symmetry

MDPI

Article

On New Decomposition Theorems for Mixed-Norm Besov Spaces with Ingredient Modulus of Smoothness

Junjian Zhao [1], Marko Kostić [2] and Wei-Shih Du [3,*]

[1] School of Mathematical Sciences, Tiangong University, Tianjin 300387, China
[2] Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21125 Novi Sad, Serbia
[3] Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan
* Correspondence: wsdu@mail.nknu.edu.tw

Abstract: In this paper, we introduce and study the concept of the ingredient modulus of smoothness in component form in $L_{\vec{p}}(\mathbb{R}^d)$ and a kind of mixed-norm Sobolev space. We obtain some new properties, inequalities, and auxiliary results in mixed-norm spaces $L_{\vec{p}}(\mathbb{R}^d)$. In addition, a new concept of mixed-norm Besov space is presented and a new decomposition theorem for mixed-norm Besov spaces is established.

Keywords: mixed-norm Lebesgue space; Sobolev space; Besov space; Littlewood–Paley decomposition; ingredient modulus of smoothness in component form; mixed-norm Besov space

MSC: 41A17; 41A35; 42B08; 42C15; 42C20

check for updates

Citation: Zhao, J.; Kostić, M.; Du, W.-S. On New Decomposition Theorems for Mixed-Norm Besov Spaces with Ingredient Modulus of Smoothness. *Symmetry* **2023**, *15*, 642. https://doi.org/10.3390/sym15030642

Academic Editors: Oluwatosin Mewomo and Qiaoli Dong

Received: 7 February 2023
Revised: 20 February 2023
Accepted: 1 March 2023
Published: 3 March 2023

1. Introduction

The classical Besov space, $B^s_{p,q}(\mathbb{R}^d)$, introduced by Besov [1,2] for $s > 0$, is a family of smoothness spaces rooted in Lebesgue spaces, $L_p(\mathbb{R}^d)$, via the modulus of continuity method. Several classical function spaces, such as Lebesgue, Sobolev, and Hölder spaces, can be recovered as special cases of Besov spaces. In view of the importance of Besov space theory in some mathematical fields, such as harmonic analysis (see [3–7]), approximation theory (see [8]), the regularity of solutions of partial differential equations (see [9]), and probability and statistics (see [10]), it has been a significant research field that has attracted attention in the past few decades.

A number of problems in financial mathematics, quantum physical chemistry, and related fields are modeled on Sobolev-type function spaces (see, for example, [11,12]). For more general versions (Bessel potential) of Sobolev-type function spaces, we refer the reader to the paper of Cleanthous–Georgiadis–Nielsen [13]. For this reason, there has been an increasing interest recently in the study of linear or nonlinear approximations of Sobolev spaces and Besov spaces. To solve these important problems, the use of the theory of classical Besov spaces may be technically very difficult or its application may not give sharp and meaningful results. Therefore, the necessity to employ modifications to classical Besov spaces has arisen. It is well known that Besov spaces have been extended and generalized in many various different directions and in a variety of settings with equivalent norms, which are defined by various moduli of smoothness or different kinds of series decompositions, see, e.g., [10,11]. In order to be widely used in function theory and approximation theory, it is necessary to study the equivalent relationship between these norms of Besov spaces defined by various moduli of smoothness. Many equivalent decompositions theorems have been investigated by several authors, see [11,14–18] and the references therein. There is a rich literature on classical equivalence decompositions; however, it is worth noting that the works on equivalent decomposition theorem on Besov spaces with mixed norms are relatively few compared to classical theories.

The notion of a mixed-norm Lebesgue space, $L_{\vec{p}}(\mathbb{R}^d)$, with $\vec{p} = (p_1, \cdots, p_d) \in [1, \infty]^d$ is a natural generalization of the classical Lebesgue space $L_p(\mathbb{R}^d)$, which was originally introduced by Hörmander [3] ((3.1.3) in Section 3.1, page 125) in 1960 and Benedek and Panzone [19] (Section 1, page 301) in 1961. In fact, mixed-norm Lebesgue spaces have significant practical significance and important applications. In PDEs, functions defined by special and time quantities may belong to some mixed-norm spaces. Recently, the studies of inhomogeneous Besov spaces, Triebel–Lizorkin spaces, and Hardy spaces has been widely developed under mixed Lebesgue norms (see [15,20–23]). In addition, sampling theory has also been studied based on mixed-norm theories (see [24,25]).

In recent years, there has been growing interest in obtaining some equivalent norms of these function spaces, see, for example, [26,27]. It is well known that the modulus of smoothness has been used to approximate the ingenious measure of the structural characteristics of functions in approximation theory. A natural and important question is how to define a mixed-norm Besov space, $B^s_{\vec{p},q}(\mathbb{R}^d)$, by various equivalent norms. In [28] (Definition 4.1), we demonstrated research on mixed-norm Besov spaces defined by iterated difference-type moduli of smoothness. With the help of Littlewood–Paley decomposition theory of space, we established a sufficient condition for characterization of mixed-norm Besov spaces (for more details, see [28]).

The main contributions of this paper can be summarized as follows:

(i) We introduce and study the concept of ingredient modulus of smoothness in component form in $L_{\vec{p}}(\mathbb{R}^d)$ and establish some properties and auxiliary results in mixed-norm spaces, $L_{\vec{p}}(\mathbb{R}^d)$ (for more details, see Section 2.2).

(ii) We introduce a kind of mixed-norm Sobolev space (see Definition 4 below) and obtain useful inequalities (see Theorem 1 below).

(iii) We present a Bernstain type inequality in $L_{\vec{p}}(\mathbb{R}^d)$ sense (see Lemma 4 below) and an auxiliary inequality (see Lemma 5 below) by using the Littlewood-Paley decomposition.

(iv) We introduce and study the concept of mixed-norm Besov space $B^s_{\vec{p},q}(\mathbb{R}^d)$ (see Definition 5 below) and establish a new decomposition theorem in $B^s_{\vec{p},q}(\mathbb{R}^d)$ (see Theorem 2 below).

The special function decomposition results for the study of ingredient differential operators and electronic wave functions may instigate further research in the future.

2. Concepts, Properties and Auxiliary Results in Mixed-Norm Spaces

2.1. Preliminaries

Let us start with some fundamental notations or definitions needed in this paper. The symbols $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ will denote the set of real numbers, the set of integers, and the set of positive integers, respectively. For convenience, let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ be the set of non-negative integers. Recall that the discrete space $l_p(\mathbb{Z})$ is defined by

$$l_p(\mathbb{Z}) = \left\{ c = (c_k)_{k\in\mathbb{Z}} : \left(\sum_{k\in\mathbb{Z}} |c_k|^p \right)^{\frac{1}{p}} < \infty \right\} \text{ if } 1 \le p < \infty,$$

and

$$l_p(\mathbb{Z}) = \left\{ c = (c_k)_{k\in\mathbb{Z}} : \sup_{k\in\mathbb{Z}} |c_k| < \infty \right\} \text{ if } p = \infty.$$

For $c = \{c_k\}_{k\in\mathbb{Z}} \in l_p(\mathbb{Z})$, we define $l_p(\mathbb{Z})$-norm of c by

$$\|c\|_{l_p} = \|\{c_k\}_k\|_{l_p} = \begin{cases} \left(\sum_{k\in\mathbb{Z}} |c_k|^p \right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty; \\ \sup_{k\in\mathbb{Z}} |c_k|, & \text{if } p = \infty. \end{cases}$$

A mixed-norm Lebesgue space is a natural generalization of the classical Lebesgue space $L_p(\mathbb{R}^d)$, in which independent variables that may have different meanings are considered.

Definition 1. *(see, e.g., [19] ((3.1.3) in Section 3.1, page 125), [29] (page 357), and [18] ((6.1.3.1) on page 726)).*

Let $\vec{p} = (p_1, \ldots, p_d) \in [1, \infty]^d$ be a mixed-norm index. The mixed-norm Lebesgue space $L_{\vec{p}}(\mathbb{R}^d)$ is the set of all measurable functions f on $\mathbb{R}^d$ such that

$$\left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, \ldots, x_d)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} dx_3 \ldots dx_d \right)^{\frac{1}{p_d}} < \infty.$$

For $f \in L_{\vec{p}}(\mathbb{R}^d)$, we define the $L_{\vec{p}}(\mathbb{R}^d)$-norm of f by

$$\begin{aligned} \|f\|_{L_{\vec{p}}(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, \ldots, x_d)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} dx_3 \ldots dx_d \right)^{\frac{1}{p_d}} \\ &= \left\| \ldots \left\| \|f(x_1, x_2, \ldots, x_d)\|_{L_{p_1}(x_1)} \right\|_{L_{p_2}(x_2)} \cdots \right\|_{L_{p_d}(x_d)}. \end{aligned}$$

If $p_i = \infty$ for $i = 1, \ldots, d$, then the relevant L_{p_i}-norms are replaced by L_∞-norms. To simplify the notation $\|f\|_{L_{\vec{p}}(\mathbb{R}^d)}$, we also abbreviate this to $\|f\|_{\vec{p}}$.

