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Copula Approach for Dependent Competing Risks of Generalized Half-Logistic Distributions under Accelerated Censoring Data

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Abstract: In medicals sciences and reliability engineering, the failure of individuals or units (I/Us) occurs due to independent causes of failure. In general, the symmetry between dependent and independent causes of failure is essential to the nature of the problem at hand. In this study, we considered the accelerated dependent competing risks model when the lifetime of I/Us was modeled using a generalized half-logistic distribution. The data were obtained with respect to constant stress accelerated life tests (ALTs) with a type-II progressive censoring scheme. The dependence structure was formulated using the copula approach (symmetric Archimedean copula). The model parameters were estimated with the maximum likelihood method; only two dependent causes of failure and bivariate Pareto copula functions were proposed. The approximate confidence intervals were constructed using both the asymptotic normality distribution of MLEs and bootstrap techniques. Additionally, an estimator of the reliability of the system under a normal stress level was constructed. The results from the estimation methods were tested by performing a Monte Carlo simulation study. Finally, an analysis of data sets from two stress levels was performed for illustrative purposes.

Keywords: accelerated life tests; bootstrap confidence intervals; competing risks model; copula function; generalized half-logistic distribution; maximum likelihood estimation; progressive censoring

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1. Introduction

The life characteristics of I/Us can be understood by analyzing time-to-failure data. However, the high survival of I/Us is achievable with modern technology. Hence, sufficient data may not be available. The requirement for a rapid source of data regarding the lifetime of I/Us in a short period of time has necessitated research and development in several directions. The first direction can be carried out with ALTs. The experiment under ALTs is exposed to higher stress levels than the normal stress level. Due to higher stress, I/Us fail earlier than the expected time. Therefore, an experimenter has more failure time data in a shorter period of time. Recently, ALTs have become an increasingly important source of data; see Nelson [1] for a key reference on ALTs. In the literature, different types of ALTs are presented, and the first type is called a constant-stress ALT. The experiment for constant-stress ALTs is run under constant stress to the final point, as seen in the studies of Bagdonavicius and Nikulin [2], Kim and Bai [3] and Ismail et al. [4]. The second type of ALT is called step stress ALT, which corresponds to the situation when stress changes at pre-specified times or after a pre-specified number of failures (see, Miller and Nelson [5], Gouno et al. [6], Fan et al. [7], Tangi et al. [8] and Almarashi and Abd-Elmougod [9]. Progressive-stress ALTs are the third type of ALT that are used if the stress continually increases. For more details on progressive-stress ALTs, see Wang and Fei [10] and Abdel-Hamid and Al-Hussaini [11]. The second direction involves the use of censoring schemes for a rapid source of data, in which the data are recorded for some I/Us and not for all I/Us under the test. In the statistical literature, the type-I and type-II censoring schemes are common censoring schemes. When the experiment runs under a constant test time, and the number of failures is obtained randomly, a type-I censoring scheme is used. However, when the experiment is run under a constant number of failures and the test time is random, a type-II censoring scheme is used. The last two types of censoring schemes do not allow the removal of I/Us from the test other than the final point. In different applications of engineering or clinical studies, I/Us may need to be removed at any step of the test. Therefore, the concept of a progressive censoring scheme (PCS) emerged (see Balakrishnan and Aggarwala [12]. The PCS with a type-II censoring scheme was called a type-II PCS, and the mechanism of the type-II PCS can be described as follows. Suppose a sample of size n is randomly selected from a life population for a test. The number of failures m needed for statistical inference and censoring schemes $\mathbf{R} = \{R_1, R_2, \ldots, R_m\}$ is proposed as a fixed prior value. When the time of first failure $T_{1;m,n}$ is observed, R_1 survival I/Us are removed from the test. When we observe the second failure time, $T_{2;m,n}$, R_2 survival I/Us are removed from the test. The experiment is continued until the failure $T_{m;m,n}$ is observed; then, the remaining R_m survival I/Us are removed from the test. Therefore, $\mathbf{T} = \{T_{1;m,n}, T_{2;m,n}, \ldots, T_{m;m,n}\}$ is a type-II PC sample that satisfies $n = m + \sum_{i=1}^{m} R_i$.

For convenience, the multiple variables in various fields of science, such as engineering, social science, medical trials, and biological science, are proposed to be independent. A competing risks model in statistical modeling can be discussed with respect to two or more independent variables see Cox [13], David and Moeschberger [14], Crowder [15], Balakrishnan [16], Modhesh and Abd-Elmougod [17], Bakoban and Abd-Elmougod [18] and Ganguly and Kundu [19]. This model was discussed recently by different authors, including Algarni et al. [20], Tahani el al. [21], Alghamdi [22], Almarashi and Abd-Elmougod [9], and Alghamdia et al. [23,24]. Most of the previous studies on competing for risk modeling have assumed the failure modes are independent for mathematical simplicity. The proposal of independent variables does not agree with the problem at hand. Hence, dependent variables have been proposed. The multivariate distributions can be used to describe the structures between dependent variables; for example, see Marshall and Olkin [25] for the case of a multivariate exponential distribution. The problem of statistical inferences, especially the estimation of the correlation structure between variables under multivariate distributions, is more restricted in a simple multivariate distribution. Therefore, modeling under a competing risks model with the assumption of a dependence variable with bivariate or multivariate distributions appears. This problem was discussed using the copula approach, in which the marginal distributions are described independently of each random variable, and the copula function describes the dependence structure formed with marginal distributions.

The half-logistic (HL) distribution was used by Balakrishnan [26] to model the absolute standard logistic random variable. The HL distribution was applied in a reliability and survival analysis with censoring data. Additionally, a generalized version of the HL distribution was presented by Balakrishnan and Hossain [27] as a generalized half-logistic (GHL) distribution. Analyzing the dependent competing risks model using the copula approach for a system tested under type-II PCS with a constant-stress ALTs model is the main objective of this study. This problem is discussed under considerations that: (i) two stress levels and (ii) the time of failure with respect to two dependent causes of failure is modeled by a generalized half-logistic distribution. The random variable T is called the GHL random variable if it has the cumulative distribution function of CDF presented by

$$F(t)=1-\left(\frac{2e^{-\frac{t}{\theta}}}{1+e^{-\frac{t}{\theta}}}\right)^{\beta}, t>0,\ \beta,\ \theta>0. \tag{1}$$

The GHL distribution was discussed by different authors, for example, Ramakrishnan [28], Arora et al. [29] and Kim et al. [30], and Chaturvedi et al. [31] considered different statistical inferential issues based on the GHL distribution. Other research results related to ALT can be found in Almarashi [32].

The structure of the paper is as follows. The model and the methodology are presented in Section 2. The result and discussion are presented in Section 3. Where in Section 3.1, the copula function with its types and properties are discussed. ML Estimation of the model parameters is presented in Section 3.2. The approximate confidence intervals based on the asymptotic normality of the MLEs are presented in Section 3.3. The two bootstrap confidence intervals are presented in Section 3.4. The reliability estimator of the system under normal stress conditions is presented in Section 3.5. The simulation results under a Monte Carlo simulation study are provided in Section 3.6. Data analysis is discussed in Section 3.7. Finally, the model considerations and their uses in real-life phenomena are discussed in Section 4.

