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A Morita-Takeuchi Context and Hopf Coquasigroup Galois Coextensions

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Abstract: For H a Hopf quasigroup and C, a left quasi H-comodule coalgebra, we show that the smash coproduct $C \rtimes H$ (as a symmetry of smash product) is linked to some quotient coalgebra $Q = C/CH^{*+}$ by a Morita-Takeuchi context (as a symmetry of Morita context). We use the Morita-Takeuchi setting to prove that for finite dimensional H, equivalent conditions for C/Q to be a Hopf quasigroup Galois coextension (as a symmetry of Galois extension). In particular, we consider a special case of quasigroup graded coalgebras as an application of our theory.

Keywords: Quasigroup; Hopf (co)quasigroup; Morita-Takeuchi context; Smash coproduct; Galois coextension

MSC: 16W50; 17A60

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1. Introduction

The concept of a Hopf algebra was invented by Borel in 1953 (see [1]), honoring the basic work of H. Hopf in algebraic topology. In the classical Hopf theory, as the symmetry (or dual) of the Morita context in the paper [2] for a left smash product $C \rtimes H$ (as a symmetry of smash coproduct), the authors in [3] introduced the Morita-Takeuchi context $(C \rtimes H, C/CH^{*+}, C, C, f, g)$ for a left H-comodule coalgebra C over a Hopf algebra H, and used it characterize finite Hopf Galois coextensions (as a symmetry of Galois extension). More studies on these aspects are referred to in the papers [4–7]).

At present, progress obtained in understanding the structure of Hopf algebras have been entwined with the development of different notions of mathematics such as weak Hopf algebras [8], quasi-Hopf algebras (symmetrically, coquasi-Hopf algebras [9]), multiplier Hopf algebras [10], Hom-Hopf algebras [11], etc. More generally, a Hopf coquasigroup (symmetrically, a Hopf quasigroup) was introduced by Klim and Majid [12], whose dual notion is a Hopf quasigroup (non-associative coalgebra), a particular case of the notion of unital counit coassociative bialgebra introduced in [13]. More studies of these aspects are referred to in the papers [14–18]. For some basic and recent papers related to non-associative BCC-algebras and B-filters in this field, we refer to [19–21].

The main purpose of this paper is to develop a Morita-Takeuchi context theory in the setting of Hopf quasigroups. This article is organized as follows.

In Section 2, we recall and investigate some basic definitions and properties of Hopf (co)quasigroups, smash coproducts, integrals, and Morita-Takeuchi contexts.

In Section 3, we first study a smash coproduct $C \rtimes H$ for a left quasi H-comodule coalgebra C over a Hopf quasigroup H. Then, we mainly construct a Morita-Takeuchi context linking a smash coproduct $C \rtimes H$ and some quotient coalgebra $Q = C/CH^{*+}$, with the connecting bicomodules being both C (see Theorem 1).

In Section 4, we introduce the notion of a Hopf coquasigroup Galois coextension and study injectivity of the Morita-Takeuchi context maps (see Theorems 2 and 3).

Finally, in Section 5, we introduce the notion of a quasigroup graded coalgbera and obtain a Morita-Takeuchi context associated with a quasigroup graded coalgebra (see Theorem 4 and Corollary 4).

Throughout this paper, k is a fixed field and all vector spaces are over k. By linear maps, we mean k-linear maps. Unadorned $\otimes$ means $\otimes_k$. If (C, Δ) is a coalgebra, we make use of the Sweedler's notation for the coproduct Δ, $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ (see [1]). We denote a category of left C-comodules by ${}^C\mathbf{Mod}$. Similarly for $\mathbf{Mod}^C$.

2. Preliminaries

In this section, we recall some basic notions and properties used in this paper.

2.1. Hopf (Co)quasigroups

Recall from [12] that a *Hopf coquasigroup* is a unital associative algebra H equipped with counital algebra homomorphisms $\Delta : H \longrightarrow H \otimes H$, $\varepsilon_H : H \longrightarrow k$ and linear map $S : H \longrightarrow H$, such that the following conditions hold:

$$\sum S(h_{(1)})h_{(2)(1)} \otimes h_{(2)(2)} = 1 \otimes h = \sum h_{(1)}S(h_{(2)(1)}) \otimes h_{(2)(2)}, \tag{1}$$

$$\sum h_{(1)(1)} \otimes S(h_{(1)(2)})h_{(2)} = h \otimes 1 = \sum h_{(1)(1)} \otimes h_{(1)(2)}S(h_{(2)}) \tag{2}$$

for all $h \in H$. Hence, a Hopf coquasigroup is a Hopf algebra if and only if its coproduct is coassociative.

We can define $H^+ = ker\varepsilon_H$. Then, H^+ is a coideal since ε_H is a coalgebra map (see Theorem 1.4.7 of [1]). Furthermore, even though H is not Hopf algebra, H^+ is a Hopf ideal in the sense of Sweedler (see Theorem 4.3.1 of [1]).

Symmetrically, a *Hopf quasigroup* is a unital counital coassociative bialgebra (H, Δ, ε) armed with a linear map $S : H \longrightarrow H$, such that

$$\sum S(h_1)(h_2g) = \varepsilon(h)g = \sum h_1(S(h_2)g), \ \sum (hg_1)S(g_2) = h\varepsilon(g) = \sum (hS(g_1))g_2 \tag{3}$$

for any $h, g \in H$. Hence, a Hopf quasigroup is a Hopf algebra if and only if its product is associative.

Recall from [22] that a quasigroup is a non-empty set G with a product, identity 1, and with the property that for each $g \in G$, there is $g^{-1} \in G$, such that

$$g^{-1}(gh) = h, \qquad (hg)g^{-1} = h, \qquad \text{for all } h \in G.$$

It is easy to see that in any quasigroup G, one has unique inverses and

$$(g^{-1})^{-1} = g, \qquad (gh)^{-1} = h^{-1}g^{-1}, \qquad \text{for all } g, h \in G.$$

An associative quasigroup is a group.

Let G be a quasigroup. Then, it follows from Proposition 4.7 of [12] that $H = k(G)$ is a Hopf quasigroup with a linear extension of the product, and $\Delta(h) = h \otimes h, \varepsilon(h) = 1$ and $S(h) = h^{-1}$ on the basis of the elements $h \in G$. If G is finite, then $k(G)^*$ is a Hopf coquasigroup (see [12]). Explicitly, a k-basis of $k(G)^*$ is the set of projections $\{p_g \mid g \in G\}$; that is, for any $g \in G$ and $x = \sum_{h \in G} \alpha_h h \in k(G)$, $p_g(x) = \alpha_g \in k$. The set $\{p_g\}$ consists of orthogonal idempotents whose sum is 1. The coproduct on $k(G)^*$ is given by $\Delta(p_g) = \sum_{h \in G} p_{gh^{-1}} \otimes p_h$, and the counit is given by $\varepsilon(p_g) = \delta_{1,g}$ (where δ denotes the Kronecker delta).

If H is the finite dimensional Hopf coquasigroup, its linear dual H^* is not the Hopf coquasigroup, but a Hopf quasigroup, and one has a non-degenerate bilinear form $\langle , \rangle :$ $H^* \times H \longrightarrow k$ given by $\langle h^*, h \rangle = h^*(h)$ for all $h^* \in H^*$ and $h \in H$.

Let $h^* \in H$ and $h \in H$. Then, the left action of h on h^* is denoted by $h \rightharpoonup h^*$ and is given by

$$h \rightharpoonup h^* = \sum \langle h^*_{(2)}, h \rangle h^*_{(1)}.$$

Similarly the right action of h^* on h is denoted by $h \leftharpoonup h^*$ and is given by

$$h \leftharpoonup h^* = \sum \langle h^*, h_{(1)} \rangle h_{(2)}. \tag{4}$$

2.2. Quasi Comodule Coalgebras and Module (Co)algebras

Definition 1. *Let H be a Hopf quasigroup and (C, Δ, ε) a coassociative and counital coalgebra. Then,*

(i) C is called a left quasi H-comodule coalgebra *if C is a left H-comodule such that, for all $a, b \in A$,*

$$\sum c_{(-1)}h \otimes c_{0(1)} \otimes c_{0(2)} = \sum c_{(1)(-1)}(c_{(2)(-1)}h) \otimes c_{(1)0} \otimes c_{(2)0}; \tag{5}$$
$$\sum c_{(-1)}\varepsilon(c_0) = \varepsilon(c)1_H \tag{6}$$

for any $c \in C$ and $h \in H$.