Throughout this article, positive constants will be denoted by C, and they may vary at every occurrence. The Schwartz class will be stated by $\mathcal{S}(\mathbb{R}^d)$, with the Fourier transform defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx$$

for every $f \in \mathcal{S}(\mathbb{R}^d)$. Other types of Fourier transform are the classical extension of this form (see [30] (pp. 96–106)). The symbol $\mathcal{S}'(\mathbb{R}^d)$ denotes the space of functionals of $\mathcal{S}(\mathbb{R}^d)$.

2.2. Ingredient Modulus of Smoothness in Component Form in $L_{\vec{p}}(\mathbb{R}^d)$

For clarity, let us first recall the definition of the classical modulus of smoothness originally below.

Definition 2. *(Classical modulus of smoothness) (see, e.g., [8]) Let $\vec{p} \in [1, \infty]^d$ and $f \in L_{\vec{p}}(\mathbb{R}^d)$. Denote*

$$\Delta_h^1 f(x) := f(x-h) - f(x)$$

and

$$\Delta_h^M f(x) := \Delta_h^1 \Delta_h^{M-1} f(x) \quad \textit{for } M \in \mathbb{N}.$$

Let $t > 0$. Then, the modulus of smoothness of M-order is defined by

$$\rho_{\vec{p}}^M(f, t) := \sup_{|h| \le t} \|\Delta_h^M f\|_{\vec{p}}.$$

In this paper, we introduce and study the concept of ingredient modulus of smoothness in component form in $L_{\vec{p}}(\mathbb{R}^d)$, which will be used for statistics and anisotropic Nikolskij spaces.

Definition 3. *(Ingredient modulus of smoothness in component form.) Let $\vec{p} \in [1,\infty]^d$ and $f \in L_{\vec{p}}(\mathbb{R}^d)$. For any $1 \le i \le d$, let e_i denote the vector with 1 in the ith coordinate and 0's elsewhere. Define*

$$\Delta^1_{he_i} f(x) := f(x - he_i) - f(x)$$

and

$$\Delta^L_{he_i} f(x) := \Delta^{L-1}_{he_i}\Delta^1_{he_i} f(x) \quad \text{for } L \in \mathbb{N} \text{ with } L \ge 2.$$

Let $t > 0$. Then, the L-order ingredient modulus of smoothness in component form of f is defined by

$$\omega^L_{\vec{p},i}(f,t) := \sup_{|h|\le t} \|\Delta^L_{he_i} f\|_{\vec{p}}. \tag{1}$$

Remark 1. *In this paper, we will use $L = 1$ and $L = 2$ in (1) for characterization, that is*

$$\omega^1_{\vec{p},i}(f,t) := \sup_{|h|\le t} \|\Delta^1_{he_i} f\|_{\vec{p}},$$

and

$$\omega^2_{\vec{p},i}(f,t) := \sup_{|h|\le t} \|\Delta^2_{he_i} f\|_{\vec{p}}.$$

We start with the following fundamental properties for the ingredient modulus of smoothness of first and second order in the setting of $L_{\vec{p}}(\mathbb{R}^d)$. The one-dimensional case will degenerate to the case in [10] (Lemma 9.1, page 100). Of course, the corresponding characterization is also the same.

Proposition 1. *Let $\vec{p} \in [1,\infty]^d$ and $f \in L_{\vec{p}}(\mathbb{R}^d)$. Then, the following statements hold.*

(1) Both functions $\omega^1_{\vec{p},i}(f,t)$ and $\omega^2_{\vec{p},i}(f,t)$ are nondecreasing in the second argument;

(2) $\omega^2_{\vec{p},i}(f,t) \le 2\omega^1_{\vec{p},i}(f,t) \le 4\|f\|_{\vec{p}}$ for all $t > 0$;

(3) $\omega^1_{\vec{p},i}(f,t) \le \sum_{j=0}^{\infty} 2^{-(j+1)}\omega^2_{\vec{p},i}(f,2^j t) \le t\int_t^\infty \frac{\omega^2_{\vec{p},i}(f,s)}{s^2} ds$ for all $t > 0$;

(4) $\omega^1_{\vec{p},i}(f,ts) \le (s+1)\omega^1_{\vec{p},i}(f,t)$ for all $s,t > 0$;

(5) $\omega^2_{\vec{p},i}(f,ts) \le (s+1)^2\omega^2_{\vec{p},i}(f,t)$ for all $s,t > 0$.

Proof. From the definitions of $\omega^1_{\vec{p},i}(f,t)$ and $\omega^2_{\vec{p},i}(f,t)$, the conclusions (1) and (2) follow. To see (3), let $t > 0$ be given. Since $2\Delta^1_{he_i} = \Delta^1_{2he_i} - \Delta^2_{he_i}$, it leads to

$$\omega^1_{\vec{p},i}(f,t) \le \frac{1}{2}\Big[\omega^2_{\vec{p},i}(f,t) + \omega^1_{\vec{p},i}(f,2t)\Big].$$

Thus, for any k, we obtain

$$\omega^1_{\vec{p},i}(f,t) \le \sum_{j=0}^{k} 2^{-(j+1)}\omega^2_{\vec{p},i}(f,2^j t) + 2^{-(k+1)}\omega^1_{\vec{p},i}(f,2^{k+1}t). \tag{2}$$

Taking the limit of k as tending to infinity in (2) yields

$$\omega^1_{\vec{p},i}(f,t) \le \sum_{j=0}^{\infty} 2^{-(j+1)}\omega^2_{\vec{p},i}(f,2^j t).$$

On the other hand, since

$$2^{-(j+1)}\omega^2_{\vec{p},i}(f,2^j t) = t2^{-(j+1)}\frac{\omega^2_{\vec{p},i}(f,2^j t)}{t}$$

$$
\begin{aligned}
&= t\int_{2^j t}^{2^{(j+1)}t} \frac{\omega^2_{\vec{p},i}(f,2^j t)}{s^2} ds \\
&\le t\int_{2^j t}^{2^{(j+1)}t} \frac{\omega^2_{\vec{p},i}(f,s)}{s^2} ds,
\end{aligned}
$$

the last inequality implies

$$
\sum_{j=0}^{\infty} 2^{-(j+1)}\omega^2_{\vec{p},i}(f,2^j t) \le t\int_t^{\infty} \frac{\omega^2_{\vec{p},i}(f,s)}{s^2} ds.
$$

We now verify (4). Note that $\Delta^1_{nhe_i}f(x) = \sum_{k=0}^{n-1} \Delta^1_{he_i}f(x-kh)$ leads to

$$
\omega^1_{\vec{p},i}(f,nt) \le n\omega^1_{\vec{p},i}(f,t) \quad \text{for } n \in \mathbb{N}.
$$

Therefore, we obtain

$$
\omega^1_{\vec{p},i}(f,st) \le \omega^1_{\vec{p},i}(f,([s]+1)t) \le ([s]+1)\omega^1_{\vec{p},i}(f,t) \le (s+1)\omega^1_{\vec{p},i}(f,t),
$$

where $[s]$ is the greatest integer less than or equal to s. Finally, we show conclusions (5). Since

$$
\begin{aligned}
\Delta^2_{nhe_i}f(x) &= \Delta^1_{nhe_i}\Delta^1_{nhe_i}f(x) \\
&= \Delta^1_{nhe_i}\left(\sum_{k=0}^{n-1}\Delta^1_{he_i}f(x-khe_i)\right) \\
&= \sum_{k'=0}^{n-1}\sum_{k=0}^{n-1}\Delta^1_{he_i}\Delta^1_{he_i}f(x-khe_i-k'he_i) \\
&= \sum_{k'=0}^{n-1}\sum_{k=0}^{n-1}\Delta^2_{he_i}f(x-khe_i-k'he_i),
\end{aligned}
$$

we obtain $\omega^2_{\vec{p},i}(f,nt) \le n^2\omega^2_{\vec{p},i}(f,t)$ for $n \in \mathbb{N}$. Hence, we obtain

$$
\omega^2_{\vec{p},i}(f,st) \le \omega^2_{\vec{p},i}(f,([s]+1)t) \le ([s]+1)^2\omega^2_{\vec{p},i}(f,t) \le (s+1)^2\omega^2_{\vec{p},i}(f,t).
$$

The proof is complete. □

In order to link ingredient modulus of smoothness in component form with the traditional modulus of smoothness defined by Δ^1_h and Δ^2_h (see, e.g., [10] (Definition 9.1, page 99)), we establish the following useful result.

Lemma 1. *Let $\vec{p} \in [1,\infty]^d$. Define $\Delta^1_h f(x) = f(x-h) - f(x)$ and $\Delta^2_h f(x) = \Delta^1_h\Delta^1_h f(x)$ for $f \in L_{\vec{p}}(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$. Then*

$$
\|\Delta^2_h f(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \le C\left(\sum_{i=1}^{d}\|\Delta^1_{he_i}f(x)\|_{L_{\vec{p}}(\mathbb{R}^d)}\right),
$$

where C is a positive constant independent of f, $\vec{p}$, and he_i.

