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Fractional Weighted Midpoint-Type Inequalities for s-Convex Functions

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Abstract: In the present paper, we first prove a new integral identity. Using that identity, we establish some fractional weighted midpoint-type inequalities for functions whose first derivatives are extended s-convex. Some special cases are discussed. Finally, to prove the effectiveness of our main results, we provide some applications to numerical integration as well as special means.

Keywords: fractional derivatives; weighted integral; midpoint formula; integral inequalities; s-convex functions

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Citation: Nasri, N.; Aissaoui, F.; Bouhali, K.; Frioui, A.; Meftah, B.; Zennir, K.; Radwan, T. Fractional Weighted Midpoint-Type Inequalities for s-Convex Functions. *Symmetry* **2023**, *15*, 612. https://doi.org/10.3390/sym15030612

Academic Editors: Sergei D. Odintsov, Junesang Choi, Francisco Martínez González and Mohammed K. A. Kaabar

Received: 1 January 2023
Revised: 7 February 2023
Accepted: 23 February 2023
Published: 28 February 2023

1. Introduction

It is well known that convexity is one of the most fundamental principles of analysis that is widely used in several fields of pure and applied sciences. Especially, in the classical theory of optimization where convexity causes it to be possible to obtain necessary and sufficient global optimality conditions; in consumer theory in economics, information theory as well as in the field of inequalities where the relationship is closely linked. For papers related to convexity and integral inequalities we refer readers to [1–5].

A real function defined on E is called convex; if for all $x, z \in E$ and all $a \in [0,1]$, we have

$$g(ax + (1-a)z) \le ag(x) + (1-a)g(z).$$

We note that all convex function on a finite interval, and $[\varrho, \varpi]$ must satisfy the so called Hermite–Hadamard inequality (see [6]).

$$g\left(\frac{\varrho+\varpi}{2}\right) \le \frac{1}{\varpi-\varrho}\int_{\varrho}^{\varpi} g(x)dx \le \frac{g(\varrho)+g(\varpi)}{2}. \tag{1}$$

Inequality (1) can be seen as a second definition of convex functions equivalent to the first one for continuous function (see [7]); it is a character of which all convex functions must satisfy at least the left- or right-hand side.

Pearce and Pečarić [8] introduced the following inequality connected with (1)

$$\left| g\left(\frac{\varrho+\varpi}{2}\right) - \frac{1}{\varpi-\varrho}\int_{\varrho}^{\varpi} g(x)dx \right| \le \frac{\varpi-\varrho}{4}\left(\frac{|g'(\varrho)|^q + |g'(\varpi)|^q}{2}\right)^{\frac{1}{q}},$$

where $q \ge 1$.

Kirmaci [9] proved that, for all function f such that $|g'|$ or $|g'|^q$ are convex, the following inequalities hold:

$$\left|g\left(\frac{\varrho+\varpi}{2}\right)-\frac{1}{\varpi-\varrho}\int_{\varrho}^{\varpi}g(x)dx\right|\leq\frac{\varpi-\varrho}{8}\left(|g'(\varrho)|+|g'(\varpi)|\right),$$

where $q\geq 1$. Furthermore, they proved the following result

$$\begin{aligned}&\left|g\left(\frac{\varrho+\varpi}{2}\right)-\frac{1}{\varpi-\varrho}\int_{\varrho}^{\varpi}g(x)dx\right|\\&\leq\frac{\varpi-\varrho}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left(\left(3|g'(\varrho)|^q+|g'(\varpi)|^q\right)^{\frac{1}{q}}+\left(|g'(\varrho)|^q+3|g'(\varpi)|^q\right)^{\frac{1}{q}}\right),\end{aligned}$$

where $q,p>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

İşcan et al. [10] showed the following midpoint inequalities for P-functions (see (3) below):

$$\left|g\left(\frac{\varrho+\varpi}{2}\right)-\frac{1}{\varpi-\varrho}\int_{\varrho}^{\varpi}g(x)dx\right|\leq\frac{\varpi-\varrho}{4}\left(|g'(\varrho)|^c+|g'(\varpi)|^c\right)^{\frac{1}{c}},$$

where $c\geq 1$. Furthermore, they proved the following result:

$$\begin{aligned}&\left|g\left(\frac{\varrho+\varpi}{2}\right)-\frac{1}{\varpi-\varrho}\int_{\varrho}^{\varpi}g(x)dx\right|\\&\leq\frac{\varpi-\varrho}{4}\left(\frac{1}{b+1}\right)^{\frac{1}{b}}\left(\left(|g'(\varrho)|^c+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^c\right)^{\frac{1}{c}}+\left(\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^c+|g'(\varpi)|^c\right)^{\frac{1}{c}}\right)\\&\leq\frac{\varpi-\varrho}{4}\left(\frac{1}{b+1}\right)^{\frac{1}{b}}\left(\left(2|g'(\varrho)|^c+|g'(\varpi)|^c\right)^{\frac{1}{c}}+\left(|g'(\varrho)|^c+2|g'(\varpi)|^c\right)^{\frac{1}{c}}\right),\end{aligned}$$

where $c,b>1$ with $\frac{1}{c}+\frac{1}{b}=1$.

Nowadays, fractional calculus has become a popular implement for scientists. It has been successfully used in various fields of science and engineering see [11–18]. Its main strength in the description of memory and genetic properties of different materials and processes has aroused great interest for researchers in different domains. This innovative idea of fractional calculus has attracted many researchers in recent years, several generalizations, extensions, refinements, and finding a counterpart have appeared (see [19–26]).

In [6], Sarikaya and Yildirim established the analogue fractional of inequality (1) as follows:

$$g\left(\frac{\varrho+\varpi}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varpi-\varrho)^{\alpha}}\left(J^{\alpha}_{(\frac{\varrho+\varpi}{2})+}g(\varpi)+J^{\alpha}_{(\frac{\varrho+\varpi}{2})-}g(\varrho)\right)\leq\frac{g(\varrho)+g(\varpi)}{2}.$$

Furthermore, the authors investigate the following fractional midpoint inequalities for convex-first derivatives

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varpi-\varrho)^{\alpha}}\left(J^{\alpha}_{(\frac{\varrho+\varpi}{2})+}g(\varpi)+J^{\alpha}_{(\frac{\varrho+\varpi}{2})-}g(\varrho)\right)-g\left(\frac{\varrho+\varpi}{2}\right)\right|\\&\leq\frac{\varpi-\varrho}{4(\alpha+1)}\left(\left(\frac{(\alpha+1)|g'(\varrho)|^q+(\alpha+3)|g'(\varpi)|^q}{2(\alpha+2)}\right)^{\frac{1}{q}}\right.\end{aligned}$$

$$+\left(\frac{(\alpha+3)|g'(\varrho)|^q+(\alpha+1)|g'(\varpi)|^q}{2(\alpha+2)}\right)^{\frac{1}{q}}\Bigg)$$

and

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varpi-\varrho)^{\alpha}}\left(J^{\alpha}_{(\frac{\varrho+\varpi}{2})+}g(\varpi)+J^{\alpha}_{(\frac{\varrho+\varpi}{2})-}g(\varrho)\right)-g\left(\frac{\varrho+\varpi}{2}\right)\right|\\ &\leq\frac{\varpi-\varrho}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\frac{|g'(\varrho)|^q+3|g'(\varpi)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|g'(\varrho)|^q+|g'(\varpi)|^q}{4}\right)^{\frac{1}{q}}\right)\\ &\leq\frac{\varpi-\varrho}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left(|g'(\varrho)|+|g'(\varpi)|\right),\end{aligned}$$

where $\alpha>0, p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1, \Gamma$ is the gamma function and $J^{\alpha}_{(\frac{\varrho+\varpi}{2})+}$ and $J^{\alpha}_{(\frac{\varrho+\varpi}{2})-}$ are the Riemann–Liouville integrals (see Definition 1 below).

Motivated by the above results, here, we first prove a new integral identity and, then, by using this identity, we establish some fractional weighted midpoint-type inequalities for functions that the first derivatives are extended s-convex functions. We also derive some known results and, state applications in numerical integration and in special means are presented to prove the effectiveness of our main results.

