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Exact Solutions of Maxwell Equations in Homogeneous Spaces with the Group of Motions $G_3(VIII)$

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Abstract: The problem of the classification of the exact solutions to Maxwell's vacuum equations for admissible electromagnetic fields and homogeneous space-time with the group of motions $G_3(VIII)$ according to the Bianchi classification is considered. All non-equivalent solutions are found. The classification problem for the remaining groups of motion, $G_3(N)$, has already been solved in other papers. All non-equivalent solutions of empty Maxwell equations for all homogeneous spaces with admissible electromagnetic fields are now known.

Keywords: Maxwell equations; algebra of symmetry operators; theory of symmetry; linear partial differential equations

1. Introduction

If the symmetry of space-time and physical fields is given by Killing fields whose number is no less than three, it is possible to reduce the field equations and the equations of motion of the tested charged particles to ordinary differential equations.

Spaces admitting complete sets of mutually commutative Killing tensor fields of rank no greater than two are of special interest in the theory of gravitation. Such spaces are called Steckel spaces. The theory of Steckel spaces was developed in [1–7] (see also [8–11] and the bibliographies given there). The equations of motion of test particles in Stackel spaces can be integrated using the commutative integration method (CIM) (or the method of complete separation of variables). Exact solutions to the gravitational equations are still actively used in the study of various aspects of gravitational theory and cosmology (see, for example, refs. [12–23]).

Another method for exact integration of the equations of motion for a test particle (the method of non-commutative integration (NCIM)) was proposed in [24]. This method is applied to spaces admitting non-commutative groups of motion $G_r(r), r \geq 3$ (see A. Petrov [25]). It allows for reducing the equations of motion to systems of ordinary differential equations. By analogy with Stackel spaces, we call them poststack spaces (PSS). PSS are also actively studied in gravitational theory and cosmology (see, e.g., [26–34]). The classification of electromagnetic fields in which the Klein–Gordon–Fock equations and Hamilton–Jacobi equations admit non-commutative algebras of symmetry operators for a charged sample particle was carried out in [35–38].

Commutative and non-commutative integration methods have a similar classification problem, namely enumerating all non-equivalent metrics and electromagnetic potentials satisfying the requirements of the given symmetry. For Stackel spaces, the problem of classifying admissible external electromagnetic fields and electrovacuum solutions of the Einstein–Maxwell equations was solved in [39].

In previous works ([40–42]), the non-null PSS of all types were considered according to the Bianchi classification, except type $VIII$. In the present work, all non-equivalent exact

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solutions of Maxwell's vacuum equations for non-null PSS of type $VIII$ are obtained. Thus, this classification is complete for all non-null PSS.

2. Admissible Electromagnetic Fields in Homogeneous Spaces

According to its definition (see [43]), the space-time V_4 is homogeneous if its metric can be represented in a semi-geodesic coordinate system as follows:

$$ds^2 = -du^{0^2} + \eta_{ab}e^a_\alpha e^b_\beta du^\alpha du^\beta, \quad g_{ij} = -\delta^0_i\delta^0_j + \delta^\alpha_i\delta^\beta_j e^a_\alpha e^b_\beta \eta_{ab}(u^0), \quad det|\eta_{ab}| = \eta^2 > 0, \quad \dot{e}^a_\alpha = 0, \tag{1}$$

where the condition

$$[Y_a, Y_b] = C^c_{ab}Y_c, \quad Y_a = e^\alpha_a \hat{\partial}_\alpha \tag{2}$$

is satisfied. Here, e^α_a are the triad of the dual vectors:

$$e^b_\alpha e^\alpha_a = \delta^b_a \tag{3}$$

and C^a_{bc} are structural constants of the group $G_3(N)$, which acts on V_4. The vectors of the frame e^a_α define a non-holonomic coordinate system in the hypersurface of transitivity V_3 of the group $G_3(N)$. Here and elsewhere, dots denote the derivatives of the variable u^0. The coordinate indices of the semi-geodesic coordinate system are denoted by the letters $i, j, k = 0, 1, \ldots 3$. The variables of the local coordinate system on V_3 are provided with indices $\alpha, \beta, \gamma = 1, \ldots 3$. Indices of a non-holonomic frame are provided with the indices $a, b, c = 1, \ldots 3$. The rule is used according to which of the the repeating upper and lower indices are summarized within the index range.

It has been proven in the paper [36] that for a charged test particle moving in the external electromagnetic field with potential A_i, the Hamilton–Jacobi equation:

$$g^{ij}(p_i + A_i)(p_j + A_j) = m^2 \quad (p_i + A_i = P_i), \tag{4}$$

and the Klein–Gordon–Fock equation:

$$\hat{H}\varphi = (g^{ij}(-i\hat{p}_l + A_l)(-i\hat{p}_j + A_j) = m^2\varphi \quad (-i\hat{p}_j + A_j = \hat{P}_j) \tag{5}$$

admit the integrals of motion

$$X_\alpha = \xi^i_\alpha p_i \quad (or \quad \hat{X}_\alpha = \xi^i_\alpha \hat{p}_i), \tag{6}$$

if and only if the condition

$$\xi^\alpha_a(\xi^\beta_b A_\beta)_{,\alpha} = C^c_{ab}\xi^\beta_b A_\beta \tag{7}$$

is satisfied. Here, $p_i = \partial_i\varphi$, $\hat{p}_k = -\imath\hat{\nabla}_k$ ($\hat{\nabla}_k$ is the covariant derivative operator corresponding to the partial derivative operator $\hat{\partial}_i$ and φ is a scalar function of the particle with mass m), ξ^i_α is the Killing vector, and C^c_{ab} are structural constants:

$$[\hat{X}_a, \hat{X}_b] = C^c_{ab}\hat{X}_c.$$

If A_i satisfies condition (7), the electromagnetic field is called admissible. All admissible electromagnetic fields for groups of motion $G_r(N)(r \geq 3)$ acting transitively on hypersurfaces of space-time have been found in [36–38].

