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A New Comprehensive Subclass of Analytic Bi-Univalent Functions Related to Gegenbauer Polynomials

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Abstract: In the current study, we provide a novel qualitative new subclass of analytical and bi-univalent functions in the symmetry domain $\mathbb{U}$ defined by the use of Gegenbauer polynomials. We derive estimates for the Fekete–Szegö functional problems and the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for the functions that belong to each of these new subclasses of the bi-univalent function classes. Some more results are revealed after concentrating on the parameters employed in our main results.

Keywords: bi-univalent functions; Gegenbauer polynomials; analytic functions; Fekete–Szegö problem; univalent functions; bi-univalent functions

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1. Introduction

Orthogonal polynomials were discovered by Legendre in 1784 [1]. Under specific model restrictions, orthogonal polynomials are frequently employed to discover solutions of ordinary differential equations. Moreover, orthogonal polynomials are a critical feature in approximation theory [2,3].

Two polynomials P_n and P_m, of order n and m, respectively, are orthogonal if

$$\langle P_n,\ P_m\rangle = \int_c^d P_n(x)P_m(x)r(x)dx = 0, \qquad \text{for} \qquad n \neq m, \tag{1}$$

where $r(x)$ is a non-negative function in the interval (c,d); therefore, all finite order polynomials $P_n(x)$ have a well-defined integral.

An example of an orthogonal polynomial is a Gegenbauer polynomial (GP). When conventional algebraic formulations are used, a symbolic relationship exists between the integral representation of typically real functions T_R and the generating function of (GP) T_R, according to [4]. This resulted in the discovery of a number of useful inequalities in the realm of (GP).

Recently, Amourah et al. [5] considered the Gegenbauer function $\mathcal{G}_\gamma(x,z)$ for a non-zero real constant γ, given by

$$\mathcal{G}_\gamma(x,z) = \frac{1}{(1-2xz+z^2)^\gamma}, \quad x \in [-1,1]\,. \tag{2}$$

For a fixed x and analytic function $\mathcal{G}_\gamma$, we can write

$$\mathcal{G}_\gamma(x,z) = \sum_{n=0}^{\infty} G_n^\gamma(x)z^n, \tag{3}$$

where $G_n^\gamma(x)$ is the (GP) of degree n (see [4]).

The recurrence relations that characterize (GP) are as follows:

$$G_n^{\gamma}(x) = \frac{1}{n}\left[2x(n+\gamma-1)G_{n-1}^{\gamma}(x) - (n+2\gamma-2)G_{n-2}^{\gamma}(x)\right], \tag{4}$$

with the initial values

$$G_0^{\gamma}(x) = 1, G_1^{\gamma}(x) = 2\gamma x \text{ and } G_2^{\gamma}(x) = 2\gamma(1+\gamma)x^2 - \gamma. \tag{5}$$

Note that, for $\gamma = 1$ or $\gamma = \frac{1}{2}$, we get the Chebyshev polynomials $G_n^1(x)$ and Legendre polynomials $G_n^{\frac{1}{2}}(x)$, respectively.

Due to the rise of quantum groups, q-orthogonal polynomials are now of considerable interest in both physics and mathematics. For example, the q-Laguerre and q-Hermite polynomials have a group-theoretic setting in the q-deformed harmonic oscillator. Jackson's q-exponential is a key component of the mathematical framework needed to describe the recurrence relations, generating functions, and orthogonality relations of these q-polynomials. Quesne [6] derived a new formulation of Jackson's q-exponential as a closed-form multiplicative series of regular exponentials with known coefficients. It is vital to consider how this discovery can impact the theory of q-orthogonal polynomials in this situation. The current work seeks to do this by obtaining brand-new nonlinear connection equations in terms of q-Gegenbauer polynomials.

This paper describes an analytical investigation into a newly constructed subclass $\mathbf{B}_{\Sigma}^{\gamma}(\sigma, \delta, \mu, x)$ of bi-univalent functions with Gegenbauer polynomials.

2. Preliminaries

Let $\mathcal{A}$ denote the class of analytic functions f in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$. As a consequence, every $f \in \mathcal{A}$ has the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad (z \in \mathbb{U}). \tag{6}$$

Further, the class of all univalent functions $f \in \mathcal{A}$ is denoted by $\mathcal{S}$ (for details, see [7]).