Proof. In fact,

$$
\begin{aligned}
&\Delta_h^1 f(x_1, x_2, \cdots, x_d) \\
&= f(x_1 - h_1, x_2 - h_2, \cdots, x_d - h_d) - f(x_1, x_2, \cdots, x_d) \\
&= f(x_1 - h_1, x_2 - h_2, \cdots, x_d - h_d) - f(x_1 - h_1, x_2 - h_2, \cdots, x_d) \\
&\quad + f(x_1 - h_1, x_2 - h_2, \cdots, x_d) + \cdots + f(x_1 - h_1, x_2, \cdots, x_d) - f(x_1, x_2, \cdots, x_d) \\
&= \Delta_{he_d}^1 f(x_1 - h_1, \cdots, x_{d-1} - h_{d-1}, x_d) + \cdots + \Delta_{he_1}^1 f(x_1, \cdots, x_d).
\end{aligned} \tag{3}
$$

Using (3), we obtain

$$
\begin{aligned}
&\Delta_h^2 f(x_1, x_2, \cdots, x_d) \\
&= \Delta_h^1 \Delta_h^1 f(x_1, x_2, \cdots, x_d) \\
&= \Delta_h^1 \Big(\Delta_{he_d}^1 f(x_1 - h_1, \cdots, x_{d-1} - h_{d-1}, x_d) + \cdots + \Delta_{he_1}^1 f(x_1, \cdots, x_d) \Big) \\
&= [\Delta_{he_d}^1 f(x_1 - h_1 - h_1, \cdots, x_{d-1} - h_{d-1} - h_{d-1}, x_d - h_d) + \cdots + \Delta_{he_1}^1 f(x_1 - h_1, \cdots, x_d - h_d)] \\
&\quad - [\Delta_{he_d}^1 f(x_1 - h_1 - h_1, \cdots, x_{d-1} - h_{d-1} - h_{d-1}, x_d) + \cdots + \Delta_{he_1}^1 f(x_1 - h_1, \cdots, x_d)] \\
&\quad + [\Delta_{he_d}^1 f(x_1 - h_1 - h_1, \cdots, x_{d-1} - h_{d-1} - h_{d-1}, x_d) + \cdots + \Delta_{he_1}^1 f(x_1 - h_1, \cdots, x_d)] \\
&\quad \cdots + [\Delta_{he_d}^1 f(x_1 - h_1 - h_1, \cdots, x_{d-1} - h_{d-1}, x_d) + \cdots + \Delta_{he_1}^1 f(x_1 - h_1, \cdots, x_d)] \\
&\quad - [\Delta_{he_d}^1 f(x_1 - h_1, \cdots, x_{d-1} - h_{d-1}, x_d) + \cdots + \Delta_{he_1}^1 f(x_1, \cdots, x_d)] \\
&= \Delta_{he_d}^1 \Big[\Delta_{he_d}^1 f(x_1 - h_1 - h_1 \cdots, x_{d-1} - h_{d-1} - h_{d-1}, x_d) \Big] + \cdots + \Delta_{he_1}^1 \Big[\Delta_{he_1}^1 f(x_1, \cdots, x_d) \Big].
\end{aligned} \tag{4}
$$

By the definition of $\Delta_{he_i}^1$ and (4), we can prove

$$
\|\Delta_h^2 f(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \le C \left(\sum_{i=1}^d \|\Delta_{he_i}^1 f(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \right) \quad \text{for some positive constant } C.
$$

□

2.3. A Kind of Mixed-Norm Sobolev Space $W_{\vec{p}}^n(\mathbb{R}^d)$

In this section, we will introduce and study a kind of mixed-norm Sobolev space, whose definition can be considered as a natural generalization of [31] ((1) on page 722). Antonić and Ivec [32] (Corollary 3, page 197) proved the boundedness of Fourier multipliers in this kind of mixed Sobolev space.

Definition 4. *Let $\vec{p} \in [1, \infty]^d$. The mixed-norm Sobolev space $W_{\vec{p}}^n(\mathbb{R}^d)$ is given by all Lebesgue measurable functions $f(x) \in L_{\vec{p}}(\mathbb{R}^d)$ with finite norms*

$$
\|f\|_{W_{\vec{p}}^n(\mathbb{R}^d)} := \|f\|_{L_{\vec{p}}(\mathbb{R}^d)} + \sum_{|\mathbf{k}|=n} \|\partial_{\mathbf{k}} f\|_{L_{\vec{p}}(\mathbb{R}^d)}
$$

for $\mathbf{k} = (k_1, k_2, \cdots, k_d)$ and $n \in \mathbb{N}$. Here, $\partial_{\mathbf{k}} f := (\partial_1^{k_1} \partial_2^{k_2} \cdots \partial_d^{k_d}) f(x_1, x_2, \cdots, x_d)$ means the $\mathbf{k}$-order partial derivative of f.

Remark 2. *It is worth mentioning that in Definition 4, if $\vec{p}$ cannot be taken as infinity, then the mixed-norm Sobolev space at this time is a generalization of the function space in [27] (Sobolev spaces, page 7). If $\vec{p}$ can be taken as infinity, then the functions in this mixed-norm Sobolev space* *are uniformly continuous and bounded after obtaining weak derivatives (see* [10] *(Remark 8.4) and* [33] *(Proposition 8.4)).*

Remark 3. *Let $\vec{p} \in (1,\infty)^d$ and $\vec{k} \in \mathbb{N}_0^d$. Recall that the mixed-norm Sobolev space $W^{\vec{k}}_{\vec{p}}$ in the sense of Lizorkin approach* [34] *(page 227) was defined as the set of all $f \in S'(\mathbb{R}^d)$ such that*

$$\|f\|_{W^{\vec{k}}_{\vec{p}}} := \|f\|_{L_{\vec{p}}(\mathbb{R}^d)} + \sum_{j=1}^{n} \left\| \frac{\partial^{k_j} f}{\partial x_j^{k_j}} \right\|_{L_{\vec{p}}(\mathbb{R}^d)} < \infty.$$

It is quite obvious that Definition 4 and Lizorkin's definition for the mixed-norm Sobolev space are different. In fact, the spaces in Remark 3 are anisotropic and of course are not the ordinary version that we presented in Definition 4.

The following mixed-norm Minkowski inequality in the setting of $L_{\vec{p}}(\mathbb{R}^d)$ is crucial in this paper.

Lemma 2. *(Mixed-norm Minkowski inequality, see* [35] *(Theorem 3, page 5).) Let $f(x,y)$ be a Borel function on $\mathbb{R}^d \times \mathbb{R}^d$ and $\vec{p} \in [1,\infty]^d$. Then*

$$\left\| \int_{\mathbb{R}^d} f(x,\cdot)dx \right\|_{L_{\vec{p}}(\mathbb{R}^d)} \le \int_{\mathbb{R}^d} \|f(x,\cdot)\|_{L_{\vec{p}}(\mathbb{R}^d)} dx.$$

Theorem 1. *Let $\vec{p} \in [1,\infty]^d$. Then, the following inequalities hold.*

(a) $\omega^1_{\vec{p},i}(f,t) \le t\|\partial^1 f\|_{\vec{p}}$ *for all $f \in W^1_{\vec{p}}(\mathbb{R}^d)$ and $t > 0$;*

(b) $\omega^2_{\vec{p},i}(f,t) \le t^2\|\partial^2 f\|_{\vec{p}}$ *for all $f \in W^2_{\vec{p}}(\mathbb{R}^d)$ and $t > 0$.*

Proof. We only verify the case $d = 2$, and a similar argument could be made for $d > 2$. Clearly, we have

$$f(x_1 - h_1, x_2) - f(x_1, x_2) = -h_1 \int_0^1 \partial_1 f(x_1 - s_1 h_1, x_2) ds_1. \tag{5}$$

By Lemma 2,

$$\begin{aligned}
\sup_{|h|\le t} \|f(x_1 - h_1, x_2) - f(x_1, x_2)\|_{L_{\vec{p}}(\mathbb{R}^d)} &\le \sup_{|h|\le t} |h_1| \int_0^1 \|\partial_1 f(x_1 - s_1 h_1, x_2)\|_{L_{\vec{p}}(\mathbb{R}^d)} ds_1 \\
&\le t\|\partial_1 f(x_1, x_2)\|_{L_{\vec{p}}(\mathbb{R}^d)} \\
&\le t\|\partial^1 f(x_1, x_2)\|_{L_{\vec{p}}(\mathbb{R}^d)}.
\end{aligned}$$

In a similar way, we obtain

$$\begin{aligned}
\sup_{|h|\le t} \|f(x_1, x_2 - h_2) - f(x_1, x_2)\|_{L_{\vec{p}}(\mathbb{R}^d)} &\le \sup_{|h|\le t} |h_2| \int_0^1 \|\partial_2 f(x_1, x_2 - s_2 h_2)\|_{L_{\vec{p}}(\mathbb{R}^d)} ds_2 \\
&\le t\|\partial_2 f(x_1, x_2)\|_{L_{\vec{p}}(\mathbb{R}^d)}.
\end{aligned}$$