Let us show that solutions of the system of Equation (7) for HPSS of type $VIII$ can be represented in the form:

$$A_\alpha = \alpha_a(u^0)e^a_\alpha \Rightarrow \mathbf{A}_a = e^\alpha_a A_\alpha = \alpha_a(u^0). \tag{8}$$

To prove this, let us find the frame vector using the metric tensor of Bianchi's $VIII$-type space (see [25]).

$$\begin{aligned}ds^2 = du^{1^2}a_{11} + 2du^1du^2(a_{11}u^{1^2} - 2a_{13}u^1 + a_{12})\exp(-u^3) + 2du^1du^3(a_{13} - a_{11}u^1)\\ +du^{2^2}(a_{11}u^{1^4} - 4a_{13}u^{1^3} + 2(a_{12} + 2a_{33})u^{1^2} - 4a_{23}u^1 + a_{22})\exp(-2u^3)\\ 2du^2du^3(-a_{11}u^{1^3} + 3a_{13}u^{1^2} - 2(a_{12} + 2a_{33})u^1 + a_{23})\exp(-u^2) + du^{3^2}(a_{11}u^{1^2} - 2a_{13}u^1 + a_{33}) + \varepsilon du^{0^2}.\end{aligned} \tag{9}$$

where a_{ab} are arbitrary functions on $u^0, \varepsilon^2 = 1$. To obtain the functions e^α_a, it is sufficient to consider the components g_{11}, g_{12}, and g_{13} from system (1). The solution can be represented in the form:

$$e^a_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ u^{1^2}\exp(-u^1) & \exp(-u^3) & -2u^1\exp(-u^3) \\ -u^1 & 0 & 1 \end{pmatrix}, e^\alpha_a = \begin{pmatrix} 1 & 0 & 0 \\ u^{1^2} & \exp(u^3) & 2u^1 \\ u^1 & 0 & 1 \end{pmatrix}. \tag{10}$$

The lower index numbers the lines. The solution of the system of Equation (7) has been found in [36]. It has the form:

$$A_1 = \alpha_0(u^0), \quad A_2 = (\alpha_0 u^{1^2} + 2\beta_0(u^0)u^1 + \gamma_0(u^0)), \quad A_3 = -(\alpha_0 u^1 + \beta_0).$$

By denoting: $\alpha_0 = \alpha_1, \quad \gamma_0 = \alpha_2, \quad \beta_0 = -\alpha_3$, we obtain (8).

3. Maxwell's Equations

All exact solutions of empty Maxwell's equations for solvable groups have been found in papers [40,41]. The present paper solves the problem for the group $G_3(VIII)$.

Consider empty Maxwell's equations for an admissible electromagnetic field in homogeneous space with a group of motions G_r:

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}F^{ij})_{,j} = 0. \tag{11}$$

The metric tensor and the electromagnetic potential are defined by relations (1) and (8). When $i = 0$, from Equation (11) it follows:

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}F^{\alpha}_{0.})_{\alpha} = \frac{1}{e}(e^\alpha_a e\eta^{ab}\dot{\alpha}_b)_{,\alpha} = \rho_a\frac{(\eta^{ab}\eta\dot{\alpha}_b)}{\eta} = 0 \quad (\rho_a = e^\alpha_{a,\alpha} + e^\alpha_a e_{,\alpha}/e). \tag{12}$$

Here, it is denoted:

$$g = -\det||g_{\alpha\beta}|| = -(\eta e)^2, \quad where \quad \eta^2 = \det||\eta_{\alpha\beta}||, \quad e = \det||e^a_\alpha||.$$

Let $i = \alpha$. Then, from Equation (11), it follows:

$$\frac{1}{\eta}(\eta F^{\alpha}_{0.})_{,0} = \frac{1}{e}(eF^{\beta\alpha})_{,\beta} \Rightarrow \frac{1}{\eta}(\eta\eta^{ab}e^\alpha_a\dot{\alpha}_b)_{,0} = \frac{1}{e}(e^\beta_b\eta^{ab}e^\alpha_{\tilde{a}}e^\gamma_{\tilde{b}}\eta^{\tilde{a}\tilde{b}}F_{\beta\gamma}ee^\nu_a)_{,\nu} \Rightarrow \tag{13}$$

$$e(\dot{\alpha}_b\eta\eta^{ab})_{,0} = \eta e^a_\alpha(ee^\beta_b e^\alpha_{\tilde{a}_1}e^\gamma_{\tilde{b}}F_{\beta\gamma})_{|a_1}\eta^{a_1 b}\eta^{\tilde{a}\tilde{b}}. \tag{14}$$

Let us find components of $F_{\alpha\beta}$ using relation (8).

$$F_{\alpha\beta} = (e^a_{\beta,\alpha} - e^a_{\beta,\alpha})\alpha_a = e^c_\beta e^\gamma_c e^d_\alpha e^\nu_d(e^a_{\gamma,\nu} - e^a_{\nu,\gamma})\alpha_a = e^b_\beta e^a_\alpha e^c_\gamma(e^\gamma_{a|b} - e^\gamma_{b|a})\alpha_c = e^b_\beta e^a_\alpha C^c_{ba}\alpha_c. \tag{15}$$

Then,

$$(eF^{\alpha\beta})_{,\beta} = \eta^{ab}\eta^{\tilde{a}\tilde{b}}C^d_{\tilde{b}b}\alpha_d((ee^\alpha_a)_{|\tilde{a}} + ee^\alpha_a e^\gamma_{\tilde{a},\gamma}). \tag{16}$$

We present the structural constants of a group G_3 in the form:

$$C^c_{ab} = C^c_{12}\varepsilon^{12}_{\tilde{a}\tilde{b}} + C^c_{13}\varepsilon^{13}_{\tilde{a}\tilde{b}} + C^c_{23}\varepsilon^{23}_{\tilde{a}\tilde{b}}, \tag{17}$$

where
$$\varepsilon^{AB}_{ab} = \delta^A_a \delta^B_b - \delta^A_b \delta^B_a.$$

Let us denote:
$$\sigma_1 = C^a_{23}\alpha_a, \quad \sigma_2 = C^a_{31}\alpha_a, \quad \sigma_3 = C^a_{12}\alpha_a;$$
$$\begin{cases} \gamma_1 = \sigma_1\eta_{11} + \sigma_2\eta_{12} + \sigma_3\eta_{13}, \\ \gamma_2 = \sigma_1\eta_{12} + \sigma_2\eta_{22} + \sigma_3\eta_{23}, \\ \gamma_3 = \sigma_1\eta_{13} + \sigma_2\eta_{23} + \sigma_3\eta_{33}. \end{cases}$$

