

Article

Some New Estimates of Fixed Point Results under Multi-Valued Mappings in G-Metric Spaces with Application

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Abstract: It is well known fact that fixed point results are very useful for finding the solution of different types of differential equations. In this paper, some new results of multi-valued functions involving rational inequality in G-metric spaces have been obtained. Some of the new concepts are also defined over G-metric spaces that are open set, interior of set, and limit point of set. Moreover, we have presented the application of our main results in the field of Homotopy. Some non-trivial examples are also provided to discuss the validation of the main finding.

Keywords: multi-valued mapping results satisfied the rational inequality; fixed point results; open ball; open set; limit point; single valued mapping results



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1. Introduction

The boundary of the fixed point was bounded using Topology and Mathematical analysis. Later on, the unique solutions of integral and differential equations were solved using the theory of fixed points. Thus, the concept of the fixed point was enlarged to integral and differential equations. Stefan Banach (1892–1945) presented the idea of locating the fixed point of self mapping in 1922. This idea is known as a Banach contraction or Banach contractive theory. Before Banach fixed theory, Brouwer presented the idea of a fixed point called Brouwer fixed point theory [1]. This theory only guaranteed the existence of a solution, but not their uniqueness or the existence of a fixed point. However, it might be helpful in calculating the zeroes of a certain function. Schauder [2] worked on a compact and bounded closed convex subset of Banach spaces and introduced a fixed point, which is known as a Schauder Fixed Point. As an application, the Schauder fixed point theory is used in Game theory, approximation theory, and other fields such as optimization theory, engineering, and economics. After these, a lot of metric spaces and fixed point theorems were introduced to the literature.

After Banach fixed theory, Kannan [3] worked on Banach fixed point theory and relaxed one of the conditions for the scholar that continuity is necessary for Kannan's fixed point theory. In 2011, A. Azam and B. Fisher [4] developed a new idea of complex valued metric spaces. Many authors arrived with this new concept and introduced many different types of fixed point theorems with the contractive conditions using single-valued mappings and multi-valued mappings, interested scholars might include ([5–11]). Along with these, Gähler [3] introduced the concept of 2-metric space, which was the extension of regular metric spaces. After detail study, some of the authors felt that the statement of generalization of regular spaces is not correct, see for detail [12], and proved that 2-distance mapping is not continuous with its domain. Furthermore, due to the lack of continuity, Banach contraction mapping theorem is not verified over here. To overcome the loss of continuity in 2-metric spaces, Dhage [13] introduced the new concept of D-metric spaces

and provided their topological structure. Moreover, he felt that D-metric spaces were the extension of the usual metric spaces and proved several fixed point results. (see [14,15].) However, Z. Mustafa [16] searched for some primary properties of topological spaces that were not satisfied by D-metric spaces, one of them is D-convergence of the sequence $\{u_n\}$ to u , i.e., $D(u_m, u_n, u) \rightarrow 0$, whenever $n, m \rightarrow \infty$ may not convergence in any fundamental topology. Z. Mustafa came across these flaws and introduced a new concept that overcame this failure and developed a more general idea called G-metric spaces and introduced many related fixed point theorems under a different sort of contractive condition, see for instance; ([17–29]).

Moreover, under the detailed study of 2-metric spaces and G-metric spaces, some fixed point theorems obeying the rational contractive condition are extracted here in this area, under G-metric spaces. Some theorems of multi-valued functions with rational inequality in G-metric spaces, which are new to the literature of fixed point theory and its applications are presented here in this article. As an application that provides the accuracy to our main result, Homotopy was introduced with readers. Furthermore, examples are provided to the undergoing discussion to strengthen the statements of theorems.

2. Preliminaries

This section recalls some classical and new definitions, results, and preliminary notions.

Definition 1. [25] Let $\mathbb{M}$ be a non-empty set. $G : \mathbb{M} \times \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{R}^+$ be a function satisfying the following conditions:
for all $u, v, w, a \in \mathbb{M}$

- (i) $G(u, v, w) = 0$, if $u = v = w$.
- (ii) $0 < G(u, u, w)$, where $u \neq v$.
- (iii) $G(u, u, v) \leq G(u, v, w)$ where $v \neq w$.
- (iv) $G(u, v, w) = G(u, w, v) = G(v, w, u) = \dots$ (i.e., is symmetric property).
- (v) $G(u, v, w) \leq G(u, a, a) + G(a, v, w)$, (i.e., is rectangular property).

Then, the mapping G is known as the generalized metric, more precisely, G -metric on $\mathbb{M}$, and the pair $(\mathbb{M}, G)$ is called G -metric spaces.

Definition 2. [25] The G -metric space is symmetric if $G(u, v, v) = G(u, u, v)$ for all $u, v \in \mathbb{M}$.

Example 1. [16] Let $(\mathbb{M}, G)$ be a G -metric spaces, and

- (i) $G_u(u, v, w) = \max[d(u, v), d(v, w), d(u, w)]$,
- (ii) $G_v(u, v, w) = d(u, v) + d(v, w) + d(u, w)$,

then $(\mathbb{M}, G_u)$ and $(\mathbb{M}, G_v)$ is symmetric G -metric spaces.

The following are the definitions of Convergence and Completeness on G -metric spaces.

Lemma 1. [16] Let $\mathbb{M}$ be a G -metric space.

- (i) A sequence $\{u_n\}_{n=0}^\infty$ is G -converges if and only if $G(u_n, u, u) \rightarrow 0$, as $n \rightarrow \infty$. More precisely, for every given $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that for all $n > N$, $G(u, u_n, u_n) < \varepsilon$, or $G(u_n, u, u) < \varepsilon$.
- (ii) A sequence $\{u_n\}_{n=0}^\infty$ in $\mathbb{M}$ is named as a G -Cauchy sequence if, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(u_n, u_m, u_l) < \varepsilon$ for all $n, m, l \geq N$.
- (iii) A space $\mathbb{M}$ is said to be G -complete if every G -Cauchy sequence in $\mathbb{M}$ is G -convergent.

Definition 3. Let $(\mathbb{M}, G)$ be a G -metric space, then the mapping $G : \mathbb{M} \times \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{R}^+$ is jointly continuous in all three variables.

Definition 4. Let $(\mathbb{M}, G)$ be a G -metric space and $\mathbb{P} : \mathbb{M} \rightarrow \mathbb{M}$ is said to be G -continuous mapping. If the sequence of $\mathbb{P}$, i.e., $\{\mathbb{P}^{n+1}h\}$ is G -convergent to $h \in \mathbb{M}$, then h is a fixed point of $\mathbb{P}$, then $\{\mathbb{P}u_n\}$ is G -convergent to $\mathbb{P}h$

Here, we define the Interior point, Open ball, Closed ball, and Limit point of a set.

Definition 5. Let $(\mathbb{M}, G)$ be a G -metric space.

(i) We say that a point $h \in \mathbb{M}$ is the interior point of a set $E \subseteq \mathbb{M}$, whenever there exists $0 < r \in \mathbb{R}$ such that

$$B(h, r) = \{k \in \mathbb{M} : G(k, h, h) < r\}.$$

(ii) We say that a point $h \in \mathbb{M}$ is the exterior point of a set $E \subseteq \mathbb{M}$, whenever there exists $0 < r \in \mathbb{R}$ such that

$$B(h, r) = \{k \in \mathbb{M} : G(k, h, h) > r\}.$$

(iv) A set denoted by $B(h; r)$ and defined as:

$$B(h; r) = \{k \in \mathbb{M}, G(k, h, h) < r\}$$

is called an open ball.

(iii) A set E is said to an open set, if for every element $h \in E$ there exist an open ball in E

$$h \in B(h; r) \subseteq E.$$

(iii) We say that a point $h \in \mathbb{M}$ is the limit point of a set $E \subseteq \mathbb{M}$ whenever for every $0 < r \in \mathbb{R}$ such that

$$B(h; r) \cap \mathbb{M} - h \neq \emptyset.$$

Definition 6. Let $(\mathbb{M}, G)$ be a G -metric space. We denote

$$s(u) = \{z \in \mathbb{M} : u \leq z\},$$

for $z \in \mathbb{M}$ and $\mathbb{E} \in CB(\mathbb{M})$

$$s(u, \mathbb{E}) = \bigcup_{n \in \mathbb{E}} s(G(u, n, n)) = \bigcup_{n \in \mathbb{E}} \{z \in \mathbb{M} : G(u, n, n) \leq z\},$$

for $\mathbb{E}, \mathbb{F} \in CB(\mathbb{M})$, we have

$$s(\mathbb{E}, \mathbb{F}) = \left(\bigcap_{n \in \mathbb{E}} s(n, \mathbb{F}) \right) \cap \left(\bigcap_{m \in \mathbb{E}} s(m, \mathbb{F}) \right).$$

Definition 7. Let $(\mathbb{M}, G)$ be a G -metric space.

(i) Let $\mathbb{P} : \mathbb{M} \rightarrow CB(\mathbb{M})$ be a multi-valued mapping. For $u \in \mathbb{M}$ and $E \in CB(\mathbb{M})$, define

$$W_u(E) = \{G(u, a, a) : a \in E\},$$

and for $u, v \in \mathbb{M}$, and $\mathbb{T}v \in CB(\mathbb{M})$, we have

$$W_u(\mathbb{P}v) = \{G(u, b, b) : b \in \mathbb{P}v\}.$$

(ii) A mapping $F : \mathbb{M} \rightarrow 2^{\mathbb{R}}$ is said to be bounded below if for each $u \in \mathbb{M}$ there exists $z_u \leq w$ for all $w \in F_u$.

(iii) For multi-valued mapping $J : \mathbb{M} \rightarrow CB(\mathbb{M})$, we say that it has a lower bound property

on $(\mathbb{M}, G)$ if for any $u \in \mathbb{M}$ the mapping $F_u : \mathbb{M} \rightarrow 2^{\mathbb{R}}$ defined by $F_u(Jv) = W_u(Fv)$ is bounded below. This means that for $u, vs. \in \mathbb{M}$, there is an element $u_u(Jv) \in \mathbb{R}$, that implies $u_l u(Jv) \leq a$ for all $a \in W_u(Jv)$, where $u_u(Jv)$ is said to be lower bound of J corresponding with (u, v) .