2. Methodology

Suppose, without loss of generality, the test begins with two stress levels, and a selected sample of size n is divided randomly into two independent sets of sizes n_1 and n_2. The two sets are placed under two stress levels $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively. Prior to the experiment, the two integers m_1 and m_2 as well as two censoring schemes $\mathbf{R}_1 = \{R_{11}, R_{12}, \ldots, R_{1m_1}\}$ and $\mathbf{R}_2 = \{R_{21}, R_{22}, \ldots, R_{2m_2}\}$ are proposed to satisfy $n_k = m_k + \sum_{i=1}^{m_k} R_{ki}$, $k=1, 2$. The failure of I/Us occurs with respect to one of two dependent failure causes $\delta_{ki} = \{1, 2\}$. The mechanism of the test under consideration is based on type-II PCS with two stress levels and two dependent causes of failure, which are described as follows.

When the first failure $T_{k1;m_k,n_k}$ and the corresponding cause of failure δ_{k1} are observed, R_{k1} survival I/Us are randomly removed from the test. Also, when the second failure $T_{k2;m_k,n_k}$ is observed with a corresponding cause of failure δ_{k2}, R_{k2} survival I/Us are randomly removed from the test. The experiment is continued until the $(T_{km_k;m_k,n_k}, \delta_{km_k})$ is observed, and the remaining R_{km_k} survival I/Us are removed from the test. Therefore, the joint competing risks type-II PC sample can be defined by

$$\mathbf{t}_k|_{k=1,2} = \{(T_{k1;m_k,n_k}, \delta_{k1}), (T_{k2;m_k,n_k}, \delta_{k2}), \ldots, (T_{km_k;m_k,n_k}, \delta_{km_k})\}.$$

The joint likelihood function under stress level $\mathbf{s}_k$ with observed data $\mathbf{t}_k$ is given by

$$L_k \propto S_k(t_{km_k})^{R_{km_k}} \prod_{i=1}^{m_k} \left\{ \left[\frac{\partial C(u,v)}{\partial u} f_{k1}(t_{ki}) \right]^{I_1(\delta_{ki}=1)} \left[\frac{\partial C(u,v)}{\partial v} f_{k2}(t_{ki}) \right]^{I_2(\delta_{k2}=2)} S_k(t_{ki})^{R_{k1}} \right\} \tag{2}$$

where $C(u,v)$ is appropriate copula function and $S_k(.)$ is the joint survival function under two dependent variable with stress level $\mathbf{S}_k$ and

$$I_j(\delta_{ki}) = \begin{cases} 1, \text{ if } \delta_{ki} = j \\ 0, \text{ if } \delta_{ki} \neq j \end{cases}. \tag{3}$$

The joint likelihood function can be defined by

$$\mathbf{L} \propto \prod_{k=1}^{2} L_k \tag{4}$$

Model considerations

1. The number of I/Us which is failed with respect to stress level $\mathbf{S}_k$ and cause j is denoted by

$$n_{kj} = \sum_{i=1}^{m_k} I_j(\delta_{ki}), k, j = 1, 2.$$

2. The number of I/Us which is failed with respect cause j is denoted by

$$J_j = \sum_{k=1}^{2} \sum_{i=1}^{m_k} I_j(\delta_{ki}), k, j = 1, 2.$$

3. For any stress levels $\mathbf{S}_k$, $k = 1, 2$, only two dependence causes of failure are exist.
4. The time-to-failure T_{kj} respected to stress level $\mathbf{S}_k$ and the cause of failure j distribute with GHL distribution with scale parameters θ_{kj} and shape parameter β_j with CDFs given by

$$F_{kj}(t|\beta_j, \theta_{kj}) = 1 - \left(\frac{2e^{-\frac{t}{\theta_{kj}}}}{1 + e^{-\frac{t}{\theta_{kj}}}} \right)^{\beta_j}, t > 0,\ \beta_j,\ \theta_{kj} > 0. \tag{5}$$

The corresponding PDFs

$$f_{kj}(t|\beta_j, \theta_{kj}) = \frac{\beta_j}{\theta_{kj}(1 + e^{-\frac{t}{\theta_{kj}}})} \left(\frac{2e^{-\frac{t}{\theta_{kj}}}}{1 + e^{-\frac{t}{\theta_{kj}}}} \right)^{\beta_j}, \tag{6}$$

$$S_{kj}(t|\beta_j, \theta_{kj}) = \left(\frac{2e^{-\frac{t}{\theta_{kj}}}}{1 + e^{-\frac{t}{\theta_{kj}}}} \right)^{\beta_j}, \tag{7}$$

and

$$H_{kj}(t|\beta_j, \theta_{kj}) = \frac{\beta_j}{\theta_{kj}(1 + e^{-\frac{t}{\theta_{kj}}})}. \tag{8}$$

5. The shape parameters is common for stress levels $\mathbf{S}_k$, $k = 1, 2$ and different for causes of failure.
6. The joint survival function under BPC is given by

$$S_k(t) = \left(\left(\frac{2e^{-\frac{t}{\theta_{k1}}}}{1 + e^{-\frac{t}{\theta_{k1}}}} \right)^{\frac{-\beta_1}{\gamma}} + \left(\frac{2e^{-\frac{t}{\theta_{k2}}}}{1 + e^{-\frac{t}{\theta_{k2}}}} \right)^{\frac{-\beta_2}{\gamma}} - 1 \right)^{-\gamma}. \tag{9}$$

7. The scale parameters θ_{k1} is log-linear function of the stress function $\phi(\mathbf{S}_k)$ of the j-th competing failure mode

$$\log\theta_{kj} = a_j + b_j\phi(\mathbf{S}_k),\ k;\ j = 1, 2, \tag{10}$$

where, a_j, $b_j > 0$ are the unknown parameters.

3. Result and Discussion

In this section, we discuss the copula function utilized in terms of its types and properties. Then, the MLEs, approximate confidence intervals based on the asymptotic normality of the MLEs, and the two bootstrap confidence intervals of the model parameters are derived. We present the ML estimation of the system's reliability under normal stress conditions. Additionally, a Monte Carlo simulation study and data analysis are introduced. Finally, we provide the conclusion and remarks.

3.1. Copula Function

The problem of modeling dependence structure between variable under dependent competing failure causes with the copula function is more convenient. The type of dependence structure is determined by the selected copula see Sklar [33]. Suppose that, F_i and

$S_i = \bar{F}_i = 1 - F_i$ are the marginal distribution and the corresponding survival functions of the random variables T_i, $i = 1, 2, \ldots, J$, respectively. Then, a unique J-dimensional copula C is used to define the joint distribution function $H(t_1, \ldots, t_J)$ by

$$H(t_1, \ldots, t_J) = C(F_1(t_1), \ldots, F_J(t_J)). \tag{11}$$

The joint distribution function H of the dependence structure depends on the choice of marginal functions $F_i, i = 1, 2, \ldots, J$ and the corresponding copula function C. The copula function of the continuos marginal functions $F_i, i = 1, 2, \ldots, J$ is constructed using the corollary of Sklar's theorem. Additionally, from the marginal m-dimensional invariance distribution functions $F_i^{-1}(u_i), i = 1, 2, \ldots, J$, the copula function is defined by,

$$C(u_1, \ldots, u_J) = H(F_1^{-1}(u_1), \ldots, F_J^{-1}(u_J)). \tag{12}$$

The multivariate survival function $S(t_1, \ldots, t_J)$ under transformation $T_i \to F_i(T_i) = 1 - S_i(T_i)$ with Sklar's theorem via an appropriate copula $\bar{C}$ called the survival copula of $(T_1, \ldots, T_J)$ and can be expressed by

$$S(t_1, \ldots, t_J) = \bar{C}(S_1(t_1), \ldots, S_J(t_J)), \tag{13}$$

where $\bar{C}$ is the appropriate survival copula of $T_i, i = 1, 2, \ldots, J$.. Then, we can say that copula functions C and $\bar{C}$ are related, with marginal distribution functions F_i and marginal survival functions S_i, $i = 1, 2, \ldots, J$, and the multivariate distribution and survival functions, respectively. Several types of copula functions are available, such as the Gumbel copula, Clayton copula, Frank copula, Student's t-copula, Gaussian copula, and so on. Then, the problem of the choice of an appropriate copula is an important sub-class, as discussed by Nelsen [34]. The marginal distribution is used to determine the right copula function. For example, the Gaussian copula is applied when the marginal distributions are normal distributions. The characteristics of the dependence between information are an appropriate way to choose the right copula function see Zhang et al. [35]. Archimedean copulas present important one and two-parameter families, such as the Gumbel family, Gumbel–Hougaard family, Gumbel–Barnett family, Clayton family, Frank family, and so on. In our problem, we consider the Gumbel copula as a sub-class of Archimedean copulas, as follows.