(ii) C is called a left (resp. right) H-module coalgebra *if C is a left (resp. right) H-quasimodule such that, for all $h \in H, c \in C$,*

$$\Delta(h \cdot c) = \sum h_{(1)} \cdot c_{(1)} \otimes h_{(2)} \cdot c_{(2)}, \quad \varepsilon(h \cdot c) = \varepsilon(h)\varepsilon(c) \tag{7}$$

(resp. $\Delta(c \cdot h) = \sum c_{(1)} \cdot h_{(1)} \otimes c_{(2)} \cdot h_{(2)}, \quad \varepsilon(c \cdot h) = \varepsilon(c)\varepsilon(h)$).

Let H be a Hopf quasigroup and let C be a left quasi H-comodule coalgebra. The smash coproduct of C by H (see [23]), denoted by $C \rtimes H$, is defined as the tensor product $C \rtimes H$, with the coproduct given by

$$\Delta(c \rtimes h) = \sum c_{(1)} \rtimes c_{(2)(-1)}h_{(1)} \otimes c_{(2)0} \rtimes h_{(2)}$$

and the counit given by

$$\varepsilon(c \rtimes h) = \varepsilon(c)\varepsilon(h)$$

where $c \in C$ and $h \in H$.

Remark 1. *If we take $h = 1_H$ in Equation (5), we have*

$$\sum c_{(-1)} \otimes c_{0(1)} \otimes c_{0(2)} = \sum c_{(1)(-1)}c_{(2)(-1)} \otimes c_{(1)0} \otimes c_{(2)0} \tag{8}$$

For any $c \in C$. This is in line with the usual definition of the left H-comodule coalgebra when H is Hopf algebra. However, if we replace Equation (5) with Equation (8), $C \rtimes H$ is not always coassociative.

Then, we have

Proposition 1. *With notation as above. Then, C is a left $C \rtimes H$-comodule with the structure*

$$\rho^l(c) = \sum (c_{(1)} \rtimes c_{(2)(-1)}) \otimes c_{(2)0} \tag{9}$$

for any $c \in C$.

Proof. For $c \in C$, it is easy to see that $(\varepsilon \otimes id)\rho^l = id$. We also have

$$\begin{aligned}
& \sum \Delta((c_{(1)} \rtimes c_{(2)(-1)})) \otimes c_{(2)0} \\
= & \sum (c_{(1)(1)} \rtimes c_{(1)(2)(-1)}c_{(2)(-1)(1)}) \otimes (c_{(1)(2)0} \rtimes c_{(2)(-1)(2)}) \otimes c_{(2)0} \\
= & \sum (c_{(1)} \rtimes c_{(2)(-1)}c_{(3)(-1)(1)}) \otimes (c_{(2)0} \rtimes c_{(3)(-1)(2)}) \otimes c_{(3)0} \\
= & \sum (c_{(1)} \rtimes c_{(2)(-1)}c_{(3)(-1)}) \otimes (c_{(2)0} \rtimes c_{(3)0(-1)}) \otimes c_{(3)00} \\
= & \sum (c_{(1)} \rtimes c_{(2)(-1)}) \otimes (c_{(2)0(1)} \rtimes c_{(2)0(2)(-1)}) \otimes c_{(2)0(2)0} \\
= & \sum (c_{(1)} \rtimes c_{(2)(-1)}) \otimes \rho^l(c_{(2)0}).
\end{aligned}$$

This completes the proof. $\square$

2.3. Morita-Takeuchi Contexts

Let M be a right C-comodule and N a left C-comodule of the structure maps ρ_M and ρ_N. Then, the cotensor product $M\square_C N$ is the kernel of the map $\rho_M \otimes id - id \otimes \rho_N$, which is a symmetry of the tensor product.

Recall from [24] that a Morita-Takeuchi context $(C, D, {}^C P^D, {}^D Q^C, f, g)$ consists of coalgebras C and D, bicomodules ${}^C P^D$, ${}^D Q^C$, and bicomodule maps $f : C \longrightarrow P\square_D Q$, $g : D \longrightarrow Q\square_C P$, making the following diagrams commute:

$$\begin{array}{ccc} P & \xrightarrow{\cong} & P\square_D D \\ \downarrow{\cong} & & \downarrow{id\square g} \\ C\square_C P & \xrightarrow[f\square id]{} & P\square_D Q\square_C P \end{array} \qquad \begin{array}{ccc} Q & \xrightarrow{\cong} & Q\square_C C \\ \cong\downarrow & & \downarrow{id\square f} \\ D\square_D Q & \xrightarrow[g\square id]{} & Q\square_C P\square_D Q \end{array}$$

The context is called *strict* if f and g are injective; hence, isomorphisms. In this case, the categories $\mathbf{Mod}^C$ and $\mathbf{Mod}^D$ of comodules over C, resp. D, are equivalent categories.

2.4. Integrals

In [25], the notion of an integral of Hopf (co)quasigroups was investigated. We will use the following slight different definition from the one given in [25].

Definition 2. *Let H be a Hopf coquasigroup. A left integral in H is an element $t \in H$, such that*

$$ht = \varepsilon(h)t, \quad \forall h \in H.$$

We denote the space of left integrals in H by $\int_H^\ell$.

Remark 2. *(1) Let H be a finite dimensional vector space. If H is a Hopf quasigroup, then H^* is a Hopf coquasigroup with natural structure induced by H. Conversely, if H^* is a Hopf coquasigroup, then $H \cong (H^*)^*$ is a Hopf quasigroup.*

(2) Although the definition given by Klim does not seem to be quite the same as Definition 1.4 formally, the both are consistent according to Lemma 3.2 of [25], and our definition here is closer to the classical form for Hopf algebras.

(3) If H is a finite-dimensional Hopf coquasigroup, then $dim(\int_H^\ell) = 1$, i.e., a left integral exists and is unique up to scale.

(4) If H is a Hopf coquasigroup, $T \in \int_{H^*}^r$, then for all $h, g \in H$,

$$T(S(h)g_{(1)})g_{(2)} = h_{(1)}T(S(h_{(2)})g). \tag{10}$$

$$\sum h_{(1)}\langle T, h_{(2)}\rangle = \langle T, h\rangle 1_H. \tag{11}$$

(5) Let T be a left integral of H^* and let t be the distinguished group-like element t of H, which satisfies: $Th^* = \langle h^*, t\rangle T$ for any $h^* \in H^*$. Then, we have

$$\sum \langle T, h_{(1)}\rangle h_{(2)} = \langle T, h\rangle t. \tag{12}$$

$$S^*(T) = t \rightharpoonup T = \sum T_{(1)}\langle T_{(2)}, t\rangle. \tag{13}$$

$$\langle T, S^{-1}(h)t\rangle = \langle T, h\rangle. \tag{14}$$

In particular, we have the Frobenius isomorphism as follows.

Proposition 2. *Let H be a finite-dimensional Hopf coquasigroup. Then, we have the Frobenius isomorphism*

$$F : H \longrightarrow H^*, \quad h \mapsto h \rightharpoonup T = \sum T_{(1)} \langle T_{(2)}, h\rangle.$$

Proof. Straightforward. □

3. The Morita-Takeuchi Context Associated with a Smash Coproduct

In this section, we will construct a Morita-Takeuchi context (a symmetry of Morita context) related to a smash coproduct $C \rtimes H$ (a symmetry of smash product).