The subordination of analytic functions f and g is denoted by $f \prec g$ if, for all $z \in \mathbb{U}$, there exists a Schwarz function ϖ with $\varpi(0) = 0$ and $|\varpi(z)| < 1$, such that

$$f(z) = g(\varpi(z)).$$

Moreover, if g is univalent in $\mathbb{U}$, then $f(z) \prec g(z)$, if, and only if, $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (see [8]).

It is known that, the inverse function $g(z) = f^{-1}(z)$ for the analytic and univalent function $f(z)$ from a domain $\mathbb{D}_1$ onto a domain $\mathbb{D}_2$ defined by

$$g(f(z)) = z, \quad (z \in \mathbb{D}_1),$$

is an analytic and univalent. Moreover (see [7]), every function $f \in \mathcal{S}$ has an inverse map f^{-1} satisfying

$$z = f^{-1}(f(z)) \quad (z \in \mathbb{U}),$$

and

$$f\left(f^{-1}(\varpi)\right) = \varpi \quad \left(|\varpi| < r_0(f); r_0(f) \geq \frac{1}{4}\right).$$

The inverse function is really given by

$$g(z) = f^{-1}(\varpi) = \varpi - a_2^2\varpi + (2a_2^2 - a_3)\varpi^3 - (5a_2^3 - 5a_2 a_3 + a_4)\varpi^4 + \cdots. \tag{7}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f^{-1}(z)$ and $f(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (6) (see [9–18]).

Lewin [19] examined the bi-univalent function class Σ and demonstrated that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [20] proposed that $|a_2| < \sqrt{2}$. On the other hand, Netanyahu [21] showed that $\max_{f\in\Sigma} |a_2| = 4/3$.

Some subclasses of the bi-univalent function classes Σ, $\mathcal{S}^*(z)$ and $\mathcal{K}(z)$ of bi-starlike and bi-convex order $(0 < z \le 1)$ were defined by Brannan and Taha [22]. They discovered non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for each of the function classes $\mathcal{S}^*_\Sigma(z)$ and $\mathcal{K}_\Sigma(z)$. In 1984, Tan [23] obtained the most well-known estimate for *Sigma* functions, that is, $|a_2|1.485$. See the groundbreaking work done by Srivastava et al. [24] for a brief history and fascinating instances of functions in the class Σ. The coefficient estimation problem for each of the Taylor–Maclaurin coefficients, $|a_n|$ $(n \ge 3; n \in \mathbb{N})$, is most likely still open.

In 1936, Robertson [25] introduced the class $\mathcal{S}^*(\varepsilon)$ of starlike functions of order ε in $\mathbb{U}$, such that:

$$\mathcal{S}^*(\varepsilon) := \left\{ f:\ f \in \mathcal{S} \ \text{ and } \ \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \varepsilon, \quad (z \in \mathbb{U}; 0 \le \varepsilon < 1) \right\}. \tag{8}$$

If both f and f^{-1} are starlike functions of order ε, a function $f \in \Sigma$ is in the class $\mathcal{S}^*_\Sigma(\varepsilon)$ of order ε.

Babalola [26] introduced the class $\mathcal{L}_\lambda$, such that:

$$\mathcal{L}_\lambda := \left\{ f:\ f \in \mathcal{S} \ \text{ and } \ \mathrm{Re}\left\{ \frac{z(f'(z))^\lambda}{f(z)} \right\} > 0, \quad (z \in \mathbb{U}; \lambda \ge 1) \right\}.$$

Ezrohi [27] introduced the class $\mathcal{U}(\varepsilon)$, such that:

$$\mathcal{U}(\varepsilon) = \{ f:\ f \in \mathcal{S} \ \text{ and } \ \mathrm{Re}\{f'(z)\} > \varepsilon, \quad (z \in \mathbb{U}; 0 \le \varepsilon < 1) \}.$$

Chen [28] introduced the class $\mathcal{ST}(\varepsilon)$, such that:

$$\mathcal{ST}(\varepsilon) = \left\{ f:\ f \in \mathcal{S} \ \text{ and } \ \mathrm{Re}\left\{ \frac{f(z)}{z} \right\} > \varepsilon, \quad (z \in \mathbb{U}; 0 \le \varepsilon < 1) \right\}.$$