(iv) For multi-valued mapping $J : \mathbb{M} \rightarrow CB(\mathbb{M})$, we say that it has the greatest lower bound property (g.l.b. property) on $(\mathbb{M}, G)$ if the g.l.b. of $W_u(Jv)$ exists in $\mathbb{M}$ for all $u, vs. \in \mathbb{M}$. We denote the g.l.b. of $W_u(Jv)$ by $G(u, Jv, Jv)$ and define it as:

$$G(u, Jv, Jv) = \inf \{G(u, a, a) : a \in Jv\}.$$

Fixed point and common fixed point of multi-valued mappings:

Definition 8. [7] Let $(\mathbb{M}, G)$ be a G -metric space and $\mathbb{P}, \mathbb{Q} : \mathbb{M} \rightarrow CB(\mathbb{M})$ be multi-valued mappings.

(i) A point $u \in \mathbb{M}$ is called a fixed point of $\mathbb{Q}$ if $u \in \mathbb{Q}u$.

(ii) A point $u \in \mathbb{M}$ is called a common fixed point of $\mathbb{P}$ and $\mathbb{Q}$ if $u \in \mathbb{S}u$ and $u \in \mathbb{Q}u$.

Example 2. [7] Let $\mathbb{M}$ be a non-empty set and define $G : \mathbb{M} \times \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{R}^+$ and define as below:

(i) $G(u, v, w) = \frac{1}{uvw}; \forall u, v, w \in (0, 1];$

(ii) $G(u, v, w) = 0 \Leftrightarrow u = v = w \forall u, v, w \in [0, 1];$

(iii) $G(u, 0, 0) = G(0, 0, u) = G(0, u, 0) = \frac{1}{u}, \forall u \in (0, 1].$

Then the pair $(\mathbb{M}, G)$ is known as a G -metric space.

Example 3. [7] Let $\mathbb{M} = [0, \infty)$ be a set and $G : \mathbb{M} \times \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{R}^+$ define

$$G(u, v, w) = \begin{cases} 0, & \text{if } u = v = w \\ 1, & \text{if } u \neq v \neq w. \end{cases}$$

Then $(\mathbb{M}, G)$ is known as G -metric space.

3. Our Main Theorem

In this section, we propose the new estimates of fixed point results for Almost Contraction on a G -metric space.

Definition 9. Let $(\mathbb{M}, G)$ be a complete G -metric space. A mapping $\mathbb{P} : \mathbb{M} \rightarrow \mathbb{M}$ is said to be almost contraction if there exists $e \in (0, 1)$ and $\mathbb{L} \geq 0$ such that for all $u, v, w \in \mathbb{M}$,

$$G(\mathbb{P}u, \mathbb{P}v, \mathbb{P}w) \leq eG(u, v, w) + \mathbb{L}G(u, \mathbb{P}v, \mathbb{P}w).$$

Theorem 1. Let $(\mathbb{M}, G)$ be a complete G -metric space and $\mathbb{P}, \mathbb{Q} : \mathbb{M} \rightarrow CB(\mathbb{M})$ be a pair of multi-valued mappings with g.l.b. property such that

$$\begin{aligned} & \lambda_1 G(u, v, v) + \lambda_2 [G(u, \mathbb{P}u, \mathbb{P}u) + G(v, \mathbb{Q}v, \mathbb{Q}v)] + \lambda_3 [G(v, \mathbb{P}u, \mathbb{P}u) + G(u, \mathbb{Q}v, \mathbb{Q}v)] \\ & + \lambda_4 \frac{G(v, \mathbb{Q}v, \mathbb{Q}v)[1 + G(u, \mathbb{P}u, \mathbb{P}u)]}{1 + G(u, v, v)} + \lambda_5 \frac{G(v, \mathbb{P}u, \mathbb{P}u)[1 + G(u, \mathbb{Q}v, \mathbb{Q}v)]}{1 + G(u, v, v)} \\ & + \lambda_6 \frac{G(u, v, v)[1 + G(u, \mathbb{P}u, \mathbb{P}u) + G(v, \mathbb{P}u, \mathbb{P}u)]}{1 + G(u, v, v)} + \mathbb{L}G(v, \mathbb{P}u, \mathbb{P}u) \in s(\mathbb{P}u, \mathbb{Q}v) \end{aligned} \quad (1)$$

for all $u, v \in \mathbb{M}$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \mathbb{L}$ are non-negative real numbers with $\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4 + \lambda_6 < 1$. Then, $\mathbb{P}$ and $\mathbb{Q}$ have a common fixed point.

Proof. Let $u_0 \in \mathbb{M}$ then $\mathbb{Q}u_0$ is non empty, so we use $u_1 \in \mathbb{Q}u_0$. Thus, from 1

$$\begin{aligned} & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 [G(u_0, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)] + \\ & \lambda_3 [G(u_1, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)] \\ & + \lambda_4 \frac{G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)[1 + G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)]}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)[1 + G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)]}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_6 \frac{G(u_0, u_1, u_1)[1 + G(u_0, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)]}{1 + G(u_0, u_1, u_1)} + \mathbb{L}G(u_1, \mathbb{P}u_0, \mathbb{P}u_0) \in s(\mathbb{P}u_0, \mathbb{Q}u_1) \end{aligned}$$

$$\begin{aligned} & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 [G(u_0, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)] + \\ & \lambda_3 [G(u_1, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)] \\ & + \lambda_4 \frac{G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)[1 + G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)]}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)[1 + G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)]}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_6 \frac{G(u_0, u_1, u_1)[1 + G(u_0, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)]}{1 + G(u_0, u_1, u_1)} + \mathbb{L}G(u_1, \mathbb{P}u_0, \mathbb{P}u_0) \in \bigcap_{a \in \mathbb{P}u_0} s(a, \mathbb{Q}u_1) \end{aligned}$$

$$\begin{aligned} & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 [G(u_0, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)] + \\ & \lambda_3 [G(u_1, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)] \\ & + \lambda_4 \frac{G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)[1 + G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)]}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)[1 + G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)]}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_6 \frac{G(u_0, u_1, u_1)[1 + G(u_0, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)]}{1 + G(u_0, u_1, u_1)} + \mathbb{L}G(u_1, \mathbb{P}u_0, \mathbb{P}u_0) \in s(a, \mathbb{Q}u_1), \\ & \forall a \in \mathbb{P}u_0. \end{aligned}$$

Since, $u_1 \in \mathbb{P}u_0$

$$\begin{aligned} & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 [G(u_0, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)] + \\ & \lambda_3 [G(u_1, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)] \\ & + \lambda_4 \frac{G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)[1 + G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)]}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)[1 + G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)]}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_6 \frac{G(u_0, u_1, u_1)[1 + G(u_0, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)]}{1 + G(u_0, u_1, u_1)} + \mathbb{L}G(u_1, \mathbb{P}u_0, \mathbb{P}u_0) \in s(u_1, \mathbb{Q}u_1) \end{aligned}$$

$$\begin{aligned} & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 [G(u_0, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)] + \\ & \lambda_3 [G(u_1, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)] \\ & + \lambda_4 \frac{G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)[1 + G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)]}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)[1 + G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)]}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_6 \frac{G(u_0, u_1, u_1)[1 + G(u_0, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)]}{1 + G(u_0, u_1, u_1)} + \mathbb{L}G(u_1, \mathbb{P}u_0, \mathbb{P}u_0) \in \bigcup_{b \in \mathbb{Q}u_1} s(u_1, b). \end{aligned}$$

Therefore, there exists $u_2 \in \mathbb{Q}u_1$, such that

$$\begin{aligned}
& \lambda_1 G(u_0, u_1, u_1) + \lambda_2 [G(u_0, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)] + \\
& \lambda_3 [G(u_1, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)] \\
& + \lambda_4 \frac{G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)[1 + G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)]}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)[1 + G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)]}{1 + G(u_0, u_1, u_1)} \\
& + \lambda_6 \frac{G(u_0, u_1, u_1)[1 + G(u_0, \mathbb{P}u_0, \mathbb{P}u_0) + G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)]}{1 + G(u_0, u_1, u_1)} + \mathbb{L}G(u_1, \mathbb{P}u_0, \mathbb{P}u_0) \in s(G(u_1, u_2, u_2)).
\end{aligned}$$

Using the definition and g.l.b. property of $\mathbb{P}$ and $\mathbb{Q}$, we have

$$\begin{aligned}
G(u_1, u_2, u_2) & \leq \lambda_1 G(u_0, u_1, u_1) + \lambda_2 [G(u_0, u_1, u_1) + G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)] \\
& + \lambda_3 [G(u_1, u_1, u_1) + G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)] \\
& + \lambda_4 \frac{G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)[1 + G(u_0, u_1, u_1)]}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, u_1, u_1)[1 + G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)]}{1 + G(u_0, u_1, u_1)} \\
& + \lambda_6 \frac{G(u_0, u_1, u_1)[1 + G(u_0, u_1, u_1) + G(u_1, u_1, u_1)]}{1 + G(u_0, u_1, u_1)} + \mathbb{L}G(u_1, u_1, u_1)
\end{aligned}$$

$$\begin{aligned}
G(u_1, u_2, u_2) & \leq \lambda_1 G(u_0, u_1, u_1) + \lambda_2 [G(u_0, u_1, u_1) + G(u_1, u_2, u_2)] + \\
& \lambda_3 [G(u_1, u_1, u_1) + G(u_0, u_2, u_2)] \\
& + \lambda_4 \frac{G(u_1, u_2, u_2)[1 + G(u_0, u_1, u_1)]}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, u_1, u_1)[1 + G(u_0, u_2, u_2)]}{1 + G(u_0, u_1, u_1)} \\
& + \lambda_6 \frac{G(u_0, u_1, u_1)[1 + G(u_0, u_1, u_1) + G(u_1, u_1, u_1)]}{1 + G(u_0, u_1, u_1)} + \mathbb{L}G(u_1, u_1, u_1).
\end{aligned}$$