Proposition 3. *Let H be a finite dimensional Hopf coquasigroup. Then,*

(1) A coalgebra C is a right H-module coalgebra if and only if C is a left quasi H^*-comodule coalgebra,

(2) If C is a right H-module coalgebra, then $CH^+ = C \triangleleft H^+$, where $H^+ = ker\varepsilon_H$, is a coideal of C, and C/CH^+ is the quotient coalgebra with a trivial right H-module structure.

Proof. Let $\{\xi_i, \xi_i^*\}_{i \in \{1,2,\cdots,dim(H)\}}$ be a dual basis for (H, H^*).

(1) If C is a right H-module coalgebra with the H-action $\cdot$, one defines

$$\rho_C : C \longrightarrow H^* \otimes C, \quad \rho_C(c) = \sum \xi_i^* \otimes c \cdot \xi_i.$$

It is a routine check that C is a left quasi H^*-comodule coalgebra with ρ_C. Conversely, if C is a left H^*-comodule coalgebra, then we define

$$\triangleleft : C \otimes H \longrightarrow C, \quad c \otimes h \mapsto c \triangleleft h := \sum \langle h, c_{(-1)}\rangle c_0.$$

It is also straightforward that C is a right H-module coalgebra with $\triangleleft$.

(2) Obviously, we have $\varepsilon(CH^+) = \varepsilon(C \triangleleft H^+) = \varepsilon(C)\varepsilon(H^+) = 0$. The remainder is straightforward. □

Remark 3. *Let C be a right H-module coalgebra. Let $\pi : C \longrightarrow C/CH^+$ be the natural projection. Then, C may be viewed as a left or right C/CH^+-comodule in a natural way, i.e., $\gamma^l : C \longrightarrow C/CH^+ \otimes C, \quad c \mapsto \sum \pi(c_{(1)}) \otimes c_{(2)}$ or $\gamma^r : C \longrightarrow C \otimes C/CH^+, \quad c \mapsto \sum c_{(1)} \otimes \pi(c_{(2)})$.*

In what follows, H always denotes a finite dimensional Hopf quasigroup and C is a left H-comodule coalgebra. Then, we have the smash coproduct $C \rtimes H$. We also have that H^* is a Hopf coquasigroup and by Proposition 3, C is a right H^*-module coalgebra. Let Q be the quotient coalgebra C/CH^{*+} and $Q = C/CH^{*+}$. Then, C/Q is a right H^*-Galois coextension.

Lemma 1. *With notations as above. Then, C is a $C \rtimes H$-Q-bicomodule.*

Proof. It follows from Proposition 3 that

$$\sum c_{(-1)} \otimes \pi(c_0) = 1 \otimes \pi(c) \tag{15}$$

for any $c \in C$. We now have

$$\begin{aligned}
(\rho^l \otimes id)\gamma^r(c) &= \sum c_{(1)} \rtimes c_{(2)(-1)} \otimes c_{(2)0} \otimes \pi(c_{(3)}) \\
&\overset{(15)}{=} \sum c_{(1)} \rtimes c_{(2)(-1)} c_{(3)(-1)} \otimes c_{(2)0} \otimes \pi(c_{(3)0}) \\
&= \sum c_{(1)} \rtimes c_{(2)(-1)} \otimes c_{(2)0(1)} \otimes \pi(c_{(3)0(2)}) \\
&= (id \otimes id \otimes \gamma^r)\rho^l(c)
\end{aligned}$$

for any $c \in C$. □

Then, we have

Proposition 4. *With notations as above. Then,*

(1) C can be regarded as a right $C \rtimes H$-comodule with the following structure

$$\rho^r(c) = \sum c_{(1)0} \otimes c_{(2)0} \rtimes S^{-1}(c_{(1)(-1)}c_{(2)(-1)})t \tag{16}$$

for any $c \in C$.

(2) C is a Q-$C \rtimes H$-bicomodule.

Proof. (1) For $c \in C$, we have

$$(id \otimes \varepsilon)\rho^r(c) = \sum c_{(1)0}\varepsilon(c_{(2)0})\varepsilon(S^{-1}(c_{(1)(-1)}c_{(2)(-1)})t) = c\varepsilon(t) = c$$

and

$$\begin{aligned}
& (id \otimes \Delta)\rho^r(c) \\
= \ & \sum c_{(1)0} \otimes \Delta(c_{(2)0} \rtimes S^{-1}(c_{(1)(-1)}c_{(2)(-1)})t) \\
= \ & \sum c_{(1)0} \otimes (c_{(2)0(1)} \rtimes c_{(2)0(2)(-1)}S^{-1}(c_{(1)(-1)}c_{(2)(-1)})_{(1)}t_{(1)}) \\
& \otimes(c_{(2)0(2)0} \rtimes S^{-1}(c_{(1)(-1)}c_{(2)(-1)})_{(2)}t_{(2)}) \\
= \ & \sum c_{(1)0} \otimes (c_{(2)0(1)} \rtimes c_{(2)0(2)(-1)}S^{-1}(c_{(1)(-1)(2)}c_{(2)(-1)(2)})t) \\
& \otimes(c_{(2)0(2)0} \rtimes S^{-1}(c_{(1)(-1)(1)}c_{(2)(-1)(1)})t) \\
= \ & \sum c_{(1)00} \otimes (c_{(2)00(1)} \rtimes c_{(2)00(2)(-1)}S^{-1}(c_{(1)0(-1)}c_{(2)0(-1)}t)) \\
& \otimes(c_{(2)00(2)0} \rtimes S^{-1}(c_{(1)(-1)}c_{(2)(-1)})t) \\
= \ & \sum c_{(1)00} \otimes (c_{(2)0(1)0} \rtimes c_{(2)0(2)0(-1)}S^{-1}(c_{(1)0(-1)}c_{(2)0(1)(-1)}c_{(2)0(2)(-1)})t) \\
& \otimes(c_{(2)0(2)00} \rtimes S^{-1}(c_{(1)(-1)}c_{(2)(-1)})t) \\
= \ & \sum c_{(1)00} \otimes (c_{(2)00} \rtimes c_{(3)00(-1)}S^{-1}(c_{(1)0(-1)}c_{(2)0(-1)}c_{(3)0(-1)})t) \\
& \otimes(c_{(3)000} \rtimes S^{-1}(c_{(1)(-1)}c_{(2)(-1)}c_{(3)(-1)})t) \\
= \ & \sum c_{(1)00} \otimes (c_{(2)00} \rtimes \underline{c_{(3)00(-1)}S^{-1}(c_{(3)0(-1)})}S^{-1}(c_{(1)0(-1)}c_{(2)0(-1)})t) \\
& \otimes(c_{(3)000} \rtimes S^{-1}(c_{(1)(-1)}c_{(2)(-1)}c_{(3)(-1)})t) \\
= \ & \sum c_{(1)00} \otimes c_{(2)00} \rtimes S^{-1}(c_{(1)0(-1)}c_{(2)0(-1)})t \\
& \otimes c_{(3)0} \rtimes S^{-1}(c_{(1)(-1)}c_{(2)(-1)}c_{(3)(-1)})t \\
= \ & \sum c_{(1)0(1)0} \otimes c_{(1)0(2)0} \rtimes S^{-1}(c_{(1)0(1)(-1)}c_{(1)0(2)(-1)})t \\
& \otimes c_{(2)0} \rtimes S^{-1}(c_{(1)(-1)}c_{(2)(-1)})t \\
= \ & \sum \rho^r(c_{(1)0}) \otimes c_{(2)0} \rtimes S^{-1}(c_{(1)(-1)}c_{(2)(-1)})t \\
= \ & (\rho^r \otimes id)\rho^r(c).
\end{aligned}$$

(2) Similar to the proof of Lemma 1. □

Let H be a finite dimensional Hopf quasigroup and C a left H-comodule coalgebra. We define a map as follows:

$$f : C \rtimes H \longrightarrow C\square_Q C, \quad c \rtimes h \mapsto \sum c_{(1)}\square_Q c_{(2)0}\langle T, c_{(2)(-1)}h\rangle \tag{17}$$

for any $c \in C$ and $h \in H$.