The paper is organized as follows. In the next section, we provide some auxiliary results as a preliminaries. In Section 3, we provide the main results and proofs. In Section 4, we will provide an applications of our analysis to illustrate our main results. In Section 5, we conclude our work.

2. Preliminaries

In this section, we recall certain notions concerning special functions, some classes of convex functions, and the Riemann–Liouville integral operator.

A non-negative function $g:E\subset[0,\infty)\to\mathbb{R}$ is said to be s-convex in the second sense for some fixed $s\in(0,1]$, if

$$g(ax+(1-a)z)\ \leq a^s g(x)+(1-a)^s g(z), \tag{2}$$

holds for all $x,z\in E$ and $a\in[0,1]$.

Whereas, a non-negative function $g:E\to\mathbb{R}$ is said to be P-convex; if for all $x,z\in E$ and all $a\in(0,1)$, we have

$$g(ax+(1-a)z)\leq g(x)+g(z). \tag{3}$$

A non-negative function $g:E\to\mathbb{R}$ is said to be s-Godunova–Levin function, where $s\in[0,1]$; if for all $x,z\in E$, and all $a\in(0,1)$, we have

$$g(ax+(1-a)z)\leq\frac{g(x)}{a^s}+\frac{g(z)}{(1-a)^s}. \tag{4}$$

A non-negative function $g:E\subset[0,\infty)\to\mathbb{R}$ is said to be an extended s-convex for some fixed $s\in[-1,1]$; if for all $x,z\in E$ and all $a\in(0,1)$, we have

$$g(ax+(1-a)z)\ \leq a^s g(x)+(1-a)^s g(z). \tag{5}$$

Definition 1 ([12]). *Let $\Omega\in L^1[\varrho,\varpi]$. The Riemann–Liouville integrals $J^{\alpha}_{\varrho+}\Omega$ and $J^{\alpha}_{\varpi-}\Omega$ of order $\alpha>0$ with $\varpi>\varrho\geq0$ are defined by*

$$J^{\alpha}_{\varrho+}\Omega(d)=\ \frac{1}{\Gamma(\alpha)}\int_{\varrho}^{d}(d-a)^{\alpha-1}\Omega(a)da,\quad d>\varrho,$$

$$J_{\varpi^-}^{\alpha}\Omega(d) = \frac{1}{\Gamma(\alpha)}\int_d^{\varpi}(a-d)^{\alpha-1}\Omega(a)da, \quad \varpi > d,$$

respectively, where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-a}a^{\alpha-1}da,$$

and $J_{\varrho^+}^{0}\Omega(d) = J_{\varpi^-}^{0}\Omega(d) = \Omega(d)$.

For any complex numbers k, l such that $Re(k) > 0$ and $Re(l) > 0$. The beta function is provided by

$$B(k,l) = \int_0^1 a^{k-1}(1-a)^{l-1}da = \frac{\Gamma(k)\Gamma(l)}{\Gamma(k+l)}.$$

3. Main Results and Proofs

To prepare the proofs of our main results, we will need the following Lemma.

Lemma 1. *Let* $g : E = [\varrho, \varpi] \to \mathbb{R}$ *be a differentiable map on* I° *(*I° *is the interior of* I*), with* $\varrho < \varpi$*, and let* $w : [\varrho, \varpi] \to \mathbb{R}$ *be symmetric as regards* $\frac{\varrho+\varpi}{2}$*. If* $g, w \in L[\varrho, \varpi]$*, then*

$$\begin{aligned}
&L^{\alpha}[w]g\left(\frac{\varrho+\varpi}{2}\right) - L^{\alpha}[wg] \\
=&\frac{(\varpi-\varrho)^2}{4}\left(\int_0^1 p_1(a)g'\left(a\varrho+(1-a)\frac{\varrho+\varpi}{2}\right)da - \int_0^1 p_2(a)g'\left((1-a)\frac{\varrho+\varpi}{2}+a\varpi\right)da\right).
\end{aligned}$$

where

$$p_1(a) = \int_a^1 (1-b)^{\alpha-1}w\left(b\varrho+(1-b)\frac{\varrho+\varpi}{2}\right)db, \tag{6}$$

$$p_2(a) = \int_a^1 (1-b)^{\alpha-1}w\left(b\varpi+(1-b)\frac{\varrho+\varpi}{2}\right)db, \tag{7}$$

and

$$L^{\alpha}[g] = \left(\frac{2}{\varpi-\varrho}\right)^{\alpha-1}\Gamma(\alpha)\left(J_{\left(\frac{\varrho+\varpi}{2}\right)^-}^{\alpha}g(\varrho) + J_{\left(\frac{\varrho+\varpi}{2}\right)^+}^{\alpha}g(\varpi)\right). \tag{8}$$

Proof. Let

$$I = I_1 - I_2, \tag{9}$$

where

$$I_1 = \int_0^1 p_1(a)g'\left(a\varrho+(1-a)\frac{\varrho+\varpi}{2}\right)da,$$

and

$$I_2 = \int_0^1 p_2(a)g'\left(a\varpi+(1-a)\frac{\varrho+\varpi}{2}\right)da.$$

Integrating by parts I_1, we obtain

$$\int_0^1 p_1(a)g'\left(a\varrho+(1-a)\frac{\varrho+\varpi}{2}\right)da$$

$$
\begin{aligned}
&= \int_0^1 \left[\int_a^1 (1-b)^{\alpha-1} w\left(b\varrho + (1-b)\frac{\varrho+\varpi}{2}\right) db \right] g'\left(a\varrho + (1-a)\frac{\varrho+\varpi}{2}\right) da \\
&= -\frac{2}{\varpi-\varrho} \left[\int_a^1 (1-b)^{\alpha-1} w\left(b\varrho + (1-b)\frac{\varrho+\varpi}{2}\right) db \right] g\left(a\varrho + (1-a)\frac{\varrho+\varpi}{2}\right) \Bigg|_{a=0}^{a=1} \\
&\quad - \frac{2}{\varpi-\varrho} \int_0^1 (1-a)^{\alpha-1} w\left(a\varrho + (1-a)\frac{\varrho+\varpi}{2}\right) g\left(a\varrho + (1-a)\frac{\varrho+\varpi}{2}\right) da \\
&= \frac{2}{\varpi-\varrho} \left[\int_0^1 (1-b)^{\alpha-1} w\left(b\varrho + (1-b)\frac{\varrho+\varpi}{2}\right) db \right] g\left(\frac{\varrho+\varpi}{2}\right) \\
&\quad - \frac{2}{\varpi-\varrho} \int_0^1 (1-a)^{\alpha-1} w\left(a\varrho + (1-a)\frac{\varrho+\varpi}{2}\right) g\left(a\varrho + (1-a)\frac{\varrho+\varpi}{2}\right) da \\
&= \left(\frac{2}{\varpi-\varrho}\right)^{\alpha+1} \left[\int_{\varrho}^{\frac{\varrho+\varpi}{2}} (u-\varrho)^{\alpha-1} w(u) du \right] g\left(\frac{\varrho+\varpi}{2}\right) \\
&\quad - \left(\frac{2}{\varpi-\varrho}\right)^{\alpha+1} \int_{\varrho}^{\frac{\varrho+\varpi}{2}} (u-\varrho)^{\alpha-1} w(u) g(u) du \qquad (10) \\
&= \left(\frac{2}{\varpi-\varrho}\right)^{\alpha+1} \Gamma(\alpha) \left(J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^-} w(\varrho) \right) g\left(\frac{\varrho+\varpi}{2}\right) - \left(\frac{2}{\varpi-\varrho}\right)^{\alpha+1} \Gamma(\alpha) J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^-} (wg)(\varrho).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_0^1 p_2(a) g'\left(a\varpi + (1-a)\frac{\varrho+\varpi}{2}\right) da \\
&= \int_0^1 \left(\left[\int_a^1 (1-b)^{\alpha-1} w\left(b\varpi + (1-b)\frac{\varrho+\varpi}{2}\right) db \right] g'\left(a\varpi + (1-a)\frac{\varrho+\varpi}{2}\right) da \right. \\
&= \frac{2}{\varpi-\varrho} \left[\int_a^1 (1-b)^{\alpha-1} w\left(b\varpi + (1-b)\frac{\varrho+\varpi}{2}\right) db \right] g\left(a\varpi + (1-a)\frac{\varrho+\varpi}{2}\right) \Bigg|_{a=0}^{a=1} \\
&\quad + \frac{2}{\varpi-\varrho} \int_0^1 (1-a)^{\alpha-1} w\left(a\varpi + (1-a)\frac{\varrho+\varpi}{2}\right) g\left(a\varpi + (1-a)\frac{\varrho+\varpi}{2}\right) da \\
&= -\frac{2}{\varpi-\varrho} \left[\int_0^1 (1-b)^{\alpha-1} w\left(b\varpi + (1-b)\frac{\varrho+\varpi}{2}\right) db \right] g\left(\frac{\varrho+\varpi}{2}\right) \\
&\quad + \frac{2}{\varpi-\varrho} \int_0^1 (1-a)^{\alpha-1} w\left(a\varpi + (1-a)\frac{\varrho+\varpi}{2}\right) g\left(a\varpi + (1-a)\frac{\varrho+\varpi}{2}\right) da \\
&= -\left(\frac{2}{\varpi-\varrho}\right)^{\alpha+1} \left[\int_{\frac{\varrho+\varpi}{2}}^{\varpi} (\varpi-u)^{\alpha-1} w(u) du \right] g\left(\frac{\varrho+\varpi}{2}\right) \\
&\quad + \left(\frac{2}{\varpi-\varrho}\right)^{\alpha+1} \int_{\frac{\varrho+\varpi}{2}}^{\varpi} (\varpi-u)^{\alpha-1} w(u) g(u) du \qquad (11)
\end{aligned}
$$