Using the rectangular inequality, we have

$$G(u_0, u_2, u_2) \leq G(u_0, u_1, u_1) + G(u_1, u_2, u_2)$$

This implies

$$\begin{aligned}
G(u_1, u_2, u_2) & \leq \lambda_1 G(u_0, u_1, u_1) + \lambda_2 [G(u_0, u_1, u_1) + G(u_1, u_2, u_2)] + \\
& \lambda_3 [G(u_0, u_1, u_1) + G(u_1, u_2, u_2)] \\
& + \lambda_4 \frac{G(u_1, u_2, u_2)[1 + G(u_0, u_1, u_1)]}{1 + G(u_0, u_1, u_1)} + \\
& \lambda_6 \frac{G(u_0, u_1, u_1)[1 + G(u_0, u_1, u_1) + G(u_1, u_1, u_1)]}{1 + G(u_0, u_1, u_1)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
(1 - \lambda_2 - \lambda_4)G(u_1, u_2, u_2) & \leq (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_6)G(u_0, u_1, u_1), \\
G(u_1, u_2, u_2) & \leq \frac{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_6)}{(1 - \lambda_2 - \lambda_3 - \lambda_4)}G(u_0, u_1, u_1).
\end{aligned}$$

Let $k = \frac{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_6)}{(1 - \lambda_2 - \lambda_3 - \lambda_4)}$

Then, we obtain

$$G(u_1, u_2, u_2) \leq kG(u_0, u_1, u_1).$$

Inductively, we can find a sequence $\{u_p\}_{p=1}^{\infty}$ in $\mathbb{M}$ such that

$$\begin{aligned} G(u_1, u_2, u_2) &\leq kG(u_0, u_1, u_1) \\ G(u_2, u_3, u_3) &\leq k^2G(u_0, u_1, u_1) \\ &\vdots \\ G(u_p, u_{p+1}, u_{p+1}) &\leq k^pG(u_0, u_1, u_1). \end{aligned}$$

Now, by triangular inequality and for $q > p$, we have the following

$$\begin{aligned} G(u_p, u_q, u_q) &\leq G(u_p, u_{p+1}, u_{p+1}) + G(u_{p+1}, u_{p+2}, u_{p+2}) + \dots + G(u_{q-1}, u_q, u_q), \\ &\leq (k^p + k^{p+1} + \dots + k^{q-1})G(u_0, u_1, u_1). \end{aligned}$$

Since, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_6 < 1 - \lambda_2 - \lambda_3 - \lambda_4$. This implies that $k < 1$.

$$\begin{aligned} G(u_p, u_q, u_q) &\leq G(u_p, u_{p+1}, u_{p+1}) + G(u_{p+1}, u_{p+2}, u_{p+2}) + \dots + G(u_{q-1}, u_q, u_q), \\ &\leq (k^p + k^{p+1} + \dots + k^{q-1})G(u_0, u_1, u_1), \\ &\leq k^p(1 + k^1 + \dots + k^{q-p-1})G(u_0, u_1, u_1), \\ &\leq \frac{1 - k^{q-p-1}}{1 - k}k^pG(u_0, u_1, u_1). \end{aligned}$$

$$G(u_p, u_q, u_q) \rightarrow 0 \text{ as } p, q \rightarrow \infty,$$

Using Lemma 1, $\{u_p\}_{p=1}^{\infty}$ is a G -Cauchy sequence in $\mathbb{M}$.

For the completeness of $\mathbb{M}$, there exist some $r \in \mathbb{M}$, such that

$$\lim_{p \rightarrow \infty} u_p = r.$$

Now, we show that $r \in \mathbb{P}r$ and $r \in \mathbb{Q}r$.

From (1), we have

$$\begin{aligned} &\lambda_1 G(u_{2p}, r, r) + \lambda_2 [G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(r, \mathbb{Q}r, \mathbb{Q}r)] + \lambda_3 [G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)] \\ &+ \lambda_4 \frac{G(r, \mathbb{Q}r, \mathbb{Q}r)[1 + G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})]}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})[1 + G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)]}{1 + G(u_{2p}, r, r)} \\ &+ \lambda_6 \frac{G(u_{2p}, r, r)[1 + G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})]}{1 + G(u_{2p}, r, r)} + \mathbb{L}G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) \in s(\mathbb{P}u_{2p}, \mathbb{Q}r). \end{aligned}$$

This implies

$$\begin{aligned} &\lambda_1 G(u_{2p}, r, r) + \lambda_2 [G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(r, \mathbb{Q}r, \mathbb{Q}r)] + \lambda_3 [G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)] \\ &+ \lambda_4 \frac{G(r, \mathbb{Q}r, \mathbb{Q}r)[1 + G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})]}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})[1 + G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)]}{1 + G(u_{2p}, r, r)} \\ &+ \lambda_6 \frac{G(u_{2p}, r, r)[1 + G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})]}{1 + G(u_{2p}, r, r)} + \mathbb{L}G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) \\ &\in \bigcap_{a \in \mathbb{P}I_{2p}} s(a, \mathbb{Q}r) \end{aligned}$$

$$\begin{aligned}
& \lambda_1 G(u_{2p}, r, r) + \lambda_2 [G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(r, \mathbb{Q}r, \mathbb{Q}r)] + \lambda_3 [G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)] \\
& + \lambda_4 \frac{G(r, \mathbb{Q}r, \mathbb{Q}r)[1 + G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})]}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})[1 + G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)]}{1 + G(u_{2p}, r, r)} \\
& + \lambda_6 \frac{G(u_{2p}, r, r)[1 + G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})]}{1 + G(u_{2p}, r, r)} + \mathbb{L}G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) \\
& \in s(a, \mathbb{Q}r), \quad \forall a \in \mathbb{P}u_{2n}.
\end{aligned}$$

Since $u_{2p+1} \in \mathbb{P}u_{2p}$, therefore

$$\begin{aligned}
& \lambda_1 G(u_{2p}, r, r) + \lambda_2 [G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(r, \mathbb{Q}r, \mathbb{Q}r)] + \lambda_3 [G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)] \\
& + \lambda_4 \frac{G(r, \mathbb{Q}r, \mathbb{Q}r)[1 + G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})]}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})[1 + G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)]}{1 + G(u_{2p}, r, r)} \\
& + \lambda_6 \frac{G(u_{2p}, r, r)[1 + G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})]}{1 + G(u_{2p}, r, r)} + \mathbb{L}G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) \in s(u_{2p+1}, \mathbb{Q}r)
\end{aligned}$$

$$\begin{aligned}
& \lambda_1 G(u_{2p}, r, r) + \lambda_2 [G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(r, \mathbb{Q}r, \mathbb{Q}r)] + \lambda_3 [G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)] \\
& + \lambda_4 \frac{G(r, \mathbb{Q}r, \mathbb{Q}r)[1 + G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})]}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})[1 + G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)]}{1 + G(u_{2p}, r, r)} \\
& + \lambda_6 \frac{G(u_{2p}, r, r)[1 + G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})]}{1 + G(u_{2p}, r, r)} + \mathbb{L}G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) \\
& \in \bigcup_{b \in \mathbb{Q}r} s(G(u_{2p+1}, b)).
\end{aligned}$$

This implies that there exists some $r_p \in \mathbb{Q}r$, such that

$$\begin{aligned}
& \lambda_1 G(u_{2p}, r, r) + \lambda_2 [G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(r, \mathbb{Q}r, \mathbb{Q}r)] + \lambda_3 [G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)] \\
& + \lambda_4 \frac{G(r, \mathbb{Q}r, \mathbb{Q}r)[1 + G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})]}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})[1 + G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)]}{1 + G(u_{2p}, r, r)} \\
& + \lambda_6 \frac{G(u_{2p}, r, r)[1 + G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})]}{1 + G(u_{2p}, r, r)} + \\
& \mathbb{L}G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) \in s(G(u_{2p+1}, r_p, r_p))
\end{aligned}$$

$$\begin{aligned}
G(u_{2p+1}, r_p, r_p) & \leq \lambda_1 G(u_{2p}, r, r) + \lambda_2 [G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(r, \mathbb{Q}r, \mathbb{Q}r)] \\
& + \lambda_3 [G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)] \\
& + \lambda_4 \frac{G(r, \mathbb{Q}r, \mathbb{Q}r)[1 + G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})]}{1 + G(u_{2p}, r, r)} + \\
& \lambda_5 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})[1 + G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)]}{1 + G(u_{2p}, r, r)} \\
& + \lambda_6 \frac{G(u_{2p}, r, r)[1 + G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})]}{1 + G(u_{2p}, r, r)} + \\
& \mathbb{L}G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})
\end{aligned}$$

$$\begin{aligned}
G(u_{2p+1}, r_p, r_p) &\leq \lambda_1 G(u_{2p}, r, r) + \lambda_2 [G(u_{2p}, u_{2p+1}, u_{2p+1}) + G(r, r_p, r_p)] \\
&\quad + \lambda_3 [G(r, u_{2p+1}, u_{2p+1}) + G(u_{2p}, r_p, r_p)] \\
&\quad + \lambda_4 \frac{G(r, r_p, r_p)[1 + G(u_{2p}, u_{2p+1}, u_{2p+1})]}{1 + G(u_{2p}, r, r)} + \\
&\quad \lambda_5 \frac{G(r, u_{2p+1}, u_{2p+1})[1 + G(u_{2p}, r_p, r_p)]}{1 + G(u_{2p}, r, r)} \\
&\quad + \lambda_6 \frac{G(u_{2p}, r, r)[1 + G(u_{2p}, u_{2p+1}, u_{2p+1}) + G(r, u_{2p+1}, u_{2p+1})]}{1 + G(u_{2p}, r, r)} + \\
&\quad \mathbb{L}G(r, u_{2p+1}, u_{2p+1})
\end{aligned}$$