Equation (16) will take the form:

$$\eta(\eta\eta^{ab}\dot{\alpha}_b)_{,0} = \delta^a_1(\gamma_1(C^1_{32}) - \gamma_2(C^1_{31} + \rho_3) + \gamma_3(C^1_{21} + \rho_2)) + \delta^a_2(\gamma_1(C^2_{32} + \rho_3) + \gamma_2 C^2_{13} - \gamma_3(C^2_{12} + \rho_1)) + \delta^a_3(-\gamma_1(C^3_{23} + \rho_2) + \gamma_2(C^3_{13} + \rho_1) + \gamma_3 C^3_{21}), \tag{18}$$

$$\rho_a \eta^{ab}\dot{\alpha}_b = 0. \tag{19}$$

To decrease the order of Equation (18), we introduce new independent functions:

$$b^a = \delta^c_a b_c = \eta\eta^{ab}\dot{\alpha}_b \quad \Rightarrow \quad \eta\dot{\alpha}_a = \eta_{ab}b^b. \tag{20}$$

Let us introduce the function:

$$n_{ab} = n_{ab}(u^0) = \frac{\eta_{ab}}{\eta} \quad \Rightarrow \quad \det n_{ab} = n = \frac{1}{\eta}. \tag{21}$$

Then, Maxwell's Equations (18) and (21) take the form of a system of linear algebraic equations on the unknown functions n_{ab}:

$$\dot{b}^a = \delta^a_1(\tilde{\gamma}_1(C^1_{32}) - \tilde{\gamma}_2(C^1_{31} + \rho_3) + \tilde{\gamma}_3(C^1_{21} + \rho_2)) + \delta^a_2(\tilde{\gamma}_1(C^2_{32} + \rho_3) + \tilde{\gamma}_2 C^2_{13} - \tilde{\gamma}_3(C^2_{12} + \rho_1)) + \delta^a_3(-\tilde{\gamma}_1(C^3_{23} + \rho_2) + \tilde{\gamma}_2(C^3_{13} + \rho_1) + \tilde{\gamma}_3 C^3_{21}) \quad (\tilde{\gamma}_a = n\gamma_a), \tag{22}$$

$$\dot{\alpha}_a = n_{ab}b^b. \tag{23}$$

Equation (19):
$$\rho_a b^a = 0 \tag{24}$$
is a restriction on the function b^a (if $\rho_a \neq 0$). Let us obtain the Maxwell's equations for the group $G_3(VIII)$. Non-zero structural constants, in this case, have the form:

$$C^3_{12} = 2, \quad C^1_{13} = 1, \quad C^2_{32} = 1 \Rightarrow \tag{25}$$

From here, it follows that

$$\sigma_1 = -\alpha_2, \quad \sigma_2 = -\alpha_1, \quad \sigma_3 = 2\alpha_3.$$

Using these relations, we obtain Maxwell's Equation (18) in the form:

$$\hat{B}\hat{n} = \hat{\omega}, \tag{26}$$

where
$$\hat{B} = \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & a_1 & 0 & a_2 & a_3 & 0 \\ 0 & b_1 & 0 & b_2 & b_3 & 0 \\ 0 & 0 & a_1 & 0 & a_2 & a_3 \\ 0 & 0 & b_1 & 0 & b_2 & b_3 \end{pmatrix}, \tag{27}$$

$$\hat{n}^T = (n_{11}, n_{12}, n_{13}, n_{22}, n_{23}, n_{33}); \quad \hat{\omega}^T = (-\dot{b}_2, \dot{a}_2, -\dot{b}_1, \dot{a}_1, \frac{\dot{b}_3}{2}, -\frac{\dot{a}_3}{2}).$$

Hereafter, the following notations are used:

$$\alpha_1 = a_2, \quad \alpha_2 = a_1, \quad \alpha_3 = -\frac{a_3}{2}. \tag{28}$$

Let us find the algebraic complement of the matrix $\hat{B}$:

$$\hat{V} = \begin{pmatrix} b_1 v_1^2 & -a_1 v_1^2 & b_2 v_1^2 & -a_2 v_1^2 & b_3 v_1^2 & -a_3 V_1^2 \\ b_1 v_1 v_2 & -a_1 v_1 v_2 & b_2 v_1 v_2 & -a_2 v_1 v_2 & b_3 v_1 v_2 & -a_3 v_1 v_2 \\ b_1 v_1 v_3 & -a_1 v_1 v_3 & b_2 v_1 V v_3 & -a_2 v_1 v_3 & b_3 v_1 v_3 & -a_3 v_1 v_3 \\ b_1 v_2^2 & -a_1 v_2^2 & b_2 v_2^2 & -a_2 v_2^2 & b_3 v_2^2 & -a_3 v_2^2 \\ b_1 v_2 v_3 & -a_1 v_2 v_3 & b_2 v_2 v_3 & -a_2 v_2 v_3 & b_3 v_2 v_3 & -a_3 v_2 v_3 \\ b_1 v_3^2 & -a_1 v_3^2 & b_2 v_3^2 & -a_2 v_3^2 & b_3 v_3^2 & -a_3 V_3^2 \end{pmatrix} \tag{29}$$

$$v_1 = a_2 b_3 - a_3 b_2, \quad v_2 = a_3 b_2 - a_2 b_3, \quad v_3 = a_1 b_2 - a_2 b_1.$$

As $\hat{B}$ is a singular matrix, $\hat{V}$ is the annulling matrix for $\hat{B}$:

$$\hat{V}\hat{B} = 0. \tag{30}$$

Therefore, when ${v_1}^2 + {v_2}^2 + {v_3}^2 \neq 0,$ one of the equations from system (26) can be replaced by the equation:

$${a_3}^2 + {b_3}^2 = 4(a_1 a_2 + b_1 b_2 + c) \quad (c = const). \tag{31}$$

Depending on the rank of the matrix $\hat{B}$, one or more functions $n_{ab}(u^0)$ are independent. It is possible to express the remaining functions n_{ab} through the functions a_a, b_a. To find non-equivalent solutions of the system (26), one should consider the following variants:

1. $a_1 \neq 0$; 2. $a_1 = 0,\ a_2 \neq 0$; 3. $a_1 = a_2 = 0, a_3 \neq 0$. Taking this observation into account, let us consider all non-equivalent options.

4. Solutions of Maxwell Equations

Since the functions a_a satisfy the condition:

$$a_1^2 + a_2^2 + a_3^2 \neq 0,$$

the *rank* of matrix (29) cannot be less than three if

$$v_1^2 + v_2^2 + v_3^2 \neq 0 \Rightarrow rank||\hat{B}|| = 5.$$

In order to obtain a complete solution to the classification problem, it is necessary:

(I) To consider all non-equivalent variants with non-zero minors of $rank = 5$ of the matrix $\hat{B}$;

(II) To consider all non-equivalent variants under the condition: $v_a = 0$ $(rank \leq 3)$.