$$= -\left(\frac{2}{\varpi-\varrho}\right)^{\alpha+1}\Gamma(\alpha)\left(J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^{+}}w(\varpi)\right)g\left(\frac{\varrho+\varpi}{2}\right)+\left(\frac{2}{\varpi-\varrho}\right)^{\alpha+1}\Gamma(\alpha)J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^{+}}(wg)(\varpi).$$

Substituting (10) and (11) into (9), then multiplying the resulting equality by $\frac{(\varpi-\varrho)^2}{4}$ and using (8), we obtain the desired result. □

Theorem 1. *Let $g:[\varrho,\varpi]\to\mathbb{R}$ be a differentiable function on (ϱ,ϖ) such that $g'\in L([\varrho,\varpi])$ with $0\le\varrho<\varpi$, and let $w:[\varrho,\varpi]\to\mathbb{R}$ be a continuous and symmetric function as regards $\frac{\varrho+\varpi}{2}$. If $|g'|$ is an extended s-convex for some fixed $s\in(-1,1]$, then we have*

$$\begin{aligned}
&\left|L^{\alpha}[w]g\left(\frac{\varrho+\varpi}{2}\right)-L^{\alpha}[wg]\right|\\
\le&\frac{(\varpi-\varrho)^2}{4\alpha}\|w\|_{[\varrho,\varpi],\infty}\\
&\times\left(\frac{\Gamma(s+1)\Gamma(\alpha+1)|g'(\varrho)|+2\Gamma(s+\alpha+1)\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|+\Gamma(s+1)\Gamma(\alpha+1)|g'(\varpi)|}{\Gamma(s+\alpha+2)}\right),
\end{aligned}$$

where Γ is the gamma function.

Proof. Using Lemma 1, the absolute value and s-convexity of $|g'|$ provide

$$\begin{aligned}
&\left|L^{\alpha}[w]g\left(\frac{\varrho+\varpi}{2}\right)-L^{\alpha}[wg]\right|\\
\le&\frac{(\varpi-\varrho)^2}{4}\left(\int_0^1|p_1(a)|\left|g'\left(a\varrho+(1-a)\frac{\varrho+\varpi}{2}\right)\right|da\right.\\
&\left.+\int_0^1|p_2(a)|\left|g'\left(a\varpi+(1-a)\frac{\varrho+\varpi}{2}\right)\right|da\right)\\
\le&\frac{(\varpi-\varrho)^2}{4}\|w\|_{[\varrho,\varpi],\infty}\left(\int_0^1\left(\int_a^1(1-b)^{\alpha-1}db\right)\left|g'\left(a\varrho+(1-a)\frac{\varrho+\varpi}{2}\right)\right|da\right.\\
&\left.+\int_0^1\left(\int_a^1(1-b)^{\alpha-1}db\right)\left|g'\left(a\varpi+(1-a)\frac{\varrho+\varpi}{2}\right)\right|da\right)\\
\le&\frac{(\varpi-\varrho)^2}{4\alpha}\|w\|_{[\varrho,\varpi],\infty}\left(\int_0^1(1-a)^{\alpha}\left(a^s|g'(\varrho)|+(1-a)^s\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|\right)da\right.\\
&\left.+\int_0^1(1-a)^{\alpha}\left(a^s|g'(\varpi)|+(1-a)^s\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|\right)da\right)\\
=&\frac{(\varpi-\varrho)^2}{4\alpha}\|w\|_{[\varrho,\varpi],\infty}\\
&\times\left(\frac{\Gamma(s+1)\Gamma(\alpha+1)|g'(\varrho)|+2\Gamma(s+\alpha+1)\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|+\Gamma(s+1)\Gamma(\alpha+1)|g'(\varpi)|}{\Gamma(s+\alpha+2)}\right).
\end{aligned}$$

Then, the proof is now completed. □

Corollary 1. *In Theorem 1, if we use:*

1. $s = 0$, *we obtain*

$$\left| L^{\alpha}[w]g\left(\frac{\varrho+\varpi}{2}\right) - L^{\alpha}[wg] \right| \leq \frac{(\varpi-\varrho)^2}{4\alpha(\alpha+1)} \|w\|_{[\varrho,\varpi],\infty} \left(|g'(\varrho)| + 2\left| g'\left(\frac{\varrho+\varpi}{2}\right) \right| + |g'(\varpi)| \right).$$

2. $s = 1$, *we obtain*

$$\left| L^{\alpha}[w]g\left(\frac{\varrho+\varpi}{2}\right) - L^{\alpha}[wg] \right| \leq \frac{(\varpi-\varrho)^2}{2\alpha(\alpha+1)} \|w\|_{[\varrho,\varpi],\infty} \left(\frac{|g'(\varrho)| + 2(\alpha+1)\left| g'\left(\frac{\varrho+\varpi}{2}\right) \right| + |g'(\varpi)|}{2(\alpha+2)} \right).$$

Corollary 2. *In Theorem 1, if we use $\alpha = 1$, we obtain*

$$\left| g\left(\frac{\varrho+\varpi}{2}\right) \int_{\varrho}^{\varpi} w(\mathrm{N})d\mathrm{N} - \int_{\varrho}^{\varpi} w(\mathrm{N})g(\mathrm{N})d\mathrm{N} \right| \leq \frac{(\varpi-\varrho)^2}{4(s+1)(s+2)} \|w\|_{[\varrho,\varpi],\infty} \left(|g'(\varrho)| + 2(s+1)\left| g'\left(\frac{\varrho+\varpi}{2}\right) \right| + |g'(\varpi)| \right).$$

Remark 1. *In Corollary 2, if we use $s \in (0,1]$, we obtain the first inequality of Corollary 2.2.1 in [27]. Moreover, if we use $s = 0$ and $s = 1$, we obtain Corollary 2 and Corollary 3 in [28] respectively.*