We know that

$$G(r, r_p, r_p) \leq G(r, u_{2p+1}, u_{2p+1}) + G(u_{2p+1}, r_p, r_p)$$

Using this inequality, we obtain

$$\begin{aligned}
G(r, r_p, r_p) &\leq G(r, u_{2p+1}, u_{2p+1}) + \lambda_1 G(u_{2p}, r, r) \\
&\quad + \lambda_2 [G(u_{2p}, u_{2p+1}, u_{2p+1}) + G(r, r_p, r_p)] \\
&\quad + \lambda_3 [G(r, u_{2p+1}, u_{2p+1}) + G(u_{2p}, r_p, r_p)] \\
&\quad + \lambda_4 \frac{G(r, r_p, r_p)[1 + G(u_{2p}, u_{2p+1}, u_{2p+1})]}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, u_{2p+1}, u_{2p+1})[1 + G(u_{2p}, r_p, r_p)]}{1 + G(u_{2p}, r, r)} \\
&\quad + \lambda_6 \frac{G(u_{2p}, r, r)[1 + G(u_{2p}, u_{2p+1}, u_{2p+1}) + G(r, u_{2p+1}, u_{2p+1})]}{1 + G(u_{2p}, r, r)} + \mathbb{L}G(r, u_{2p+1}, u_{2p+1})
\end{aligned}$$

By letting $p \rightarrow \infty$, we obtain

$$G(r, r_p, r_p) \rightarrow 0 \text{ as } p \rightarrow \infty.$$

Using Lemma 1, $r_p \rightarrow r$ as $p \rightarrow \infty$ also since $\mathbb{Q}r$ is closed, thus $r \in \mathbb{Q}r$. Similarly, we have $r \in \mathbb{P}r$. Hence, $\mathbb{P}$ and $\mathbb{Q}$ have a common fixed point. $\square$

We deduce the following results from above theorem. For $\lambda_6 = 0$, we have corollary as follows:

Corollary 1. Let $(\mathbb{M}, G)$ be a complete G -metric space and $\mathbb{P}, \mathbb{Q} : \mathbb{M} \rightarrow CB(\mathbb{M})$ be a pair of multi-valued mappings with g.l.b. property such that

$$\begin{aligned}
&\lambda_1 G(u, v, v) + \lambda_2 [G(u, \mathbb{P}u, \mathbb{P}u) + G(v, \mathbb{Q}v, \mathbb{Q}v)] + \lambda_3 [G(v, \mathbb{P}u, \mathbb{P}u) + G(u, \mathbb{Q}v, \mathbb{Q}v)] \\
&\quad + \lambda_4 \frac{G(v, \mathbb{Q}v, \mathbb{Q}v)[1 + G(u, \mathbb{P}u, \mathbb{P}u)]}{1 + G(u, v, v)} + \lambda_5 \frac{G(v, \mathbb{P}u, \mathbb{P}u)[1 + G(u, \mathbb{Q}v, \mathbb{Q}v)]}{1 + G(u, v, v)} \\
&\quad + \mathbb{L}G(v, \mathbb{P}u, \mathbb{P}u) \in s(\mathbb{P}u, \mathbb{Q}v)
\end{aligned}$$

for all $u, v \in \mathbb{M}$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \mathbb{L}$ are non-negative real numbers with $\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4 < 1$, then, $\mathbb{P}$ and $\mathbb{Q}$ have a common fixed point.

For $\lambda_i = 0$, where $i = 5, 6$, we have corollary as follows:

Corollary 2. Let $(\mathbb{M}, G)$ be a complete complex valued extended b -metric space and $\mathbb{P}, \mathbb{Q} : \mathbb{M} \rightarrow CB(\mathbb{M})$ be a pair of multi-valued mappings with g.l.b. property such that

$$\begin{aligned} & \lambda_1 G(u, v, v) + \lambda_2 [G(u, \mathbb{P}u, \mathbb{P}u) + G(v, \mathbb{Q}v, \mathbb{Q}v)] + \lambda_3 [G(v, \mathbb{P}u, \mathbb{P}u) + G(u, \mathbb{Q}v, \mathbb{Q}v)] \\ & + \lambda_4 \frac{G(v, \mathbb{Q}v, \mathbb{Q}v)[1 + G(u, \mathbb{P}u, \mathbb{P}u)]}{1 + G(u, v, v)} + \\ & + \mathbb{L}G(v, \mathbb{P}u, \mathbb{P}u) \in s(\mathbb{P}u, \mathbb{Q}v) \end{aligned}$$

for all $l, m \in \mathbb{M}$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mathbb{L}$ are non-negative real numbers with $\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4 < 1$. Then, $\mathbb{P}$ and $\mathbb{Q}$ have a common fixed point.

For $\lambda_i = 0$, where $i = 4, 5, 6$, we have a corollary as follows:

Corollary 3. Let $(\mathbb{M}, G)$ be a complete G -metric space and $\mathbb{P}, \mathbb{Q} : \mathbb{M} \rightarrow CB(\mathbb{M})$ be a pair of multi-valued mappings with g.l.b. property such that

$$\begin{aligned} & \lambda_1 G(u, v, v) + \lambda_2 [G(u, \mathbb{P}u, \mathbb{P}u) + G(v, \mathbb{Q}v, \mathbb{Q}v)] + \lambda_3 [G(v, \mathbb{P}u, \mathbb{P}u) + G(u, \mathbb{Q}v, \mathbb{Q}v)] \\ & + \mathbb{L}G(v, \mathbb{P}u, \mathbb{P}u) \in s(\mathbb{P}u, \mathbb{Q}v) \end{aligned}$$

for all $u, v \in \mathbb{M}$ and $\lambda_1, \lambda_2, \lambda_3, \mathbb{L}$ are non-negative real numbers with $\lambda_1 + 2\lambda_2 + 2\lambda_3 < 1$. Then, $\mathbb{P}$ and $\mathbb{Q}$ have a common fixed point.

For $\mathbb{P} = \mathbb{Q}$, we have the following corollary:

Corollary 4. Let $(\mathbb{M}, G)$ be a complete G -metric space and $\mathbb{Q} : \mathbb{M} \rightarrow CB(\mathbb{M})$ be a multi-valued map with g.l.b. property such that

$$\begin{aligned} & \lambda_1 G(u, v, v) + \lambda_2 [G(u, \mathbb{Q}u, \mathbb{Q}u) + G(v, \mathbb{Q}v, \mathbb{Q}v)] + \lambda_3 [G(v, \mathbb{Q}u, \mathbb{Q}u) + G(u, \mathbb{Q}v, \mathbb{Q}v)] \\ & + \lambda_4 \frac{G(v, \mathbb{Q}v, \mathbb{Q}v)[1 + G(u, \mathbb{Q}u, \mathbb{Q}u)]}{1 + G(u, v, v)} + \lambda_5 \frac{G(v, \mathbb{Q}u, \mathbb{Q}u)[1 + G(u, \mathbb{Q}v, \mathbb{Q}v)]}{1 + G(u, v, v)} \\ & + \lambda_6 \frac{G(u, v, v)[1 + G(u, \mathbb{Q}u, \mathbb{Q}u) + G(v, \mathbb{Q}u, \mathbb{Q}u)]}{1 + G(u, v, v)} + \mathbb{L}G(v, \mathbb{Q}u, \mathbb{Q}u) \in s(\mathbb{Q}u, \mathbb{Q}v) \end{aligned} \quad (2)$$

for all $l, m \in \mathbb{M}$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \mathbb{L}$ are non-negative real numbers with $\lambda_1 + 2\lambda_2 + 2\lambda_3\theta(l, m) + \lambda_4 + \lambda_5 + \lambda_6 < 1$.

Then, $\mathbb{Q}$ has a fixed point.

Remark 1. From Corollary 4, we deduced several corollaries by putting $\lambda_6 = 0$, $\lambda_6, \lambda_5 = 0$ and $\lambda_6, \lambda_5, \lambda_4 = 0$.

Example 4. Let $\mathbb{M} = [0, 1]$. Define $G : \mathbb{M} \times \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{R}^+$ by

$$G(u, v, w) = \max[d(u, v), d(v, w), d(u, w)]$$

where $(\mathbb{M}, d)$ is a usual metric space.

Then, $(\mathbb{M}, G)$ is a G -metric space. Let $\mathbb{P}, \mathbb{Q} : \mathbb{M} \rightarrow \mathbb{M}$ be a multi-valued mapping defined by:

$$\mathbb{P}u = [0, \frac{u}{5}], \quad \mathbb{Q}v = [0, \frac{v}{7}].$$

By using $u = v = 0$, the contractive condition of Theorem 1 is fulfilled. For non-zero $u, v \in \mathbb{M}$. Then,

$$\begin{aligned} G(u, v, w) &= \max[d(u, v), d(v, w), d(u, w)], \\ G(u, v, v) &= \max[d(u, v), d(v, v), d(u, v)], \\ &= \max[2d(u, v), 0], \\ &= 2d(u, v). \end{aligned}$$

Similarly,

$$\begin{aligned} G(u, \mathbb{P}u, \mathbb{P}u) &= \max[d(u, \mathbb{P}u), d(\mathbb{P}u, \mathbb{P}u), d(u, \mathbb{P}u)], \\ &= \max[2d(u, \mathbb{P}u), 0], \\ &= 2d(u, \mathbb{P}u), \\ &= 2|u - \frac{u}{5}|. \end{aligned}$$

Now, define

$$\begin{aligned} G(v, \mathbb{P}u, \mathbb{P}u) &= 2|v - \frac{u}{5}|, \\ G(u, \mathbb{Q}u, \mathbb{Q}u) &= 2|u - \frac{u}{7}|, \\ G(v, \mathbb{Q}v, \mathbb{Q}v) &= 2|v - \frac{v}{7}|, \\ G(u, \mathbb{Q}v, \mathbb{Q}v) &= 2|u - \frac{v}{7}|, \\ s(\mathbb{P}u, \mathbb{Q}v) &= 2|\frac{u}{5} - \frac{v}{7}|. \end{aligned}$$

Letting $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 = 0$, $\lambda_1 = \frac{1}{2}$ and $\mathbb{L} \geq 1$, it can directly seem that all the assertions of Theorem 1 are obeyed. In such a situation, 0 is a common fixed point of $\mathbb{P}$ and $\mathbb{Q}$.