In asddition, Singh [29] introduced the class $\mathcal{B}_1(\alpha)$, such that:

$$\mathcal{B}_1(\alpha) = \left\{ f:\ f \in \mathcal{S} \ \text{ and } \ \mathrm{Re}\left\{ zf'(z)\frac{f^{\alpha-1}(z)}{z^\alpha} \right\} > 0, (z \in \mathbb{U}; \alpha \ge 0) \right\}.$$

Motivated by Robertson [25], Babalola [26], Ezrohi [27], Chen [28] and Singh [29], we defined a new comprehensive subclass of the function class $\mathcal{S}$ as follows:

Definition 1. *Let $\delta, \sigma \ge 1$, $\mu \in \mathbb{C}$, $\mathrm{Re}(\mu) \ge 0$. A function $f \in \mathcal{S}$ given by (6) is said to be in the class $\mathbf{B}^\varepsilon_\Sigma(\sigma, \delta, \mu)$ if*

$$Re\left\{ (1-\sigma)f'(z) + \sigma(f'(z))^\delta \left(\frac{f(z)}{z} \right)^{\mu-1} \right\} > \varepsilon, \quad (z \in \mathbb{U}, 0 \le \varepsilon < 1). \tag{9}$$

Remark 1. *By taking specific values for the parameters δ, μ, and σ in Definition 1, we get various well-known subclasses of $\mathcal{S}$ that have been studied by several authors. For example, if $\sigma = \delta = 1$, we get the class $\mathcal{B}_1(\alpha)$, if $\sigma = 1$ and $\mu = \varepsilon = 0$, we get the class $\mathcal{L}_\lambda$, if $\sigma = \delta = 1$ and $\mu = 0$, we get the class $\mathcal{S}^*(\varepsilon)$, if $\sigma = \mu = \delta = 1$, we get the class $\mathcal{U}(\varepsilon)$, and if $\sigma = 1, \delta = 0$ and $\mu = 2$, we get the class $\mathcal{ST}(\varepsilon)$.*

Very recently, Amourah et al. [30] introduced three subclasses of analytic and bi-univalent functions using $q-$Gegenbauer polynomials. Additionally, Alsoboh et al. [31] used the $q-$Gegenbauer polynomials connected to the generalization of the neutrosophic $q-$Poisson distribution series to develop a new subclass of bi-univalent functions. Coefficient bounds $|a_2|$ and $|a_3|$, as well as Fekete–Szegö inequalities, are determined for functions belonging to these subclasses (see also [32–34]).

Several researchers, including [35–47], have recently investigated bi-univalent functions associated with orthogonal polynomials.

The primary aim of this paper is to study the properties of a new subclass of bi-univalent functions related to Gegenbauer polynomials.

Definition 2. *Let $\delta, \sigma \geq 1$, $\mu \in \mathbb{C}$, $\mathrm{Re}(\mu) \geq 0$ and γ is a non-zero real constant. A function $f \in \Sigma$ given by (6) is said to be in the class $\mathbf{B}_{\Sigma}^{\gamma}(\sigma, \delta, \mu, x)$ if the following subordinations are satisfied:*

$$(1-\sigma)f'(z) + \sigma\left(f'(z)\right)^{\delta}\left(\frac{f(z)}{z}\right)^{\mu-1} \prec \mathcal{G}_{\gamma}(x,z) \tag{10}$$

and

$$(1-\sigma)g'(\varpi) + \sigma\left(g'(\varpi)\right)^{\delta}\left(\frac{g(\varpi)}{\varpi}\right)^{\mu-1} \prec \mathcal{G}_{\gamma}(x,\varpi), \tag{11}$$

where $x \in \left(\frac{1}{2}, 1\right]$, γ is a non-zero real constant, the function $\mathcal{G}_{\gamma}$ is of the form (2) and $g(\varpi) = f^{-1}(\varpi)$ is defined by (7).

Special cases:

1. Let $\delta \geq 1$, $\mu \in \mathbb{C}$ and $\mathrm{Re}(\mu) \geq 0$. A function $f \in \Sigma$ given by (6) is in the class $\mathbf{B}_{\Sigma}^{\gamma}(1, \delta, \mu, x)$ if

$$\left(f'(z)\right)^{\delta}\left(\frac{f(z)}{z}\right)^{\mu-1} \prec \mathcal{G}_{\gamma}(x,z)$$

and

$$\left(g'(\varpi)\right)^{\delta}\left(\frac{g(\varpi)}{\varpi}\right)^{\mu-1} \prec \mathcal{G}_{\gamma}(x,\varpi),$$

where $x \in \left(\frac{1}{2}, 1\right]$ and γ is a non-zero real constant.