Hence, $\omega^1_{\vec{p},i}(f,t) \le t\|\partial^1 f\|_{\vec{p}}$ follows and conclusion (a) is proved. To see (b), by (5), we obtain

$$\begin{aligned}
\Delta^2_{he_1} f(x_1, x_2) &= \Delta^1_{he_1} \Delta^1_{he_1} f(x_1, x_2) \\
&= -h_1 \int_0^1 \partial_1 (\Delta^1_{he_1} f)(x_1 - s_1 h_1, x_2) ds_1
\end{aligned}$$

$$= -h_1 \int_0^1 \Delta^1_{he_i}(\partial_1 f)(x_1 - s_1 h_1, x_2) ds_1$$
$$= -h_1 \int_0^1 (-h_1) \int_0^1 (\partial_1 \partial_1 f)(x_1 - s_1 h_1 - l_1 h_1, x_2) dl_1 ds_1.$$

Therefore, using Lemma 2, we obtain

$$\sup_{|h|\le t} \|\Delta^2_{he_1} f(x_1, x_2)\|_{L_{\vec{p}}(\mathbb{R}^d)} \le \sup_{|h|\le t} (h_1)^2 \|\partial_1\partial_1 f\|_{L_{\vec{p}}(\mathbb{R}^d)} \le t^2 \|\partial^1 \partial_1 f\|_{\vec{p}} \le t^2 \|\partial^2 f\|_{L_{\vec{p}}(\mathbb{R}^d)}$$

and

$$\sup_{|h|\le t} \|\Delta^2_{he_2} f(x_1, x_2)\|_{L_{\vec{p}}(\mathbb{R}^d)} \le \sup_{|h|\le t} (h_2)^2 \|\partial_2\partial_2 f\|_{L_{\vec{p}}(\mathbb{R}^d)} \le t^2 \|\partial_2 \partial_2 f\|_{\vec{p}} \le t^2 \|\partial^2 f\|_{L_{\vec{p}}(\mathbb{R}^d)}.$$

Therefore, we prove $\omega^2_{\vec{p},i}(f,t) \le t^2 \|\partial^2 f\|_{L_{\vec{p}}(\mathbb{R}^d)}$ for any d and $f \in W^2_{\vec{p}}(\mathbb{R}^d)$. The proof is complete. □

2.4. Littlewood–Paley Decomposition in $L_{\vec{p}}(\mathbb{R}^d)$

The convolution $f * g$ is defined by the formula $f * g(x) := \int_{\mathbb{R}^d} f(x-y) g(y) dy$ for $f \in L_p$ ($1 \le p \le \infty$) and $g \in L_1$. In this section, we first recall the well-known Littlewood–Paley decomposition. Let $\mathcal{D}(\mathbb{R}^d)$ be a space with infinitely many times differentiable compactly supported functions. Define γ by $\hat{\gamma}(\xi) \in \mathcal{D}(\mathbb{R}^d)$ with supp $\hat{\gamma}(\xi) \subset [-A, A]^d$ for some $A > 0$, and

$$\hat{\gamma}(\xi) = 1 \ \text{ for } \xi \in \left[-\frac{3A}{4}, \frac{3A}{4}\right]^d.$$

Meanwhile, a function β is given by

$$\hat{\beta}(\xi) := \hat{\gamma}\left(\frac{\xi}{2}\right) - \hat{\gamma}(\xi).$$

Let $\beta_k(x) := 2^{kd} \beta(2^k x)$ for $k \in \mathbb{N}_0$. Then

$$\widehat{\beta_k}(\xi) = \hat{\beta}\left(\frac{\xi}{2^k}\right)$$

and

$$\hat{\gamma}(\xi) + \sum_{k=0}^{\infty} \hat{\beta}\left(\frac{\xi}{2^k}\right) = 1.$$

Hence, the Littlewood–Paley decomposition operators are defined by

$$\mathcal{P}_k f(x) := \beta_k * f(x) \ \text{ for } k \in \mathbb{N}_0$$

and

$$\mathcal{P}_{-1} f(x) := \gamma * f(x).$$

Therefore, we have the Littlewood–Paley decomposition

$$f = \sum_{k=-1}^{\infty} \mathcal{P}_k f \left(\overset{def}{=} \lim_{n\to\infty} \sum_{k=-1}^{n} \mathcal{P}_k f \right) \tag{6}$$

for all $f \in \mathcal{S}'(\mathbb{R}^d)$.

The following result is a generalized convolution inequality in mixed-norm Lebesgue spaces.

Lemma 3. *(Mixed-norm convolution inequality, see* [36] *((2.5) in page 4) or* [35] *(Theorem 4, page 6).) Let $f(x) \in L_{\vec{p}}(\mathbb{R}^d)$ and $g(x) \in L_1(\mathbb{R}^d)$ for $\vec{p} \in [1,\infty]^d$. Then*

$$\|f * g\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq \|f\|_{L_{\vec{p}}(\mathbb{R}^d)} \|g\|_{L_1(\mathbb{R}^d)}.$$

The next lemma is a Bernstain-type inequality in $L_{\vec{p}}(\mathbb{R}^d)$ sense inspired by the Littlewood–Paley decomposition, which is important in the decomposition estimation for the partial modulus of smoothness in component form. Note that the definition of the derivative (the definition of derivative in this paper is also based on this meaning) is based on a distributional or is called the weak derivative [10] (Proposition 8.1, page 70). Here, we give a concise and quick proof.

Lemma 4. *Let $\vec{p} \in [1,\infty]^d$ and $f \in L_{\vec{p}}(\mathbb{R}^d)$. If supp $\hat{f} \in [-M, M]^d$ for some $M > 0$, then there exists a constant $C > 0$, such that for $K \in \mathbb{N}$,*

$$\|\partial^K f\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq C M^K \|f\|_{L_{\vec{p}}(\mathbb{R}^d)}.$$

Proof. Assume $A = 2$. Then, $\mathrm{supp}\hat{\gamma}(\xi) \subset [-2,2]^d$ and $\hat{\gamma}(\xi) = 1$ for $\xi \in [-\frac{3}{2}, \frac{3}{2}]^d$. Define $\gamma^*(x) := M^d \gamma(Mx)$. Thus, $\widehat{\gamma^*}(\xi) = \hat{\gamma}(\frac{\xi}{M})$, which means $\widehat{\gamma^*}(\xi) = 1$ for $\xi \in [-\frac{3M}{2}, \frac{3M}{2}]^d$. Therefore, it follows that

$$\hat{f}(\xi) = \widehat{\gamma^*}(\xi)\hat{f}(\xi),$$
$$f(x) = \gamma^* * f(x)$$

and

$$\partial^K f(x) = (\partial^K \gamma^*) * f(x).$$

Using the mixed-norm convolution inequality in Lemma 3, we obtain

$$\|\partial^K f\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq \|\partial^K \gamma^*\|_{L_1(\mathbb{R}^d)} \|f\|_{L_{\vec{p}}(\mathbb{R}^d)} = M^K \|\partial^K \gamma\|_{L_1(\mathbb{R}^d)} \|f\|_{L_{\vec{p}}(\mathbb{R}^d)}.$$

The proof is complete. □

We can also obtain the above lemma by referring to Proposition 4 of [22] (with $\vec{p} = \vec{r}$ and $R_1 = \cdots = R_d = M$).

Applying Littlewood–Paley decomposition theory, we establish the following important lemma which will be used for the sufficiency estimation in our new decomposition theorem (see Theorem 2 below).

Lemma 5. *Let $\vec{p} \in [1,\infty]^d$ and $f \in L_{\vec{p}}(\mathbb{R}^d)$. If $\sum_{j=-1}^{\infty} \|\mathcal{P}_j(\partial^n f)\|_{L_{\vec{p}}(\mathbb{R}^d)} < \infty$ for some $n \in \mathbb{N}$. Then, $\partial^n f \in L_{\vec{p}}(\mathbb{R}^d)$ and*

$$\omega^2_{\vec{p},i}(\partial^n f, t) \leq \sum_{j=-1}^{\infty} \omega^2_{\vec{p},i}(\mathcal{P}_j(\partial^n f), t).$$

Proof. Using the Littlewood–Paley decomposition in (6) (the detailed proof is left to the readers), we obtain

$$\partial^n f = \sum_{j=-1}^{\infty} \mathcal{P}_j(\partial^n f)$$

and

$$\|\partial^n f\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq \sum_{j=-1}^{\infty} \|\mathcal{P}_j(\partial^n f)\|_{L_{\vec{p}}(\mathbb{R}^d)} < \infty.$$

Hence, we obtain $\partial^n f \in L_{\vec{p}}(\mathbb{R}^d)$. Moreover, since

$$\|\Delta^2_{he_i}(\partial^n f)(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq \sum_{j=-1}^{\infty} \|\Delta^2_{he_i}(\mathcal{P}_j(\partial^n f))\|_{L_{\vec{p}}(\mathbb{R}^d)},$$

by Proposition 1 (2), we have $\omega^2_{\vec{p},i}(\mathcal{P}_j(\partial^n f)), t) < \infty$. With the help of

$$\sum_{j=-1}^{\infty} \|\mathcal{P}_j(\partial^n f)\|_{L_{\vec{p}}(\mathbb{R}^d)} < \infty,$$

we can finish the proof. □

3. New Decomposition Theorem of $B^s_{\vec{p},q}(\mathbb{R}^d)$

In this section, we first introduce the following concept of mixed-norm Besov spaces induced by the ingredient modulus of smoothness.