Lemma 2. *With notations as above. Then, f is a left and right $C \rtimes H$-comodule map.*

Proof. We firstly check that f is well-defined. In fact, for any $c \in C$ and $h \in H$,

$$\begin{aligned}
(\gamma^r \otimes id)f(c \rtimes h) &= \sum \gamma^r(c_{(1)}) \otimes c_{(2)0}\langle T, c_{(2)(-1)}h\rangle \\
&= \sum c_{(1)} \otimes \pi(c_{(2)}) \otimes c_{(3)0}\langle T, c_{(3)(-1)}h\rangle \\
&\overset{(15)}{=} \sum c_{(1)} \otimes \pi(c_{(2)0}) \otimes c_{(3)0}\langle T, c_{(2)(-1)}c_{(3)(-1)}h\rangle \\
&= \sum c_{(1)} \otimes \pi(c_{(2)0(1)}) \otimes c_{(2)0(2)}\langle T, c_{(2)(-1)}h\rangle \\
&= \sum c_{(1)} \otimes \gamma^l(c_{(2)0})\langle T, c_{(2)(-1)}h\rangle \\
&= (id \otimes \gamma^l)f(c \rtimes h)
\end{aligned}$$

Then, we show that f is left $C \rtimes H$-collinear. Actually, for any $c \in C$ and $h \in H$,

$$\begin{aligned}
&(\rho^l_C \otimes id)f(c \rtimes h) \\
= {} & \sum \rho^l(c_{(1)})\Box_Q c_{(2)0}\langle T, c_{(2)(-1)}h\rangle \\
= {} & \sum (c_{(1)(1)} \rtimes c_{(1)(2)(-1)}) \otimes c_{(1)(2)0}\Box_Q c_{(2)0}\langle T, c_{(2)(-1)}h\rangle \\
= {} & \sum (c_{(1)} \rtimes c_{(2)(-1)}) \otimes c_{(2)0}\Box_Q c_{(3)0}\langle T, c_{(3)(-1)}h\rangle \\
= {} & \sum c_{(1)} \rtimes c_{(2)(-1)}(c_{(3)(-1)}h)_{(1)} \otimes c_{(2)0}\Box_Q c_{(3)00}\langle T, (c_{(3)(-1)}h)_{(2)}\rangle \quad \text{by Remark 2(4)} \\
= {} & \sum c_{(1)} \rtimes c_{(2)(-1)}(c_{(3)(-1)}h_{(1)}) \otimes c_{(2)0}\Box_Q c_{(3)00}\langle T, c_{(3)0(-1)}h_{(2)}\rangle \\
\overset{(4)}{=} {} & \sum c_{(1)} \rtimes c_{(2)(-1)}h_{(1)} \otimes c_{(2)0(1)}\Box_Q c_{(2)0(2)0}\langle T, c_{(2)0(2)(-1)}h_{(2)}\rangle \\
= {} & \sum c_{(1)} \rtimes c_{(2)(-1)}h_{(1)} \otimes f(c_{(2)0} \rtimes h_{(2)}) \\
= {} & (id \otimes f)\Delta(c \rtimes h)
\end{aligned}$$

Finally, we prove that f is right $C \rtimes H$-collinear. For any $c \in C$ and $h \in H$, we have

$$\begin{aligned}
&(id \otimes \rho^r_C)f(c \rtimes h) \\
= {} & \sum c_{(1)}\Box_Q \rho^r c_{(2)0})\langle T, c_{(2)(-1)}h\rangle \\
= {} & \sum c_{(1)}\Box_Q c_{(2)0(1)0} \otimes c_{(2)0(2)0} \rtimes S^{-1}(c_{(2)0(1)(-1)}c_{(2)0(2)(-1)})t\langle T, c_{(2)(-1)}h\rangle \\
= {} & \sum c_{(1)}\Box_Q c_{(2)00} \otimes c_{(3)00} \rtimes h_{(3)}S^{-1}((c_{(2)0(-1)}c_{(3)0(-1)})h_{(2)})\langle T, (c_{(2)(-1)}c_{(3)(-1)})h_{(1)}\rangle t \\
= {} & \sum c_{(1)}\Box_Q c_{(2)0} \otimes c_{(3)0} \rtimes h_{(3)}S^{-1}(((c_{(2)(-1)}c_{(3)(-1)})h)_{(2)})\langle T, ((c_{(2)(-1)}c_{(3)(-1)})h)_{(1)}\rangle t \\
= {} & \sum c_{(1)}\Box_Q c_{(2)0} \otimes c_{(3)0} \rtimes h_{(2)}S^{-1}(t)t\langle T, (c_{(2)(-1)}c_{(3)(-1)})h_{(1)}\rangle \quad \text{by Remark 2(5)} \\
= {} & \sum c_{(1)}\Box_Q c_{(2)0} \otimes c_{(3)0} \rtimes h_{(2)}\langle T, c_{(2)(-1)}(c_{(3)(-1)}h_{(1)})\rangle \\
= {} & \sum c_{(1)(1)}\Box_Q c_{(1)(2)0}\langle T, c_{(1)(2)(-1)}(c_{(2)(-1)}h_{(1)})\rangle \otimes c_{(2)0} \rtimes h_{(2)} \\
= {} & \sum f(c_{(1)} \rtimes c_{(2)(-1)}h_{(1)}) \otimes c_{(2)0} \rtimes h_{(2)} \\
= {} & (f \otimes id)\Delta(c \rtimes h)
\end{aligned}$$

This completes the proof. □

We now are to define another map g, as follows:

$$g : Q \longrightarrow C\Box_{C\rtimes H}C, \quad \bar{c} \mapsto \sum c_{(1)0}\Box_{C\rtimes H}c_{(2)0}\langle T, c_{(1)(-1)}c_{(2)(-1)}\rangle \tag{18}$$

for any $c \in C$ and $h \in H$.

Lemma 3. *With notations as above. Then, g is Q-bicollinear.*

Proof. We check that g is well-defined. For any $c \in C$,

$$
\begin{aligned}
& (\rho_C^r \otimes id)g(\bar{c}) \\
= \;& \sum \rho_C^r(c_{(1)0}) \otimes c_{(2)0}\langle T, c_{(1)(-1)}c_{(2)(-1)}\rangle \\
= \;& \sum c_{(1)0(1)0} \otimes c_{(1)0(2)0} \rtimes S^{-1}(c_{(1)0(1)(-1)}c_{(1)0(2)(-1)})t \otimes c_{(2)0}\langle T, c_{(1)(-1)}c_{(2)(-1)}\rangle \\
= \;& \sum c_{(1)00} \otimes c_{(2)00} \rtimes S^{-1}(c_{(1)0(-1)}c_{(2)0(-1)})t \otimes c_{(3)0}\langle T, (c_{(1)(-1)}c_{(2)(-1)})c_{(3)(-1)}\rangle \\
= \;& \sum c_{(1)0} \otimes c_{(2)0} \rtimes c_{(3)(-1)(3)}S^{-1}((c_{(1)(-1)(2)}c_{(2)(-1)(2)})c_{(3)(-1)(2)})t \\
& \otimes c_{(3)0}\langle T, (c_{(1)(-1)(1)}c_{(2)(-1)(1)})c_{(3)(-1)(1)}\rangle \\
= \;& \sum c_{(1)0} \otimes c_{(2)0} \rtimes c_{(3)(-1)(3)}S^{-1}((c_{(1)(-1)}c_{(2)(-1)})_{(2)}c_{(3)(-1)(2)})t \\
& \otimes c_{(3)0}\langle T, (c_{(1)(-1)}c_{(2)(-1)})_{(1)}c_{(3)(-1)(1)}\rangle \\
= \;& \sum c_{(1)0} \otimes (c_{(2)0} \rtimes c_{(3)0(-1)}) \otimes c_{(3)00}\langle T, (c_{(1)(-1)}c_{(2)(-1)})c_{(3)(-1)}\rangle \\
\overset{(4)}{=} \;& \sum (c_{(1)0} \otimes c_{(2)0}) \rtimes c_{(3)0(-1)} \otimes c_{(3)00}\langle T, c_{(1)(-1)}(c_{(2)(-1)}c_{(3)(-1)})\rangle \\
= \;& \sum c_{(1)0} \otimes (c_{(2)0(1)} \rtimes c_{(2)0(2)(-1)}) \otimes c_{(2)0(2)0}\langle T, c_{(1)(-1)}c_{(2)(-1)}\rangle \\
= \;& \sum c_{(1)0} \otimes \rho_C^l(c_{(2)0})\langle T, c_{(1)(-1)}c_{(2)(-1)}\rangle \\
= \;& (id \otimes \rho_C^l)g(\bar{c}).
\end{aligned}
$$

If $\bar{c} = \bar{d}$, then $c - d \in CH^{*+}$ with $c, d \in C$. Observe that for any $h^* \in H^*, c \in C$, one has $g(c \cdot h^*) = \langle h^*, 1\rangle g(c)$ and $CH^{*+} \subseteq Ker(g)$. Therefore, $g(c) = g(d)$ and so g is well-defined.