2. Let $\mu \in \mathbb{C}$, and $\mathrm{Re}(\mu) \geq 0$. A function $f \in \Sigma$ given by (6) is in the class $\mathbf{B}_{\Sigma}^{\gamma}(1, 1, \mu, x)$ if

$$\left(f'(z)\right)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec \mathcal{G}_{\gamma}(x,z)$$

and

$$\left(g'(\varpi)\right)\left(\frac{g(\varpi)}{\varpi}\right)^{\mu-1} \prec \mathcal{G}_{\gamma}(x,\varpi),$$

where $x \in \left(\frac{1}{2}, 1\right]$ and γ is a non-zero real constant.

Unless otherwise stated, we will assume in this paper that $\delta, \sigma \geq 1$, $\mu \in \mathbb{C}$, $\mathrm{Re}(\mu) \geq 0$, $x \in \left(\frac{1}{2}, 1\right]$ and γ is a non-zero real constant.

3. Coefficient Bounds of the Subclass $\mathbf{B}_{\Sigma}^{\gamma}(\sigma, \delta, \mu, x)$

In this section, we find the initial coefficient bounds of the subclass $\mathbf{B}_{\Sigma}^{\gamma}(\sigma, \delta, \mu, x)$.

Theorem 1. *Let $f \in \Sigma$ given by (6) belong to the class $\mathbf{B}_{\Sigma}^{\gamma}(\sigma,\delta,\mu,x)$. Then*

$$|a_2| \leq \frac{2|\gamma|x\sqrt{2|\gamma|x}}{\sqrt{\begin{vmatrix}2\gamma^2x^2[\sigma((\mu+2)(\mu-3)+4\delta(\mu-1)+2\delta(2\delta+1))+6] \\ -(2\gamma(\gamma+1)x^2-\gamma)(\sigma(2\delta+\mu-3)+2)^2\end{vmatrix}}},$$

and

$$|a_3| \leq \frac{4\gamma^2x^2}{|\sigma(2\delta+\mu-3)+2|^2} + \frac{2|\gamma|x}{|\sigma(3\delta+\mu-4)+3|}.$$

Proof. Let $f \in \mathbf{B}_{\Sigma}^{\gamma}(\sigma,\delta,\mu,x)$. From Equations (10) and (11), for all $z,\varpi \in \mathbb{U}$ and analytic functions r, s, such that $r(0) = s(0) = 0$ and $|r(z)| < 1$, $|s(\varpi)| < 1$,, we can write

$$\begin{aligned}&(1-\sigma)f'(z)+\sigma\left(f'(z)\right)^{\delta}\left(\frac{f(z)}{z}\right)^{\mu-1} \\ &= \mathcal{G}_{\gamma}(x,r(z))\end{aligned} \tag{12}$$

and

$$\begin{aligned}&(1-\sigma)g'(\varpi)+\sigma\left(g'(\varpi)\right)^{\delta}\left(\frac{g(\varpi)}{\varpi}\right)^{\mu-1} \\ &= \mathcal{G}_{\gamma}(x,s(\varpi)).\end{aligned} \tag{13}$$

Thus, we have

$$\begin{aligned}&(1-\sigma)f'(z)+\sigma\left(f'(z)\right)^{\delta}\left(\frac{f(z)}{z}\right)^{\mu-1} \\ &= 1+G_1^{\gamma}(x)b_1z+\left[G_1^{\gamma}(x)b_2+G_2^{\gamma}(x)b_1^2\right]z^2+\cdots\end{aligned} \tag{14}$$

and

$$\begin{aligned}&(1-\sigma)g'(\varpi w)+\sigma\left(g'(\varpi)\right)^{\delta}\left(\frac{g(\varpi)}{\varpi}\right)^{\mu-1} \\ &= 1+G_1^{\gamma}(x)d_1\varpi+\left[G_1^{\gamma}(x)d_2+G_2^{\gamma}(x)d_1^2\right])\varpi^2+\cdots.\end{aligned} \tag{15}$$