Definition 5. *Let $1 \leq q \leq \infty$, $\vec{p} \in [1,\infty]^d$, and $s = n + \alpha$ for $n \in \mathbb{N}_0$ and $0 < \alpha < 1$. The mixed-norm Besov space $B^s_{\vec{p},q}(\mathbb{R}^d)$ is the space of all Lebesgue measurable functions f on $\mathbb{R}^d$ such that $f \in W^n_{\vec{p}}(\mathbb{R}^d)$ with*

$$\sum_{i=1}^{d}\left(\int_0^\infty \left|\frac{\omega^2_{\vec{p},i}(\partial^n f,t)}{t^\alpha}\right|^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty \text{ for } 1 \leq q < \infty,$$

and

$$esssup_t \left|\frac{\omega^2_{\vec{p},i}(\partial^n f,t)}{t^\alpha}\right| < \infty \text{ for } q = \infty.$$

For $f \in B^s_{\vec{p},q}(\mathbb{R}^d)$, we define an integral version of $B^s_{\vec{p},q}(\mathbb{R}^d)$-norm of f by

$$\|f\|^{(int)}_{B^s_{\vec{p},q}} := \begin{cases} \|f\|_{W^n_{\vec{p}}(\mathbb{R}^d)} + \sum_{i=1}^{d}\left(\int_0^\infty \left|\frac{\omega^2_{\vec{p},i}(\partial^n f,t)}{t^\alpha}\right|^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \|f\|_{W^n_{\vec{p}}(\mathbb{R}^d)} + \sum_{i=1}^{d} esssup_t \left|\frac{\omega^2_{\vec{p},i}(\partial^n f,t)}{t^\alpha}\right|, & q = \infty. \end{cases}.$$

We begin by proving a discrete version of $B^s_{\vec{p},q}(\mathbb{R}^d)$-norm which will be used to establish a new decomposition theorem in mixed-norm Besov spaces.

Lemma 6. *Let $1 \leq q \leq \infty$, $\vec{p} \in [1,\infty]^d$, and $s = n + \alpha$ for $n \in \mathbb{N}_0$ and $0 < \alpha < 1$. For any $f \in B^s_{\vec{p},q}(\mathbb{R}^d)$, we define*

$$\|f\|^{(dis)}_{B^s_{\vec{p},q}} := \|f\|_{W^n_{\vec{p}}} + \sum_{i=1}^{d} \left\| \{2^{j\alpha}\omega^2_{\vec{p},i}(\partial^n f, 2^j)\}_j \right\|_{l_q}.$$

Then, $\|f\|^{(dis)}_{B^s_{\vec{p},q}}$ is also a $B^s_{\vec{p},q}(\mathbb{R}^d)$-norm of f and is equivalent to $\|f\|^{(int)}_{B^s_{\vec{p},q}}$.

Proof. It is easy to see that

$$\int_0^\infty \left|\frac{\omega^2_{\vec{p},i}(\partial^n f,t)}{t^\alpha}\right|^q \frac{dt}{t} = \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left|\frac{\omega^2_{\vec{p},i}(\partial^n f,t)}{t^\alpha}\right|^q \frac{dt}{t}$$

and

$$\log(2)\left|\frac{\omega^2_{\vec{p},i}(\partial^n f, 2^j)}{2^{(j+1)\alpha}}\right|^q \leq \int_{2^j}^{2^{j+1}}\left|\frac{\omega^2_{\vec{p},i}(\partial^n f, t)}{t^\alpha}\right|^q \frac{dt}{t} \leq \log(2)\left|\frac{\omega^2_{\vec{p},i}(\partial^n f, 2^{j+1})}{2^{j\alpha}}\right|^q.$$

Hence, $\|f\|^{(\text{dis})}_{B^s_{\vec{p},q}}$ and $\|f\|^{(\text{int})}_{B^s_{\vec{p},q}}$ are two equivalent norms of $B^s_{\vec{p},q}(\mathbb{R}^d)$. $\square$

We can describe the relationship between $\omega^1_{\vec{p},i}(\partial^n f, 2^j)$ and $\omega^2_{\vec{p},i}(\partial^n f, 2^j)$ as follows.

Lemma 7. *Let $1 \leq q \leq \infty$, $\vec{p} \in [1,\infty]^d$ and $s = n + \alpha$ for $n \in \mathbb{N}_0$ and $0 < \alpha < 1$. Then*

$$C_1 \sum_{i=1}^d \left\|\left\{2^{j\alpha}\omega^1_{\vec{p},i}(\partial^n f, 2^j)\right\}_j\right\|_{l_q} \leq \sum_{i=1}^d \left\|\left\{2^{j\alpha}\omega^2_{\vec{p},i}(\partial^n f, 2^j)\right\}_j\right\|_{l_q} \leq C_2 \sum_{i=1}^d \left\|\left\{2^{j\alpha}\omega^1_{\vec{p},i}(\partial^n f, 2^j)\right\}_j\right\|_{l_q}$$

for some positive constants C_1 and C_2.

Proof. By Proposition 1 (2), we can prove the right inequality immediately. To verify the left inequality, by Proposition 1 (3) and the Hardy inequality in the spirit of Remark 9.1 in [10], we can come to the desired conclusion. $\square$

The following discrete Hardy inequality is very crucial for computing some discrete inequalities in this paper.

Lemma 8. *(Discrete Hardy inequality, see* [37] *(Lemma 3.4, page 27) or* [10] *(Lemma 9.2, page 102).) Let $\{a_j\} \in l_1$ and $\{b_j\} \in l_p$ for $1 \leq p \leq \infty$. Then, $\{c_k\} \in l_p$ and $\{d_k\} \in l_p$ for $c_k = \sum_{j=k}^\infty a_j b_{j-k}$ and $d_k = \sum_{j=0}^k a_{k-j} b_j$.*

We now prove the following new decomposition theorem in mixed-norm Besov spaces, which is the main result of the section.

Theorem 2. *Let $1 \leq q \leq \infty$, $\vec{p} \in [1,\infty]^d$, and $s = n + \alpha$ for $n \in \mathbb{N}_0$ and $0 < \alpha < 1$. Then, $f \in B^s_{\vec{p},q}(\mathbb{R}^d)$ if and only if $\mathcal{P}_{-1} f \in L_{\vec{p}}(\mathbb{R}^d)$ and $\left\{2^{js}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)}\right\}_{j=0}^\infty \in l_q$.*

Proof. *Necessity*. Suppose $f \in B^s_{\vec{p},q}(\mathbb{R}^d)$. Then, $\mathcal{P}_{-1} f \in L_{\vec{p}}(\mathbb{R}^d)$. Therefore, it remains to prove $\left\{2^{js}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)}\right\}_{j=0}^\infty \in l_q$. By using the Fourier transform, we obtain

$$(i\xi_1)^n \hat{\beta}\left(\frac{\xi}{2^j}\right)\hat{f}(\xi) = \mathcal{F}(\partial_1^n(\beta_j * f)(\xi)),$$

where the symbol $\partial_1^n f$ is used to denote n-order partial derivatives for the 1st variable of f. It is worth noting that $\partial_1^n f$ in the above equality can be replaced by $\partial_2^n f$, $\partial_3^n f$, or any other type of n-order partial derivatives $\partial_k f$ with $|k| = n$, which would not have any impact on the proof that we will carry out. Since $\operatorname{supp}\hat{\beta} \subset [-2A, 2A]^d \setminus \left[-\frac{3A}{4}, \frac{3A}{4}\right]^d$, $\hat{\beta}\left(\frac{\xi}{2^j}\right)\hat{f}(\xi)$ is compactly supported. Hence, we have

$$\begin{aligned}
\hat{\beta}\left(\frac{\xi}{2^j}\right)\hat{f}(\xi) &= (2^{-jn})\left(\frac{2^j}{i\xi_1}\right)^n (i\xi_1)^n \hat{\beta}\left(\frac{\xi}{2^j}\right)\hat{f}(\xi) \\
&= (2^{-jn})(-i)^n \frac{\hat{\eta}\left(\frac{\xi_1}{2^j}\right)}{\left(\frac{i\xi_1}{2^j}\right)^n}\hat{\eta}\left(\frac{\xi_2}{2^j}\right)\cdots\hat{\eta}\left(\frac{\xi_d}{2^j}\right)\mathcal{F}(\partial_1^n(\beta_j * f)(\xi)) \\
&= (2^{-jn})(-i)^n \hat{\eta}_n\left(\frac{\xi_1}{2^j}\right)\hat{\eta}\left(\frac{\xi_2}{2^j}\right)\cdots\hat{\eta}\left(\frac{\xi_d}{2^j}\right)\mathcal{F}(\partial_1^n(\beta_j * f)(\xi)),
\end{aligned}$$

where $\hat{\eta}_n$ is given by $\hat{\eta}_n(\xi_1) := \frac{\hat{\eta}(\xi_1)}{\xi_1^n}$, $\hat{\eta}(\xi_1), \cdots, \hat{\eta}(\xi_d) \in \mathcal{D}(\mathbb{R}^d)$ with

$$\hat{\eta}(\xi_1)\cdots\hat{\eta}(\xi_d) = 1$$

on the support $\hat{\beta}$ and 0 in the neighborhood of original point. It follows that

$$\begin{aligned}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)} &\le \|\eta_n\|_{L_1(\mathbb{R}^d)}\|\eta\|_{L_1(\mathbb{R}^d)}^{n-1}2^{-jn}\|\partial_1^n(\beta_j * f)\|_{L_{\vec{p}}(\mathbb{R}^d)}\\ &= \|\eta_n\|_{L_1(\mathbb{R}^d)}\|\eta\|_{L_1(\mathbb{R}^d)}^{n-1}2^{-jn}\|\beta_j * \partial_1^n f\|_{L_{\vec{p}}(\mathbb{R}^d)}.\end{aligned}$$