Corollary 3. *In Theorem 1, if we choose:*

1. $w(u) = \frac{1}{\varpi-\varrho}$, *we obtain*

$$\left| g\left(\frac{\varrho+\varpi}{2}\right) - \frac{2^{\alpha-1}}{(\varpi-\varrho)^{\alpha}}\Gamma(\alpha+1)\left(J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^-} g(\varrho) + J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^+} g(\varpi) \right) \right| \leq \frac{\varpi-\varrho}{4\Gamma(s+\alpha+2)} \Big(\Gamma(s+1)\Gamma(\alpha+1)|g'(\varrho)| + 2\Gamma(s+\alpha+1)\left| g'\left(\frac{\varrho+\varpi}{2}\right) \right| + \Gamma(s+1)\Gamma(\alpha+1)|g'(\varpi)| \Big).$$

2. $w(u) = \frac{1}{\varpi-\varrho}$ *and $\alpha = 1$, we obtain*

$$\left| g\left(\frac{\varrho+\varpi}{2}\right) - \frac{1}{\varpi-\varrho}\int_{\varrho}^{\varpi} g(u)du \right| \leq \frac{\varpi-\varrho}{4(s+2)(s+1)} \left(|g'(\varrho)| + 2(s+1)\left| g'\left(\frac{\varrho+\varpi}{2}\right) \right| + |g'(\varpi)| \right).$$

Corollary 4. *In Theorem 1, using the s-convexity of $|g'|$, i.e.,*

$$\left| g'\left(\frac{\varrho+\varpi}{2}\right) \right| \leq \frac{|g'(\varrho)| + |g'(\varpi)|}{2^{s-1}(1+s)},$$

we obtain

$$\left| L^{\alpha}[w]g\left(\frac{\varrho+\varpi}{2}\right) - L^{\alpha}[wg] \right|$$

$$\leq \frac{(\varpi-\varrho)^2}{4\alpha(1+s)}\|w\|_{[\varrho,\varpi],\infty}\left(\frac{2^{2-s}\Gamma(s+\alpha+1)+\Gamma(s+2)\Gamma(\alpha+1)}{\Gamma(s+\alpha+2)}\right)(|g'(\varrho)|+|g'(\varpi)|).$$

Corollary 5. *In Corollary 4, if we use:*

1. $\alpha=1$, *we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right)\int_{\varrho}^{\varpi}w(\mathrm{N})d\mathrm{N}-\int_{\varrho}^{\varpi}w(\mathrm{N})g(\mathrm{N})d\mathrm{N}\right|$$
$$\leq \frac{(2^{2-s}+1)(\varpi-\varrho)^2}{4(1+s)(s+2)}\|w\|_{[\varrho,\varpi],\infty}(|g'(\varrho)|+|g'(\varpi)|).$$

2. $w(u)=\frac{1}{\varpi-\varrho}$, *we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right)-\frac{2^{\alpha-1}}{(\varpi-\varrho)^{\alpha}}\Gamma(\alpha+1)\left(J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^-}g(\varrho)+J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^+}g(\varpi)\right)\right|$$
$$\leq \frac{\varpi-\varrho}{4(1+s)}\left(\frac{2^{2-s}\Gamma(s+\alpha+1)+\Gamma(s+2)\Gamma(\alpha+1)}{\Gamma(s+\alpha+2)}\right)(|g'(\varrho)|+|g'(\varpi)|).$$

3. $w(u)=\frac{1}{\varpi-\varrho}$ *and* $\alpha=1$, *we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right)-\frac{1}{\varpi-\varrho}\int_{\varrho}^{\varpi}g(u)du\right|\leq\frac{(2^{2-s}+1)(\varpi-\varrho)}{4(1+s)(s+2)}(|g'(\varrho)|+|g'(\varpi)|).$$

Remark 2. *Corollary 5, the third point will be reduced to Theorem 2.2 in [9] when* $s=1$.

Theorem 2. *Let* $g:[\varrho,\varpi]\to\mathbb{R}$ *be a differentiable function on* (ϱ,ϖ) *such that* $g'\in L([\varrho,\varpi])$ *with* $0\leq\varrho<\varpi$, *and let* $w:[\varrho,\varpi]\to\mathbb{R}$ *be a continuous and symmetric function with respect to* $\frac{\varrho+\varpi}{2}$. *If* $|g'|^q$ *is an extended s-convex for some fixed* $s\in(-1,1]$ *and* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then we have*

$$\left|L^{\alpha}[w]g\left(\frac{\varrho+\varpi}{2}\right)-L^{\alpha}[wg]\right|$$
$$\leq\frac{(\varpi-\varrho)^2}{4\alpha(p\alpha+1)^{\frac{1}{p}}}\|w\|_{[\varrho,\varpi],\infty}\left(\left(\frac{|g'(\varrho)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|g'(\varpi)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right),$$

where $B(.,.)$ *is the beta function.*

Proof. Using Lemma 1, the absolute value, Hölder's inequality, and s-convexity of $|g'|$, we obtain

$$\left|L^{\alpha}[w]g\left(\frac{\varrho+\varpi}{2}\right)-L^{\alpha}[wg]\right|$$
$$\leq\frac{(\varpi-\varrho)^2}{4}\left(\int_0^1|p_1(a)|\left|g'\left(a\varrho+(1-a)\frac{\varrho+\varpi}{2}\right)\right|da\right.$$
$$\left.+\int_0^1|p_2(a)|\left|g'\left(a\varpi+(1-a)\frac{\varrho+\varpi}{2}\right)\right|da\right)$$

$$\leq \frac{(\varpi-\varrho)^2}{4}\left(\left(\int_0^1 |p_1(a)|^p da\right)^{\frac{1}{p}}\left(\int_0^1 \left|g'\left(a\varrho+(1-a)\frac{\varrho+\varpi}{2}\right)\right|^q da\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 |p_2(a)|^p da\right)^{\frac{1}{p}}\left(\int_0^1 \left|g'\left(a\varpi+(1-a)\frac{\varrho+\varpi}{2}\right)\right|^q da\right)^{\frac{1}{q}}\right)$$

$$\leq \frac{(\varpi-\varrho)^2}{4}\|w\|_{[\varrho,\varpi],\infty}\left(\int_0^1\left(\int_a^1 (1-b)^{\alpha-1}db\right)^p da\right)^{\frac{1}{p}}$$

$$\times\left(\left(\int_0^1\left(a^s|g'(\varrho)|^q+(1-a)^s\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)da\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left(a^s|g'(\varpi)|^q+(1-a)^s\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)da\right)^{\frac{1}{q}}\right)$$

$$\leq \frac{(\varpi-\varrho)^2}{4\alpha}\|w\|_{[\varrho,\varpi],\infty}\left(\int_0^1 (1-a)^{p\alpha}da\right)^{\frac{1}{p}}$$

$$\times\left(\left(\frac{|g'(\varrho)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|g'(\varpi)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)$$

$$=\frac{(\varpi-\varrho)^2}{4\alpha(p\alpha+1)^{\frac{1}{p}}}\|w\|_{[\varrho,\varpi],\infty}\left(\left(\frac{|g'(\varrho)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|g'(\varpi)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right).$$

The proof is now finished. □

Corollary 6. *In Theorem 2, if we use:*

1. $s=0$, *we obtain*

$$\left|L^\alpha[w]g\left(\frac{\varrho+\varpi}{2}\right)-L^\alpha[wg]\right|$$
$$\leq \frac{(\varpi-\varrho)^2}{4\alpha(p\alpha+1)^{\frac{1}{p}}}\|w\|_{[\varrho,\varpi],\infty}$$
$$\times\left(\left(|g'(\varrho)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(|g'(\varpi)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\right).$$

2. $s=1$, *we obtain*

$$\left|L^\alpha[w]g\left(\frac{\varrho+\varpi}{2}\right)-L^\alpha[wg]\right|$$
$$\leq \frac{(\varpi-\varrho)^2}{4\alpha(p\alpha+1)^{\frac{1}{p}}}\|w\|_{[\varrho,\varpi],\infty}$$

$$\times\left(\left(\frac{|g'(\varrho)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|g'(\varpi)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right).$$