It is widely understood that if

$$|r(z)| = \left|b_1z+b_2z^2+b_3z^3+\cdots\right| < 1, \ \ (z \in \mathbb{U})$$

and

$$|s(\varpi)| = \left|d_1\varpi+d_2^2\varpi+d_3^3\varpi+\cdots\right| < 1, \ \ (\varpi \in \mathbb{U}),$$

then

$$|b_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{16}$$

Comparing the coefficients in (14) and (15), we get

$$(\sigma(2\delta+\mu-3)+2)a_2 = G_1^{\gamma}(x)b_1, \tag{17}$$

$$\begin{aligned}&\left[\sigma\left(\frac{(\mu-1)(\mu-2)}{2}+2\delta(\mu-1)+2\delta(\delta-1)\right)\right]a_2^2+[\sigma(3\delta+\mu-4)+3]a_3 \\ &= G_1^{\gamma}(x)b_2+G_2^{\gamma}(x)b_1^2,\end{aligned} \tag{18}$$

$$-(\sigma(2\delta+\mu-3)+2)a_2 = G_1^{\gamma}(x)d_1, \tag{19}$$

and

$$\left[\sigma\left(\frac{(\mu-2)(\mu+3)}{2}+2\delta(\mu-1)+2\delta(\delta+2)-4\right)+6\right]a_2^2-[\sigma(3\delta+\mu-4)+3]a_3$$
$$= G_1^{\gamma}(x)d_2 + G_2^{\gamma}(x)d_1^2. \tag{20}$$

From (17) and (19), it follows that

$$b_1 = -d_1 \tag{21}$$

and

$$2(\sigma(2\delta+\mu-3)+2)^2 a_2^2 = \left[G_1^{\gamma}(x)\right]^2\left(b_1^2+d_1^2\right). \tag{22}$$

If we add (18) and (20), we get

$$[\sigma((\mu+2)(\mu-3)+4\delta(\mu-1)+2\delta(2\delta+1))+6]a_2^2$$
$$= G_1^{\gamma}(x)(b_2+d_2) + G_2^{\gamma}(x)\left(b_1^2+d_1^2\right). \tag{23}$$

Substituting the value of $\left(b_1^2+d_1^2\right)$ from (22) in the right-hand side of (23), we get

$$\left|[\sigma((\mu+2)(\mu-3)+4\delta(\mu-1)+2\delta(2\delta+1))+6] - \frac{2(\sigma(2\delta+\mu-3)+2)^2 G_2^{\gamma}(x)}{\left[G_1^{\gamma}(x)\right]^2}\right| a_2^2 = G_1^{\gamma}(x)(b_2+d_2). \tag{24}$$

We discover using (15), (16) and (24) that

$$|a_2| \leq \frac{2|\gamma|x\sqrt{2|\gamma|x}}{\sqrt{\begin{array}{c}\left|2\gamma^2x^2[\sigma((\mu+2)(\mu-3)+4\delta(\mu-1)+2\delta(2\delta+1))+6]\right.\\ \left.-\left(2\gamma(\gamma+1)x^2-\gamma\right)(\sigma(2\delta+\mu-3)+2)^2\right|\end{array}}}$$

Moreover, if we subtract (20) from (18), we have

$$2[\sigma(3\delta+\mu-4)+3]\left(a_3-a_2^2\right) = G_1^{\gamma}(x)(b_2-d_2) + G_2^{\gamma}(x)\left(b_1^2-d_1^2\right). \tag{25}$$

Then, in view of (5) and (22), Equation (25) becomes

$$a_3 = \frac{\left[G_1^{\gamma}(x)\right]^2}{2[\sigma(2\delta+\mu-3)+2]^2}\left(b_1^2+d_1^2\right) + \frac{G_1^{\gamma}(x)}{2[\sigma(3\delta+\mu-4)+3]}(b_2-d_2).$$

Thus, applying (5), we get

$$|a_3| \leq \frac{4\gamma^2x^2}{|\sigma(2\delta+\mu-3)+2|^2} + \frac{2|\gamma|x}{|\sigma(3\delta+\mu-4)+3|}.$$

□

4. Fekete–Szegö Problem for the Subclass $\mathbf{B}_{\Sigma}^{\gamma}(\sigma,\delta,\mu,x)$

The Fekete–Szegö inequality is one of the most well-known problems involving coefficients of univalent analytic functions.