The components of the matrix $\hat{\eta}$ and the functions α_a are given by formulae (21) and (28). In view of these circumstances, let us list all exact solutions of empty Maxwell equations for PSS of type VIII.

I. $rank||\hat{B}|| = 5$.

1. $a_1 v_1 \neq 0 \Rightarrow$ the minor $\hat{B}_{12}$ and its inverse matrix $\hat{P} = \hat{B}_{12}^{-1}$ have the form:

$$\hat{B}_{12} = \begin{pmatrix} a_2 & a_3 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & a_3 & 0 \\ b_1 & 0 & b_2 & b_3 & 0 \\ 0 & a_1 & 0 & a_2 & a_3 \\ 0 & b_1 & 0 & b_2 & b_3 \end{pmatrix}, \tag{32}$$

$$\hat{P} = \begin{pmatrix} -\frac{v_2}{a_1 v_1} & -\frac{a_3 b_2}{\alpha_1 v_1} & \frac{a_2 a_3}{a_1 v_1} & -\frac{a_3 b_3}{a_1 v_1} & \frac{a_3^2}{a_1 v_1} \\ -\frac{V_3}{a_1 v_1} & \frac{a_2 b_2}{a_1 v_1} & -\frac{a_2^2}{a_1 v_1} & \frac{a_2 b_3}{a_1 v_1} & -\frac{a_2 a_3}{a_1 v_1} \\ -\frac{V_2^2}{a_1 v_1^2} & \frac{(a_3 b_1 v_1 - a_2 b_3 v_3)}{a_1 v_1^2} & \frac{a_3(a_2 v_2 - a_1 v_1)}{a_1 v_1^2} & -\frac{a_3 b_3 v_2}{a_1 v_1^2} & \frac{a_2^2 v_2}{a_1 v_1^2} \\ -\frac{v_2 v_3}{a_1 v_1^2} & \frac{a_2 b_2 v_2}{a_1 v_1^2} & -\frac{a_2^2 V_2}{a_1 v_1^2} & -\frac{a_3 b_3 v_3}{a_1 v_1^2} & \frac{a_3^2 v_3}{a_1 v_1^2} \\ -\frac{v_3^2}{a_1 v_1^2} & \frac{a_2 b_2 v_3}{a_1 v_1^2} & -\frac{a_2^2 v_3^2}{a_1 v_1^2} & \frac{(a_3 b_2 v_3 - a_2 b_1 v_1)}{a_1 v_1^2} & \frac{a_2(a_1 v_1 - a_3 v_3)}{a_1 v_1^2} \end{pmatrix} \tag{33}$$

Then, the solution to Equation (26) is as follows:

$$\hat{n}_1 = \hat{P}_1 \hat{\omega}_1, \tag{34}$$

where

$$\hat{n}_1^T = (n_{12}, n_{13}, n_{22}, n_{23}, n_{33});$$

$$\hat{\omega}_1^T = (-(\dot{b}_2 + a_1 n_{11}), -\dot{b}_1, \dot{a}_1, \frac{\dot{b}_3}{2}, -\frac{\dot{a}_3}{2}).$$

Functions n_{11}, a_a, and b_a are arbitrary functions of u^0 that obey condition (31).

2. $a_2 v_1 \neq 0$. Obviously, we obtain a non-equivalent solution to the previous one only if $a_1 = 0$. In order to implement the classification, a similar choice should be made for all other variants. The matrix $\hat{B}_{14}$ and its inverse matrix $\hat{P}_2 = \hat{B}_{14}^{-1}$ have the form:

$$\hat{B}_{14} = \begin{pmatrix} a_2 & \alpha_3 & 0 & 0 & 0 \\ b_2 & b_3 & 0 & 0 & 0 \\ 0 & 0 & a_2 & a_3 & 0 \\ 0 & 0 & 0 & a_2 & a_3 \\ 0 & b_1 & 0 & b_2 & b_3 \end{pmatrix}, \quad \hat{P}_2 = \begin{pmatrix} \frac{b_3}{v_1} & -\frac{a_3}{v_1} & 0 & 0 & 0 \\ -\frac{b_2}{v_1} & \frac{\alpha_2}{v_1} & 0 & 0 & 0 \\ \frac{a_3^2 b_1 b_2}{a_2 v_1^2} & -\frac{a_3^2 b_1}{v_1^2} & \frac{1}{a_2} & -\frac{a_3 b_3}{a_2 v_1} & \frac{a_3^2}{a_2 v_1} \\ -\frac{a_3 b_1 b_2}{v_1^2} & \frac{a_2 a_3 b_1}{v_1^2} & 0 & \frac{b_3}{v_1} & -\frac{a_3}{v_1} \\ \frac{a_2 b_1 b_2}{v_1^2} & -\frac{a_2^2 b_1}{v_1^2} & 0 & -\frac{b_2}{v_1} & \frac{a_2}{v_1} \end{pmatrix} \tag{35}$$

Then, the solution to Equation (26) is as follows:

$$\hat{n}_2 = \hat{P}_2 \hat{\omega}_2, \tag{36}$$

where

$$\hat{n}_2^T = (n_{12}, n_{13}, n_{22}, n_{23}, n_{33});$$

$$\hat{\omega}_2 = (-\dot{b}_2, (\dot{a}_2 - b_1 n_{11}), -\dot{b}_1, \frac{\dot{b}_3}{2}, -\frac{\dot{a}_3}{2}).$$

Functions n_{11}, a_a, and β_a are arbitrary functions of u^0 that obey condition (31).