Archimedean copula

Under a two-dimensional copula, the function Ω, which is defined to satisfy

$$\Omega(C(u, v)) = \Omega(u) + \Omega(v), \tag{14}$$

is called Archimedean copula. Furthermore, the inverse transformation of the function Ω provides the solution of the copula function C to satisfy

$$C(u, v) = \Omega^{[-1]}(\Omega(u) + \Omega(v)). \tag{15}$$

The Archimedean copula $\Omega(t) = t^{-1/\gamma} - 1, \gamma \geq 1$, define the bivariate Pareto copula (BPC) by

$$C_\gamma(u, v) = (u^{-1/\gamma} + v^{-1/\gamma} - 1)^{-\gamma} \tag{16}$$

Measure of association

In the literature, there are several types of copulas with different parameter values. Hence, there is no comparable function; the comparable case under Kendall's tau is also defined from the copula function by

$$\tau = 4 \int_0^1 \int_0^1 C(u, v) c(u, v) du dv = 4E[C(U, V)] - 1.$$

In which, under BPC τ is reduced to

$$\tau = 4\int_0^1 \frac{\Omega(t)}{\Omega'(t)}dt + 1 = 1/(2\gamma+1). \tag{17}$$

3.2. The Point ML Estimate

When considering two stress levels $\mathbf{S}_k$, $k = 1, 2$, the joint likelihood function (2) after using the density function (6), survival function (9), and bivariate Pareto copula function, which is defined by (16), is reduced to

$$\begin{aligned} L_k = {} & \beta_1^{n_{k1}}\beta_2^{n_{k2}}\theta_{k1}^{-n_{k1}}\theta_{k2}^{-n_{k2}}\left(\left(\frac{2e^{-\frac{t_{m_k}}{\theta_{k1}}}}{1+e^{-\frac{t_{m_k}}{\theta_{k1}}}}\right)^{\frac{-\beta_1}{\gamma}} + \left(\frac{2e^{-\frac{t_{m_k}}{\theta_{k2}}}}{1+e^{-\frac{t_{m_k}}{\theta_{k2}}}}\right)^{\frac{-\beta_2}{\gamma}} - 1\right)^{-\gamma R_{km_k}} \\ & \times\prod_{i=1}^{m_k}\left\{\left[\frac{1}{(1+e^{-\frac{t_{ki}}{\theta_{k1}}})}\left(\frac{2e^{-\frac{t_{ki}}{\theta_{k1}}}}{1+e^{-\frac{t_{ki}}{\theta_{k1}}}}\right)^{-\frac{\beta_1}{\gamma}}\right]^{I(\delta_{ki}=1)}\left[\frac{1}{(1+e^{-\frac{t_{ki}}{\theta_{k2}}})}\left(\frac{2e^{-\frac{t_{ki}}{\theta_{k2}}}}{1+e^{-\frac{t_{ki}}{\theta_{k2}}}}\right)^{-\frac{\beta_2}{\gamma}}\right]^{I(\delta_{ki}=2)}\right. \\ & \left.\times\left(\left(\frac{2e^{-\frac{t_{ki}}{\theta_{k1}}}}{1+e^{-\frac{t_{ki}}{\theta_{k1}}}}\right)^{\frac{-\beta_1}{\gamma}} + \left(\frac{2e^{-\frac{t_{ki}}{\theta_{k2}}}}{1+e^{-\frac{t_{ki}}{\theta_{k2}}}}\right)^{\frac{-\beta_2}{\gamma}} - 1\right)^{-\gamma(R_{ki}+1)-1}\right\} \end{aligned} \tag{18}$$

The natural log-likelihood function of (18), under joint likelihood function (11) is reduced to

$$\begin{aligned} \ell &= \sum_{k=1}^{2}\log\mathbf{L} = J_1\log\beta_1 + J_2\log\beta_2 + n_{11}\log\theta_{11} + n_{12}\log\theta_{12} + n_{21}\log\theta_{21} \\ &+ n_{22}\log\theta_{22} - \sum_{k=1}^{2}\left\{\sum_{i=1}^{m_k} I(\delta_{ki}=1)\left[\frac{\beta_1}{\gamma}\log[Z_{1ki}] + \log[Y_{1ki}]\right] + \sum_{i=1}^{m_k} I(\delta_{ki}=2)\right. \\ &\times \left[\frac{\beta_2}{\gamma}\log[Z_{2ki}] + \log[Y_{2ki}]\right] + \sum_{i=1}^{m_k}(\gamma(R_{ki}+1)+1)\log\left[Z_{1ki}^{-\frac{\beta_1}{\gamma}} + Z_{2ki}^{-\frac{\beta_2}{\gamma}} - 1\right] \\ &+ \left.\gamma R_{km_k}\log\left[Z_{1m_k}^{-\frac{\beta_1}{\gamma}} + Z_{2m_k}^{-\frac{\beta_2}{\gamma}} - 1\right]\right\}, \end{aligned} \tag{19}$$

where

$$Z_{lki} = \frac{2e^{-\frac{t_{ki}}{\theta_{kl}}}}{1+e^{-\frac{t_{ki}}{\theta_{kl}}}},\ l = 1, 2, \tag{20}$$

and

$$Y_{lki} = 1 + e^{-\frac{t_{ki}}{\theta_{kl}}}. \tag{21}$$

The likelihood Equations are obtained from (19) by zero-value of the first partially derivative of ℓ with respect to parameters vector $\Omega = \{\beta_1, \beta_2, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}\}$ as follows.

$$\begin{aligned}\frac{\partial \ell}{\partial \beta_j} &= \frac{J_j}{\beta_j} - \frac{1}{\gamma}\sum_{k=1}^{2}\Bigg\{\sum_{i=1}^{m_k} I(\delta_{ki}=j)\log\Big[Z_{jki}\Big] - \sum_{i=1}^{m_k}(\gamma(R_{ki}+1)+1)\frac{Z_{jki}^{-\frac{\beta_j}{\gamma}}\log Z_{jki}}{Z_{1ki}^{-\frac{\beta_1}{\gamma}} + Z_{2ki}^{-\frac{\beta_2}{\gamma}} - 1} \\ &- \gamma R_{km_k}\frac{Z_{jm_k}^{-\frac{\beta_j}{\gamma}}\log Z_{jm_k}}{Z_{1m_k}^{-\frac{\beta_1}{\gamma}} + Z_{2m_k}^{-\frac{\beta_2}{\gamma}} - 1}\Bigg\} = 0,\ j = 1,2,\end{aligned} \tag{22}$$