Now, we will estimate $\|\beta_j * \partial_1^n f\|_{L_{\vec{p}}(\mathbb{R}^d)}$. From the definition of β, β_j is an even function (i.e., $\beta_j(-x) = \beta_j(x)$) and, by $\hat{\beta}_j(0) = 0$, we can easily obtain $\int \beta_j(y)dy = 0$. Thus, we deduce

$$\begin{aligned}&\beta_j * \partial_1^n f(x)\\ &= \textstyle\int_{\mathbb{R}^d}\beta_j(y)\partial_1^n f(x-y)dy\\ &= \tfrac{1}{2}\textstyle\int_{\mathbb{R}^d}\beta_j(y)\{\partial_1^n f(x-y) - 2\partial_1^n f(x) + \partial_1^n f(x+y)\}dy\\ &= \tfrac{1}{2}\textstyle\int_{\mathbb{R}^d}\beta(y)\{\partial_1^n f(x-2^{-j}y) - 2\partial_1^n f(x) + \partial_1^n f(x+2^{-j}y)\}dy.\end{aligned}$$

Applying Lemmas 2 and 1, we obtain

$$\begin{aligned}&\|\beta_j * \partial_1^n f(x)\|_{L_{\vec{p}}(\mathbb{R}^d)}\\ &\le \tfrac{1}{2}\|\textstyle\int_{\mathbb{R}^d}|\beta(y)||\{\partial_1^n f(x-2^{-j}y) - 2\partial_1^n f(x) + \partial_1^n f(x+2^{-j}y)\}|dy\|_{L_{\vec{p}}(\mathbb{R}^d)}\\ &= \tfrac{1}{2}\|\textstyle\int_{\mathbb{R}^d}|\beta(y)||\Delta^2_{2^{-j}y}\partial_1^n f(x+2^{-j}y)|dy\|_{L_{\vec{p}}(\mathbb{R}^d)}\\ &\le \tfrac{1}{2}\textstyle\int_{\mathbb{R}^d}|\beta(y)|\|\Delta^2_{2^{-j}y}\partial_1^n f(x+2^{-j}y)\|_{L_{\vec{p}}(\mathbb{R}^d)}dy\\ &\le \tfrac{1}{2}\textstyle\int_{\mathbb{R}^d}|\beta(y)|\sup_{|h|\le 2^{-j}|y|}\|\Delta_h^2\partial_1^n f(z)\|_{L_{\vec{p}}(\mathbb{R}^d)}dy,\end{aligned} \tag{7}$$

where $\Delta^2_{2^{-j}y}$ and Δ_h^2 come from Lemma 1. Therefore, by Proposition 1 (4) and Lemma 1, we conclude that

$$\begin{aligned}\|\beta_j * \partial_1^n f(x)\|_{L_{\vec{p}}(\mathbb{R}^d)} &\le \frac{C}{2}\int|\beta(y)|\sup_{|h|\le 2^{-j}|y|}\left(\sum_{i=1}^d\|\Delta^1_{he_i}\partial_1^n f(x)\|_{L_{\vec{p}}(\mathbb{R}^d)}\right)dy\\ &\le \frac{C}{2}\int|\beta(y)|\left(\sum_{i=1}^d\omega^1_{\vec{p},i}(\partial_1^n f, 2^{-j}|y|)\right)dy\\ &\le \frac{C}{2}\left(\int|\beta(y)|(1+|y|)^2dy\right)\left(\sum_{i=1}^d\omega^1_{\vec{p},i}(\partial_1^n f, 2^{-j})\right)\\ &:= C'\sum_{i=1}^d\omega^1_{\vec{p},i}(\partial_1^n f, 2^{-j}).\end{aligned}$$

Thus, we have

$$\begin{aligned}2^{js}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)} &\le C'\|\eta_n\|_{L_1(\mathbb{R}^d)}\|\eta\|_{L_1(\mathbb{R}^d)}^{d-1}2^{j(s-n)}\left(\sum_{i=1}^d\omega^1_{\vec{p},i}(\partial_1^n f, 2^{-j})\right)\\ &:= C''2^{j\alpha}\sum_{i=1}^d\omega^1_{\vec{p},i}(\partial_1^n f, 2^{-j}).\end{aligned}$$

By applying Lemmas 6 and 7, $\left\{2^{js}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)}\right\}_{j=0}^{\infty} \in l_q$ is verified.

Sufficiency. Assume $\mathcal{P}_{-1}f \in L_{\vec{p}}(\mathbb{R}^d)$ and $\left\{2^{js}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)}\right\}_{j=0}^{\infty} \in l_q$. We claim $f \in B^s_{\vec{p},q}(\mathbb{R}^d)$. Using a similar argument as in the proof of the necessity part, we obtain

$$\begin{aligned}\mathcal{F}[\partial_1^n(\beta_j * f)](\xi) &= (i\xi_1)^n \hat{\beta}\left(\frac{\xi}{2^j}\right)\hat{f}(\xi)\\ &= i^n \frac{\hat{\eta}\left(\frac{\xi_1}{2^j}\right)}{\left(\frac{\xi_1}{2^j}\right)^{-n}}\left(\frac{i\xi_1}{2^j}\right)^n \hat{\eta}\left(\frac{\xi_2}{2^j}\right)\cdots\hat{\eta}\left(\frac{\xi_d}{2^j}\right)2^{jn}\hat{\beta}\left(\frac{\xi}{2^j}\right)\hat{f}(\xi)\\ &= i^n \widehat{\eta_{-n}}\left(\frac{\xi_1}{2^j}\right)\left(\frac{i\xi_1}{2^j}\right)^n \hat{\eta}\left(\frac{\xi_2}{2^j}\right)\cdots\hat{\eta}\left(\frac{\xi_d}{2^j}\right)2^{jn}\hat{\beta}\left(\frac{\xi}{2^j}\right)\hat{f}(\xi)\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{P}_j[\partial_1^n f]\|_{L_{\vec{p}}(\mathbb{R}^d)} &= \|\partial_1^n(\beta_j * f)\|_{L_{\vec{p}}(\mathbb{R}^d)}\\ &= \|\beta_j * \partial_1^n f\|_{L_{\vec{p}}(\mathbb{R}^d)}\\ &\leq 2^{jn}\|\eta_{-n}\|_{L_1(\mathbb{R}^d)}\|\eta\|^{d-1}_{L_1(\mathbb{R}^d)}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)}\\ &= 2^{j(n-s)}\|\eta_{-n}\|_{L_1(\mathbb{R}^d)}\|\eta\|^{d-1}_{L_1(\mathbb{R}^d)}2^{js}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)}\\ &= 2^{-j\alpha}\|\eta_{-n}\|_{L_1(\mathbb{R}^d)}\|\eta\|^{d-1}_{L_1(\mathbb{R}^d)}2^{js}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)}.\end{aligned} \tag{8}$$

Since $\left\{2^{js}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)}\right\}_{j=0}^{\infty} \in l_q$, from (8), we obtain $\sum_{j=-1}^{\infty}\|\mathcal{P}_j[\partial_1^n f]\|_{\vec{p}} < \infty$. This yields $\|\partial_1^n f\|_{\vec{p}} < \infty$ by Lemma 5. Other estimations of such type of $\|\partial^n f\|_{\vec{p}}$ can also be verified; therefore, we finally conclude $\|f\|_{W^n_{\vec{p}}(\mathbb{R}^d)} < \infty$. Next, we will prove

$$\left\{2^{k\alpha}\omega^2_{\vec{p},i}(\partial^n f, 2^{-k})\right\}_{k\in\mathbb{Z}} \in l_q.$$

We consider three separate cases below:
Case 1. If $k = 0$, then we are done.
Case 2. Assume $k < 0$. Then, for $1 \leq q < \infty$, we have

$$2^{k\alpha}\omega^2_{\vec{p},i}(\partial^n f, 2^{-k}) \leq 4(2^{k\alpha})\|\partial^n f\|_{L_{\vec{p}}(\mathbb{R}^d)}$$

and

$$\sum_{j=-\infty}^{-1}(2^{k\alpha}\omega^2_{\vec{p},i}(\partial^n f, 2^{-k}))^q < \infty.$$