Next, we check that g is left Q-collinear. For ant $\bar{c} \in Q, c \in C$, we have

$$
\begin{aligned}
& (\gamma^l \otimes id)g(\bar{c}) \\
= \;& \sum \gamma^l(c_{(1)0})\square_{C\rtimes H}c_{(2)0}\langle T, c_{(1)(-1)}c_{(2)(-1)}\rangle \\
= \;& \sum \overline{c_{(1)0(1)}} \otimes c_{(1)0(2)}\square_{C\rtimes H}c_{(2)0}\langle T, c_{(1)(-1)}c_{(2)(-1)}\rangle \\
= \;& \sum \overline{c_{(1)0}} \otimes c_{(2)0}\square_{C\rtimes H}c_{(3)0}\langle T, (c_{(1)(-1)}c_{(2)(-1)})c_{(3)(-1)}\rangle \\
\overset{(4)}{=} \;& \sum \overline{c_{(1)0}} \otimes c_{(2)0}\square_{C\rtimes H}c_{(3)0}\langle T, c_{(1)(-1)}(c_{(2)(-1)}c_{(3)(-1)})\rangle \\
\overset{(15)}{=} \;& \sum \overline{c_{(1)}} \otimes c_{(2)0}\square_{C\rtimes H}c_{(3)0}\langle T, c_{(2)(-1)}c_{(3)(-1)}\rangle \\
= \;& \sum \overline{c_{(1)}} \otimes c_{(2)(1)0}\square_{C\rtimes H}c_{(2)(2)0}\langle T, c_{(2)(1)(-1)}c_{(2)(2)(-1)}\rangle \\
= \;& \sum \overline{c_{(1)}} \otimes g(\overline{c_{(2)}}) \\
= \;& (id \otimes g)\Delta(\bar{c}).
\end{aligned}
$$

Similarly, one checks that g is right Q-collinear. In fact, for ant $\bar{c} \in Q, c \in C$,

$$
\begin{aligned}
(id \otimes \gamma^r)g(\bar{c}) &= \sum c_{(1)0}\square_{C\rtimes H}\gamma^r(c_{(2)0})\langle T, c_{(1)(-1)}c_{(2)(-1)}\rangle \\
&= \sum c_{(1)0}\square_{C\rtimes H}c_{(2)0(1)} \otimes \overline{c_{(2)0(2)}}\langle T, c_{(1)(-1)}c_{(2)(-1)}\rangle \\
&= \sum c_{(1)0}\square_{C\rtimes H}c_{(2)0} \otimes \overline{c_{(3)0}}\langle T, c_{(1)(-1)}(c_{(2)(-1)}c_{(3)(-1)})\rangle \\
&= \sum c_{(1)0}\square_{C\rtimes H}c_{(2)0} \otimes \overline{c_{(3)0}}\langle T, c_{(1)(-1)}(c_{(2)(-1)}c_{(3)(-1)})\rangle \\
&= \sum c_{(1)0}\square_{C\rtimes H}c_{(2)0}\langle T, c_{(1)(-1)}c_{(2)(-1)}\rangle \otimes \overline{c_{(3)}} \\
&= \sum c_{(1)(1)0}\square_{C\rtimes H}c_{(1)(2)0}\langle T, c_{(1)(1)(-1)}c_{(1)(2)(-1)}\rangle \otimes \overline{c_{(2)}} \\
&= \sum g(\overline{c_{(1)}}) \otimes \overline{c_{(2)}} \\
&= (g \otimes id)\Delta(\bar{c}).
\end{aligned}
$$

This finishes the proof. □

We have the main result of this paper as follows.

Theorem 1. *Let H be a finite dimensional Hopf quasigroup and C a left H-comodule coalgebra. With notations as above, we have a Morita-Takeuchi context $(C \rtimes H, Q, C, C, f, g)$ with the maps f and g as defined by Equations (17) and (18).*

Proof. According to Lemmas 2 and 3, it remains to prove that the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\gamma^r} & C\Box_Q Q \\ \downarrow \rho^l & & \downarrow id\Box_Q g \\ C\rtimes H\Box_{C\rtimes H}C & \xrightarrow[f\Box_{C\rtimes H}id]{} & C\Box_Q C\Box_{C\rtimes H}C \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\rho^r} & C\Box_{C\rtimes H}C\rtimes H \\ \gamma^l \downarrow & & \downarrow id\Box_{C\rtimes H}f \\ Q\Box_Q C & \xrightarrow[g\Box_Q id]{} & C\Box_{C\rtimes H}C\Box_Q C \end{array}$$

For the commutativity of the first diagram, we have, for $c \in C$

$$\begin{aligned} ((id\Box_Q g) \circ \gamma^r)(c) &= \sum c_{(1)}\Box_Q g(\overline{c_{(2)}}) \\ &= \sum c_{(1)}\Box_Q c_{(2)(1)0}\Box_{C\rtimes H}c_{(2)(2)0}\langle T, c_{(2)(1)(-1)}c_{(2)(2)(-1)}\rangle \\ &= \sum c_{(1)}\Box_Q c_{(2)0}\Box_{C\rtimes H}c_{(3)0}\langle T, c_{(2)(-1)}c_{(3)(-1)}\rangle \\ &= \sum c_{(1)(1)}\Box_Q c_{(1)(2)0}\langle T, c_{(1)(2)(-1)}c_{(2)(-1)}\rangle\Box_{C\rtimes H}c_{(2)0} \\ &= \sum f(c_{(1)} \rtimes c_{(2)(-1)})\Box_{C\rtimes H}c_{(2)0} \\ &= ((f\Box_{C\rtimes H}id)\circ\rho^l)(c). \end{aligned}$$

For the second one, we have

$$\begin{aligned} ((id\Box_{C\rtimes H} f)\circ\rho^r)(c) &= \sum c_{(1)0}\Box_{C\rtimes H}f(c_{(2)0} \rtimes S^{-1}(c_{(1)(-1)}c_{(2)(-1)})t) \\ &= \sum c_{(1)0}\Box_{C\rtimes H}c_{(2)0(1)}\Box_Q c_{(2)0(2)0}\langle T, c_{(2)0(2)(-1)}S^{-1}(c_{(1)(-1)}c_{(2)(-1)})t\rangle \\ &= \sum c_{(1)0}\Box_{C\rtimes H}c_{(2)0}\Box_Q c_{(3)00}\langle T, c_{(3)0(-1)}S^{-1}(c_{(1)(-1)}(c_{(2)(-1)}c_{(3)(-1)}))t\rangle \\ &= \sum c_{(1)0}\Box_{C\rtimes H}c_{(2)0}\Box_Q c_{(3)0}\langle T, c_{(3)(-1)(2)}S^{-1}((c_{(1)(-1)}c_{(2)(-1)})c_{(3)(-1)(1)})t\rangle \\ &= \sum c_{(1)0}\Box_{C\rtimes H}c_{(2)0}\Box_Q c_{(3)}\langle T, S^{-1}(c_{(1)(-1)}c_{(2)(-1)})t\rangle \\ &= \sum c_{(1)0}\Box_{C\rtimes H}c_{(2)0}\Box_{C\rtimes H}c_{(3)}\langle T, c_{(1)(-1)}c_{(2)(-1)}\rangle \quad \text{by Remark 2 (5)} \\ &= \sum c_{(1)(1)0}\Box_{C\rtimes H}c_{(1)(2)0}\langle T, c_{(1)(1)(-1)}c_{(1)(2)(-1)}\rangle\Box_{C\rtimes H}c_{(2)} \\ &= \sum g(\overline{c_{(1)}})\Box_{C\rtimes H}c_{(2)} \\ &= ((g\Box_{C\rtimes H}id)\circ\gamma^l)(c). \end{aligned}$$

This completes the proof. □

4. Finite Hopf Coquasigroup Galois Coextensions

In this section, we study the injectivity of the Morita-Takeuchi maps f and g (the symmetry of Morita maps) in Theorem 1.