4. Banach Type Contractive Mapping

Theorem 2. Let $(\mathbb{M}, G)$ be a complete G -metric space and $\mathbb{P}, \mathbb{Q} : \mathbb{M} \rightarrow CB(\mathbb{M})$ be a pair multi-valued mappings satisfying the g.l.b. property such that

$$\lambda_1 G(u, v, v) + \frac{\lambda_2 G(u, \mathbb{P}u, \mathbb{P}u) G(v, \mathbb{Q}v, \mathbb{Q}v)}{1 + G(u, v, v)} + \frac{\lambda_3 G(v, \mathbb{P}u, \mathbb{P}u) G(u, \mathbb{Q}v, \mathbb{Q}v)}{1 + G(u, v, v)} \in s(\mathbb{P}u, \mathbb{Q}v) \quad (3)$$

for all $u, v \in \mathbb{M}$ and $\lambda_1, \lambda_2, \lambda_3$ are non-negative real numbers with $\lambda_1 + \lambda_2 + \lambda_3 < 1$ and, assuming that $k(1 - \lambda_2) = \lambda_1$, where $k \in [0, 1)$. Then, $\mathbb{P}$ and $\mathbb{Q}$ have a common fixed point.

Proof. Let $u_0 \in \mathbb{M}$ then Tu_0 is not empty so we use $u_1 \in Tu_0$, thus from (3)

$$\begin{aligned} &\lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0) G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ &+ \lambda_3 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0) G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \in s(\mathbb{P}u_0, \mathbb{Q}u_1) \end{aligned}$$

This implies that

$$\begin{aligned} & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_3 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \in \bigcap_{a' \in \mathbb{P}u_0} s(a', \mathbb{Q}u_1) \\ & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_3 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \in s(a', \mathbb{Q}u_1), \quad \forall a' \in \mathbb{P}u_0. \end{aligned}$$

Since $u_1 \in \mathbb{P}u_0$, we have

$$\begin{aligned} & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_3 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \in s(u_1, \mathbb{Q}u_1) \\ & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_3 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \in \bigcup_{b' \in \mathbb{Q}u_1} s(G(u_1, b', b')). \end{aligned}$$

Therefore, there exist $u_2 \in \mathbb{Q}u_1$ such that

$$\begin{aligned} & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_3 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \in s(G(u_1, u_2, u_2)). \end{aligned}$$

Using Definition 6 and g.l.b. property of $\mathbb{P}$ and $\mathbb{Q}$, we obtain

$$\begin{aligned} G(u_0, u_1, u_1) & \leq \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_3 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)}, \\ G(u_0, u_1, u_1) & \leq \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, u_1, u_1)G(u_1, u_2, u_2) + \lambda_3 G(u_1, u_1, u_1)G(u_0, u_2, u_2)}{1 + G(u_0, u_1, u_1)}. \end{aligned}$$

This implies

$$\begin{aligned} G(u_1, u_2, u_2) & \leq \lambda_1 G(u_0, u_1, u_1) + \frac{\lambda_2 G(u_0, u_1, u_1)G(u_1, u_2, u_2)}{1 + G(u_0, u_1, u_1)} \\ & = \lambda_1 G(u_0, u_1, u_1) + \lambda_2 G(u_1, u_2, u_2) \frac{G(u_0, u_1, u_1)}{1 + G(u_0, u_1, u_1)}, \end{aligned}$$

$$\begin{aligned}
G(u_1, u_2, u_2) &\leq \lambda_1 G(u_0, u_1, u_1) + \lambda_2 G(u_0, u_1, u_1) \\
(1 - \lambda_2)G(u_1, u_2, u_2) &\leq \lambda_1 G(u_0, u_1, u_1) \\
G(u_1, u_2, u_2) &\leq \frac{\lambda_1}{(1 - \lambda_2)} G(u_0, u_1, u_1).
\end{aligned}$$

Inductively, we develop a sequence $\{u_p\}_{p=1}^{\infty}$ in $\mathbb{M}$, such that

$$\begin{aligned}
G(u_1, u_2, u_2) &\leq kG(u_0, u_1, u_1), \\
G(u_2, u_3, u_3) &\leq k^2G(u_0, u_1, u_1), \\
&\vdots \\
G(u_p, u_{p+1}, u_{p+1}) &\leq k^pG(u_0, u_1, u_1),
\end{aligned}$$

Now, using triangular inequality and for $q > p$, we have

$$\begin{aligned}
G(u_p, u_q, u_q) &\leq G(u_p, u_{p+1}, u_{p+1}) + G(u_{p+1}, u_{p+2}, u_{p+2}) + \dots + G(u_{q-1}, u_q, u_q), \\
&= (k^p + k^{p+1} + k^{p+2} + \dots + k^{q-1})G(u_0, u_1, u_1), \\
&= (1 + k^p + k^{p+1} + k^{p+2} + \dots + k^{q-p-1})k^pG(u_0, u_1, u_1), \\
&= \frac{1 - k^{q-p-1}}{1 - k}k^pG(u_0, u_1, u_1).
\end{aligned}$$

$G(u_p, u_q, u_q) \rightarrow 0$ as $p, q \rightarrow \infty$; using Lemma 1, $\{u_p\}_{p=1}^{\infty}$ is a G -Cauchy sequence in $\mathbb{M}$.

For the completeness of $\mathbb{M}$, there exist some $r \in \mathbb{M}$, such that

$$\lim_{p \rightarrow \infty} u_p = r.$$

Now, we show that $r \in \mathbb{P}r$ and $r \in \mathbb{Q}r$.

From (3), we have

$$\begin{aligned}
&\lambda_1 G(u_{2p}, r, r) + \frac{\lambda_2 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \\
&+ \lambda_3 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \in s(\mathbb{P}u_{2p}, \mathbb{Q}r) \\
&\lambda_1 G(u_{2p}, r, r) + \frac{\lambda_2 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \\
&+ \lambda_3 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \in \bigcap_{a' \in \mathbb{P}u_{2p}} s(a', \mathbb{Q}r).
\end{aligned}$$

Since $u_{2p+1} \in \mathbb{P}u_{2p}$, we have

$$\begin{aligned}
&\lambda_1 G(u_{2p}, r, r) + \frac{\lambda_2 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \\
&+ \lambda_3 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \in s(u_{2p+1}, \mathbb{Q}r)
\end{aligned}$$

$$\lambda_1 G(u_{2p}, r, r) + \frac{\lambda_2 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \lambda_3 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \in \bigcup_{b' \in \mathbb{Q}r} s(G(u_{2p+1}, b', b')).$$

This implies there exists $r_p \in \mathbb{Q}r$ such that

$$\lambda_1 G(u_{2p}, r, r) + \lambda_2 \frac{G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \lambda_3 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \in s(G(u_{2p+1}, r_p, r_p)).$$

This implies

$$G(u_{2p+1}, r_p, r_p) \leq \lambda_1 G(u_{2p}, r, r) + \lambda_2 \frac{G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \lambda_3 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)},$$

$$G(u_{2p+1}, r_p, r_p) \leq \lambda_1 G(u_{2p}, r, r) + \lambda_2 \frac{G(u_{2p}, u_{2p+1}, u_{2p+1}) G(r, r_p, r_p)}{1 + G(u_{2p}, r, r)} + \lambda_3 \frac{G(r, u_{2p+1}, u_{2p+1}) G(u_{2p}, r_p, r_p)}{1 + G(u_{2p}, r, r)}.$$

Now, using rectangular property

$$G(r, r_p, r_p) \leq [G(r, u_{2p+1}, u_{2p+1}) + G(u_{2p+1}, r_p, r_p)]$$

$$G(r, r_p, r_p) \leq G(r, u_{2p+1}, u_{2p+1}) + \lambda_1 G(u_{2p}, r, r) + \frac{\lambda_2 G(u_{2p}, u_{2p+1}, u_{2p+1}) G(r, r_p, r_p) + \lambda_3 G(r, u_{2p+1}, u_{2p+1}) G(u_{2p}, r_p, r_p)}{1 + G(u_{2p}, r, r)}$$

Letting the limit $p \rightarrow \infty$, we obtain $G(r, r_p, r_p) \rightarrow 0$.

By using Lemma 1, we have $r_p \rightarrow r$. Since $\mathbb{Q}r$ is closed, $r \in \mathbb{Q}r$. By a similar way, we can prove that $r \in \mathbb{P}r$. Thus, $\mathbb{Q}$ and $\mathbb{P}$ have a common fixed point. $\square$

By putting $\lambda_3 = 0$ in above the above we have

Corollary 5. Let $(\mathbb{M}, G)$ be a complete G -metric space and $\mathbb{P}, \mathbb{Q} : \mathbb{M} \rightarrow CB(\mathbb{M})$ be a pair of multi-valued mappings satisfying the g.l.b. property such that

$$\lambda_1 G(u, v, v) + \frac{\lambda_2 G(u, \mathbb{P}u, \mathbb{P}u) G(v, \mathbb{Q}v, \mathbb{Q}v)}{1 + G(u, v, v)} \in s(\mathbb{P}u, \mathbb{Q}v) \quad (4)$$

for all $u, v \in \mathbb{M}$ and λ_1, λ_2 are non-negative reals with $\lambda_1 + \lambda_2 < 1$. Then, $\mathbb{P}$ and $\mathbb{Q}$ have a common fixed point.

By putting $\mathbb{P} = \mathbb{Q}$ in Theorem (5.1), we have the corollary as follow.