Corollary 7. *In Theorem 2, if we use $\alpha=1$, we obtain*

$$\begin{aligned}&\left|g\left(\frac{\varrho+\varpi}{2}\right)\int_{\varrho}^{\varpi}w(\mathrm{N})d\mathrm{N}-\int_{\varrho}^{\varpi}w(\mathrm{N})g(\mathrm{N})d\mathrm{N}\right|\\&\leq\frac{(\varpi-\varrho)^2}{4(p+1)^{\frac{1}{p}}}\|w\|_{[\varrho,\varpi],\infty}\left(\left(\frac{|g'(\varrho)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|g'(\varpi)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Remark 3. *In Corollary 7, if we assume that $s\in(0,1]$, we obtain Theorem 2.4 in [27]. Moreover, if we use $s=1$, we obtain Corollary 7 in [28], respectively.*

Corollary 8. *In Theorem 2, if we choose*

1. $w(u)=\frac{1}{\varpi-\varrho}$, *we obtain*

$$\begin{aligned}&\left|g\left(\frac{\varrho+\varpi}{2}\right)-\frac{2^{\alpha-1}}{(\varpi-\varrho)^{\alpha}}\Gamma(\alpha+1)\left(J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^-}g(\varrho)+J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^+}g(\varpi)\right)\right|\\&\leq\frac{\varpi-\varrho}{4(p\alpha+1)^{\frac{1}{p}}}\left(\left(\frac{|g'(\varrho)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|g'(\varpi)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right).\end{aligned}$$

2. $w(u)=\frac{1}{\varpi-\varrho}$ *and $\alpha=1$, we obtain*

$$\begin{aligned}&\left|g\left(\frac{\varrho+\varpi}{2}\right)-\frac{1}{\varpi-\varrho}\int_{\varrho}^{\varpi}g(u)du\right|\\&\leq\frac{\varpi-\varrho}{4(p+1)^{\frac{1}{p}}}\left(\left(\frac{|g'(\varrho)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|g'(\varpi)|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Remark 4. *Corollary 8, the second point will be reduced to Corollary 6 in [10] when $s=0$.*

Corollary 9. *In Theorem 2, using the s-convexity of $|g'|^q$, i.e.,*

$$\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\leq\frac{|g'(\varrho)|^q+|g'(\varpi)|^q}{2^{s-1}(1+s)},$$

we obtain

$$\begin{aligned}&\left|L^{\alpha}[w]g\left(\frac{\varrho+\varpi}{2}\right)-L^{\alpha}[wg]\right|\\&\leq\frac{(\varpi-\varrho)^2}{4\alpha(p\alpha+1)^{\frac{1}{p}}}\|w\|_{[\varrho,\varpi],\infty}\left(\left(\frac{(1+s+2^{1-s})|g'(\varrho)|^q+2^{1-s}|g'(\varpi)|^q}{(1+s)^2}\right)^{\frac{1}{q}}\right.\end{aligned}$$

$$+\left(\frac{2^{1-s}|g'(\varrho)|^q+\left(1+s+2^{1-s}\right)|g'(\varpi)|^q}{(1+s)^2}\right)^{\frac{1}{q}}\Bigg).$$

Corollary 10. *In Corollary 9:*

1. *If we use $\alpha = 1$, we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right)\int_{\varrho}^{\varpi} w(\mathrm{N})d\mathrm{N} - \int_{\varrho}^{\varpi} w(\mathrm{N})g(\mathrm{N})d\mathrm{N}\right|$$
$$\leq \frac{(\varpi-\varrho)^2}{4(p+1)^{\frac{1}{p}}}\|w\|_{[\varrho,\varpi],\infty}\left(\left(\frac{\left(1+s+2^{1-s}\right)|g'(\varrho)|^q+2^{1-s}|g'(\varpi)|^q}{(1+s)^2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{2^{1-s}|g'(\varrho)|^q+\left(1+s+2^{1-s}\right)|g'(\varpi)|^q}{(1+s)^2}\right)^{\frac{1}{q}}\right).$$

2. *If we choose $w(u) = \frac{1}{\varpi-\varrho}$, we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right) - \frac{2^{\alpha-1}}{(\varpi-\varrho)^{\alpha}}\Gamma(\alpha+1)\left(J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^-}g(\varrho) + J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^+}g(\varpi)\right)\right|$$
$$\leq \frac{\varpi-\varrho}{4(p\alpha+1)^{\frac{1}{p}}}\left(\left(\frac{\left(1+s+2^{1-s}\right)|g'(\varrho)|^q+2^{1-s}|g'(\varpi)|^q}{(1+s)^2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{2^{1-s}|g'(\varrho)|^q+\left(1+s+2^{1-s}\right)|g'(\varpi)|^q}{(1+s)^2}\right)^{\frac{1}{q}}\right).$$

3. *If we choose $w(u) = \frac{1}{\varpi-\varrho}$ and $\alpha = 1$, we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right) - \frac{1}{\varpi-\varrho}\int_{\varrho}^{\varpi} g(u)du\right|$$
$$\leq \frac{\varpi-\varrho}{4(p+1)^{\frac{1}{p}}}\left(\left(\frac{\left(1+s+2^{1-s}\right)|g'(\varrho)|^q+2^{1-s}|g'(\varpi)|^q}{(1+s)^2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{2^{1-s}|g'(\varrho)|^q+\left(1+s+2^{1-s}\right)|g'(\varpi)|^q}{(1+s)^2}\right)^{\frac{1}{q}}\right).$$

Remark 5.

1. *Corollary 10, the first point will be reduced to Corollary 2.3 in [9] when $s = 1$.*
2. *The second point of Corollary 10 will be reduced to Theorem 6 in [6] when $s = 1$.*
3. *Corollary 10, the third point will be reduced to Theorem 2.3 in [9] when $s = 1$.*

Corollary 11. *In Corollary 9, if we use the discrete power mean inequality, we obtain*

$$\left|L^{\alpha}[w]g\left(\frac{\varrho+\varpi}{2}\right) - L^{\alpha}[wg]\right|$$
$$\leq \frac{(\varpi-\varrho)^2}{2\alpha(p\alpha+1)^{\frac{1}{p}}}\|w\|_{[\varrho,\varpi],\infty}\left(\frac{1+s+2^{2-s}}{(1+s)^2}\right)^{\frac{1}{q}}\left(\frac{|g'(\varrho)|^q+|g'(\varpi)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 12. *In Corollary 11:*

1. *If we use $\alpha = 1$, we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right)\int_{\varrho}^{\varpi} w(\mathrm{N})d\mathrm{N} - \int_{\varrho}^{\varpi} w(\mathrm{N})g(\mathrm{N})d\mathrm{N}\right|$$
$$\leq \frac{(\varpi-\varrho)^2}{2(p+1)^{\frac{1}{p}}}\|w\|_{[\varrho,\varpi],\infty}\left(\frac{1+s+2^{2-s}}{(1+s)^2}\right)^{\frac{1}{q}}\left(\frac{|g'(\varrho)|^q+|g'(\varpi)|^q}{2}\right)^{\frac{1}{q}}.$$

2. *If we choose $w(u) = \frac{1}{\varpi-\varrho}$, we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right) - \frac{2^{\alpha-1}}{(\varpi-\varrho)^{\alpha}}\Gamma(\alpha+1)\left(J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^-}g(\varrho) + J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^+}g(\varpi)\right)\right|$$
$$\leq \frac{\varpi-\varrho}{2(p\alpha+1)^{\frac{1}{p}}}\left(\frac{1+s+2^{2-s}}{(1+s)^2}\right)^{\frac{1}{q}}\left(\frac{|g'(\varrho)|^q+|g'(\varpi)|^q}{2}\right)^{\frac{1}{q}}.$$

3. *If we choose $w(u) = \frac{1}{\varpi-\varrho}$ and $\alpha = 1$, we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right) - \frac{1}{\varpi-\varrho}\int_{\varrho}^{\varpi} g(u)du\right|$$
$$\leq \frac{\varpi-\varrho}{2(p+1)^{\frac{1}{p}}}\left(\frac{1+s+2^{2-s}}{(1+s)^2}\right)^{\frac{1}{q}}\left(\frac{|g'(\varrho)|^q+|g'(\varpi)|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 3. *Let $g : [\varrho,\varpi] \to \mathbb{R}$ be a differentiable function on (ϱ,ϖ) such that $g' \in L([\varrho,\varpi])$ with $0 \leq \varrho < \varpi$, and let $w : [\varrho,\varpi] \to \mathbb{R}$ be a continuous and symmetric function with respect to $\frac{\varrho+\varpi}{2}$. If $|g'|^q$ is an extended s-convex for some fixed $s \in (-1,1]$ and $q \geq 1$, then we have*

$$\left|L^{\alpha}[w]g\left(\frac{\varrho+\varpi}{2}\right) - L^{\alpha}[wg]\right|$$
$$\leq \frac{(\varpi-\varrho)^2}{4\alpha(\alpha+1)^{1-\frac{1}{q}}}\|w\|_{[\varrho,\varpi],\infty}\left(\left(B(s+1,\alpha+1)|g'(\varrho)|^q + \frac{1}{\alpha+s+1}\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(B(s+1,\alpha+1)|g'(\varpi)|^q + \frac{1}{\alpha+s+1}\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\right),$$

where $B(.,.)$ is the beta function.