It was given for the first time by [48], who stated that, if $f \in \Sigma$ and τ is a real number, then

$$|a_3 - \tau a_2^2| \leq 1 + 2e^{-2\tau/(1-\mu)}.$$

This bound is sharp.

In this section, we provide the Fekete–Szegö inequalities for functions in the class $\mathbf{B}_{\Sigma}^{\gamma}(\sigma, \delta, \mu, x)$.

Theorem 2. *Let $f \in \Sigma$ given by (6) belong to the class $\mathbf{B}_{\Sigma}^{\gamma}(\sigma, \delta, \mu, x)$. Then,*

$$\left|a_3 - \tau a_2^2\right| \leq \begin{cases} \frac{2|\gamma|x}{|\sigma(3\delta+\mu-4)+3|}, & |\tau - 1| \leq k(x) \\ \dfrac{8\gamma^2 x^3|1-\tau|}{\begin{array}{c}\left|2\gamma[\sigma((\mu+2)(\mu-3)+4\delta(\mu-1)+2\delta(2\delta+1))+6]x^2\right. \\ \left.+\left[1-2(\gamma+1)x^2\right][\sigma(2\delta+\mu-3)+2]^2\right|\end{array}}, & |\tau - 1| \geq k(x) \end{cases}$$

where

$$k(x) = \left|\frac{\begin{array}{c}2\gamma[\sigma((\mu+2)(\mu-3)+4\delta(\mu-1)+2\delta(2\delta+1))+6]x^2 \\ +\left[1-2(\gamma+1)x^2\right][\sigma(2\delta+\mu-3)+2]^2\end{array}}{4[\sigma(3\delta+\mu-4)+3]x^2}\right|.$$

Proof. From (24) and (25)

$$\begin{aligned} a_3 - \tau a_2^2 &= \frac{(1-\tau)(b_2+d_2)\left[G_1^{\gamma}(x)\right]^3}{\begin{array}{c}\left[[\sigma((\mu+2)(\mu-3)+4\delta(\mu-1)+2\delta(2\delta+1))+6]\left[G_1^{\gamma}(x)\right]^2\right. \\ \left.-2(\sigma(2\delta+\mu-3)+2)^2 G_2^{\gamma}(x)\right]\end{array}} \\ &+ \frac{G_1^{\gamma}(x)}{2[\sigma(3\delta+\mu-4)+3]}(b_2 - d_2) \\ &= G_1^{\gamma}(x)\left[\left(h(\tau) + \frac{1}{2(\sigma(3\delta+\mu-4)+3)}\right)b_2 + \left(h(\tau) - \frac{1}{2(\sigma(3\delta+\mu-4)+3)}\right)d_2\right], \end{aligned}$$

where

$$h(\tau) = \frac{(1-\tau)\left[G_1^{\gamma}(x)\right]^2}{\begin{array}{c}\left[[\sigma((\mu+2)(\mu-3)+4\delta(\mu-1)+2\delta(2\delta+1))+6]\left[G_1^{\gamma}(x)\right]^2\right. \\ \left.-2(\sigma(2\delta+\mu-3)+2)^2 G_2^{\gamma}(x)\right]\end{array}},$$

Then, in view of (5), we have

$$\left|a_3 - \tau a_2^2\right| \leq \begin{cases} \frac{\left|G_1^{\gamma}(x)\right|}{|\sigma(3\delta+\mu-4)+3|} & 0 \leq |h(\tau)| \leq \frac{1}{2[\sigma(3\delta+\mu-4)+3]}, \\ 2\left|G_1^{\gamma}(x)\right||h(\tau)| & |h(\tau)| \geq \frac{1}{2[\sigma(3\delta+\mu-4)+3]}. \end{cases}$$

Which completes the proof. □

5. Corollaries and Consequences

Each of the new corollaries and consequences that come next is derived using our main findings from this section.