If $q = \infty$, we have $\max_{-\infty<k\leq -1}(2^{k\alpha}\omega^2_{\vec{p},i}(\partial^n f, 2^{-k})) \leq 4\|\partial^n f\|_{L_{\vec{p}}(\mathbb{R}^d)} < \infty$. Hence

$$\left\{2^{k\alpha}\omega^2_{\vec{p},i}(\partial^n f, 2^{-k})\right\}_{k\in\mathbb{Z}} \in l_q.$$

Case 3. Assume $k > 0$. Since $\mathrm{supp}\mathcal{F}[\mathcal{P}_j\partial^n f] \subset [-2^{j+1}A, 2^{j+1}A]^d$, by Lemma 4, we obtain

$$\left\|\partial^2[\mathcal{P}_j(\partial^n f)]\right\|_{L_{\vec{p}}(\mathbb{R}^d)} \leq C2^{2j}\|\mathcal{P}_j\partial^n f\|_{L_{\vec{p}}(\mathbb{R}^d)}.$$

Therefore, it follows that

$$\begin{aligned}\omega^2_{\vec{p},i}(\mathcal{P}_j(\partial^n f), 2^{-k}) &\leq (2^{-k})^2\|\partial^2[\mathcal{P}_j(\partial^n f)]\|_{L_{\vec{p}}(\mathbb{R}^d)}\\ &\leq C2^{-2k}2^{2j}\|\mathcal{P}_j\partial^n f\|_{L_{\vec{p}}(\mathbb{R}^d)}\\ &\leq C2^{-2(k-j)}2^{-j\alpha}2^{js}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)}\end{aligned} \tag{9}$$

for all $j \in \mathbb{N}$ and $k \in \mathbb{N}$, where the first inequlaity in (9) comes from Theorem 1 (b) and the last inequality in (9) comes from (8). In addition, by using Lemma 5, we also have

$$\begin{aligned}\omega^2_{\vec{p},i}(\partial^n f, 2^{-k}) &\le \sum_{j=-1}^{\infty} \omega^2_{\vec{p},i}(\mathcal{P}_j \partial^n f, 2^{-k}) \\ &= \omega^2_{\vec{p},i}(\mathcal{P}_{-1}\partial^n f, 2^{-k}) + \sum_{j=0}^{k-1} \omega^2_{\vec{p},i}(\mathcal{P}_j \partial^n f, 2^{-k}) + \sum_{j=k}^{\infty} \omega^2_{\vec{p},i}(\mathcal{P}_j \partial^n f, 2^{-k}).\end{aligned}$$

In order to finish the proof, we will proceed with the following steps.

Step 1. We show $\left\{\omega^2_{\vec{p},i}(\mathcal{P}_{-1}\partial^n f, 2^{-k})\right\}_k \in l_q$.

By Theorem 1 (b), it is easy to obtain

$$\begin{aligned}\omega^2_{\vec{p},i}(\mathcal{P}_{-1}\partial^n f, 2^{-k}) &\le C2^{-2k}\|\partial^2(\mathcal{P}_{-1}\partial^n f)\|_{L_{\vec{p}}(\mathbb{R}^d)} \\ &\le C2^{-2k}\|\mathcal{P}_{-1}\partial^n f\|_{L_{\vec{p}}(\mathbb{R}^d)} \\ &\le C2^{-2k}\|\partial^n f\|_{L_{\vec{p}}(\mathbb{R}^d)},\end{aligned}$$

which implies $\left\{\omega^2_{\vec{p},i}(\mathcal{P}_{-1}\partial^n f, 2^{-k})\right\}_k \in l_q$.

Step 2. We prove $\left\{\sum_{j=0}^{k-1} \omega^2_{\vec{p},i}(\mathcal{P}_j \partial^n f, 2^{-k})\right\}_k \in l_q$.

Using (9), we have

$$\begin{aligned}\sum_{j=0}^{k-1} \omega^2_{\vec{p},i}(\mathcal{P}_j \partial^n f, 2^{-k}) &\le C\sum_{j=0}^{k-1} 2^{-2(k-j)}2^{-j\alpha}2^{js}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)} \\ &\le 2^{-k\alpha}\sum_{j=0}^{k-1} 2^{-2(k-j)}2^{\alpha(k-j)}2^{js}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)} \\ &= 2^{-k\alpha}\sum_{j=0}^{k-1} 2^{-(k-j)(2-\alpha)}2^{js}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)}.\end{aligned}$$

Taking into account Lemma 8 and the last inequalities, we deduce

$$\left\{\sum_{j=0}^{k-1} \omega^2_{\vec{p},i}(\mathcal{P}_j \partial^n f, 2^{-k})\right\}_k \in l_q.$$

Step 3. We prove $\left\{\sum_{j=k}^{\infty} \omega^2_{\vec{p},i}(\mathcal{P}_j \partial^n f, 2^{-k})\right\}_k \in l_q$.

Using (8) again, we obtain

$$\begin{aligned}\sum_{j=k}^{\infty} \omega^2_{\vec{p},i}(\mathcal{P}_j \partial^n f, 2^{-k}) &\le 4\sum_{j=k}^{\infty} \|\mathcal{P}_j(\partial^n f)\|_{L_{\vec{p}}(\mathbb{R}^d)} \\ &\le 4C\sum_{j=k}^{\infty} 2^{-j\alpha}2^{js}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)} \\ &= 4C2^{-k\alpha}\sum_{j=k}^{\infty} 2^{-\alpha(j-k)}2^{js}\|\mathcal{P}_j f\|_{L_{\vec{p}}(\mathbb{R}^d)}.\end{aligned}$$

Applying Lemma 8, we show $\left\{\sum_{j=k}^{\infty} \omega^2_{\vec{p},i}(\mathcal{P}_j \partial^n f, 2^{-k})\right\}_k \in l_q$.

Therefore, by Steps 1–3, we verify $\left\{2^{k\alpha}\omega^2_{\vec{p},i}(\partial^n f, 2^{-k})\right\}_{k\in\mathbb{Z}} \in l_q$. The proof is complete. □

Remark 4. *The result of Theorem 3.5 shows that our definition of a Besov space (defined by ingredient modulus of smoothness) is equivalent with inhomogeneous mixed-norm Besov spaces in* [21]*. Therefore, one of the significant findings of this new decomposition theorem is that it is an extension of Frazier and Jawerth?* [16] *construction of the discrete decomposition transformation.*

Remark 5. *(i) It is known that a Besov space defined by the modulus of smoothness is a kind of very practical space (see* [8,37–41] *and their references), which can be used to analyze turbulence* [8]*, the solution of some equations* [42]*, and so on. Hence, we study Besov spaces defined by moduli of continuity.*

(ii) If we can characterize this kind of mixed-norm Besov space by using discrete coefficients, such as the characterization in [14]*, then the mixed-norm Besov space studied in this paper is the same as a classical mixed-norm Besov space* [14,36] *in the sense of equivalent norms. Of course, this requires further work. Moreover, the theory of homogeneous mixed-norm Besov spaces is also worth studying, see* [14]*.*

(iii) It can be seen from the proof process of (7) *that the following conclusion is not tenable:*

$$\begin{aligned}
&\|\beta_j * \partial_1^n f\|_{L_{\vec{p}}(\mathbb{R}^d)} \\
&\le C\sum_{i=1}^{d}\frac{1}{2}\int|\beta(y)|\sup_{|h|\le 2^{-j}|y|}\|\Delta^2_{he_i}\partial_1^n f(x)\|_{L_{\vec{p}}(\mathbb{R}^d)}dy \\
&\le C\omega^2_{\vec{p},i}(\partial_1^n f(x), 2^{-j}).
\end{aligned}$$

In other words, although it is obvious that formula

$$\omega^2_{\vec{p},i}(\partial_1^n f(x), 2^{-j}) \le \omega^2_{\vec{p}}(\partial_1^n f(x), 2^{-j})$$

holds, the reverse is not true. Therefore, in the previous definition of a Besov space, we use the definition of a one-order modulus of smoothness instead of a two-order modulus of smoothness. This corresponds to why we used the equivalent mixed-norm of Besov spaces under the definition of introducing a one-order modulus of smoothness in Lemma 6. The cost is that we cannot choose α to be 0 or 1, but only between 0 and 1.

4. Conclusions

The main purpose of this paper is to study the concept of ingredient modulus of smoothness in component form in $L_{\vec{p}}(\mathbb{R}^d)$ and a kind of mixed-norm Sobolev space. We obtain some new properties, inequalities, and auxiliary results in mixed-norm spaces $L_{\vec{p}}(\mathbb{R}^d)$. A new concept of mixed-norm Besov space is presented and a new decomposition theorem for mixed-norm Besov spaces is established. In theory, difference operators may be applied to the study of partial differential equations. The ingredient-type difference operator we introduced can analyze a certain component, so it can be used to solve some partial differential equations. However, in general, we are also exploring these studies, because the application of mixed norm in this field is a relatively new subject. In fact, some differential operators have been studied in these places, see [21]. We hope that our new results will have broad and sustainable applications in nonlinear analysis, mathematical physics, mechanics, biology, and future related fields.