For our definition of the Hopf coquasigroup Galois coextensions, it follows from Proposition 3 (2) that we first have

$$\begin{aligned} \sum c_{(1)} \otimes \pi(c_{(2)}) \otimes c_{(3)} \triangleleft h &= \sum c_{(1)} \otimes \pi(c_{(2)})\varepsilon(h_{(1)}) \otimes c_{(3)} \triangleleft h_{(2)} \\ &= \sum c_{(1)} \otimes \pi(c_{(2)}) \cdot h_{(1)} \otimes c_{(3)} \triangleleft h_{(2)} \\ &= \sum c_{(1)} \otimes \pi(c_{(2)} \triangleleft h_{(1)}) \otimes c_{(3)} \triangleleft h_{(2)} \\ &= \sum c_{(1)} \otimes \pi((c_{(2)} \triangleleft h)_{(1)}) \otimes (c_{(2)} \triangleleft h)_{(2)} \end{aligned}$$

for any $c \in C$ and $h \in H$. Then, we have the following definition which generalizes the one in [3] (see [7]) to the case of a Hopf coquasigroup.

As a symmetry of the Galois extension of the Hopf quasigroup appeared in [26], we have:

Definition 3. *Let H be a Hopf coquasigroup and C a right H-module coalgebra. Then, C/Q, where $Q = C/CH^+$ is said to be of right H-Galois coextension if the map*

$$\gamma : C \otimes H \longrightarrow C\Box_Q C, \quad c \otimes h \mapsto \sum c_{(1)}\Box_Q c_{(2)} \triangleleft h$$

is injective.

Let H be a finite dimensional Hopf quasigroup and C a left quasi H-comodule coalgebra. Then, H^* is a Hopf coquasigroup and C is a right H^*-module coalgebra. Then, C^* is a right H-comodule algebra having coinvariant subalgebra $C^{*coH} \cong (C^*)^{H^*} \overset{v}{\cong} Q^*$. Let $Q = C/CH^{*+}$. Then, C/Q is a right H^*-Galois coextension.

Theorem 2. *With notations above. The following are equivalent:*
(i) C/Q is a right H^*-Galois coextension.
(ii) The canonical map

$$\gamma : C \otimes H^* \longrightarrow C\Box_Q C, \quad c \otimes h^* \mapsto \sum c_{(1)}\Box_Q c_{(2)} \triangleleft h^*$$

is injective.
(iii) The canonical map f in Theorem 1 is injective.

Proof. For any $c \in C$ and $h \in H$, we have

$$\begin{aligned}
& (\gamma \circ (id \otimes F))(c \rtimes h) \\
= \ & \gamma(c \otimes h \rightharpoonup T) = \sum \gamma(c \otimes T_{(1)})\langle T_{(2)}, h\rangle \\
= \ & \sum c_{(1)}\Box_Q c_{(2)} \triangleleft T_{(1)}\langle T_{(2)}, h\rangle \\
= \ & \sum c_{(1)}\Box_Q c_{(2)0}\langle T_{(1)}, c_{(2)(-1)}\rangle\langle T_{(2)}, h\rangle \\
= \ & \sum c_{(1)}\Box_Q c_{(2)0}\langle T, c_{(2)(-1)}h\rangle \\
= \ & f(c \rtimes h).
\end{aligned}$$

Since $id \otimes F$ is an isomorphism, γ is injective if and only if f is injective. □

Corollary 1. *Let C/Q be a right H^*-Galois coextension. Then,*
(i) The left Q-comodule C and the right Q-comodule C are quasi-finitely injective comodules.
(ii) The left $C \rtimes H$-comodule C and the right $C \rtimes H$-comodule C are cogenerators.
(iii) The map g in Theorem 1 induces bicomodule isomorphisms:

$$hom_{-Q}(C, Q) \cong C \quad \text{and} \quad hom_{Q-}(C, Q) \cong C$$

(iv) We have $End_{-Q}(C) \cong C \rtimes H$ and $End_{Q-}(C) \cong C \rtimes H$.

Proof. It follows from Theorem 2.5 of [24]. □

Proposition 5. *With notations above. If C^*/Q^* is a right H-Galois extension, then C/Q is a right H^*-Galois coextension.*

Proof. By Definition 3 of [26], we let β be the canonical map

$$\beta : C^* \otimes_{Q^*} C^* \longrightarrow C^* \otimes H, \quad x \otimes y \mapsto \sum xy_0 \otimes y_{(1)}.$$

For any $c \in C, h^* \in H^*$ and $x, y \in C^*$, we have

$$\begin{aligned} \gamma^*(x \otimes y)(c \otimes h^*) &= \langle x \otimes y, \gamma(c \otimes h^*)\rangle \\ &= \sum \langle x \otimes y, c_{(1)} \Box_Q c_{(2)} \triangleleft h^*\rangle \\ &= \sum \langle x, c_{(1)}\rangle \langle y, c_{(2)} \triangleleft h^*\rangle \\ &= \sum \langle x, c_{(1)}\rangle \langle h^* \triangleright y, c_{(2)}\rangle \\ &= \sum \langle x, c_{(1)}\rangle \langle y_0, c_{(2)}\rangle \langle h^*, y_{(1)}\rangle \\ &= \sum \langle x y_0, c\rangle \langle h^*, y_{(1)}\rangle = \beta(x \otimes y)(c \otimes h^*) \end{aligned}$$

and so $\gamma^* = \beta$. Obviously, if β is surjective, γ^* is surjective. By assumption, β is an isomorphism. Thus, γ^* is surjective and γ is injective. It follows from Theorem 1 that C/Q is a right H^*-Galois coextension. $\Box$

We do not know if the converse of Proposition 5 is true. We have a partial answer, as follows.

Proposition 6. *With notations above. Let C/Q be a right H^*-Galois coextension. If C as a left or right Q-comodule is finitely cogenerated, then C^*/Q^* is a right H-Galois extension.*

Proof. By Theorem 2, we have an injective $C \rtimes H$-bicomodule map $f : C \rtimes H \longrightarrow C\Box_Q C$. If C as a left Q-comodule is finitely cogenerated, then there is a finite dimensional space V with $dim(V) = n$, such that $C \hookrightarrow V \otimes Q$ as Q-comodules. This produces an injective right $C \rtimes H$-comodule composite map:

$$C \rtimes H \xrightarrow{f} C\Box_Q C \hookrightarrow V \otimes C \cong C^n.$$

Dualizing (or symmetrizing) the above comodule map, we obtain a $C^*\#H^* \overset{\xi}{\cong} (C \rtimes H)^*$-linear map $(C^*)^n \longrightarrow C^*\#H^*$, that is, C^* is a left $C^*\#H^*$-generator. It follows from Theorem 5 of [26] that C^*/Q^* is a right H-Galois extension. $\Box$

In the end of this section, we discuss the connnection between our Morita-Takeuchi context and the Morita context from Theorem 1 of [26].

Theorem 3. *Let $(C \rtimes H, Q, C, C, f, g)$ be the Morita-Takeuchi context in Theorem 1, and let $((C^*)^{H^*}, C^*\#H^*, {}_{A^*\#H^*}C^*_{(C^*)^{H^*}}, {}_{(C^*)^{H^*}}C^*_{C^*\#H^*}, \tau = (,), \mu = [,])$ be the Morita context from Theorem 1 of [26]. Then*

(i) If μ is surjective, then f is injective.