3. $a_3 v_1 \neq 0 \Rightarrow a_1 = a_2 = 0 \Rightarrow$ the minor $\hat{B}_{16}^{-1}$ and its inverse matrix $\hat{P}_3 = \hat{B}_{16}^{-1}$ have the form:

$$\hat{B}_{16} = \begin{pmatrix} 0 & a_3 & 0 & 0 & 0 \\ b_2 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 \\ b_1 & 0 & b_2 & b_3 & 0 \\ 0 & 0 & 0 & 0 & a_3 \end{pmatrix}, \quad \hat{P}_3 = \begin{pmatrix} -\frac{b_3}{a_3 b_2} & \frac{1}{b_3} & 0 & 0 & 0 \\ \frac{1}{a_3} & 0 & 0 & 0 & 0 \\ \frac{b_1 b_3}{a_3 b_2^2} & -\frac{b_1}{b_2^2} & -\frac{b_3}{b_2 a_3} & \frac{1}{b_2} & 0 \\ 0 & 0 & \frac{1}{a_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a_3} \end{pmatrix} \tag{37}$$

Then, the solution to Equation (26) is as follows:

$$\hat{n}_3 = \hat{P}_3 \hat{\omega}_3, \tag{38}$$

where

$$\hat{n}_3^T = (n_{12}, n_{13}, n_{22}, n_{23}, n_{33}), \quad \hat{\omega}_3 = (-\dot{b}_2, (\dot{a}_2 - b_1 n_{11}), -\dot{b}_1, 0, \frac{\dot{b}_3}{2}). \tag{39}$$

4. $\alpha_1 v_2 \neq 0, \Rightarrow v_1 = 0 \Rightarrow$ the minor $\hat{B}_{24}^{-1}$ and its inverse matrix $\hat{P}_4 = \hat{B}_{24}^{-1}$ have the form:

$$\hat{B}_{24} = \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_1 & 0 & a_3 & 0 \\ 0 & b_1 & 0 & b_3 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 \\ 0 & 0 & b_1 & b_2 & b_3 \end{pmatrix}, \quad \hat{P}_4 = \begin{pmatrix} \frac{1}{\alpha_1} & \frac{a_2 b_3}{a_1 v_2} & -\frac{a_2 a_3}{a_1 v_2} & \frac{a_3 b_3}{a_1 v_2} & -\frac{a_3^2}{a_1 v_2} \\ 0 & -\frac{b_3}{v_2} & \frac{a_3}{v_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{b_3}{v_2} & \frac{a_3}{v_2} \\ 0 & \frac{b_1}{v_2} & -\frac{a_1}{v_2} & 0 & 0 \\ 0 & \frac{b_1 v_3}{v_2^2} & -\frac{a_1 v_3}{v_2} & \frac{b_1}{v_2} & -\frac{a_1}{v_2} \end{pmatrix} \tag{40}$$

Then, the solution to Equation (26) is as follows:

$$\hat{n}_4 = \hat{P}_4 \hat{\omega}_4, \tag{41}$$

where

$$\hat{n}_4^T = (n_{11}, n_{12}, n_{13}, n_{23}, n_{33}), \quad \hat{\omega}_4 = (-\dot{b}_2, -(\dot{b}_1 + a_2 n_{22}), (\dot{a}_1 - b_2 n_{22}), \frac{\dot{b}_3}{2}, -\frac{\dot{a}_3}{2}) \tag{42}$$

Functions n_{22}, a_a, and β_a are arbitrary functions of u^0 that obey condition (31) and $a_2 \beta_3 = a_3 \beta_2$.

5. $\alpha_2 V_2 \neq 0, \Rightarrow a_1 = V_1 = 0 \Rightarrow$ the minor $\hat{B}_{44}^{-1}$ and its inverse matrix $\hat{P}_5 = \hat{W}_{44}^{-1}$ have the form:

$$\hat{B}_{44} = \begin{pmatrix} 0 & a_2 & a_3 & 0 & 0 \\ b_1 & b_2 & b_3 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_2 & a_3 \\ 0 & 0 & b_1 & b_2 & b_3 \end{pmatrix}, \quad \hat{P}_5 = \begin{pmatrix} -\frac{b_2}{b_1 a_2} & \frac{1}{b_1} & 0 & 0 & 0 \\ \frac{1}{a_2} & 0 & 0 & \frac{b_3}{a_2 b_1} & -\frac{a_3}{a_2 b_1} \\ 0 & 0 & 0 & -\frac{b_3}{a_3 b_1} & \frac{1}{b_1} \\ 0 & 0 & \frac{1}{a_3} & 0 & 0 \\ 0 & 0 & -\frac{a_2}{a_3^2} & \frac{1}{a_3} & 0 \end{pmatrix} \tag{43}$$

Then, the solution to Equation (26) is as follows:

$$\hat{n}_5 = \hat{P}_2 \hat{\omega}_5, \tag{44}$$

where

$$\hat{n}_5^T = (n_{11}, n_{12}, n_{13}, n_{23}, n_{33}); \quad \hat{\omega}_5 = (-\dot{b}_2, \dot{\alpha}_2, -(\dot{b}_1 + a_2 n_{22}), \frac{\dot{b}_3}{2}, -\frac{\dot{\alpha}_3}{2}) \tag{45}$$

Functions n_{22}, a_a, and b_a are arbitrary functions of u^0 that obey condition (31) and $a_2 b_3 = a_3 b_2$.

6. $a_3 v_2 \neq 0,\ v_1 = 0, \Rightarrow a_1 = a_2 = b_2 = 0$. From condition (31) it follows:

$$a_3 = c \cos 2\varphi, \quad b_3 = c \sin 2\varphi,$$

where φ is an arbitrary function of u^0. The minor $\hat{B}_{46}^{-1}$ and its inverse matrix $\hat{P}_6 = \hat{B}_{46}^{-1}$ have the form:

$$\hat{B}_{64} = \begin{pmatrix} 0 & 0 & c\cos\varphi & 0 & 0 \\ b_1 & 0 & & 0 & 0 \\ 0 & 0 & 0 & c\cos\varphi & 0 \\ 0 & b_1 & 0 & c\sin\varphi & 0 \\ 0 & 0 & 0 & 0 & c\cos\varphi \end{pmatrix}, \hat{\Omega}_6 = \begin{pmatrix} -\frac{\sin\varphi}{b_1\cos\varphi} & \frac{1}{b_1} & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sin\varphi}{b_1\cos\varphi} & \frac{1}{b_1} & 0 \\ \frac{1}{c\cos\varphi} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{c\cos\varphi} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c\cos\varphi} \end{pmatrix}. \tag{46}$$

Then, the solution to Equation (26) is as follows:

$$\hat{n}_6 = \hat{P}_6\hat{\omega}_6, \tag{47}$$

where

$$\hat{n}_6^T = (n_{11}, n_{12}, n_{13}, n_{23}, n_{33}); \quad \hat{\omega}_6 = (0, 0, -\dot{b}_1, 0, c\dot{\varphi}\cos\varphi).$$

Functions n_{22}, b_1, and φ are arbitrary functions of u^0.