Corollary 6. Let $(\mathbb{M}, G)$ be a complete G -metric space and $\mathbb{Q} : \mathbb{M} \rightarrow CB(\mathbb{M})$ be a multi-valued mappings satisfying the g.l.b. property such that

$$\lambda_1 G(u, v, v) + \frac{\lambda_2 G(u, Qu, Qu)G(v, Qv, Qv) + \lambda_3 G(v, Qu, Qu)G(u, Qv, Qv)}{1 + G(u, v, v)} \in s(Qu, Qv) \quad (5)$$

for all $u, v \in \mathbb{M}$ and $\lambda_1, \lambda_2, \lambda_3$ are non-negative reals with $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Then, Q has a common fixed point.

5. Kannan Type Contractive Mapping

Theorem 3. Let $(\mathbb{M}, G)$ be a G -metric space and let $\mathbb{P}, Q : \mathbb{M} \rightarrow CB(\mathbb{M})$ be a pair of multi-valued mappings satisfying the g.l.b property such that

$$\lambda_1 G(u, Pu, Pu) + \lambda_2 G(v, Qv, Qv) + \frac{\lambda_3 G(u, Pu, Pu)G(v, Qv, Qv)}{1 + G(u, v, v)} \in s(Pu, Qv) \quad (6)$$

for all $u, v \in \mathbb{M}$ and $\lambda_1, \lambda_2, \lambda_3$ are non-negative reals with $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Then, $\mathbb{P}$ and Q have a common fixed point.

Proof. Let $u_0 \in \mathbb{M}$, then Tu_0 is non-empty, so we use $u_1 \in Tu_0$; thus, from (6)

$$\begin{aligned} \lambda_1 G(u_0, Pu_0, Pu_0) + \lambda_2 G(u_1, Qu_1, Qu_1) + \frac{\lambda_3 G(u_0, Pu_0, Pu_0)G(u_1, Qu_1, Qu_1)}{1 + G(u_0, u_1, u_1)} &\in s(Pu_0, Qu_1), \\ \lambda_1 G(u_0, Pu_0, Pu_0) + \lambda_2 G(u_1, Qu_1, Qu_1) + \frac{\lambda_3 G(u_0, Pu_0, Pu_0)G(u_1, Qu_1, Qu_1)}{1 + G(u_0, u_1, u_1)} &\in \bigcap_{a' \in Pu_0} s(a', Qu_1), \\ \lambda_1 G(u_0, Pu_0, Pu_0) + \lambda_2 G(u_1, Qu_1, Qu_1) + \frac{\lambda_3 G(u_0, Pu_0, Pu_0)G(u_1, Qu_1, Qu_1)}{1 + G(u_0, u_1, u_1)} &\in s(a', Qu_1), \quad \forall a' \in Pu_0. \end{aligned}$$

Since $u_1 \in Pu_0$, thus

$$\begin{aligned} \lambda_1 G(u_0, Pu_0, Pu_0) + \lambda_2 G(u_1, Qu_1, Qu_1) + \frac{\lambda_3 G(u_0, Pu_0, Pu_0)G(u_1, Qu_1, Qu_1)}{1 + G(u_0, u_1, u_1)} &\in s(u_1, Qu_1), \\ \lambda_1 G(u_0, Pu_0, Pu_0) + \lambda_2 G(u_1, Qu_1, Qu_1) + \frac{\lambda_3 G(u_0, Pu_0, Pu_0)G(u_1, Qu_1, Qu_1)}{1 + G(u_0, u_1, u_1)} &\in \bigcup_{b' \in Qu_1} s(G(u_1, b', b')). \end{aligned}$$

Therefore, there exist $u_2 \in Qu_1$

$$\lambda_1 G(u_0, Pu_0, Pu_0) + \lambda_2 G(u_1, Qu_1, Qu_1) + \frac{\lambda_3 G(u_0, Pu_0, Pu_0)G(u_1, Qu_1, Qu_1)}{1 + G(u_0, u_1, u_1)} \in s(G(u_1, u_2, u_2)).$$

By definition, we obtain

$$G(u_1, u_2, u_2) \leq \lambda_1 G(u_0, Pu_0, Pu_0) + \lambda_2 G(u_1, Qu_1, Qu_1) + \frac{\lambda_3 G(u_0, Pu_0, Pu_0)G(u_1, Qu_1, Qu_1)}{1 + G(u_0, u_1, u_1)}$$

$$G(u_1, u_2, u_2) \leq \lambda_1 G(u_0, u_1, u_1) + \lambda_2 G(u_1, u_2, u_2) + \frac{\lambda_3 G(u_0, u_1, u_1) G(u_1, u_2, u_2)}{1 + G(u_0, u_1, u_1)},$$

$$G(u_1, u_2, u_2) \leq \lambda_1 G(u_0, u_1, u_1) + \lambda_2 G(u_1, u_2, u_2) + \lambda_3 G(u_1, u_2, u_2),$$

$$(1 - \lambda_2 - \lambda_3) G(u_1, u_2, u_2) \leq \lambda_1 G(u_0, u_1, u_1),$$

$$G(u_1, u_2, u_2) \leq \frac{\lambda_1}{(1 - \lambda_2 - \lambda_3)} G(u_0, u_1, u_1).$$

Inductively, we develop a sequence $\{u_p\}$ in $\mathbb{M}$, such that

$$G(u_1, u_2, u_2) \leq k G(u_0, u_1, u_1)$$

$$G(u_2, u_3, u_3) \leq k^2 G(u_0, u_1, u_1)$$

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$$G(u_p, u_{p+1}, u_{p+1}) \leq k^p G(u_0, u_1, u_1).$$

Now, by triangular inequality and for $q > p$, we have the following

$$\begin{aligned} G(u_p, u_q, u_q) &\leq G(u_p, u_{p+1}, u_{p+1}) + G(u_{p+1}, u_{p+2}, u_{p+2}) + \dots + G(u_{q-1}, u_q, u_q), \\ &= (k^p + k^{p+1} + k^{p+2} + \dots + k^{q-1}) G(u_0, u_1, u_1), \\ &= (1 + k^p + k^{p+1} + k^{p+2} + \dots + k^{q-p-1}) k^p G(u_0, u_1, u_1), \\ &= \frac{1 - k^{q-p-1}}{1 - k} k^p G(u_0, u_1, u_1). \end{aligned}$$

$$G(u_p, u_q, u_q) \rightarrow 0 \text{ as } p, q \rightarrow \infty,$$

Using Lemma 1, $\{u_p\}_{p=1}^\infty$ is a G -Cauchy sequence in $\mathbb{M}$.

For the completeness of $\mathbb{M}$, there exist some $r \in \mathbb{M}$, such that

$$\lim_{p \rightarrow \infty} u_p = r.$$

Now, we show that $r \in \mathbb{P}r$ and $r \in \mathbb{Q}r$.

From 6, we have

$$\lambda_1 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + \lambda_2 G(r, \mathbb{Q}r, \mathbb{Q}r) + \frac{\lambda_3 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \in s(\mathbb{P}u_{2p}, \mathbb{Q}r),$$

$$\lambda_1 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + \lambda_2 G(r, \mathbb{Q}r, \mathbb{Q}r) + \frac{\lambda_3 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \in \bigcap_{a' \in Su_{2p}} s(a', \mathbb{Q}r).$$

Since $u_{2p+1} \in \mathbb{P}u_{2p}$, we have

$$\lambda_1 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + \lambda_2 G(r, \mathbb{Q}r, \mathbb{Q}r) + \frac{\lambda_3 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \in s(u_{2p+1}, \mathbb{Q}r)$$

$$\lambda_1 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + \lambda_2 G(r, \mathbb{Q}r, \mathbb{Q}r) + \frac{\lambda_3 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \\ \in \bigcup_{b' \in \mathbb{Q}r} s(G(u_{2p+1}, b', b')).$$

So, there exist some $r_p \in \mathbb{Q}r$

$$\lambda_1 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + \lambda_2 G(r, \mathbb{Q}r, \mathbb{Q}r) + \frac{\lambda_3 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \in s(G(u_{2p+1}, r_p, r_p)).$$

Using Definition 6,

$$G(u_{2p+1}, r_p, r_p) \leq \lambda_1 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) + \lambda_2 G(r, \mathbb{Q}r, \mathbb{Q}r) + \frac{\lambda_3 G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p}) G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)}.$$

Using the g.l.b. property of $\mathbb{P}$ and $\mathbb{Q}$, we have

$$G(u_{2p+1}, r_p, r_p) \leq \lambda_1 G(u_{2p}, u_{2p+1}, u_{2p+1}) + \lambda_2 G(r, r_p, r_p) + \frac{\lambda_3 G(u_{2p}, u_{2p+1}, u_{2p+1}) G(r, r_p, r_p)}{1 + G(u_{2p}, r, r)}.$$

Now,

$$G(r, r_p, r_p) \leq [G(r, u_{2p+1}, u_{2p+1}) + G(u_{2p+1}, r_p, r_p)]$$

$$G(r, r_p, r_p) \leq G(r, u_{2p+1}, u_{2p+1}) + \lambda_1 G(u_{2p}, u_{2p+1}, u_{2p+1}) + \lambda_2 G(r, r_p, r_p) \\ + \lambda_3 \frac{G(u_{2p}, u_{2p+1}, u_{2p+1}) G(r, r_p, r_p)}{1 + G(u_{2p}, r, r)}.$$

Using the limit $p \rightarrow \infty$, we obtain $G(r, r_p, r_p) \rightarrow 0$.

By Lemma 1, we have $r_p \rightarrow r$. Since $\mathbb{Q}$ is closed, then $r \in \mathbb{Q}r$. Using a similar way, we show $r \in \mathbb{P}r$. Thus, $\mathbb{P}$ and $\mathbb{Q}$ have a common fixed point. $\square$

By setting $\mathbb{P} = \mathbb{Q}$ in above Theorem 3, we obtain:

Corollary 7. Let $(\mathbb{M}, G)$ be a complete G -metric space and $T : \mathbb{M} \rightarrow CB(\mathbb{M})$ be multi-valued mappings satisfying the g.l.b. property such that

$$\lambda_1 G(u, \mathbb{Q}u, \mathbb{Q}u) + \lambda_2 G(v, \mathbb{Q}v, \mathbb{Q}v) + \frac{\lambda_3 G(u, \mathbb{Q}u, \mathbb{Q}u) G(v, \mathbb{Q}v, \mathbb{Q}v)}{1 + G(u, v, v)} \in s(\mathbb{Q}u, \mathbb{Q}v) \quad (7)$$

for all $u, v \in \mathbb{M}$ and $\lambda_1, \lambda_2, \lambda_3$ are non-negative reals with $\lambda_1 + \lambda_2 + \lambda_3 < 1$.

