_{q,t} + W_{3,t} \, B_q + W_3 \, B_{q,t}. \tag{54}$$

Substituting (18) and (20) into (54). Hence (52) holds.

Another relation between the velocity fields and acceleration fields can be given as follows: Taking the $s$-derivative of (51), we obtain

$$\alpha_{tts} = (\varpi_{1,s} + \varpi_2 k_1 + \varpi_3 k_2)T + (\varpi_{2,s} + \varpi_1 k_1 - \varpi_3 k_3)N_q + (\varpi_{3,s} + \varpi_1 k_2 + \varpi_2 k_3)B_q. \tag{55}$$

Since $\alpha_s = T$, then $\alpha_{stt} = T_{tt}$. Taking the second derivative of the first equation of (18) with respect to $t$ and using (18), we have:

$$\alpha_{stt} = T_{tt} = (\psi_1^2 + \psi_2^2)T + \psi_{1,t} N_q + \psi_{2,t} B_q. \tag{56}$$

Since the QTIC is inextensible, the compatibility condition $\alpha_{tts} = \alpha_{stt}$ is satisfied, equating (55) and (56). Hence, (53) holds. □

*An Application for the Motion of the QTIC by the Acceleration Fields*

Assume that $\alpha(s,t)$ is the QTIC evolving by the accelerations functions:

$$\varpi_1 = 0, \; \varpi_2 = k_1, \; \varpi_3 = k_2. \tag{57}$$

Assume $k_3 = 0$; then, the TEE described by (51) takes the form:

$$\frac{\partial^2 \alpha}{\partial t^2} = k_1 \ N_q + k_2 \ B_q \ . \tag{58}$$

Substituting (57) into (53) gives the form:

$$\begin{aligned} \psi_{1,t} &= k_{1,s} \ , \\ \psi_{2,t} &= k_{2,s} \ , \end{aligned} \tag{59}$$

with the following condition:

$$\psi_1^2 + \psi_2^2 = k_1^2 + k_2^2 \ . \tag{60}$$

Since $k_3 = 0$, then the third equation of (42) gives:

$$\psi_2 = \frac{\kappa_2}{\kappa_1} \psi_1. \tag{61}$$

Substituting (61) into (60), hence $\psi_1 = k_1$, and $\psi_2 = k_2$, and the PDE system (59) takes the form of transport equations:

$$\begin{aligned} k_{1,t} &= k_{1,s} \ , \\ k_{2,t} &= k_{2,s} \ . \end{aligned} \tag{62}$$

This system of transport equations has the following general solution:

$$\begin{aligned} k_1(s,t) &= \eta_1(s+t) \ , \\ k_2(s,t) &= \eta_2(s+t) \ , \end{aligned} \tag{63}$$

where $\eta_1(s+t)$ and $\eta_2(s+t)$ are arbitrary functions.

Since $\alpha_s = T$, by taking the $s$-derivative of this equation and using (10),

$$\alpha_{ss} = k_1 N_q + k_2 B_q. \tag{64}$$

Comparing (58) and (64), hence

$$\alpha_{tt} = \alpha_{ss}. \tag{65}$$

This represents the one-dimensional wave equation. If we consider the initial conditions $\alpha(s,0) = h(s)$ and $\alpha_t(s,0) = f(s)$, we obtain the general solution of the form:

$$\alpha(s,t) = \frac{1}{2}\bigg(h(s+t) + h(s-t) + \int_{s-t}^{s+t} f(x)dx\bigg). \tag{66}$$

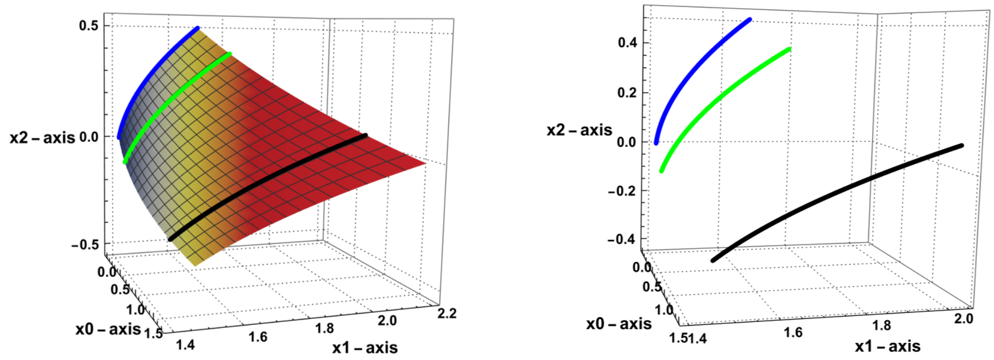
In this application, we take the following initial conditions:

$$\begin{aligned} \alpha(s,0) &= h(s) = (\sqrt{2} \sinh s, \sqrt{2} \cosh s, s) \ , \\ \alpha_t(s,0) &= f(s) = (\sqrt{2} \cosh s, \sqrt{2} \sinh s, -1) \ . \end{aligned} \tag{67}$$

Then, we obtain the general solution:

$$\alpha(s,t) = (\sqrt{2} \sinh(s+t), \sqrt{2} \cosh(s+t), s-t). \tag{68}$$

The evolution $\alpha(s,t)$ represents the family of quasi-timelike curves for different time values $t$. This family of QTIC is plotted for different time values in Figure 4.




(**a**) (**b**)

**Figure 4.** The evolution of the QTIC via accelerations ($\varpi_1 = 0, \varpi_2 = k_1$ and $\varpi_3 = k_2$) for $s \in [0, 0.5]$ and $t = [0, 0.5]$. The blue, green, and black curves represent the quasi-curvatures of the QTIC at time $t = 0, 0.1, 0.4$, respectively. (**a**) The evolution of the QTIC for $s \in [0, 0.5]$ and time $t = [0, 0.5]$; (**b**) The evolution of the QTIC for $s \in [0, 0.5]$ at time $t = 0, 0.1, 0.4$.

Figure 4a represents the evolution of the QTIC described by the acceleration functions ($\varpi_1 = 0, \varpi_2 = k_1$ and $\varpi_3 = k_2$) for $s \in [0, 0.5]$ and $t = [0, 0.5]$. The blue, green, and black curves in Figure 4b represent the evolution of the QTIC at time $t = 0, 0.1, 0.4$, respectively.

We consider the QTIC with the parametrization (68); the first and second quasi-curvatures $k_1$ and $k_2$ given by (63) can be computed using (3) and (8); hence:

$$\begin{aligned} k_1(s,t) &= \eta_1(s+t) = \frac{\sqrt{2}\cosh(s+t)}{\sqrt{\cosh(2(s+t))}}\,, \\ k_2(s,t) &= \eta_2(s+t) = \frac{\sqrt{2}\sinh(s+t)}{\sqrt{\cosh(2(s+t))}}\,. \end{aligned} \tag{69}$$

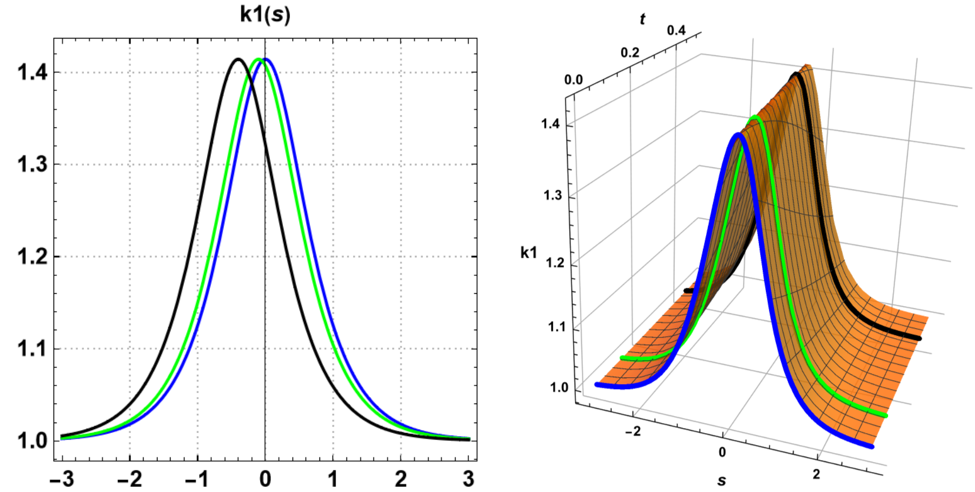
We can verify the solutions by using the properties of the q-frame in **Definition** 8, where the QTIC (68) satisfies the following PDEs:

$$\begin{aligned} -\alpha_{1,s}^2 + \alpha_{2,s}^2 + \alpha_{3,s}^2 &= -1\,, \\ -\alpha_{1,ss}^2 + \alpha_{2,ss}^2 + \alpha_{3,ss}^2 &= k_1^2 + k_2^2 = \eta_1^2(s+t) + \eta_2^2(s+t) = 2\,. \end{aligned} \tag{70}$$