Proof. Using Lemma 1, the absolute value, power mean inequality, and s-convexity of $|g'|$, we obtain

$$\left|L^{\alpha}[w]g\left(\frac{\varrho+\varpi}{2}\right) - L^{\alpha}[wg]\right|$$
$$\leq \frac{(\varpi-\varrho)^2}{4}\left(\int_0^1 |p_1(a)|\left|g'\left(a\varrho+(1-a)\frac{\varrho+\varpi}{2}\right)\right|da\right.$$
$$\left.+\int_0^1 |p_2(a)|\left|g'\left(a\varpi+(1-a)\frac{\varrho+\varpi}{2}\right)\right|da\right)$$

$$\leq \frac{(\varpi-\varrho)^2}{4}\left(\left(\int_0^1 |p_1(a)|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |p_1(a)|\left|g'\left(a\varrho+(1-a)\frac{\varrho+\varpi}{2}\right)\right|^q da\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 |p_2(a)|da\right)^{1-\frac{1}{q}}\left(\int_0^1 |p_2(a)|\left|g'\left(a\varpi+(1-a)\frac{\varrho+\varpi}{2}\right)\right|^q da\right)^{\frac{1}{q}}\right)$$
$$\leq \frac{(\varpi-\varrho)^2}{4}\|w\|_{[\varrho,\varpi],\infty}\left(\int_0^1\left(\int_a^1 (1-b)^{\alpha-1}db\right)da\right)^{1-\frac{1}{q}}$$
$$\times\left(\left(\int_0^1\left(\int_a^1 (1-b)^{\alpha-1}db\right)\left(a^s|g'(\varrho)|^q+(1-a)^s\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)da\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left(\int_a^1 (1-b)^{\alpha-1}db\right)\left(a^s|g'(\varpi)|^q+(1-a)^s\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)da\right)^{\frac{1}{q}}\right)$$
$$=\frac{(\varpi-\varrho)^2}{4\alpha}\|w\|_{[\varrho,\varpi],\infty}\left(\int_0^1 (1-a)^{\alpha}da\right)^{1-\frac{1}{q}}$$
$$\times\left(\left(\int_0^1 (1-a)^{\alpha}\left(a^s|g'(\varrho)|^q+(1-a)^s\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)da\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 (1-a)^{\alpha}\left(a^s|g'(\varpi)|^q+(1-a)^s\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)da\right)^{\frac{1}{q}}\right)$$
$$=\frac{(\varpi-\varrho)^2}{4\alpha(\alpha+1)^{1-\frac{1}{q}}}\|w\|_{[\varrho,\varpi],\infty}$$
$$\times\left(\left(|g'(\varrho)|^q\int_0^1 (1-a)^{\alpha}a^s da+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\int_0^1 (1-a)^{\alpha+s}da\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|g'(\varpi)|^q\int_0^1 (1-a)^{\alpha}a^s da+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\int_0^1 (1-a)^{\alpha+s}da\right)^{\frac{1}{q}}\right)$$
$$=\frac{(\varpi-\varrho)^2}{4\alpha(\alpha+1)^{1-\frac{1}{q}}}\|w\|_{[\varrho,\varpi],\infty}\left(\left(B(s+1,\alpha+1)|g'(\varrho)|^q+\frac{1}{\alpha+s+1}\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(B(s+1,\alpha+1)|g'(\varpi)|^q+\frac{1}{\alpha+s+1}\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\right).$$

The proof is now completed. □

Corollary 13. *In Theorem 3, if we use:*

1. $s=0$, *we get*

$$\left|L^{\alpha}[w]g\left(\frac{\varrho+\varpi}{2}\right)-L^{\alpha}[wg]\right|$$

$$\leq \frac{(\varpi-\varrho)^2}{4\alpha(\alpha+1)}\|w\|_{[\varrho,\varpi],\infty}\left(\left(\left|g'(\varrho)\right|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|g'(\varpi)\right|^q+\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\right).$$

2. *If we use $s=1$, we obtain*

$$\left|L^\alpha[w]g\left(\frac{\varrho+\varpi}{2}\right)-L^\alpha[wg]\right|$$
$$\leq \frac{(\varpi-\varrho)^2}{4\alpha(\alpha+1)}\|w\|_{[\varrho,\varpi],\infty}\left(\left(\frac{1}{\alpha+2}\left|g'(\varrho)\right|^q+\frac{\alpha+1}{\alpha+2}\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{\alpha+2}\left|g'(\varpi)\right|^q+\frac{\alpha+1}{\alpha+2}\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\right).$$

3. *If we choose $\alpha=1$, we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right)\int_{\varrho}^{\varpi}w(\mathrm{N})d\mathrm{N}-\int_{\varrho}^{\varpi}w(\mathrm{N})g(\mathrm{N})d\mathrm{N}\right|$$
$$\leq \frac{(\varpi-\varrho)^2}{8}\|w\|_{[\varrho,\varpi],\infty}\left(\frac{2}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left(\left(\left|g'(\varrho)\right|^q+(s+1)\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|g'(\varpi)\right|^q+(s+1)\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\right).$$

Remark 6. *In the third point of Corollary 13, if we assume that $s\in(0,1]$, we obtain Theorem 2.2 in [27]. Moreover, if we use $s=1$, we obtain Corollary 12 in [28].*

Corollary 14. *In Theorem 3, if we choose:*

1. $w(u)=\frac{1}{\varpi-\varrho}$, *we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right)-\frac{2^{\alpha-1}}{(\varpi-\varrho)^\alpha}\Gamma(\alpha+1)\left(J^\alpha_{\left(\frac{\varrho+\varpi}{2}\right)^-}g(\varrho)+J^\alpha_{\left(\frac{\varrho+\varpi}{2}\right)^+}g(\varpi)\right)\right|$$
$$\leq \frac{\varpi-\varrho}{4(\alpha+1)^{1-\frac{1}{q}}}\left(\left(B(s+1,\alpha+1)\left|g'(\varrho)\right|^q+\frac{1}{\alpha+s+1}\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(B(s+1,\alpha+1)\left|g'(\varpi)\right|^q+\frac{1}{\alpha+s+1}\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\right).$$

2. *If we choose $w(u)=\frac{1}{\varpi-\varrho}$ and $\alpha=1$, we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right)-\frac{1}{\varpi-\varrho}\int_{\varrho}^{\varpi}g(u)du\right|$$
$$\leq \frac{\varpi-\varrho}{8}\left(\frac{2}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left(\left(\left|g'(\varrho)\right|^q+(s+1)\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\left|g'(\varpi)\right|^q+(s+1)\left|g'\left(\frac{\varrho+\varpi}{2}\right)\right|^q\right)^{\frac{1}{q}}\Bigg).$$