Then, $\mathbb{Q}$ has a common fixed point.

Theorem 4. Let $(\mathbb{M}, G)$ be a complete G -metric space and let $\mathbb{P}, \mathbb{Q} : \mathbb{M} \rightarrow CB(\mathbb{M})$ be a pair of multi-valued mappings fulfilling the g.l.b. property such that

$$\lambda_1 G(u, v, v) + \lambda_2 \frac{G(u, \mathbb{P}u, \mathbb{P}u) G(v, \mathbb{Q}v, \mathbb{Q}v)}{1 + G(u, v, v)} + \lambda_3 \frac{G(v, \mathbb{P}u, \mathbb{P}u) G(u, \mathbb{Q}v, \mathbb{Q}v)}{1 + G(u, v, v)} \\ + \lambda_4 \frac{G(u, \mathbb{P}u, \mathbb{P}u) G(u, \mathbb{Q}v, \mathbb{Q}v)}{1 + G(u, v, v)} + \lambda_5 \frac{G(v, \mathbb{P}u, \mathbb{P}u) G(v, \mathbb{Q}v, \mathbb{Q}v)}{1 + G(u, v, v)} \in s(\mathbb{P}u, \mathbb{Q}v) \quad (8)$$

for all $u, v \in \mathbb{M}$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, are non-negative real numbers with $\lambda_1 + \lambda_2 + \lambda_3 + 2\theta\lambda_4 + \lambda_5 < 1$.

Then, $\mathbb{P}$ and $\mathbb{Q}$ have common fixed point.

Proof. Let $u_0 \in \mathbb{M}$, then Tu_0 is non-empty, so we use $u_1 \in Tu_0$; thus, from (8)

$$\begin{aligned} & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} + \lambda_3 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_4 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \in s(\mathbb{P}u_0, \mathbb{Q}u_1). \end{aligned}$$

This implies

$$\begin{aligned} & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} + \lambda_3 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_4 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \in \bigcap_{a \in \mathbb{P}u_0} s(a, \mathbb{Q}u_1) \end{aligned}$$

$$\begin{aligned} & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} + \lambda_3 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_4 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ & \in s(a, \mathbb{P}u_1), \quad \forall a \in \mathbb{P}u_0. \end{aligned}$$

Since $u_1 \in \mathbb{P}x_0$, we obtain

$$\begin{aligned} & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} + \lambda_3 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_4 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \in s(u_1, \mathbb{Q}x_1), \end{aligned}$$

$$\begin{aligned} & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} + \lambda_3 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_4 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \in \bigcup_{b \in \mathbb{Q}x_1} s(G(u_1, b, b)). \end{aligned}$$

Thus, there exists some $u_2 \in \mathbb{Q}x_1$, such that

$$\begin{aligned} & \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} + \lambda_3 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ & + \lambda_4 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \in s(G(u_1, u_2, u_2)). \end{aligned}$$

Using Definition 6

$$\begin{aligned} G(u_1, u_2, u_2) &\leq \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} + \\ &\quad \lambda_3 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} \\ &\quad + \lambda_4 \frac{G(u_0, \mathbb{P}u_0, \mathbb{P}u_0)G(u_0, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, \mathbb{P}u_0, \mathbb{P}u_0)G(u_1, \mathbb{Q}u_1, \mathbb{Q}u_1)}{1 + G(u_0, u_1, u_1)}. \end{aligned}$$

Using the g.l.b. property of $\mathbb{P}$ and $\mathbb{Q}$

$$\begin{aligned} G(u_1, u_2, u_2) &\leq \lambda_1 G(u_0, u_1, u_1) + \lambda_2 \frac{G(u_0, u_1, u_1)G(u_1, u_2, u_2)}{1 + G(u_0, u_1, u_1)} + \lambda_3 \frac{G(u_1, u_1, u_1)G(u_0, u_2, u_2)}{1 + G(u_0, u_1, u_1)} \\ &\quad + \lambda_4 \frac{G(u_0, u_1, u_1)G(u_0, u_2, u_2)}{1 + G(u_0, u_1, u_1)} + \lambda_5 \frac{G(u_1, u_1, u_1)G(u_1, u_2, u_2)}{1 + G(u_0, u_1, u_1)}. \end{aligned}$$

$$G(u_1, u_2, u_2) \leq \lambda_1 G(u_0, u_1, u_1) + \lambda_2 G(u_1, u_2, u_2) + \lambda_4 G(u_0, u_2, u_2)$$

Using rectangular inequality i.e., $G(u_0, u_2, u_2) \leq G(u_0, u_1, u_1) + G(u_1, u_2, u_2)$.

We obtain

$$\begin{aligned} G(u_1, u_2, u_2) &\leq \lambda_1 G(u_0, u_1, u_1) + \lambda_2 G(u_1, u_2, u_2) \\ &\quad + \lambda_4 G(u_0, u_1, u_1) + \lambda_4 G(u_1, u_2, u_2), \\ (1 - \lambda_2 - \lambda_4)G(u_1, u_2, u_2) &\leq (\lambda_1 + \lambda_4)G(u_0, u_1, u_1), \\ G(u_1, u_2, u_2) &\leq \frac{(\lambda_1 + \lambda_4)}{(1 - \lambda_2 - \lambda_4)} G(u_0, u_1, u_1). \end{aligned}$$

letting $\frac{(\lambda_1 + \lambda_4)}{(1 - \lambda_2 - \lambda_4)} = k$, we obtain

$$G(u_1, u_2, u_2) \leq kG(u_0, u_1, u_1).$$

Inductively, we find a sequence u_p in $\mathbb{M}$, such that

$$\begin{aligned} G(u_1, u_2, u_2) &\leq kG(u_0, u_1, u_1), \\ G(u_2, u_3, u_3) &\leq k^2G(u_0, u_1, u_1), \\ &\vdots \\ G(u_p, u_{p+1}, u_{p+1}) &\leq k^pG(u_0, u_1, u_1). \end{aligned}$$

Now, using triangular inequality and for $q > p$, we have the following

$$\begin{aligned} G(u_p, u_q, u_q) &\leq G(u_p, u_{p+1}, u_{p+1}) + G(u_{p+1}, u_{p+2}, u_{p+2}) + \dots + G(u_{q-1}, u_q, u_q), \\ &= (k^p + k^{p+1} + k^{p+2} + \dots + k^{q-1})G(u_0, u_1, u_1), \\ &= (1 + k^p + k^{p+1} + k^{p+2} + \dots + k^{q-p-1})k^pG(u_0, u_1, u_1), \\ &= \frac{1 - k^{q-p-1}}{1 - k} k^pG(u_0, u_1, u_1). \end{aligned}$$

$$G(u_p, u_q, u_q) \rightarrow 0 \text{ as } p, q \rightarrow \infty,$$

Using Lemma 1, $\{u_p\}_{p=1}^\infty$ is a G -Cauchy sequence in $\mathbb{M}$.

For the completeness of $\mathbb{M}$, there exist some $r \in \mathbb{M}$, such that

$$\lim_{p \rightarrow \infty} u_p = r.$$

Now, we show that $r \in \mathbb{P}r$ and $r \in \mathbb{Q}r$.

From (8), we have

$$\begin{aligned} & \lambda_1 G(u_{2p}, r, r) + \lambda_2 \frac{G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \lambda_3 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \\ & + \lambda_4 \frac{G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \in s(\mathbb{P}u_{2p}, \mathbb{Q}r) \\ & \lambda_1 G(u_{2p}, r, r) + \lambda_2 \frac{G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \lambda_3 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \\ & + \lambda_4 \frac{G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \in \bigcap_{a \in \mathbb{P}u_{2p}} s(a, \mathbb{Q}r). \end{aligned}$$

Since $u_{2p+1} \in \mathbb{P}u_{2p}$, we obtain

$$\begin{aligned} & \lambda_1 G(u_{2p}, r, r) + \lambda_2 \frac{G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \lambda_3 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \\ & + \lambda_4 \frac{G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \in s(u_{2p+1}, \mathbb{Q}r) \\ & \lambda_1 G(u_{2p}, r, r) + \lambda_2 \frac{G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \lambda_3 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \\ & + \lambda_4 \frac{G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \\ & \in \bigcup_{b \in \mathbb{Q}r} s(G(u_{2p+1}, b, b)) \end{aligned}$$

So, there exists some $r_p \in \mathbb{Q}r$, such that

$$\begin{aligned} & \lambda_1 G(u_{2p}, r, r) + \lambda_2 \frac{G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \lambda_3 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \\ & + \lambda_4 \frac{G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \in s(G(u_{2p+1}, r_p, r_p)). \end{aligned}$$

Therefore, by definition we have

$$\begin{aligned} G(u_{2p+1}, r_p, r_p) & \leq \lambda_1 G(u_{2p}, r, r) + \lambda_2 \frac{G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \\ & \lambda_3 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} \\ & + \lambda_4 \frac{G(u_{2p}, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(u_{2p}, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, \mathbb{P}u_{2p}, \mathbb{P}u_{2p})G(r, \mathbb{Q}r, \mathbb{Q}r)}{1 + G(u_{2p}, r, r)}. \end{aligned}$$