and

$$\begin{aligned}\frac{\partial \ell}{\partial \theta_{kj}} &= \frac{n_{k1}}{\theta_{kj}} - \sum_{k=1}^{2}\Bigg\{\sum_{i=1}^{m_k} I(\delta_{ki}=j)\frac{t_{ki}}{\theta_{kj}^2}\left[\frac{2\beta_j}{\gamma Y_{jki}} + \frac{1}{2}Z_{jki}\right] - \frac{\beta_j}{\gamma\theta_{kj}^2}\sum_{i=1}^{m_k}(\gamma(R_{ki}+1)+1) \\ &\times \frac{t_{ki}Z_{jki}^{-\frac{\beta_j}{\gamma}}}{Y_{jki}\left(Z_{1ki}^{-\frac{\beta_1}{\gamma}} + Z_{2ki}^{-\frac{\beta_2}{\gamma}} - 1\right)} - R_{km_k}\frac{\beta_j t_{km_k} Z_{1m_k}^{-\frac{\beta_j}{\gamma}}}{\theta_{kj}^2 Y_{1m_k}\left(Z_{1m_k}^{-\frac{\beta_1}{\gamma}} + Z_{2m_k}^{-\frac{\beta_2}{\gamma}} - 1\right)}\Bigg\} = 0, \\ &,\ k,\ j = 1,\ 2\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \gamma} &= \sum_{k=1}^{2}\Bigg\{\frac{\beta_1}{\gamma^2}\sum_{i=1}^{m_k} I(\delta_{ki}=1)[\log[Z_{1ki}]] + \frac{\beta_2}{\gamma^2}\sum_{i=1}^{m_k} I(\delta_{ki}=2)[\log[Z_{2ki}]] \\ &\times\ + \sum_{i=1}^{m_k}(\gamma(R_{ki}+1)+1)\log\left[Z_{1ki}^{-\frac{\beta_1}{\gamma}} + Z_{2ki}^{-\frac{\beta_2}{\gamma}} - 1\right] \\ &+ \gamma R_{km_k}\log\left[Z_{1m_k}^{-\frac{\beta_1}{\gamma}} + Z_{2m_k}^{-\frac{\beta_2}{\gamma}} - 1\right]\Bigg\},\end{aligned} \tag{24}$$

The likelihood Equations (22)–(24) cannot be solved analytically. The Newton Raphson method can be applied to obtain the parameter estimate $\hat{\Omega} = \{\beta_1, \hat{\beta}_2, \hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{21}, \hat{\theta}_{22}\}$.

3.3. Approximate Confidence Intervals (ACIs)

In this section, we consider the ACIs based on the asymptotic normality of MLEs. We first obtain the asymptotic variance–covariance matrix of the MLEs by inverting the Fisher information matrix. The Fisher information matrix is defined as the mines expectation of the second-order mixed partial derivatives of the log-likelihood function defined by

$$I_F(\Omega) = -E\left(\frac{\partial^2 \ell}{\partial\Omega_i\partial\Omega_l}\right),\ i,l = 1,2,\ldots,6, \tag{25}$$

where $\Omega = \{\beta_1, \beta_2, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}\}$. In this case, the expectation of the second mixed partial derivative of the log-likelihood function under the copula approach is complicated to calculate. Instead, we use the observed Fisher information matrix, defined by

$$I(\Omega) = -\left(\frac{\partial^2 \ell}{\partial\Omega_i\partial\Omega_l}\right),\ i,l = 1,2,\ldots,6. \tag{26}$$

Based on the property of the asymptotic distribution of the MLE $\hat{\Omega} = \{\beta_1, \hat{\beta}_2, \hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{21}, \hat{\theta}_{22}\}$ and for some regularity conditions, the MLEs are approximated to bivariate normal distribution, as follows: $(\hat{\Omega} - \Omega) \longrightarrow N_2(0, I_0^{-1}(\Omega))$, where the value of inverse observed information matrix $I_0^{-1}(\varphi)$ is computed with the ML estimates of the parameters and is defined as

$$I_0^{-1}(\varphi) = -\left(\frac{\partial^2 \ell}{\partial\Omega_i \partial\Omega_l}\right)^{-1}\Big|_{\hat{\Omega}=\{\hat{\beta}_1,\hat{\beta}_2,\hat{\theta}_{11},\hat{\theta}_{12},\hat{\theta}_{21},\hat{\theta}_{22}\}},\ i,l = 1,2,\ldots,6. \tag{27}$$

Hence, $100(1-2\alpha)\%$ approximate two-side confidence intervals for theparameters $\Omega = \{\beta_1, \beta_2, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}\}$, are given by

$$\begin{cases} \hat{\beta}_j \mp Z_\alpha \sqrt{\mathrm{var}(\hat{\beta}_j)} \\ \hat{\theta}_{1j} \mp Z_\alpha \sqrt{\mathrm{var}(\hat{\theta}_{1j})} \\ \hat{\theta}_{2j} \mp Z_\alpha \sqrt{\mathrm{var}(\hat{\theta}_{2j})} \end{cases},\ j = 1,2, \tag{28}$$

where, the value Z_α is calculated from standard normal distribution, with α right-tail probability. The model parameters have a positive range, which may contradict the lower bounded of the intervals defined by Equation (28). Hence, to avoid this contradiction delta method with log-transformation are used as follows.

As described by Meeker and Escobar [36], the log-transformation of the estimte $\hat{\Omega}$ ($\log\hat{\Omega}$) is approximated with normal distribution. Hence, the pivotal $\Phi_i = \frac{\log\hat{\Omega}_i - \log\Omega_i}{\mathrm{Var}(\log\hat{\Omega}_i)}$, $i = 1,2,\ldots,6$ is approximated with standard normal distribution. Then, the $100(1-2\alpha)\%$ approximate confidence interval of the model parameters $\Omega = \{\beta_1, \beta_2, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}\}$ can be formulated by

$$\frac{\hat{\Omega}_i\left(1,\ \exp\left(2Z_\alpha\sqrt{\mathrm{Var}(\log\hat{\Omega}_i)}\right)\right)}{\exp\left(Z_\alpha\sqrt{\mathrm{Var}(\log\hat{\Omega}_i)}\right)}, \tag{29}$$

where $\mathrm{Var}(\log\hat{\Omega})=\frac{\mathrm{Var}(\hat{\Omega})}{\hat{\Omega}}$ and i =1, 2, …, 6.For more details, see: [37].

3.4. Bootstrap Confidence Intervals (BCIs)

The bootstrap techniques are used to formulate not only parameter estimations but also to estimate the bias and variance between estimators as well as calibrate the hypothesis testing. As presented by Davison and Hinkley [38] and Efron and Tibshirani [39], the bootstrap technique has been presented as a parametric and non-parametric technique. In this subsection, we construct the BCIs for the unknown parameters using a parametric bootstrap method (bootstrap-p and bootstrap-t approaches). The following algorithm is implemented to generate the bootstrap samples using ML estimates; for more details, see Hall [40] and Efron [41].

Algorithms 1 (Generate bootstrap sample of estimates)

Step 1: For given n_1, n_2, m_1, m_2, stress levels $\mathbf{S}_1$ and $\mathbf{S}_2$ and two censoring schemes $\mathbf{R}_1 = \{R_{11}, R_{12}, \ldots, R_{1m_1}\}$ and $\mathbf{R}_2 = \{R_{21}, R_{22}, \ldots, R_{2m_2}\}$ with the original competing risks type-II PCS $\mathbf{t}_k|_{k=1,2} = \{(T_{k1;m_k,n_k}, \delta_{k1}), (T_{k2;m_k,n_k}, \delta_{k2}), \ldots, (T_{km_k;m_k,n_k}, \delta_{km_k})\}$ compute $J_1, J_2, n_{11}, n_{12}, n_{11}$ and n_{12}. Then, the estimate values of the model parameters $\hat{\Omega} = \{\beta_1, \hat{\beta}_2, \hat{\theta}_{11}, \hat{\theta}_{12}, \hat{\theta}_{21}, \hat{\theta}_{22}\}$ are computed.