7. $a_1 v_3 \neq 0 \Rightarrow v_1 = v_2 = 0$, otherwise, we obtain a solution equivalent to the previous ones. As $v_3 \neq 0 \Rightarrow a_3 = b_3 = 0$, the minor $\hat{B}_{26}$ and its inverse matrix $\hat{P}_7 = \hat{B}_{26}^{-1}$ have the form:

$$\hat{B}_{26} = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & a_2 & 0 \\ 0 & b_1 & 0 & b_2 & 0 \\ 0 & 0 & \alpha_1 & 0 & \alpha_2 \\ 0 & 0 & b_1 & 0 & b_2 \end{pmatrix}, \quad \hat{P}_7 = \begin{pmatrix} \frac{1}{\alpha_1} & -\frac{\alpha_2 b_2}{\alpha_1 v_3} & \frac{\alpha_2^2}{\alpha_1 v_3} & 0 & 0 \\ 0 & \frac{b_2}{v_3} & -\frac{\alpha_2}{v_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{b_2}{v_3} & -\frac{\alpha_2}{v_3} \\ 0 & -\frac{b_1}{v_3} & \frac{\alpha_1}{v_3} & 0 & 0 \\ 0 & 0 & 0 & -\frac{b_1}{v_3} & \frac{\alpha_1}{v_3} \end{pmatrix}. \tag{48}$$

Then, the solution to Equation (26) is as follows:

$$\hat{n}_{3a} = \hat{P}_7\hat{\omega}_7. \tag{49}$$

where

$$\hat{n}_7^T = (n_{11}, n_{12}, n_{13}, n_{22}, n_{23});$$
$$\hat{\omega}_7^T = (-\dot{b}_2, -\dot{b}_1, \dot{a}_1, 0, 0).$$

8. $a_2 v_3 \neq 0 \Rightarrow a_1 = v_1 = v_2 = 0$, otherwise, we obtain a solution equivalent to the previous ones. As $v_3 \neq 0 \Rightarrow \alpha_3 = b_3 = 0$, the minor $\hat{B}_{64}$ and its inverse matrix $\hat{P}_8 = \hat{B}_{64}^{-1}$ have the form:

$$\hat{B}_{64} = \begin{pmatrix} 0 & \alpha_2 & 0 & 0 & 0 \\ b_1 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & b_1 & 0 & b_2 \end{pmatrix}, \quad \hat{P}_8 = \begin{pmatrix} -\frac{b_2}{a_2 b_1} & -\frac{1}{b_1} & 0 & 0 & 0 \\ \frac{1}{a_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{b_2}{b_1 a_2} & \frac{1}{b_1} \\ 0 & 0 & \frac{1}{a_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a_2} & 0 \end{pmatrix}. \tag{50}$$

Then, the solution to Equation (26) is as follows:

$$\hat{n}_8 = \hat{P}_8\hat{\omega}_8. \tag{51}$$

where

$$\hat{n}_8^T = (n_{11}, n_{12}, n_{13}, n_{22}, n_{23}), \quad \hat{\omega}_8^T = (-\dot{b}_2, -\dot{b}_1, 0, 0, 0).$$

Functions n_{33}, $a_2\beta_1$, and β_2 are arbitrary functions of u^0 that obey condition (31).

II. $rank||\hat{B}|| < 5$

9. $v_a = 0$. Let us represent the system of Maxwell's equations in the form:

$$\hat{Q}\hat{n}_I = \hat{\omega}_I, \tag{52}$$

where

$$\hat{Q} = \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 \\ 0 & a_1 & 0 & a_2 & a_3 & 0 \\ 0 & 0 & a_1 & 0 & a_2 & a_3 \\ b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & b_1 & 0 & b_2 & b_3 & 0 \\ 0 & 0 & b_1 & 0 & b_2 & b_3 \end{pmatrix},$$

$$\hat{\omega}_I = (\hat{\omega}_\beta, \hat{\omega}_\alpha); \quad \hat{\omega}_\beta^T = (-\dot{b}_2, -\dot{b}_1, \frac{\dot{b}_3}{2}), \quad \hat{\omega}_\alpha^T = (\dot{a}_2, \dot{a}_1, -\frac{\dot{a}_3}{2})$$

$$\hat{n}_I = (\hat{n}_\alpha, \hat{b}_\alpha); \quad \hat{n}_\alpha^T = (n_{11}, n_{12}, n_{13}), \quad \hat{n}_\beta^T = (n_{22}, n_{23}, n_{33})$$

Consider all possible options.

(a) $a_1 \neq 0 \Rightarrow b_a = \frac{\alpha_a b_1}{\alpha_1}$. Maxwell's Equation (52) take the form:

$$\hat{B}_I \hat{n}_\alpha = (\hat{\omega}_\beta - \hat{B}_{II}\hat{n}_\beta) \Rightarrow \hat{n}_\alpha = \hat{B}_I^{-1}(\hat{\omega}_\beta - \hat{B}_{II}\hat{n}_\beta),$$

$$b_1 \hat{B}_I \hat{n}_\alpha = a_1 \hat{\omega}_\alpha - b_1 \hat{B}_{II}\hat{n}_\beta \Rightarrow b_1\hat{\omega}_\beta - a_1\hat{\omega}_\alpha = 0 \Rightarrow$$

$$\begin{cases} a_1\dot{a}_2 + b_1\dot{b}_2 = 0, \\ a_1\dot{a}_3 + b_1\dot{b}_3 = 0, \\ a_1\dot{a}_1 + b_1\dot{b}_1 = 0. \end{cases} \tag{53}$$

Here,

$$\hat{B}_I = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{pmatrix}, \hat{B}_I^{-1} = \begin{pmatrix} \frac{1}{a_1} & -\frac{a_2}{a_1^2} & -\frac{a_3}{a_1^2} \\ 0 & \frac{1}{a_1} & 0 \\ 0 & 0 & \frac{1}{a_1} \end{pmatrix}, \hat{B}_{II} = \begin{pmatrix} 0 & 0 & 0 \\ a_2 & a_3 & 0 \\ 0 & a_2 & a_3, \end{pmatrix}$$

From the last equation of system (53) it follows that

$$a_1 = e_0 \sin\varphi, \quad b_1 = e_0 \cos\varphi, \quad e_0 = const.$$

Thus, $b_2 = a_2 \frac{\cos\varphi}{\sin\varphi}$ and $b_3 = a_3 \frac{\cos\varphi}{\sin\varphi}$, and from the previous equations, it follows that

$$a_a = e_0 q_a \sin\varphi, \quad b_a = e_0 q_a \cos\varphi, \quad q_a = const, \quad q_1 = 1.$$