Author Contributions: Writing—original draft, J.Z., M.K. and W.-S.D.; writing—review and editing, J.Z., M.K. and W.-S.D. All authors contributed equally to the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: The first author is partially supported by the Natural Science Foundation of Tianjin City, China (grant no. 18JCYBJC16300). The second author is partially supported by grant 451-

03-68/2020/14/200156 of Ministry of Science and Technological Development, Republic of Serbia. The third author is partially supported by grant no. MOST 111-2115-M-017-002 of the National Science and Technology Council of the Republic of China.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors wish to express a heartfelt thank you to the anonymous referees for their valuable suggestions and comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Besov, O.V. On a family of function spaces, embedding theorems and extensions. *Dokl. Akad. Nauk SSSR* **1959**, *126*, 1163–1165. (In Russian)
2. Besov, O.V. On a family of function spaces in connection with embeddings and extensions. *Tr. Mat. Inst. Steklova* **1961**, *60*, 42–81. (In Russian)
3. Hörmander, L. Estimates for translation invariant operators in L_p spaces. *Acta Math. 1960*, *104*, 93–140. [CrossRef]
4. Hovemann, M.; Sickel, W. Besov-type spaces and differences. *Eurasian Math. J.* **2020**, *11*, 25–56. [CrossRef]
5. Jia, H.; Tao, J.; Yang, D.; Yuan, W.; Zhang, Y. Boundedness of Calderón-Zygmund operators on special John–Nirenberg–Campanato and Hardy-type spaces via congruent cubes. *Anal. Math. Phys.* **2022**, *12*, 15. [CrossRef]
6. Liu, J.; Weisz, F.; Yang, D.; Yuan, W. Littlewood-Paley and finite atomic characterizations of anisotropic variable Hardy-Lorentz spaces and their applications. *J. Fourier Anal. Appl.* **2019**, *25*, 874–922. [CrossRef]
7. Zhuo, C.; Sickel, W.; Yang, D.; Yuan, W. Characterizations of Besov-type and Triebel-Lizorkin-type spaces via averages on balls. *Can. Math. Bull.* **2017**, *60*, 655–672. [CrossRef]
8. Bittner, K.; Urban, K. On interpolatory divergence-free wavelets. *Math. Comput.* **2007**, *76*, 903–929. [CrossRef]
9. Dahlke, S.; Hansen, M.; Schneider, C.; Sickel, W. On Besov regularity of solutions to nonlinear elliptic partial differential equations. *Nonlinear Anal.* **2020**, *192*, 111686. [CrossRef]
10. Hardle, W.; Kerkyacharian, G.; Picard, D.; Tsybakov, A. *Wavelets, Approximation, and Statistical Applications, Lecture Notes in Statistics*; Springer: New York, NY, USA, 1998; Volume 129.
11. Triebel, H. *Theory of Function Spaces III*; Monographs in Mathematics; Birkhäuser: Basel, Switzerland, 2006; Volume 100.
12. Yserentant, H. *Regularity and Approximability of Electronic Wave Functions*; Springer: Berlin/Heidelberg, Germany, 2010.
13. Cleanthous, G.; Georgiadis, A.G.; Nielsen, M. Fourier multipliers on anisotropic mixed-norm spaces of distributions. *Math. Scand.* **2019**, *124*, 289–304. [CrossRef]
14. Cleanthous, G.; Georgiadis, A.G.; Nielsen, M. Discrete decomposition of homogeneous mixed-norm Besov spaces. *Contemp. Math.* **2017**, *693*, 167–184.
15. Cleanthous, G.; Georgiadis, A.G.; Nielsen, M. Anisotropic mixed-norm Hardy spaces. *J. Geom. Anal.* **2017**, *27*, 2758–2787. [CrossRef]
16. Frazier, M.; Jawerth, B. Decomposition of Besov Spaces. *Indiana Univ. Math. J.* **1985**, *34*, 777–799. [CrossRef]
17. Garrigós, G.; Tabacco, A. Wavelet decompositions of anisotropic Besov spaces. *Math. Nachrichten* **2002**, *239*, 80–102. [CrossRef]
18. Sawano, Y. *Theory of Besov Spaces*; Developments in Mathematics; Springer: Singapore, 2018; Volume 56.
19. Benedek, A.; Panzone, R. The space L^P, with mixed norm. *Duke Math. J.* **1961**, *28*, 301–324. [CrossRef]
20. Huang, L.; Liu, J.; Yang, D.; Yuan, W. Atomic and Littlewood-Paley decompositions of anisotropic mixed-norm hardy spaces and their applications. *J. Geom. Anal.* **2019**, *29*, 1991–2067. [CrossRef]
21. Georgiadis, A.G.; Nielsen, M. Pseudodifferential operators on mixed-norm Besov and Triebel-Lizorkin spaces. *Math. Nachrichten* **2016**, *289*, 2019–2036. [CrossRef]
22. Johnsen, J.; Sickel, W. A direct proof of Sobolev embeddings for quasi-homogeneous Lizorkin-Triebel spaces with mixed norms. *J. Funct. Spaces Appl.* **2007**, *5*, 183–198. [CrossRef]
23. Johnsen, J.; Sickel, W. On the trace problem for Lizorkin-Triebel spaces with mixed norms. *Math. Nachrichten* **2008**, *281*, 669–696. [CrossRef]
24. Li, R.; Liu, B.; Liu, R.; Zhang, Q. Nonuniform sampling in principal shift-invariant subspaces of mixed Lebesgue spaces $L_{p,q}(R^{d+1})$. *J. Math. Anal. Appl.* **2017**, *453*, 928–941. [CrossRef]
25. Zhao, J.; Kostić, M.; Du, W.-S. On generalizations of sampling theorem and stability theorem in shift-invariant subspaces of Lebesgue and Wiener amalgam spaces with mixed-norms. *Symmetry* **2021**, *13*, 331. [CrossRef]
26. Drihem, D. Characterization of variable Besov-type spaces by ball means of diferences. *Kyoto J. Math.* **2016**, *56*, 655–680. [CrossRef]
27. Yuan, W.; Sickel, W.; Yang, D. *Morrey and Campanato Meet Besov, Lizorkin and Triebel*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 2010.

28. Zhao, J.; Chen, Y. New decomposition theorem and new inequalities induced by synthesis-type modulus of smoothness. *J. Nonlinear Covnex Anal.* **2022**, *23*, 1151–1166.
29. Benedek, A.; Calderón, A.P.; Panzone, R. Convolution operators on Banach space valued functions. *Proc. Natl. Acad. Sci. USA* **1962**, *48*, 356–365. [CrossRef]
30. Grafakos, L. *Classical Fourier Ananlysis*, 2nd ed.; Springer: New York, NY, USA, 2008.
31. Besov, O.V.; Il'in, V.P.; Nikol'skiĭ, S.M. *Integral Representation of Functions and Embedding Theorems*; V.H. Winston and Sons: Washington, DC, USA, 1979; Volume 2.
32. Antonić, N.; Ivec, I. On the Hörmander-Mihlin theorem for mixed-norm Lebesgue spaces. *J. Math. Anal. Appl.* **2016**, *433*, 176–199. [CrossRef]
33. Brezis, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*; Springer: New York, NY, USA, 2011.
34. Lizorkin, P.I. Multipliers of fourier integrals and bounds of convolutions in spaces with mixed norms. *Math. USSR* **1970**, *4*, 225. [CrossRef]
35. Zhao, J.; Du, W.-S.; Chen, Y. New generalizations and results in shift-invariant subspaces of mixed-norm Lebesgue spaces $L_{\vec{p}}(\mathbb{R}^d)$. *Mathematics* **2021**, *9*, 227. [CrossRef]
36. Cleanthous, G.; Georgiadis, A.G. Mixed-norm α-modulation spaces. *Trans. Am. Math. Soc.* **2020**, *373*, 3323–3356. [CrossRef]
37. DeVore, R.A.; Lorentz, G.G. *Constructive Approximation*; Springer: New York, NY, USA, 1993.
38. Cohen, A. Wavelet methods in numerical analysis. In *Handbook of Numerical Analysis*; North-Holland Publishing Company: Amsterdam, The Netherlands, 2000; Volume 7, pp. 417–711.
39. Donoho, D.L. Interpolating Wavelet Transforms. Tech. Rep. Dept. Statistics, Standford University, Standford, CA, USA, 1992. Available online: http://laurent-duval.eu/Documents-WITS-starlet/Interpolets/Donoho_D_1992_tr_interpolating_wt-interpolets.pdf (accessed on 15 October 1992).
40. Liu, Y.; Zhao, J. An extension of Bittner and Urban's theorem. *Math. Comput.* **2013**, *82*, 401–411. [CrossRef]
41. Liu, Y.; Zhao, J. Convergence of Hermite interpolatory operators. *Sci. China Math.* **2010**, *53*, 2115–2126. [CrossRef]
42. Brzezniak, Z.; Ferrario, B. 2D Navier–Stokes equation in Besov spaces of negative order. *Nonlinear Anal. Theory Methods Appl.* **2009**, *70*, 3902–3916. [CrossRef]