(ii) If τ is surjective, then g is injective.

Proof. Let D be coalgebra. If $P \in \mathbf{Mod}^D$ and $W \in {}^D\mathbf{Mod}$, we will denote by $\Xi : P^* \otimes_{D^*} W^* \longrightarrow (P\Box_D W)^*$ the canonical map given by $\Xi(p^* \otimes w^*)(p \otimes w) = \langle p^*, p\rangle \langle w^*, w\rangle$.

Then, we have, for any $c, d \in C, h \in H$ and $c^*, d^* \in C^*$

$$\begin{array}{rcl}[(f^* \circ \Xi)(c^* & \otimes_{C^*\#H^*} & d*)](c \rtimes h) = [\Xi(c^* \otimes_{C^*\#H^*} d^*)]f(c \otimes h) \\ & = & \sum [\Xi(c^* \otimes_{C^*\#H^*} d^*)]c_{(1)} \Box_Q c_{(2)0} \langle T, c_{(2)(-1)} h \rangle \\ & = & \sum \langle c^*, c_{(1)} \rangle \langle d^*, c_{(2)0} \rangle \langle T, c_{(2)(-1)} h \rangle \\ & = & \sum \langle c^*, c_{(1)} \rangle \langle T_{(1)}, c_{(2)(-1)} \rangle \langle d^*, c_{(2)0} \rangle \langle T_{(2)}, h \rangle \\ & = & \sum \langle c^*, c_{(1)} \rangle \langle T_{(1)} \cdot d^*, c_{(2)} \rangle \langle T_{(2)}, h \rangle \\ & = & \sum \langle c^*(T_{(1)} \cdot d^*), c \rangle \langle T_{(2)} \varepsilon, h \rangle \\ & = & \sum \langle c^*(T_{(1)} \cdot d^*) \# T_{(2)} \varepsilon, c \rtimes h \rangle \\ & = & \langle c^* \# T)(d^* \# \varepsilon, c \rtimes h \rangle \\ & = & [(\xi \circ \mu)(c^* \otimes_{C^*\#H^*} d^*)](c \rtimes h)\end{array}$$

and so $f^* \circ \Xi = \xi \circ \mu$. It is obvious that if μ is surjective, then f is injective.

Similarly, we have, $c, d \in C, \bar{c} \in Q$ and $c^*, d^* \in C^*$

$$\begin{array}{rcl}[(g^* \circ \Xi)(c^* & \otimes_{C^*\#H^*} & d*)](\bar{c}) = [\Xi(c^* \otimes_{C^*\#H^*} d*)]g(\bar{c}) \\ & = & \sum [\Xi(c^* \otimes_{C^*\#H^*} d*)]c_{(1)0} \Box_{C \rtimes H} c_{(2)0} \langle T, c_{(1)(-1)} c_{(2)(-1)} \rangle \\ & = & \sum \langle c^*, c_{(1)0} \rangle \langle d^*, c_{(2)0} \rangle \langle T, c_{(1)(-1)} c_{(2)(-1)} \rangle \\ & = & \langle c^*, c_{0(1)} \rangle \langle d^*, c_{0(2)} \rangle \langle T, c_{(-1)} \rangle \\ & = & \langle c^* d^*, c_0 \rangle \langle T, c_{(-1)} \rangle \\ & = & \langle [T \cdot (c^* d^*)], \bar{c} \rangle \\ & = & [(\nu \circ \tau)(c^* \otimes_{C^*\#H^*} d^*)](\bar{c})\end{array}$$

and thus, $g^* \circ \Xi = \nu \circ \tau$. Then, the result follows.

This finishes the proof. □

Corollary 2. *Let the map g in the Morita-Takeuchi context in Theorem 1 be injective. Then,*

*(i) The category ${}^{C \rtimes H}$**Mod** is equivalent to a quotient category of ^{Q}**Mod**.*

(ii) The map f in the Morita-Takeuchi context in Theorem 1 is injective (i.e., the context is strict) if and only if C is a faithfully coflat left Q-comodule.

Proof. (i) It follows from Theorem 1 and Proposition 2.2 of [4].

(ii) It follows from Theorem 1 and Corollary 2.3 of [4]. □

5. A Special Case: Quasigroup Graded Coalgebras

In this section, we will treat a special case of quasigroup graded coalgebra. The main purpose is to illustrate the results in Section 4 about our Morita-Takeuchi context theory (a symmetric theory of Morita context).

Through a coalgebra C, we always understand a coassociative coalgebra with counit ε. We denote the group of all coalgebra automorphisms of C by $Aut(C)$. We fix a multiplicative quasigroup G with identity $1 = 1_G$.

Definition 4. *(1) A coalgebra (C, Δ, ε) is graded by G if C is a direct sum of subspaces,*

$$C = \bigoplus_{\sigma \in G} C_\sigma, \tag{19}$$

such that:

$$\Delta(C_\sigma) \subseteq \sum_{xy=\sigma} C_x \otimes C_y, \qquad \text{for all} \quad \sigma \in G, \tag{20}$$

and $\varepsilon(C_\sigma) = 0$ for $\sigma \neq 1$. Furthermore, C is said to be strongly graded if the canonical map $\lambda_{x,y} : C_{xy} \longrightarrow C_x \otimes C_y, c \mapsto \sum \Pi_x(c_{(1)}) \otimes \Pi_y(c_{(2)})$ is injective for all $x, y \in G$.

(2) A left C-comodule M with structure map $\rho^l : M \longrightarrow C \otimes M$ is a graded C-comodule if $M = \bigoplus_{\sigma\in G} M_\sigma$, as k-subspaces, such that $\rho^l(M_\sigma) \subseteq \sum_{xy=\sigma} C_x \otimes M_y$ for all $\sigma \in G$. For G-graded left C-comodules M and N, a G-graded comodule morphism is a C-comodule morphism $\phi : M \longrightarrow N$, such that $\phi(M_\sigma) \subseteq N_\sigma$ for $\sigma \in G$.

Remark 4. *(1) C_1 is a coalgebra with coproduct $\Delta_1 : C_1 \longrightarrow C_1 \otimes C_1$ given by $\Delta_1(c) = \sum \Pi(c_{(1)}) \otimes \Pi(c_{(2)}) = \sum \Pi(c_{(1)}) \otimes c_{(2)} = \sum c_{(1)} \otimes \Pi(c_{(2)})$ for $c \in C_1$, where $\Pi : C \longrightarrow C_1$ is the natural projection. The counit of C_1 is just ε_C restricted to C_1.*

(2) Every coalgebra C is a trivially graded coalgebra by letting $C_1 = C$ and $C_\sigma = 0$ for all $\sigma \neq 1_G$.

(3) For some G-graded coalgbera C, we denote a category of left G-graded C-comodules by ${}^C\mathbf{Gr}_G$, which is the Grothendieck category.

Proposition 7. *A coalgebra C graded by a quasigroup G may be viewed as a $k(G)$-comodule coalgebra; conversely, every $k(G)$-comodule coalgbera is a G-graded coalgebra.*

Proof. For a quasigroup G-graded coalgebra C, the linear map $\rho : C \longrightarrow k(G) \otimes C$, $c \mapsto \sigma \otimes c$ for any $\sigma \in G, c \in C_\sigma$, defines a left quasi $k(G)$-comodule coalgebra structure on C. Conversely, if C is a $k(G)$-comodule coalgebra, then any element c in C has a unique representation $\rho(c) = \sum_{g\in G} g \otimes c_g$. Put $C_g = \{c_g, c \in C\}, g \in G$, then C_g is a subspace of C. From the counital property of C, i.e., $(\varepsilon \otimes id)\rho(c) = 1 \otimes c$, we derive that $c = \sum_{g\in G} c_g$ and $C = \sum_{g\in G} C_g$. For any $c \in C$ and $g \in G$, we have that $c \in C_g$ if and only if $\rho(c) = g \otimes c$. If $\sum_{g\in G} c_g = 0$ for some $c_g \in C_g$, then we have $\sum g \otimes c_g = 0$ by applying ρ, it furthermore implies that $c_g = 0$ for all $g \in G$. Therefore, $C = \bigoplus_{g\in G} C_g$. Consider now $c \in C_g$ and $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ with homogeneous $c'_{(1)}s$ and $c'_{(2)}s$. From Equation (7), we retain that $\sum \sigma \otimes \sum c_{(1)} \otimes c_{(2)}$ equals to $\sum deg(c_{(1)})deg(c_{(2)}) \otimes c_{(1)} \otimes c_{(2)}$, or in other words, $\Delta(c)$ is the sum of all terms with $\sigma = deg(c_{(1)})deg(c_{(2)})$, with the fact that C is a G-graded coalgebra.