Then, matrices $\hat{B}_I$, $\hat{B}_I^{-1}$, and $\hat{B}_{II}$ and line $\hat{\omega}^T$ take the form:

$$\hat{B}_I = \hat{w}_1 \sin\varphi, \quad \hat{B}_I^{-1} = \frac{1}{\sin\varphi}\hat{w}_1^{-1}, \quad \hat{B}_{II} = \hat{w}_2 \sin\varphi.$$

$$\hat{w}_1 = \begin{pmatrix} 1 & q_2 & q_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{w}_1^{-1} = \begin{pmatrix} 1 & -q_2 & -q_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{w}_2 = \begin{pmatrix} 0 & 0 & 0 \\ q_2 & q_3 & 0 \\ 0 & q_2 & q_3, \end{pmatrix}$$

$$\hat{\omega}_\beta^T = \dot{\varphi}\hat{c}^T = \dot{\varphi}\sin\varphi(q_2, 1, -\frac{q_3}{2})$$

Then, the solution to Equation (26) is as follows:

$$\hat{n}_\alpha = \hat{w}^{-1}(\dot{\varphi}\hat{c} - \hat{q}\hat{n}_\beta)$$

(b) $a_1 = 0 \Rightarrow a_2 \neq 0$. Let us use the previous results, in which the indices 1 and 2 are reversed: $1 \Leftrightarrow 2$. The solution of Maxwell's equation has the form:

$$\hat{n}_\alpha = \hat{w}^{-1}(\dot{\varphi}\hat{c} - \hat{q}\hat{n}_\beta)$$

$$\hat{n}^T_\alpha = (n_{22}, n_{12}, n_{23}), \quad \hat{n}^T_\beta = (n_{11}, n_{13}, n_{33}),$$

$$\hat{w}^{-1} = \begin{pmatrix} 1 & 0 & -q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{q} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q, \end{pmatrix}, \quad \hat{c}^T = (0, 1, -\frac{q}{2}).$$

$$a_2 = e_0 \sin\varphi, \quad b_2 = e_0 \cos\varphi, \quad a_3 = e_0 q \sin\varphi, \quad b_3 = e_0 q \cos\varphi, \quad q = const, \quad \varphi = \varphi(u^0).$$

(c) $a_3 \neq 0$. The solutions, which are not equivalent to the previous ones, can be obtained under the conditions $a_1 = a_2 = 0 \Rightarrow b_1 = b_2 = 0$. From Maxwell's equations it follows that

$$a_3 n_{13} = a_3 n_{23} = 0, \quad a_3 n_{33} = \frac{\dot{b}_3}{2}, \quad b_3 n_{33} = -\frac{\dot{a}_3}{2} \Rightarrow a_3\dot{a}_3 + b_3\dot{b}_3 = 0.$$

The solution has the form

$$n_{33} = \dot{\varphi}, \quad n_{13} = n_{23} = a_1 = a_2 = b_1 = b_2 = 0, \quad a_3 = q\cos 2\varphi, \quad b_3 = q \sin 2\varphi.$$

Functions φ, n_{11}, n_{12}, and n_{22} are arbitrary functions on u^0.

5. Conclusions

In previous works [40–42], all non-equivalent solutions of Maxwell's empty equations for admissible electromagnetic fields in homogeneous space-time metrics of all types according to Bianchi's classification (except type $VIII$) were found. The present work completes the first stage of the classification problem formulated in the introduction. The next step is the classification of the corresponding exact solutions of the Einstein–Maxwell equations. All solutions obtained in the completed classification have a form suitable for further use and have sufficient arbitrariness so that the Einstein–Maxwell equations have nontrivial solutions. The use of the triad of frame vectors (see [43]) allows us to reduce the Einstein–Maxwell equations with the energy-momentum tensor of the admissible electromagnetic field to an overcrowded system of ordinary differential equations. To perform the classification, we need to study the coexistence conditions of these systems of equations. It is possible to use additional symmetries of homogeneous spaces and admissible electromagnetic fields (see [38]). In the future, we will begin to solve this classification problem.

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References

1. Stackel, P. Uber die intagration der Hamiltonschen differentialechung mittels separation der variablen. *Math. Ann.* **1897**, *49*, 145–147. [CrossRef]
2. Eisenhart, L.P. Separable systems of stackel. *Ann. Math.* **1934**, *35*, 284–305. [CrossRef]
3. Levi-Civita, T. Sulla Integrazione Della Equazione Di Hamilton–Jacobi Per Separazione Di Variabili. *Math. Ann.* **1904**, *59*, 383–397. [CrossRef]