Corollary 1. *Let $f \in \Sigma$ given by (6) belong to the class $\mathbf{B}_{\Sigma}^{\frac{1}{2}}(\sigma, \delta, \mu, x)$. Then*

$$|a_2| \leq \frac{x\sqrt{x}}{\sqrt{\left|\begin{array}{c}\frac{1}{2}[\sigma((\mu+2)(\mu-3)+4\delta(\mu-1)+2\delta(2\delta+1))+6]x^2 \\ -\frac{1}{2}\left(3x^2-1\right)(\sigma(2\delta+\mu-3)+2)^2\end{array}\right|}},$$

$$|a_3| \leq \frac{x^2}{|\sigma(2\delta+\mu-3)+2|^2} + \frac{x}{|\sigma(3\delta+\mu-4)+3|},$$

and

$$\left|a_3 - \tau a_2^2\right| \leq \begin{cases} \frac{x}{|\sigma(3\delta+\mu-4)+3|}, & |\tau-1| \leq k(x) \\ \frac{2x^3|1-\tau|}{\left|\begin{array}{c}[\sigma((\mu+2)(\mu-3)+4\delta(\mu-1)+2\delta(2\delta+1))+6]x^2 \\ +\left[1-3x^2\right][\sigma(2\delta+\mu-3)+2]^2\end{array}\right|}, & |\tau-1| \geq k(x) \end{cases}$$

where

$$k(x) = \left|\frac{[\sigma((\mu+2)(\mu-3)+4\delta(\mu-1)+2\delta(2\delta+1))+6]x^2 + \left[1-3x^2\right][\sigma(2\delta+\mu-3)+2]^2}{2[\sigma(3\delta+\mu-4)+3]x^2}\right|.$$

Corollary 2. *Let $f \in \Sigma$ given by (6) belong to the class $\mathbf{B}_{\Sigma}^{1}(\sigma, \delta, \mu, x)$. Then*

$$|a_2| \leq \frac{2x\sqrt{2x}}{\sqrt{\left|\begin{array}{c}2[\sigma((\mu+2)(\mu-3)+4\delta(\mu-1)+2\delta(2\delta+1))+6]x^2 \\ -\left(4x^2-1\right)(\sigma(2\delta+\mu-3)+2)^2\end{array}\right|}},$$

$$|a_3| \leq \frac{4x^2}{|\sigma(2\delta+\mu-3)+2|^2} + \frac{2x}{|\sigma(3\delta+\mu-4)+3|},$$

and

$$\left|a_3 - \tau a_2^2\right| \leq \begin{cases} \frac{2x}{|\sigma(3\delta+\mu-4)+3|}, & |\tau-1| \leq k(x) \\ \frac{8x^3|1-\tau|}{\left|\begin{array}{c}2[\sigma((\mu+2)(\mu-3)+4\delta(\mu-1)+2\delta(2\delta+1))+6]x^2 \\ +\left[1-4x^2\right][\sigma(2\delta+\mu-3)+2]^2\end{array}\right|}, & |\tau-1| \geq k(x) \end{cases}$$

where

$$k(x) = \left|\frac{\begin{array}{c}2[\sigma((\mu+2)(\mu-3)+4\delta(\mu-1)+2\delta(2\delta+1))+6]x^2 \\ +\left[1-4x^2\right][\sigma(2\delta+\mu-3)+2]^2\end{array}}{4[\sigma(3\delta+\mu-4)+3]x^2}\right|.$$

Remark 2. *By taking specific values for δ, μ, and σ in Definition 1, we get various well-known subclasses of Σ which have been studied by several authors, for instance, but not limited to, Bulut et al. [49], Altinkaya and Yalcin [50], and Amourah et al. [5].*

6. Conclusions

The new subclasses of the class of bi-univalent functions in the $\mathbb{U}$, $\mathbf{B}_{\Sigma}^{\gamma}(\sigma,\delta,\mu,x)$, $\mathbf{B}_{\Sigma}^{\gamma}(1,\delta,\mu,x)$ and $\mathbf{B}_{\Sigma}^{\gamma}(1,1,\mu,x)$ have all been presented and their coefficient problems have been examined Accordingly, special cases (i) and (ii) characterize these bi-univalent function classes. We have estimated the Taylor—Maclaurin coefficients $|a_2|$ and $|a_3|$ and the Fekete–Szegö functional problems for functions in each of these bi-univalent function classes. After focusing on the parameters used in our main results, several other new results were discovered.

The results obtained are not sharp and it is open for others to prove their sharpness. At the moment, this is the best possible result that we can have. In addition, we believe that this study will inspire a other researchers to extend this concept to harmonic functions and symmetric q-calculus. This concept can also be applied when using the symmetry q-sine domain and the symmetry q-cosine domain instead of the given domain.

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