Step 2: Based on n_1, m_1 and $\mathbf{R}_1$ using the algorithms given by Balakrishnan and Sandhu [42], we generate two type-II PC samples of size m_1 from GHL distributions with parameters $(\beta_1, \hat{\theta}_{11})$ and $(\beta_2, \hat{\theta}_{12})$, respectively. The competing risks type-II PC sample is difened by $(T_{1i}, \delta_{1i}) = \min(T_{11i}, T_{12i})$, i =1, 2, …, m_1.

Step 3: Based on n_2, m_2 and $\mathbf{R}_2$ generate two type-II PC samples of size m_2 from GHL distributions with parameters $(\beta_1, \hat{\theta}_{21})$ and $(\beta_2, \hat{\theta}_{22})$, respectively. The competing risks type-II PC sample is difened by $(T_{2i}, \delta_{2i}) = \min(T_{21i}, T_{22i})$, i =1, 2, …, m_2.

Step 4: From two Step 2 and 3 the joint sample $\mathbf{t}_k^*|_{k=1,2} = \{(T^*_{k1;m_k,n_k}, \delta^*_{k1}), (T^*_{k2;m_k,n_k}, \delta^*_{k2}), \ldots, (T^*_{km_k;m_k,n_k}, \delta^*_{km_k})\}$ is formulated.

Step 5: Based on $\mathbf{t}_k^*|_{k=1,2} = \{(T^*_{k1;m_k,n_k}, \delta^*_{k1}), (T^*_{k2;m_k,n_k}, \delta^*_{k2}), \ldots, (T^*_{km_k;m_k,n_k}, \delta^*_{km_k})\}$ compute the MLE estimate $\hat{\Omega}^* = \{\hat{\beta}^*_1, \hat{\beta}^*_2, \hat{\theta}^*_{11}, \hat{\theta}_{12}, \hat{\theta}^*_{21}, \hat{\theta}^*_{22}\}$.

Repeat steps from 2 to 5 **M** time s and put the estimate in ascending order, we obtain the bootstrap sample as

$$\begin{cases} \hat{\beta}_j^{*[1]}, \hat{\beta}_j^{*[2]}, \ \ldots, \ \hat{\beta}_j^{*[\mathbf{M}]} \\ \hat{\theta}_{1j}^{*[1]}, \hat{\theta}_{1j}^{*[2]}, \ldots, \hat{\theta}_{1j}^{*[\mathbf{M}]} \\ \hat{\theta}_{2j}^{*[1]}, \hat{\theta}_{2j}^{*[2]}, \ldots, \hat{\theta}_{2j}^{*[\mathbf{M}]} \end{cases}, \ j = 1,2 \tag{30}$$

Bootstrap-p confidence interval (Boot-P CIs)

From the bootstrap samples estimate (29), the $100(1-2\alpha)\%$ approximate bootstrap-p confidence intervals of the model parameters $\Omega = \{\beta_1, \beta_2, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}\}$, respectively given by

$$\begin{cases} (\hat{\beta}_j^{*[M\alpha]}, \hat{\beta}_j^{*[M(1-\alpha)]}) \\ (\hat{\theta}_{1j}^{*[M\alpha]}, \hat{\theta}_{1j}^{*[M(1-\alpha)]}) \\ (\hat{\theta}_{2j}^{*[M\alpha]}, \hat{\theta}_{2j}^{*[M(1-\alpha)]}) \end{cases}, \ j = 1,2. \tag{31}$$

Bootstrap-t confidence intervals (Boot-t CIs)

For each the order samples (29) of the estimte $\hat{\Omega}^* = \{\hat{\beta}^*_1, \hat{\beta}^*_2, \hat{\theta}^*_{11}, \hat{\theta}_{12}, \hat{\theta}^*_{21}, \hat{\theta}^*_{22}\}$, we built the order statistics values $\Pi_l^{*(1)} < \Pi_l^{*(2)} < \ldots < \Pi_l^{*(\mathbf{M})}$, where

$$\Pi_l^{*[i]} = \frac{\hat{\Omega}_l^{*[i]} - \hat{\Omega}_l}{\sqrt{\mathrm{var}\left(\hat{\Omega}_l^{*[i]}\right)}}, \ i = 1,2,\ldots,\mathbf{M}, \ l = 1,2,3,4,5,6. \tag{32}$$

There the $100(1-2\alpha)\%$ Boot-t CIs is given by

$$\left(\tilde{\Omega}^*_{l\text{boot-t}(\alpha)}, \ \tilde{\Omega}^*_{l\text{boot-t(1-}\alpha)}\right), \tag{33}$$

where the value $\tilde{\Omega}^*_{l\text{boot-t}}$ is given by

$$\tilde{\Omega}^*_{l\text{boot-t}} = \hat{\Omega}_l + \sqrt{\mathrm{Var}(\hat{\Omega}_l)}F^{-1}(x), \tag{34}$$

and $F^{-1}(x) = P(\Pi_l^* \leqslant x)$ be the cumulative distribution function of Π_l^*.

3.5. Reliability Estimation

In this section, we provide the estimated value of the reliability function of I/Us at any mission time t under normal stress level. The reliability function is defined by

$$S_0(t) = \left(\left(\frac{2e^{-\frac{t}{\theta_{01}}}}{1+e^{-\frac{t}{\theta_{01}}}}\right)^{\frac{-\beta_1}{\gamma}} + \left(\frac{2e^{-\frac{t}{\theta_{02}}}}{1+e^{-\frac{t}{\theta_{02}}}}\right)^{\frac{-\beta_2}{\gamma}} - 1\right)^{-\gamma}. \tag{35}$$

The estimate value of $S_0(t)$ is denoted by $\hat{S}_0(t) = S_0(t)|_{\hat{\theta}_{0j},\hat{\beta}}$. From (17), the estimate values of $\hat{\theta}_{0j}$ under normal stress level are obtained from: $\log\hat{\theta}_{0j} = \hat{a}_j + \hat{b}_j\phi(s_0)$. Using the least-square method, the estimate values $\hat{a}_j$, $\hat{b}_j$ of a_j, b_j are obtained as:

$$\hat{a}_j = \frac{\sum\limits_{k=1}^{2} \ln\hat{\theta}_{kj} - \hat{b}_j \sum\limits_{k=1}^{2} \phi(s_l)}{2}, \tag{36}$$

and

$$\hat{b}_j = \frac{2\sum\limits_{k=1}^{2} \ln\hat{\theta}_{kj}\phi(s_k) - \sum\limits_{k=1}^{2} \ln\hat{\theta}_{kj} \sum\limits_{k=1}^{2} \phi(s_k)}{2\sum\limits_{k=1}^{2} \phi^2(s_k) - \left(\sum\limits_{2=1}^{2} \phi(s_k)\right)^2}. \tag{37}$$