Corollary 15. *In Theorem 3, using the s-convexity of $|g'|$, we obtain*

$$\left|L^\alpha[w]g\left(\frac{\varrho+\varpi}{2}\right)-L^\alpha[wg]\right|\leq\frac{(\varpi-\varrho)^2}{4\alpha(\alpha+1)^{1-\frac{1}{q}}}\|w\|_{[\varrho,\varpi],\infty}$$
$$\times\left(\left(\frac{(1+s)(\alpha+s+1)B(s+1,\alpha+1)+2^{1-s}}{(1+s)(\alpha+s+1)}\left|g'(\varrho)\right|^q+\frac{2^{1-s}}{(1+s)(\alpha+s+1)}\left|g'(\varpi)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{2^{1-s}}{(1+s)(\alpha+s+1)}\left|g'(\varrho)\right|^q+\frac{(1+s)(\alpha+s+1)B(s+1,\alpha+1)+2^{1-s}}{(1+s)(\alpha+s+1)}\left|g'(\varpi)\right|^q\right)^{\frac{1}{q}}\right).$$

Corollary 16. *In Corollary 9, if we use:*

1. $\alpha=1$, *we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right)\int_\varrho^\varpi w(\mathrm{N})d\mathrm{N}-\int_\varrho^\varpi w(\mathrm{N})g(\mathrm{N})d\mathrm{N}\right|$$
$$\leq\frac{(\varpi-\varrho)^2}{8}\|w\|_{[\varrho,\varpi],\infty}\left(\frac{2}{(1+s)(s+2)}\right)^{\frac{1}{q}}\left(\left(\left(1+2^{1-s}\right)\left|g'(\varrho)\right|^q+2^{1-s}\left|g'(\varpi)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(2^{1-s}\left|g'(\varrho)\right|^q+\left(1+2^{1-s}\right)\left|g'(\varpi)\right|^q\right)^{\frac{1}{q}}\right).$$

2. $w(u)=\frac{1}{\varpi-\varrho}$, *we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right)-\frac{2^{\alpha-1}}{(\varpi-\varrho)^\alpha}\Gamma(\alpha+1)\left(J^\alpha_{\left(\frac{\varrho+\varpi}{2}\right)^-}g(\varrho)+J^\alpha_{\left(\frac{\varrho+\varpi}{2}\right)^+}g(\varpi)\right)\right|$$
$$\leq\frac{\varpi-\varrho}{4(\alpha+1)^{1-\frac{1}{q}}}\left(\left(\frac{(1+s)(\alpha+s+1)B(s+1,\alpha+1)+2^{1-s}}{(1+s)(\alpha+s+1)}\left|g'(\varrho)\right|^q\right.\right.$$
$$\left.+\frac{2^{1-s}}{(1+s)(\alpha+s+1)}\left|g'(\varpi)\right|^q\right)^{\frac{1}{q}}+\left(\frac{2^{1-s}}{(1+s)(\alpha+s+1)}\left|g'(\varrho)\right|^q\right.$$
$$\left.\left.+\frac{(1+s)(\alpha+s+1)B(s+1,\alpha+1)+2^{1-s}}{(1+s)(\alpha+s+1)}\left|g'(\varpi)\right|^q\right)^{\frac{1}{q}}\right).$$

3. *If we choose* $w(u)=\frac{1}{\varpi-\varrho}$ *and* $\alpha=1$, *we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right)-\frac{1}{\varpi-\varrho}\int_\varrho^\varpi g(u)du\right|$$
$$\leq\frac{\varpi-\varrho}{8}\left(\frac{2}{(1+s)(s+2)}\right)^{\frac{1}{q}}\left(\left(\left(1+2^{1-s}\right)\left|g'(\varrho)\right|^q+2^{1-s}\left|g'(\varpi)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(2^{1-s}\left|g'(\varrho)\right|^q+\left(1+2^{1-s}\right)\left|g'(\varpi)\right|^q\right)^{\frac{1}{q}}\right).$$

Remark 7. *Corollary 16, the second point will be reduced to Theorem 5 in [6] when $s=1$.*

Corollary 17. *In Corollary 15, if we use the discrete power mean inequality, we obtain*

$$\begin{aligned}&\left|L^{\alpha}[w]g\left(\tfrac{\varrho+\varpi}{2}\right)-L^{\alpha}[wg]\right|\\ \leq\ & \frac{(\varpi-\varrho)^2}{2\alpha(\alpha+1)^{1-\frac{1}{q}}}\|w\|_{[\varrho,\varpi],\infty}\left(\frac{(1+s)(\alpha+s+1)B(s+1,\alpha+1)+2^{2-s}}{(1+s)(\alpha+s+1)}\right)^{\frac{1}{q}}\left(\frac{|g'(\varrho)|^q+|g'(\varpi)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 18. *In Corollary 17, if we use:*

1. $\alpha=1$, *we obtain*

$$\begin{aligned}&\left|g\left(\frac{\varrho+\varpi}{2}\right)\int_{\varrho}^{\varpi}w(\mathrm{N})d\mathrm{N}-\int_{\varrho}^{\varpi}w(\mathrm{N})g(\mathrm{N})d\mathrm{N}\right|\\ \leq&\frac{(\varpi-\varrho)^2}{4}\|w\|_{[\varrho,\varpi],\infty}\left(\frac{1+2^{2-s}}{(1+s)(s+2)}\right)^{\frac{1}{q}}\left(|g'(a)|^q+|g'(\varpi)|^q\right)^{\frac{1}{q}}.\end{aligned}$$

2. $w(u)=\frac{1}{\varpi-\varrho}$, *we obtain*

$$\begin{aligned}&\left|g\left(\frac{\varrho+\varpi}{2}\right)-\frac{2^{\alpha-1}}{(\varpi-\varrho)^{\alpha}}\Gamma(\alpha+1)\left(J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^-}g(\varrho)+J^{\alpha}_{\left(\frac{\varrho+\varpi}{2}\right)^+}g(\varpi)\right)\right|\\ \leq&\frac{\varpi-\varrho}{2(\alpha+1)^{1-\frac{1}{q}}}\left(\frac{(1+s)(\alpha+s+1)B(s+1,\alpha+1)+2^{2-s}}{(1+s)(\alpha+s+1)}\right)^{\frac{1}{q}}\left(\frac{|g'(\varrho)|^q+|g'(\varpi)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

3. $w(u)=\frac{1}{\varpi-\varrho}$ *and* $\alpha=1$, *we obtain*

$$\left|g\left(\frac{\varrho+\varpi}{2}\right)-\frac{1}{\varpi-\varrho}\int_{\varrho}^{\varpi}g(u)du\right|\leq\frac{\varpi-\varrho}{4}\left(\frac{1+2^{2-s}}{(1+s)(s+2)}\right)^{\frac{1}{q}}\left(|g'(\varrho)|^q+|g'(\varpi)|^q\right)^{\frac{1}{q}}.$$

Remark 8. *Corollary 18, the first point will be reduced to Theorem 2 in [8] when* $s=1$.

4. Applications

4.1. Weighted Midpoint Quadrature

Let Y be the partition of the points $\varrho=\wp_0<\wp_1<\ldots<\wp_n=\varpi$ of the interval $[\varrho,\varpi]$, and consider the quadrature formula

$$\int_{\varrho}^{\varpi}w(u)g(u)du=\lambda_w(g,\mathrm{Y})+R_w(g,\mathrm{Y}),$$

where

$$\lambda_w(g,\mathrm{Y})=\sum_{i=0}^{n-1}g\left(\frac{\wp_i+\wp_{i+1}}{2}\right)\int_{\wp_i}^{\wp_{i+1}}w(u)du$$

and $R_w(g,\mathrm{Y})$ is the associated approximation error.