This finishes the proof. □

Definition 5. *We say that the quasigroup G* quasiacts *on the coalgebra C whenever there is a linear map $\varphi : G \longrightarrow Aut(C)$, such that $\varphi(1) = \varphi(aa^{-1}) = \varphi(a)\varphi(a^{-1})$ for any $a \in G$.*

Remark 5. *It follows immediately from Definition 4.4 that $\varphi(a)^{-1} = \varphi(a^{-1})$ for any $a \in G$ and $\varphi(1) = 1$.*

Proposition 8. *If G quasiacts on the coalgebra C then C has the structure of a $k(G)$-module coalgebra; conversely, any $k(G)$-module coalgebra has a natural G-quasiaction.*

Proof. Assume that $\varphi : G \longrightarrow Aut(C)$ determines that G quasiacts on C. Then, the map $k(G) \otimes C \longrightarrow C$, $a \otimes c \mapsto \varphi(a)(c)$ defines a $k(G)$-quasimodule structure on C as desired. Conversely, if C is a $k(G)$-quasimodule coalgebra, then we may define a G-quasiaction on C via $\varphi : G \longrightarrow Aut(C), \varphi(g)(c) = g \cdot c$ for any $g \in G$ and $c \in C$. □

In what follows, let G be a quasigroup and $C = \bigoplus_{\sigma\in G} C_\sigma$ be a quasigroup G-graded coalgebra. Then, we observe that $Q = C_1$. Additionally, the coalgebra structure of $C \rtimes k(G)$ is given by

$$\Delta(c \rtimes g) = \sum c_{(1)} \rtimes deg(c_{(2))}g \otimes c_{(2)} \rtimes g \quad \text{and} \quad \varepsilon(c \rtimes g) = \varepsilon_C(c)$$

for any homogeneous $c \in C$ and $g \in G$. We notice that $\Pi : C \longrightarrow C_1$ is the natural coalgebra projection. Then, C becomes the C_1-bicomodule via the structure maps $\Gamma^l : C \longrightarrow C_1 \otimes C, c \mapsto \sum \Pi(c_{(1)}) \otimes c_{(2)}$ and $\Gamma^r : C \longrightarrow C \otimes C_1, c \mapsto \sum c_{(1)} \otimes \Pi(c_{(2)})$.

By Proposition 1, C is a left $C \rtimes k(G)$-comodule with the structure

$$\rho^l(c) = \sum(c_{(1)} \rtimes deg(c_{(2)}) \otimes c_{(2)}$$

for any homogeneous $c \in C$, and C becomes a $(C \rtimes k(G), C_1)$-bicomodule.

In a similar way, by according to Lemma 2, C is a right $C \rtimes k(G)$-comodule with the following structure

$$\rho^r(c) = \sum c_{(1)} \otimes c_{(2)} \rtimes deg(c)^{-1}$$

for all homogeneous $c \in C$, and C becomes a $(C_1, C \rtimes k(G))$-bicomodule.

Let $\Pi_g : C \longrightarrow C_g$ denote the projection from C to C_g with $g \in G$. We can define the following maps:

$$f : C \rtimes k(G) \longrightarrow C\Box_{C_1}C, \quad c \rtimes g \mapsto \sum c_{(1)} \otimes \Pi_{g^{-1}}(c_{(2)}),$$
$$g : C_1 \longrightarrow C\Box_{C\rtimes k(G)}C, \quad c \mapsto \Delta_C(c) = \sum c_{(1)} \otimes c_{(2)}$$

for any $c \in C_1$ and $g \in G$.

As a corollary of Theorem 1, we have:

Theorem 4. *With notations as above. We can form a Morita-Takeuchi context $(C_1, C \rtimes k(G), {}_{C_1}C_{C\rtimes k(G)}, {}_{C\rtimes k(G)}C_{C_1}, f, g)$. The map g is injective; hence, it is an isomorphism.*

As some further applications, one can obtain:

Corollary 3. *The category ${}^{C_1}\mathbf{Mod}$ is equivalent to a quotient category of ${}^{C}\mathbf{Gr}_G$.*

Corollary 4. *Let G be a quasigroup and $C = \bigoplus_{\sigma\in G} C_\sigma$ a quasigroup G-graded coalgebra. Then, the following assertions are equivalent:*

(i) The Morita-Takeuchi context in Theorem 4.6. is strict.

(ii) C is strongly G-graded.

(iii) C is faithfully coflat as a left $C \rtimes k(G)$-comodule.

Proof. (i) $\Longrightarrow$ (ii). Take $x, y \in G$ and $c \in C_{xy}$, if $\lambda_{x,y}(c) = \sum \Pi_x(c_{(1)}) \otimes \Pi_y(c_{(2)}) = 0$, then $f(c \rtimes y^{-1}) = \sum c_{(1)} \otimes \Pi_y(c_{(2)} = \sum \Pi_x(c_{(1)}) \otimes \Pi_y(c_{(2)} = 0$, so $c \rtimes y^{-1} = 0$ and $c = 0$. Hence, C is strongly G-graded.

(ii) $\Longrightarrow$ (i). Notice that $C \otimes C = \bigoplus_{x,y\in G} C_x \otimes C_y$. Take $a = \sum c_i \rtimes g_i \in C \rtimes k(G)$ with $deg(c_i) = \sigma_i$. Suppose that $(\sigma_i, g_i) \neq (\sigma_j, g_j)$ with $i \neq j$.

If $f(a) = 0$, then $\sum c_{i(1)} \otimes \Pi_{g^{-1}i}(c_{i(2)}) = 0$, which implies

$$\sum \Pi_{\sigma_i g_i} c_{i(1)} \otimes \Pi_{g^{-1}i}(c_{i(2)}) = 0.$$

On the other hand, $\sum \Pi_{\sigma_i g_i} c_{i(1)} \otimes \Pi_{g^{-1}i}(c_{i(2)}) \in C_{\sigma_i g_i} \otimes C_{g_i^{-1}}$, so we have, for fixed i

$$\sum \Pi_{\sigma_i g_i} c_{i(1)} \otimes \Pi_{g^{-1}i}(c_{i(2)}) = 0,$$

which yields $\lambda_{\sigma_i g_i, g_i^{-1}}(c_i) = 0$, and hence, $c_i = 0$ for any choice of i; that is, $a = 0$ follows. Therefore, f is injective.

(i) $\Longleftrightarrow$ (iii). Follows from Corollary 2(ii).

This completes the proof. □

Corollary 5. *The G-graded coalgebra C is strongly graded if and only if the induced functor $-\Box_{C_1}C : {}^{C_1}\mathbf{Mod} \longrightarrow {}^{C}\mathbf{Gr}_G$ is an equivalence of categories.*

Corollary 6. *Let G be a quasigroup. If C is a strongly G-graded coalgebra then G is a finite quasigroup.*

Proof. It follows the proof by contradiction. In fact, we could select a non-zero homogeneous $c \in C$ and $g \in G$ such that $g \neq deg(c_{(2)})^{-1}$ for all $c_{(2)}$ in $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ if G is infinite. Then, $f(c \rtimes g) = \sum c_{i(1)} \otimes \Pi_{g^{-1}i}(c_{i(2)}) = 0$, but that would contradict the injectivity of f. $\square$

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