3.6. Simulation Study

In this section, we describe a simulation study conducted to assess and compare the proposed methods of estimation and to test the effect of the selection censoring schemes. In our study, without loss of generality, only two stress levels were considered for the present constant-stress ALTs. Additionally, two dependent causes of failure present a competing risks model under type-II PCS. The samples were generated by using the algorithms presented by Balakrishnan and Sandhu [42]. Suppose that, under a normal stress level with $\mathbf{S}_0$ = 5 °C = 278 K, the two stress levels $\mathbf{S}_1$ = 30 °C = 303 K, $\mathbf{S}_2$ = 60 °C = 333 K are applied. Different values of sample sizes n_1 and n_2 and affect sample sizes m_1 and m_2 are used with proposing stress level $\mathbf{S}_k$, i = 1, 2 shown in Tables 1–4. The shape parameters are taken to be $\beta_1 = 0.2$, $\beta_2 = 0.5$, and the scale parameters θ_{kj} are computed from

$$\log\theta_{kj} = a_j + b_j\phi(\mathbf{S}_k) = a_j + \frac{b_j}{\mathbf{S}_k},\ k = 1,2,\ j = 1,2,$$

where $a_1 = -5$, $a_2 = -8$, $b_1 = 1600$, and $b_2 = 2700$. The value of θ_{kj}, $i = 1,2; j = 1,2$ is given by: $(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}) = (1.3238, 2.4865, 0.82267, 1.114168)$. Generate 1000 samples, and for each sample, compute the point MLE, the corresponding asymptotic confidence interval, and two bootstrap confidence intervals (bootstrap-p and bootstrap-t). The parameter of the copula function describes the dependence structure, which is considered to be γ = 1, 2 or equivalently, Kendall's τ association τ = 1/3, 1/5. The results of the mean squared error (MSEs) are shown in Tables 1 and 3. The value of the coverage percentage (CP) of the ACIs and Boot-P and Boot-t CIs are reported in Tables 2 and 4.

Discussion: The results of the numerical computation for the Monte Carlo simulation study have revealed the following points:

1. The proposed model and the proposed methods of estimation serve well for all of the parameter values and censoring schemes.
2. The values of MSEs decrease when the sample size and affected sample size increase.
3. The results show that the value of the copula parameter $\gamma = 2$ has a small MSEs than value $\gamma = 1$. Hence, a stronger dependent serves better than a weaker dependent.
4. Finally, the coverage percentages of ACIs are always less than the nominal level when the sample size is less or equivalent to 60. For a sample size as large as 70, the coverage percentages of ACIs improve, which can maintain the pre-fixed nominal level.
5. Bootstrap-t serve well than Bootstarp-p and MLE.

Table 1. Estimated MESs when $\gamma = 1$ and Ω = (0.2, 0.5, 1.3238, 2.4865, 0.8227, 1.1142).

(n_1, m_1)	(n_2, m_2)	Scheme	β_1	β_2	θ_{11}	θ_{12}	θ_{21}	θ_{22}
(25,10)	(25,10)	$\mathbf{R}_1 = (6, 1, \ldots, 1)$ $\mathbf{R}_2 = (6, 1, \ldots, 1)$	0.0873	0.1242	0.3214	0.5621	0.2741	0.2987
(25,20)	(25,20)	$\mathbf{R}_1 = (2, 2, 1, 0, \ldots, 0)$ $\mathbf{R}_2 = (2, 2, 1, 0, \ldots, 0)$	0.0745	0.1115	0.3098	0.5428	0.2622	0.2777
(50,20)	(50,20)	$\mathbf{R}_1 = (2, 1, 2, 1, \ldots, 2, 1)$ $\mathbf{R}_2 = (2, 1, 2, 1, \ldots, 2, 1)$	0.0768	0.1103	0.3111	0.5414	0.2611	0.2792
(50,20)	(50,20)	$\mathbf{R}_1 = (30, 0, 0, \ldots, 0)$ $\mathbf{R}_2 = (30, 0, 0, \ldots, 0)$	0.0722	0.1089	0.3102	0.5399	0.2601	0.2774
(50,20)	(50,35)	$\mathbf{R}_1 = (0, 0, \ldots, 0, 30)$ $\mathbf{R}_2 = (0, 0, \ldots, 0, 15)$	0.0715	0.1045	0.3111	0.5389	0.2541	0.2730
(50,35)	(50,20)	$\mathbf{R}_1 = (0, 0, \ldots, 0, 15)$ $\mathbf{R}_2 = (0, 0, \ldots, 0, 30)$	0.0692	0.1093	0.3045	0.5352	0.2613	0.2771
(80,40)	(80,40)	$\mathbf{R}_1 = (1^{40})$ $\mathbf{R}_2 = (1^{40})$	0.0601	0.0875	0.3003	0.5211	0.2492	0.2665
(80,40)	(80,40)	$\mathbf{R}_1 = (0^{20}, 2^{20})$ $\mathbf{R}_2 = (0^{20}, 2^{20})$	0.0614	0.0879	0.3012	0.5209	0.2489	0.2671
(80,60)	(80,40)	$\mathbf{R}_1 = (1^{20}, 0^{40})$ $\mathbf{R}_2 = (1^{40})$	0.0541	0.0869	0.2985	0.5154	0.2494	0.2653
(80,40)	(80,60)	$\mathbf{R}_1 = (1^{40})$ $\mathbf{R}_2 = (1^{20}, 0^{40})$	0.0608	0.0833	0.3007	0.5207	0.2448	0.2618
(80,60)	(80,60)	$\mathbf{R}_1 = (0^{40}, 1^{20})$ $\mathbf{R}_2 = (0^{40}, 1^{20})$	0.0518	0.0782	0.2945	0.5105	0.2399	0.2559

Table 2. 95% CPs when $\gamma = 1$ and Ω = (0.2, 0.5, 1.3238, 2.4865, 0.8227, 1.1142).

(n_1, m_1)	(n_2, m_2)	Scheme	Method	β_1	β_2	θ_{11}	θ_{12}	θ_{21}	θ_{22}
(25,10)	(25,10)	$\mathbf{R}_1 = (6, 1, \ldots, 1)$ $\mathbf{R}_2 = (6, 1, \ldots, 1)$	MLE Boot-p Boot-t	0.87 0.87 0.89	0.89 0.88 0.89	0.86 0.89 0.90	0.88 0.89 0.90	0.88 0.86 0.89	0.89 0.89 0.90
(25,20)	(25,20)	$\mathbf{R}_1 = (2, 2, 1, 0, \ldots, 0)$ $\mathbf{R}_2 = (2, 2, 1, 0, \ldots, 0)$	MLE Boot-p Boot-t	0.89 0.89 0.93	0.91 0.88 0.92	0.88 0.91 0.92	0.89 0.89 0.92	0.91 0.79 0.93	0.91 0.94 0.94
(50,20)	(50,20)	$\mathbf{R}_1 = (2, 1, 2, 1, \ldots, 2, 1)$ $\mathbf{R}_2 = (2, 1, 2, 1, \ldots, 2, 1)$	MLE Boot-p Boot-t	0.91 0.91 0.94	0.90 0.88 0.93	0.89 0.89 0.92	0.88 0.90 0.92	0.92 0.91 0.91	0.90 0.92 0.93
(50,20)	(50,20)	$\mathbf{R}_1 = (30, 0, 0, \ldots, 0)$ $\mathbf{R}_2 = (30, 0, 0, \ldots, 0)$	MLE Boot-p Boot-t	0.91 0.89 0.93	0.91 0.89 0.93	0.89 0.90 0.95	0.91 0.92 0.92	0.91 0.92 0.92	0.91 0.92 0.92
(50,20)	(50,35)	$\mathbf{R}_1 = (0, 0, \ldots, 0, 30)$ $\mathbf{R}_2 = (0, 0, \ldots, 0, 15)$	MLE Boot-p Boot-t	0.91 0.91 0.93	0.90 0.91 0.93	0.92 0.89 0.94	0.90 0.92 0.94	0.93 0.90 0.91	0.92 0.91 0.92
(50,35)	(50,20)	$\mathbf{R}_1 = (0, 0, \ldots, 0, 15)$ $\mathbf{R}_2 = (0, 0, \ldots, 0, 30)$	MLE Boot-p Boot-t	0.90 0.90 0.94	0.91 0.91 0.93	0.89 0.88 0.93	0.90 0.91 0.92	0.91 0.92 0.92	0.91 0.91 0.93
(80,40)	(80,40)	$\mathbf{R}_1 = (1^{40})$ $\mathbf{R}_2 = (1^{40})$	MLE Boot-p Boot-t	0.91 0.91 0.94	0.92 0.90 0.93	0.90 0.92 0.92	0.92 0.96 0.94	0.91 0.90 0.92	0.92 0.92 0.93
(80,40)	(80,40)	$\mathbf{R}_1 = (0^{20}, 2^{20})$ $\mathbf{R}_2 = (0^{20}, 2^{20})$	MLE Boot-p Boot-t	0.92 0.91 0.91	0.92 0.92 0.92	0.91 0.93 0.92	0.91 0.91 0.92	0.94 0.92 0.94	0.92 0.91 0.90
(80,60)	(80,40)	$\mathbf{R}_1 = (1^{20}, 0^{40})$ $\mathbf{R}_2 = (1^{40})$	MLE Boot-p Boot-t	0.92 0.92 0.92	0.92 0.92 0.92	0.92 0.92 0.95	0.94 0.92 0.91	0.91 0.92 0.92	0.94 0.93 0.94
(80,40)	(80,60)	$\mathbf{R}_1 = (1^{40})$ $\mathbf{R}_2 = (1^{20}, 0^{40})$	MLE Boot-p Boot-t	0.90 0.91 0.94	0.90 0.93 0.95	0.92 0.91 0.92	0.91 0.90 0.92	0.95 0.94 0.92	0.92 0.90 0.93
(80,60)	(80,60)	$\mathbf{R}_1 = (0^{40}, 1^{20})$ $\mathbf{R}_2 = (0^{40}, 1^{20})$	MLE Boot-p Boot-t	0.91 0.92 0.94	0.97 0.90 0.92	0.91 0.92 0.95	0.93 0.94 0.93	0.91 0.92 0.95	0.92 0.91 0.94