Proposition 1. *Let* $g:[\varrho,\varpi]\to\mathbb{R}$ *be a differentiable function on* (ϱ,ϖ) *with* $0\leq\varrho<\varpi$ *and* $g'\in L^1[\varrho,\varpi]$, *and let* $w:[\varrho,\varpi]\to\mathbb{R}$ *be symmetric as regards* $\frac{\varrho+\varpi}{2}$. *If* $|g'|$ *is s-convex function, then for* $n\in\mathbb{N}$ *we have*

$$|R_w(g,\mathrm{Y})|\leq\frac{(2^{2-s}+1)}{4(1+s)(s+2)}\|w\|_{[\varrho,\varpi],\infty}\sum_{i=0}^{n-1}(\wp_{i+1}-\wp_i)^2\left(|g'(\wp_i)|+|g'(\wp_{i+1})|\right).$$

Proof. Applying Corollary 5 on the subintervals $[\wp_i, \wp_{i+1}]$ $(i = 0, 1, ..., n-1)$ of the partition Y, we obtain

$$\left| g\left(\frac{\wp_i + \wp_{i+1}}{2}\right) \int_{\wp_i}^{\wp_{i+1}} w(u)du - \int_{\wp_i}^{\wp_{i+1}} w(u)g(u)du \right|$$
$$\leq \frac{(2^{2-s}+1)(\wp_{i+1} - \wp_i)^2}{4(1+s)(s+2)} \|w\|_{[\wp_i,\wp_{i+1}],\infty} \left(|g'(\wp_i)| + |g'(\wp_{i+1})|\right).$$

Add the above inequalities for all $i = 0, 1, ..., n-1$ and using the triangular inequality to obtain the desired result. □

Proposition 2. *Let $g : [\varrho, \varpi] \to \mathbb{R}$ be a differentiable function on (ϱ, ϖ) with $0 \leq \varrho < \varpi$ and $g' \in L^1[\varrho, \varpi]$, and let $w : [\varrho, \varpi] \to \mathbb{R}$ be symmetric as regards $\frac{\varrho+\varpi}{2}$. If $|g'|^q$ is a s-convex function, then for $n \in \mathbb{N}$ we have*

$$|R(g, Y)| \leq \frac{\|w\|_{[\varrho,\varpi],\infty}}{2(p+1)^{\frac{1}{p}}} \sum_{i=0}^{n-1} (\wp_{i+1} - \wp_i)^2 \left(\frac{1+s+2^{2-s}}{(1+s)^2}\right)^{\frac{1}{q}} \left(\frac{|g'(\wp_i)|^q + |g'(\wp_{i+1})|^q}{2}\right)^{\frac{1}{q}}.$$

Proof. Applying Corollary 12 on the subintervals $[\wp_i, \wp_{i+1}]$ $(i = 0, 1, ..., n-1)$ of the partition Y, we obtain

$$\left| g\left(\frac{\wp_i + \wp_{i+1}}{2}\right) \int_{\wp_i}^{\wp_{i+1}} w(u)du - \int_{\wp_i}^{\wp_{i+1}} w(u)g(u)du \right|$$
$$\leq \frac{(\wp_{i+1} - \wp_i)^2}{2(p+1)^{\frac{1}{p}}} \|w\|_{[\wp_i,\wp_{i+1}],\infty} \left(\frac{1+s+2^{2-s}}{(1+s)^2}\right)^{\frac{1}{q}} \left(\frac{|g'(\wp_i)|^q + |g'(\wp_{i+1})|^q}{2}\right)^{\frac{1}{q}}.$$

Add the above inequalities for all $i = 0, 1, ..., n-1$ and using the triangular inequality to obtain the desired result. □

Proposition 3. *Let $g : [\varrho, \varpi] \to \mathbb{R}$ be a differentiable function on (ϱ, ϖ) with $0 \leq \varrho < \varpi$ and $g' \in L^1[\varrho, \varpi]$, and let $w : [\varrho, \varpi] \to \mathbb{R}$ be symmetric as regards $\frac{\varrho+\varpi}{2}$. If $|g'|^q$ is a s-convex function, then, for $n \in \mathbb{N}$, we have*

$$|R(g, Y)| \leq \frac{\|w\|_{[a_i,b],\infty}}{4} \left(\frac{1+2^{2-s}}{(1+s)(s+2)}\right)^{\frac{1}{q}} \sum_{i=0}^{n-1} (\wp_{i+1} - \wp_i)^2 \left(|g'(\wp_i)|^q + |g'(\wp_{i+1})|^q\right)^{\frac{1}{q}}.$$

Proof. Applying Corollary 18 on the subintervals $[\wp_i, \wp_{i+1}]$ $(i = 0, 1, ..., n-1)$ of the partition Y, we obtain

$$\left| g\left(\frac{\wp_i + \wp_{i+1}}{2}\right) \int_{\wp_i}^{\wp_{i+1}} w(u)du - \int_{\wp_i}^{\wp_{i+1}} w(u)g(u)du \right|$$
$$\leq \frac{(\wp_{i+1} - \wp_i)^2}{4} \|w\|_{[\wp_i,\wp_{i+1}],\infty} \left(\frac{1+2^{2-s}}{(1+s)(s+2)}\right)^{\frac{1}{q}} \left(|g'(\wp_i)|^q + |g'(\wp_{i+1})|^q\right)^{\frac{1}{q}}.$$

Add the above inequalities for all $i = 0, 1, ..., n-1$ and using the triangular inequality to obtain the desired result. □

4.2. Application to Special Means

Let ϱ, ϖ be two arbitrary real numbers:

The Arithmetic mean:

$$A(\varrho, \varpi) = \frac{\varrho + \varpi}{2}.$$

The Logarithmic mean:

$$L(\varrho,\varpi)=\frac{\varpi-\varrho}{\ln\varpi-\ln\varrho},\ \varrho,\varpi>0,\ \varrho\neq\varpi.$$

The p-Logarithmic mean:

$$L_p(\varrho,\varpi)=\left(\frac{\varpi^{p+1}-\varrho^{p+1}}{(p+1)(\varpi-\varrho)}\right)^{\frac{1}{p}},\ \varrho,\varpi>0,\ \varrho\neq\varpi\ \text{ and } p\in\mathbb{R}\setminus\{-1,0\}.$$

Proposition 4. *Let $\varrho,\varpi\in\mathbb{R}$ with $0<\varrho<\varpi$, then we have*

$$\left|A^{\frac{3}{2}}(\varrho,\varpi)-L_{\frac{3}{2}}^{\frac{3}{2}}(\varrho,\varpi)\right|\leq\frac{\varpi-\varrho}{10}\left(\varrho^{\frac{1}{2}}+3\left(\frac{\varrho+\varpi}{2}\right)^{\frac{1}{2}}+\varpi^{\frac{1}{2}}\right).$$

Proof. Using Corollary 3 for function $g(k)=k^{\frac{3}{2}}$ whose derivative $g'(k)=\frac{3}{2}k^{\frac{1}{2}}$ is $\frac{1}{2}$-convex. □

Proposition 5. *Let $\varrho,\varpi\in\mathbb{R}$ with $0<\varrho<\varpi$, then we have*

$$\left|A^{-1}(\varrho,\varpi)-L^{-1}(\varrho,\varpi)\right|\leq\frac{(\varpi-\varrho)\sqrt{3}}{12}\left(\left(\frac{2\varrho+\varpi}{\varrho\varpi}\right)^{\frac{1}{2}}+\left(\frac{\varpi+2\varrho}{\varrho\varpi}\right)^{\frac{1}{2}}\right).$$

Proof. Applying Corollary 17 with $q=2$ to the function $g(k)=\frac{1}{k}$ whose derivative $|g'(k)|^2=\frac{1}{k}$ is P-function. □

5. Conclusions

In this study, we considered the weighted midpoint-type integral inequalities for s-convex first derivatives using Riemann–Liouville integrals operators, where the main novelties of the paper are provided by a new identity regarding the weighted midpoint-type inequalities being presented and some new fractional weighted midpoint-type inequalities for functions whose first derivatives are s-convex being established. Some special cases are derived and the applications of our results are provided.

Author Contributions: Writing—original draft preparation, N.N., F.A. and A.F.; writing—review and editing, K.B.; visualization, B.M.; supervision, K.Z.; project administration, T.R.; funding acquisition, T.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: Researchers would like to thank the Deanship of Scientific Research, Qassim University for funding the publication of this project.

Conflicts of Interest: The authors declare no conflict of interest.

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