The quasi-curvatures $k_1$ and $k_2$ are plotted at different time values in Figures 5 and 6.

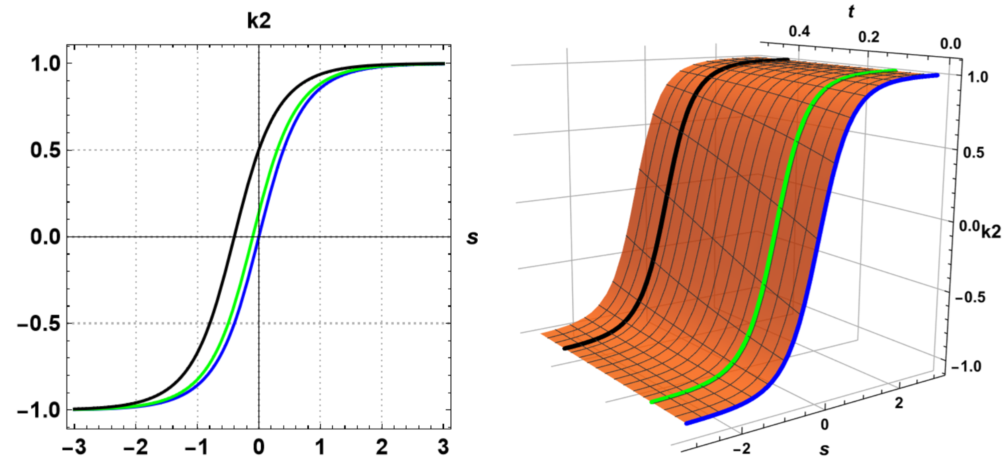
Figure 5a represents the evolution of the first quasi-curvature $k_1$ of the QTIC for different time values at $t = 0, 0.1, 0.4$ in two dimensions. Figure 5b represents the three-dimensional graph of the flows of the first quasi-curvature $k_1$ for $s \in [-3, 3]$ and $t = [0, 0.5]$. The blue, green, and black curves represent the evolution of the first quasi-curvature at time $t = 0, 0.1, 0.4$, respectively. It is obvious that the evolution of the first quasi-curvature $k_1$ had a right shift with an increase in time, and the shape did not change.

Figure 6a represents the evolution of the second quasi-curvature $k_2$ of the QTIC for some different time values at $t = 0, 0.1, 0.4$ in two dimensions. Figure 6b represents the three-dimensional graph of the flows of the second quasi-curvature $k_2$ for $s \in [-3, 3]$ and $t = [0, 0.5]$. The blue, green, and black curves represent the evolution of the second quasi-curvature at time $t = 0, 0.1, 0.4$, respectively. There was not any obvious shift for the evolution of the second quasi-curvature $k_2$ with the increase in time, and the shape did not change.




(**a**) (**b**)

**Figure 5.** The time evolution of the quasi-curvatures of the QTIC, $k_1(s,t) = \frac{\sqrt{2}\cosh(s+t)}{\sqrt{\cosh(2(s+t)}}$ in 2-D and 3-D for $s \in [-3,3]$ and $t = [0,0.5]$. The blue, green, and black curves represent the quasi-curvature of the QTIC at time $t = 0, 0.1, 0.4$, respectively. (**a**) The 2-D graph for the evolution of $k_1(s)$ at $t = 0, 0.1, 0.4$; (**b**) the 3-D graph for the evolution of $k_1(s,t)$ for $s \in [-3,3]$ and $t = [0,0.5]$.