Table 3. Estimated MESs when $\gamma = 2$ and Ω = (0.2, 0.5, 1.3238, 2.4865, 0.8227, 1.1142).

(n_1, m_1)	(n_2, m_2)	Scheme	β_1	β_2	θ_{11}	θ_{12}	θ_{21}	θ_{22}
(25,10)	(25,10)	$\mathbf{R}_1 = (6, 1, \ldots, 1)$ $\mathbf{R}_2 = (6, 1, \ldots, 1)$	0.0825	0.1200	0.3162	0.5572	0.2741	0.2987
(25,20)	(25,20)	$\mathbf{R}_1 = (2, 2, 1, 0, \ldots, 0)$ $\mathbf{R}_2 = (2, 2, 1, 0, \ldots, 0)$	0.0701	0.1072	0.3045	0.5401	0.2584	0.2719
(50,20)	(50,20)	$\mathbf{R}_1 = (2, 1, 2, 1, \ldots, 2, 1)$ $\mathbf{R}_2 = (2, 1, 2, 1, \ldots, 2, 1)$	0.0725	0.1055	0.3061	0.5382	0.2562	0.2701
(50,20)	(50,20)	$\mathbf{R}_1 = (30, 0, 0, \ldots, 0)$ $\mathbf{R}_2 = (30, 0, 0, \ldots, 0)$	0.0682	0.1051	0.349	0.5354	0.2571	0.2748
(50,20)	(50,35)	$\mathbf{R}_1 = (0, 0, \ldots, 0, 30)$ $\mathbf{R}_2 = (0, 0, \ldots, 0, 15)$	0.0677	0.1002	0.349	0.5341	0.2500	0.2701
(50,35)	(50,20)	$\mathbf{R}_1 = (0, 0, \ldots, 0, 15)$ $\mathbf{R}_2 = (0, 0, \ldots, 0, 30)$	0.0651	0.1048	0.3007	0.5313	0.2582	0.2729
(80,40)	(80,40)	$\mathbf{R}_1 = (1^{40})$ $\mathbf{R}_2 = (1^{40})$	0.0571	0.0824	0.2890	0.5142	0.2433	0.2619
(80,40)	(80,40)	$\mathbf{R}_1 = (0^{20}, 2^{20})$ $\mathbf{R}_2 = (0^{20}, 2^{20})$	0.0572	0.0841	0.2975	0.5162	0.2417	0.2614
(80,60)	(80,40)	$\mathbf{R}_1 = (1^{20}, 0^{40})$ $\mathbf{R}_2 = (1^{40})$	0.0508	0.0821	0.2929	0.5118	0.2451	0.2614
(80,40)	(80,60)	$\mathbf{R}_1 = (1^{40})$ $\mathbf{R}_2 = (1^{20}, 0^{40})$	0.0555	0.0800	0.2952	0.5144	0.2403	0.2581
(80,60)	(80,60)	$\mathbf{R}_1 = (0^{40}, 1^{20})$ $\mathbf{R}_2 = (0^{40}, 1^{20})$	0.0488	0.0728	0.2901	0.5044	0.2362	0.2511

Table 4. 95% CPs when $\gamma = 2$ and Ω = (0.2, 0.5, 1.3238, 2.4865, 0.8227, 1.1142).

(n_1, m_1)	(n_2, m_2)	Scheme	Method	β_1	β_2	θ_{11}	θ_{12}	θ_{21}	θ_{22}
(25,10)	(25,10)	$\mathbf{R}_1 = (6, 1, \ldots, 1)$	MLE	0.88	0.89	0.87	0.88	0.86	0.90
		$\mathbf{R}_2 = (6, 1, \ldots, 1)$	Boot-p	0.87	0.89	0.88	0.88	0.86	0.89
			Boot-t	0.90	0.89	0.90	0.91	0.89	0.90
(25,20)	(25,20)	$\mathbf{R}_1 = (2, 2, 1, 0, \ldots, 0)$	MLE	0.89	0.90	0.90	0.89	0.91	0.90
		$\mathbf{R}_2 = (2, 2, 1, 0, \ldots, 0)$	Boot-p	0.89	0.88	0.90	0.89	0.90	0.90
			Boot-t	0.91	0.92	0.90	0.91	0.91	0.93
(50,20)	(50,20)	$\mathbf{R}_1 = (2, 1, 2, 1, \ldots, 2, 1)$	MLE	0.90	0.90	0.89	0.90	0.91	0.91
		$\mathbf{R}_2 = (2, 1, 2, 1, \ldots, 2, 1)$	Boot-p	0.91	0.90	0.89	0.90	0.90	0.92
			Boot-t	0.92	0.93	0.96	0.92	0.91	0.94
(50,20)	(50,20)	$\mathbf{R}_1 = (30, 0, 0, \ldots, 0)$	MLE	0.90	0.90	0.89	0.90	0.91	0.91
		$\mathbf{R}_2 = (30, 0, 0, \ldots, 0)$	Boot-p	0.89	0.90	0.91	0.92	0.91	0.90
			Boot-t	0.92	0.91	0.89	0.96	0.93	0.92
(50,20)	(50,35)	$\mathbf{R}_1 = (0, 0, \ldots, 0, 30)$	MLE	0.90	0.90	0.89	0.90	0.91	0.91
		$\mathbf{R}_2 = (0, 0, \ldots, 0, 15)$	Boot-p	0.91	0.90	0.89	0.92	0.91	0.90
			Boot-t	0.93	0.92	0.94	0.92	0.91	0.95
(50,35)	(50,20)	$\mathbf{R}_1 = (0, 0, \ldots, 0, 15)$	MLE	0.88	0.90	0.89	0.89	0.91	0.90
		$\mathbf{R}_2 = (0, 0, \ldots, 0, 30)$	Boot-p	0.90	0.91	0.90	0.91	0.91	0.90
			Boot-t	0.94	0.92	0.93	0.94	0.92	0.91
(80,40)	(80,40)	$\mathbf{R}_1 = (1^{40})$	MLE	0.92	0.93	0.90	0.96	0.92	0.92
		$\mathbf{R}_2 = (1^{40})$	Boot-p	0.91	0.90	0.90	0.96	0.92	0.93
			Boot-t	0.96	0.93	0.92	0.96	0.92	0.93
(80,40)	(80,40)	$\mathbf{R}_1 = (0^{20}, 2^{20})$	MLE	0.93	0.92	0.92	0.90	0.94	0.93
		$\mathbf{R}_2 = (0^{20}, 2^{20})$	Boot-p	0.90	0.92	0.91	0.90	0.92	0.91
			Boot-t	0.92	0.92	0.93	0.92	0.94	0.92
(80,60)	(80,40)	$\mathbf{R}_1 = (1^{20}, 0^{40})$	MLE	0.95	0.90	0.92	0.94	0.91	0.90
		$\mathbf{R}_2 = (1^{40})$	Boot-p	0.91	0.92	0.92	0.91	0.91	0.93
			Boot-t	0.93	0.92	0.94	0.91	0.92	0.94