4. Jarov-Jrovoy, M.S. Integration of Hamilton–Jacobi equation by complete separation of variables method. *J. Appl. Math. Mech.* **1963**, *27*, 173–219. [CrossRef]
5. Carter, B. A New family of Einstein spaces. *Phys. Lett. A* **1968**, *26*, 399–400. [CrossRef]
6. Shapovalov, V.N. Symmetry and separation of variables in the Hamilton–Jacobi equation. *Sov. Phys. J.* **1978**, *21*, 1124–1132. [CrossRef]
7. Shapovalov, V.N. Stackel's spaces. *Sib. Math. J.* **1979**, *20*, 1117–1130.; [CrossRef]
8. Miller, W. *Symmetry Furthermore, Separation Of Variables*; Cambridge University Press: Cambridge, UK, 1984; p. 318.
9. Obukhov, V.V. Hamilton–Jacobi equation for a charged test particle in the Stackel space of type (2.0). *Symmetry* **2020**, *12*, 1289. [CrossRef]
10. Obukhov, V.V. Hamilton–Jacobi equation for a charged test particle in the Stackel space of type (2.1). *Int. J. Geom. Meth. Mod. Phys.* **2020**, *14*, 2050186. [CrossRef]
11. Obukhov, V.V. Separation of variables in Hamilton–Jacobi and Klein-Gordon-Fock equations for a charged test particle in the stackel spaces of type (1.1). *Int. J. Geom. Meth. Mod. Phys.* **2021**, *18*, 2150036. [CrossRef]
12. Mitsopoulos, A.; Tsamparlis, M.; Leon, G.; Paliathanasis, A. New conservation laws and exact cosmological solutions in Brans-Dicke cosmology with an extra scalar field. *Symmetry* **2021**, *13*, 1364. [CrossRef]
13. Dappiaggi, C.; Juárez-Aubry, B.A.; Marta, A. Ground State for the Klein-Gordon field in anti-de Sitter spacetime with dynamical Wentzell boundary conditions. *Phys. Rev. D* **2022**, *105*, 105017. [CrossRef]
14. Astorga, F.; Salazar, J.F.; Zannias, T. On the integrability of the geodesic flow on a Friedmann-Robertson-Walker spacetime. *Phys. Scr.* **2018**, *93*, 085205. [CrossRef]
15. Capozziello, S.; De Laurentis, M.; Odintsov, D. Hamiltonian dynamics and Noether symmetries in extended gravity cosmology. *Eur. Phys. J. C* **2012**, *72*, 2068. [CrossRef]
16. Salih, K.; Oktay, C. Generalized cosmological constant from gauging Maxwell-conformal algebra. *Phys. Lett. B* **2020**, *803*, 135295. [CrossRef]
17. Cebecioğlu, O.; Kibaroğlu, S. Maxwell-modified metric affine gravity. *Eur. Phys. J.* **2021**, *81*, 900. [CrossRef]
18. Ildes, M.; Arik, M. Analytic solutions of Brans-Dicke cosmology: Early inflation and late time accelerated expansion. *Int. J. Mod. Phys.* **2023**, *32*, 2250131. [CrossRef]
19. Nojiri, S.; Odintsov, S.D.; Faraoni, V. Searching for dynamical black holes in various theories of gravity. *Phys. Rev. D*. **2021**, *103*, 044055. [CrossRef]
20. Epp, V.; Pervukhina, O. The Stormer problem for an aligned rotator. *MNRAS* **2018**, *474*, 5330–5339. [CrossRef]
21. Epp, V.; Masterova, M.A. Effective potential energy for relativistic particles in the field of inclined rotating magnetized sphere. *Astrophys. Space Sci.* **2014**, *353*, 473–483. [CrossRef]
22. Kumaran, Y.; Ovgun, A. Deflection angle and shadow of the reissner-nordstrom black hole with higher-order magnetic correction in einstein-nonlinear-maxwell fields. *Symmetry* **2022**, *14*, 2054. [CrossRef]
23. Osetrin, K.; Osetrin, E. Shapovalov wave-like spacetimes. *Symmetry* **2020**, *12*, 1372. [CrossRef]
24. Shapovalov, A.V.; Shirokov I.V. Noncommutative integration method for linear partial differential equations. functional algebras and dimensional reduction. *Theoret. Math. Phys.* **1996**, *106*, 1–10. [CrossRef]
25. Petrov A.Z. *Einstein Spaces*; Pergamon Press: Oxford, UK, 1969.
26. Breev A.; Shapovalov A.; Gitman D. Noncommutative eduction of Nonlinear Schredinger Equation on Lie Groups. *Universe* **2022**, 8, 445. [CrossRef]
27. Breev, A.I.; Shapovalov, A.V. Non-commutative integration of the Dirac equation in homogeneous spaces. *Symmetry* **2020**, *12*, 1867. [CrossRef]
28. Breev, A.I.; Shapovalov, A.V. Yang–Mills gauge fields conserving the symmetry algebra of the Dirac equation in a homogeneous space. *J. Phys. Conf. Ser.* **2014**, 563, 012004. [CrossRef]
29. Magazev, A.A.; Boldyreva, M.N. Schrodinger equations in electromagnetic fields: Symmetries and noncommutative integration. *Symmetry* **2021**, *13*, 1527. [CrossRef]
30. Osetrin, E.; Osetrin, K.; Filippov, A. Plane Gravitational Waves in Spatially-Homogeneous Models of type-(3.1) Stackel Spaces. *Russ. Phys. J.* **2019**, *62*, 292–301. [CrossRef]
31. Osetrin, K.; Osetrin, E.; Osetrina, E. Geodesic deviation and tidal acceleration in the gravitational wave of the Bianchi type IV universe. *Eur. Phys. J. Plus.* **2022**, *137*, 856. [CrossRef]
32. Osetrin, K.; Osetrin, E.; Osetrina, E. Gravitational wave of the Bianchi VII universe: Particle trajectories, geodesic deviation and tidal accelerations. *Eur. Phys. J. C* **2022**, *82*, 1–16. [CrossRef]
33. Alam, M.N. Cemil Tunc Constructions of the optical solitons and other solitons to the conformable fractional Zakharov-Kuznetsov equation with power law nonlinearity. *J. Taibah Univ. Sci.* **2020**, *14*, 94–100. [CrossRef]
34. Al-Asad, M.F.; Alam, N.; Tunç, C.; Sarker, M.S. Heat transport exploration of free convection flow inside enclosure having vertical wavy walls. *J. Appl. Comput. Mech.* **2021**, *7*, 520–527.
35. Magazev, A.A. Integrating Klein-Gordon-Fock equations in an extremal electromagnetic field on Lie groups. *Theor. Math. Phys.* **2012**, *173*, 1654–1667. [CrossRef]
36. Obukhov V.V. Algebra of symmetry operators for Klein-Gordon-Fock Equation. *Symmetry* **2021**, *13*, 727. [CrossRef]

37. Obukhov V.V. Algebra of the symmetry operators of the Klein-Gordon-Fock equation for the case when groups of motions G_3 act transitively on null subsurfaces of spacetime. *Symmetry* **2022**, *14*, 346. [CrossRef]
38. Obukhov V.V. Algebras of integrals of motion for the Hamilton–Jacobi and Klein-Gordon-Fock equations in spacetime with a four-parameter groups of motions in the presence of an external electromagnetic field. *J. Math. Phys.* **2022**, *63*. [CrossRef]
39. Odintsov, S.D. Editorial for Feature Papers 2021–2022. *Symmetry* **2023**, *15*, 32. [CrossRef]
40. Obukhov V.V. Maxwell Equations in Homogeneous Spaces for Admissible Electromagnetic Fields. *Universe* **2022**, *8*, 245. [CrossRef]
41. Obukhov, V.V. Maxwell Equations in Homogeneous Spaces with Solvable Groups of Motions. *Symmetry* **2022**, *14*, 2595. [CrossRef]
42. Obukhov V.V. Exact Solutions of Maxwell Equations in Homogeneous Spaces with the Group of Motions G3(IX). *Axioms* **2023**, *12*, 135. [CrossRef]
43. Landau, L.D.; Lifshits, E.M. *Theoretical Physics, Field Theory*, 7th ed.; Nauka, Chief Editorial Board for Physical and Mathematical Literature: Moscow, Russia, 1988; 512p, Volume II, ISBN 5-02-014420-7.