(**a**) (**b**)

**Figure 6.** The time evolution of the quasi-curvatures of the QTIC, $k_2(s,t) = \frac{\sqrt{2}\sinh(s+t)}{\sqrt{\cosh(2(s+t)}}$ in 2-D and 3-D for $s \in [-3,3]$ and $t = [0,0.5]$. The blue, green, and black curves represent the quasi-curvature of the QTIC at time $t = 0, 0.1, 0.4$, respectively. (**a**) The 2-D graph for the evolution of $k_2(s)$ at $t = 0, 0.1, 0.4$; (**b**) the 3-D graph for the evolution of $k_2$ for $s \in [-3,3]$ and $t = [0,0.5]$.

## 6. Discussion

The study of the evolution of curves is an attractive research topic in differential geometry. In this paper, we investigated the flows of timelike curves in Minkowski space according to a quasi-frame, which we called "quasi-timelike curves". The quasi-frame is an important frame in the study of the evolution of curves; it is very effective at the points where the curvature of the curve vanishes. In this work, the motion of quasi-timelike curves was specified by velocity and acceleration. The equations of the evolution of the quasi-frame and the evolution of the curvatures (we called them quasi-curvatures) were derived, and some applications were given. In these applications, partial differential equations played an important role in describing the motion of the QTIC. They arose in the form of heat equations and transport equations. Moreover, the flows of the QTIC satisfied the one-dimensional wave equation. With the aid of Wolfram Mathematica 12, we obtained the solutions of these partial differential equations and graphed them.

## 7. Conclusions

The present paper investigated the motion of the quasi-timelike curve (QTIC) in Minkowski space $\mathbb{R}^{2,1}$. The results of this paper are summarized as follows:

1. We studied the motion of the QTIC by the velocity fields $W_1, W_2$, and $W_3$, with the equation of motion $\frac{\partial\alpha(s,t)}{\partial t} = W_1 T + W_2 N_q + W_3 B_q$, where $W_1, W_2$, and $W_3$ represented the velocity functions in the direction of the $q$- frame $T, N_q, B_q$.
2. The time evolution equations (TEEs) for the $q$-frame $T, N_q, B_q$ of the QTIC in Minkowski space $\mathbb{R}^{2,1}$ were derived, and the TEEs for the quasi-curvatures $k_1, k_2$, and $k_3$ were obtained as a system of PDEs (Theorem 2).
3. We gave an application of the motion of the QTIC by the velocity functions $W_1 = 0, W_2 = k_2$, and $W_3 = -k_1$, with the quasi-curvatures $k_1(s,t) = A_0 \cos(c_1 s + c_1^2 t + c_2)$, $k_2(s,t) = A_0 \sin(c_1 s + c_1^2 t + c_2)$, and $k_3 = 0$. We plotted the evolution of the quasi-curvatures $k_1$ and $k_2$ ( Figures 1 and 2) and the evolution of the QTIC (Figure 3).
4. We studied the motion of the QTIC described by the acceleration fields with the equation of motion $\frac{\partial^2\alpha}{\partial t^2} = \varpi_1\ T + \varpi_2\ N_q + \varpi_3\ B_q$.
5. We gave an application of the motion of the QTIC by the acceleration functions $\varpi_1 = 0, \varpi_2 = k_1$, and $\varpi_3 = k_2$, with the quasi-curvatures $k_1(s,t) = \frac{\sqrt{2}\cosh(s+t)}{\sqrt{\cosh(2(s+t)}}$, $k_2(s,t) = \frac{\sqrt{2}\sinh(s+t)}{\sqrt{\cosh(2(s+t)}}$, and $k_3 = 0$. We plotted the evolution of the quasi-curvatures $k_1$ and $k_2$ as illustrated in Figures 5 and 6 and the evolution of the QTIC, shown in Figure 4.
6. Through the given applications, we presented the description of the graphs, which indicated the flows of the quasi-timelike curves and their first and second quasi-curvatures.

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## Abbreviations

The following abbreviations are used in this manuscript:

| | |
|---|---|
| (2-D) | Two dimensions. |
| (3-D) | Three dimensions. |
| IFC | Inextensible flows of curves. |
| PDE(s) | Partial Differential Equation(s) |
| q-frame | Quasi-frame. |
| QTIC | Quasi-timelike curve. |
| TEEs | Time evolution equations. |

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