Table 4. *Cont.*

(n_1, m_1)	(n_2, m_2)	Scheme	Method	β_1	β_2	θ_{11}	θ_{12}	θ_{21}	θ_{22}
(80,40)	(80,60)	$\mathbf{R}_1 = (1^{40})$	MLE	0.93	0.92	0.92	0.94	0.95	0.93
		$\mathbf{R}_2 = (1^{20}, 0^{40})$	Boot-p	0.91	0.92	0.91	0.90	0.92	0.90
			Boot-t	0.94	0.92	0.93	0.92	0.92	0.91
(80,60)	(80,60)	$\mathbf{R}_1 = (0^{40}, 1^{20})$	MLE	0.94	0.97	0.92	0.93	0.92	0.94
		$\mathbf{R}_2 = (0^{40}, 1^{20})$	Boot-p	0.91	0.90	0.92	0.92	0.92	0.92
			Boot-t	0.93	0.92	0.92	0.93	0.95	0.94

3.7. Data Analysis

In this section, we describe the application of the results developed in this study to a set of data generated from the proposed model for illustration purposes. The data were generated under the consideration $\beta_1 = 0.2$, $\beta_2 = 0.3$, $\log\theta_{k1} = -5 + \frac{1500}{\mathbf{S}_k}$ and $\log\theta_{k2} = -3 + \frac{1000}{\mathbf{S}_k}$. The values of the stress levels are considered as given in the simulation section, $\mathbf{S}_1$ = 30 °C = 303 K, $\mathbf{S}_2$ = 60 °C = 333 K and normal stress under $\mathbf{S}_0$ = 5 °C = 278 K. Hence, the value of the parameter vector $\Omega = (\beta_1, \beta_2, \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}) =$ (0.2, 0.3, 0.9517, 1.3503, 0.6093, 1.003). For the censoring scheme, we consider $n_1 = n_2 = 50$, $m_1 = m_2 = 30$ and $\mathbf{R}_1 \equiv \mathbf{R}_2 = \{2, 0, 0, 2, 0, 0, 0, 2, 0, 0, 2, 0, 0, 0, 2, 0, 0, 2, 0, 0, 2, 0, 2, 0, 0, 0, 0, 2, 0, 2\}$. The generated data are reported in Table 5 and the corresponding point and interval estimate are reported in Table 6. Under normal stress conditions with $\mathbf{S}_0$ = 5°C = 278 K; the results of the reliability estimates are reported in Table 7.

Table 5. The set of generated data.

$\mathbf{S}_1$	0.0512	0.3533	0.5253	0.5297	0.7615	0.7737	0.7941	1.2752	1.3143	1.8658
	1	2	1	1	1	1	1	1	1	1
	2.0817	2.6984	2.7525	2.8935	2.9355	3.0838	3.9839	4.3776	4.4603	4.9003
	1	1	1	1	1	1	1	2	2	2
	5.0907	5.1854	5.2346	5.2654	5.4079	5.8034	5.9266	5.9658	6.7407	7.1575
	2	2	2	2	2	2	2	2	2	2
$\mathbf{S}_2$	0.0793	0.1826	0.2967	0.3620	0.5833	0.7460	0.8963	1.0052	1.0378	1.1951
	2	2	2	2	1	1	1	1	1	1
	1.1955	1.3332	1.4629	1.5213	1.5498	1.7281	1.8321	1.9088	2.0058	2.2701
	1	1	1	1	1	1	1	1	1	1
	2.3247	2.3734	2.4229	2.8080	3.5315	3.7945	3.9892	4.1753	4.1884	7.7951
	1	1	1	1	1	1	1	1	1	2

Table 6. The point MLEs and the coresponding 95% Aproximate, boot-p and boot-t CIs.

	Exact	MLE	95% ACI	95% Boot-p	95%Boot-t
β_1	0.2000	0.0792	(0.0094, 0.6644)	(0.0478, 1.4254)	(0.0113, 0.5478)
β_2	0.3000	0.4475	(0.1034, 1.9365)	(0.1220, 2.8412)	(0.0047, 0.8745)
θ_{11}	0.9517	0.6328	(0.0878, 4.5614)	(0.2345, 4.9994)	(0.2473, 2.9982)
θ_{12}	1.3503	2.8562	(0.8957, 9.1080)	(0.7845, 13.1457)	(0.7412, 5.6547)
θ_{21}	0.6093	0.2959	(0.0421, 2.0789)	(0.1240, 4.2145)	(0.2314, 1.9879)
θ_{22}	1.0030	4.8317	(0.9872, 20.3486)	(0.4521, 22.3874)	(0.5462, 10.8754)

Table 7. The relability of the system under normal stress conditions for given time t.

t	$S(t)$	t	$S(t)$
0.5	0.916947	3.0	0.515836
1.0	0.831423	3.5	0.451519
1.5	0.746312	4.0	0.394082
2.0	0.664164	4.5	0.343289
2.5	0.586922	5.0	0.298688

4. Conclusions

The lifetime distribution of highly reliable materials and components was assessed using accelerated life tests (ALTs). A life test under accelerated environmental conditions may be fully accelerated or partially accelerated. Additionally, the products can fail due to one of several possible causes of failure, which need not be independent. Therefore, in this paper, we focused on the problem of the statistical inference of dependent competing risks for generalized half-logistic distributions under constant-stress ALTs with type-II PCS. The dependence structure between lifetimes was measured under a bivariate Pareto copula function. The model parameters were estimated using the ML method, and the corresponding approximate confidence intervals were constructed using the asymptotic properties of the MLEs. Additionally, parametric bootstrap confidence intervals were obtained. The developed methods were explained using a Monte Carlo simulation study and a numerical example. The results show that the dependence structure is very important in competing risk models. The estimated values of the model parameters become closer to the true values when the effective sample size increases. We found that the MLEs of the parameters become closer to the true values when the dependence on competing failure modes becomes stronger. Overall, the Boot-t confidence intervals have good stability with satisfactory coverage percentages and, hence, can be used when the exact confidence intervals cannot be obtained. Based on our knowledge, this study is the first to introduce the dependent competing risks for generalized half-logistic distributions under constant-stress ALTs with type-II PCS.

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