Using the g.l.b. property

$$\begin{aligned} G(u_{2p+1}, r_p, r_p) &\leq \lambda_1 G(u_{2p}, r, r) + \lambda_2 \frac{G(u_{2p}, u_{2p+1}, u_{2p+1})G(r, r_p, r_p)}{1 + G(u_{2p}, r, r)} + \\ &\quad \lambda_3 \frac{G(r, u_{2p+1}, u_{2p+1})G(u_{2p}, r_p, r_p)}{1 + G(u_{2p}, r, r)} \\ &\quad + \lambda_4 \frac{G(u_{2p}, u_{2p+1}, u_{2p+1})G(u_{2p}, r_p, r_p)}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, u_{2p+1}, u_{2p+1})G(r, r_p, r_p)}{1 + G(u_{2p}, r, r)}. \end{aligned}$$

Using rectangular inequality, i.e.,

$$G(r, r_p, r_p) \leq [G(r, u_{2p+1}, u_{2p+1}) + G(u_{2p+1}, r_p, r_p)].$$

We obtain

$$\begin{aligned} G(r, r_p, r_p) &\leq G(r, u_{2p+1}, u_{2p+1}) + \lambda_1 G(u_{2p}, r, r) \\ &\quad + \lambda_2 \frac{G(u_{2p}, u_{2p+1}, u_{2p+1})G(r, r_p, r_p)}{1 + G(u_{2p}, r, r)} + \lambda_3 \frac{G(r, u_{2p+1}, u_{2p+1})G(u_{2p}, r_p, r_p)}{1 + G(u_{2p}, r, r)} \\ &\quad + \lambda_4 \frac{G(u_{2p}, u_{2p+1}, u_{2p+1})G(u_{2p}, r_p, r_p)}{1 + G(u_{2p}, r, r)} + \lambda_5 \frac{G(r, u_{2p+1}, u_{2p+1})G(r, r_p, r_p)}{1 + G(u_{2p}, r, r)}. \end{aligned}$$

Using limit as $p \rightarrow \infty$, we obtain

$$G(r, r_p, r_p) \rightarrow 0.$$

Using Definition 6, $r_p \rightarrow r$ as $p \rightarrow \infty$ also since $\mathbb{Q}r$ is closed then $r \in \mathbb{Q}r$. Similarly, $r \in \mathbb{P}r$. Thus, $\mathbb{P}$ and $\mathbb{Q}$ have a common fixed point. $\square$

By using $\mathbb{Q} = \mathbb{P}$, we have the following corollary.

Corollary 8. Let $(\mathbb{M}, G)$ be a complete C.V. extended b-metric space and $\mathbb{Q} : \mathbb{M} \rightarrow CB(\mathbb{M})$ be a multi-valued mapping fulfilling the g.l.b. property such that

$$\begin{aligned} \lambda_1 G(u, v, v) + \lambda_2 \frac{G(u, \mathbb{Q}u, \mathbb{Q}u)G(v, \mathbb{Q}v, \mathbb{Q}v)}{1 + d(u, v, v)} + \lambda_3 \frac{G(v, \mathbb{Q}u, \mathbb{Q}u)G(u, \mathbb{Q}v, \mathbb{Q}v)}{1 + G(u, v, v)} \\ + \lambda_4 \frac{G(u, \mathbb{Q}u, \mathbb{Q}u)G(u, \mathbb{Q}v, \mathbb{Q}v)}{1 + G(u, v, v)} + \lambda_5 \frac{G(v, \mathbb{Q}u, \mathbb{Q}u)G(v, \mathbb{Q}v, \mathbb{Q}v)}{1 + G(u, v, v)} \in s(\mathbb{Q}u, \mathbb{Q}v) \end{aligned} \quad (9)$$

for all $u, v \in \mathbb{M}$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are non-negative real numbers with $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_5 < 1$. Then, $\mathbb{Q}$ has a fixed point.

6. Application to Homotopy Result

Here, we apply the Homotopy result to our main theorem. Before applying the result, we recall some of the familiar definitions.

Definition 10. A relation $\leq$ is a total order on a set $\mathbb{M}$ if for every $u, v, w \in \mathbb{M}$, the following assertion is to satisfy:

- (i) Reflexivity : $u \leq u$.
- (ii) Anti-symmetry: if $u \leq v$ and $v \leq u$, then $s = u$.
- (iii) Transitive: if $u \leq v$ and $v \leq w$, then $u \leq w$.
- (iv) Comparability: for every $u, v \in \mathbb{M}$, either $u \leq v$ or $v \leq u$.

Remark 2. If the conditions (i), (ii), and (iii) are satisfied, then $\mathbb{M}$ is called partially ordered. A totally ordered set is called a chain.

Lemma 2. Kuratowski–Zorn’s

If $\mathbb{M}$ is a non-empty partially ordered set in which every chain has an upper bound, the $\mathbb{M}$ has a maximal element.

Definition 11. Let $\mathbb{E}$ and $\mathbb{F}$ be topological spaces and $\varphi, \psi : \mathbb{E} \rightarrow \mathbb{F}$ are continuous functions. A function $H : \mathbb{E} \times [0, 1] \rightarrow \mathbb{F}$ such that if $u \in \mathbb{E}$, then $H(u, 0) = \varphi(u)$ and $H(u, 1) = \psi(u)$, are called a homotopy between φ and ψ

Theorem 5. Let $(\mathbb{M}, G)$ be complete a G –metric space and O be an open subset of $\mathbb{M}$. Let $H : [0, 1] \times \overline{O} \rightarrow CB(\mathbb{M})$ be a multi-valued mapping with g.l.b. property. Assume that there exists $a \in \mathbb{M}$ and $0 < e \in \mathbb{R}$ such that the following conditions are satisfied:

- (a) $a \in H(t, a)$, for each $a \in Bd(O)$ and for all $t \in [0, 1]$;
- (b) $H(t, \cdot) : \overline{O} \rightarrow CB(\mathbb{M})$ be a multi-valued mapping satisfying:

$$\begin{aligned} & \lambda_1 G(a, b, b) + \frac{\lambda_2 G(a, H(t, a), H(t, a)) G(b, H(t_0, b), H(t_0, b))}{1 + G(a, b, b)} \\ & + \frac{\lambda_3 G(b, H(t, a), H(t, a)) G(a, H(t_0, b), H(t_0, b))}{1 + G(a, b, b)} \in s(H(t, a), H(t_0, a)) \end{aligned}$$

and

$$\frac{(1-k)r}{\tau} \in s(a_0, H(t_0, a_0)), \quad (10)$$

where $k = \frac{\lambda_1}{1-\lambda_2} < 1$, for every $\tau \geq 1$,

- (c) there exist a continuous non-decreasing function $g : (0, 1] \rightarrow A \cup O$ such that:

$$g(s) - g(t) \in s(H(s, a), H(t, b)), \quad g(s) \in g(t)$$

for every $s, t \in [0, 1]$ and each $a \in \overline{G}$, where $A = \{z \in \mathbb{M} : 0 < z\}$. Then, $H(0, \cdot)$ has a fixed point if and only if $H(1, \cdot)$ has a fixed point.

Proof. Suppose that $H(0, \cdot)$ has a fixed point u , this implies that $u \in H(0, u)$. From (a), $u \in O$. Define

$$B = \{(t, a) \in [0, 1] \times O : a \in H(a, t)\}.$$

Clearly, B is a non-empty set. Define a partial ordering in B as:

$$\begin{aligned} (t, a) \leq (s, b) & \Leftrightarrow t \leq s \quad \text{and} \quad G(a, b, b) \leq \frac{2}{1-k} (2g(s) - g(t) - g(t)) \\ & \leq \frac{2}{1-k} (2g(s) - 2g(t)). \end{aligned}$$

Let Z be a chain in B and $t_0 = \sup\{t, (t, a) \in Z\}$ and let $\{(t_n, a_n)\}_{n=1}^\infty$ be a sequence in Z such that $(t_n, a_n) \leq (t_{n+1}, a_{n+1})$ and $t_n \rightarrow t_0$ as $n \rightarrow \infty$. Then, for $n < m$, we have

$$\begin{aligned} G(a_m, a_n, a_n) & \leq \frac{2}{1-k} (2g(t_m) - g(t_n) - g(t_n)) \\ & \leq \frac{2}{1-k} (2g(t_m) - 2g(t_n)) \rightarrow 0 \quad n, m \rightarrow \infty, \end{aligned}$$

this means that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{M}$. Then, the completeness of $\mathbb{M}$ will lead us to the existing of point $a_0 \in \mathbb{M}$ such that $a_n \rightarrow a_0$ as $n \rightarrow \infty$. by condition (b) we have

$$\lambda_1 G(a_n, a_0, a_0) + \frac{\lambda_2 G(a_n, H(t_n, a_n), H(t_n, a_n)) G(a_0, H(t_0, a_0), H(t_0, a_0))}{1 + G(a_n, a_0, a_0)} + \frac{\lambda_3 G(a_0, H(t_n, a_n), H(t_n, a_n)) G(a_n, H(t_0, a_0), H(t_0, a_0))}{1 + G(a_n, a_0, a_0)} \in s(H(t_n, a_n), H(t_0, a_0)).$$

Since, $a_n \in H(t_n, a_n)$, then we have

$$\lambda_1 G(a_n, a_0, a_0) + \frac{\lambda_2 G(a_n, H(t_n, a_n), H(t_n, a_n)) G(a_0, H(t_0, a_0), H(t_0, a_0))}{1 + G(a_n, a_0, a_0)} + \frac{\lambda_3 G(a_0, H(t_n, a_n), H(t_n, a_n)) G(a_n, H(t_0, a_0), H(t_0, a_0))}{1 + G(a_n, a_0, a_0)} \in s(a_n, H(t_0, a_0)).$$

Therefore, there exist $a_p \in H(t_0, a_0)$ such that

$$\lambda_1 G(a_n, a_0, a_0) + \frac{\lambda_2 G(a_n, H(t_n, a_n), H(t_n, a_n)) G(a_0, H(t_0, a_0), H(t_0, a_0))}{1 + G(a_n, a_0, a_0)} + \frac{\lambda_3 G(a_0, H(t_n, a_n), H(t_n, a_n)) G(a_n, H(t_0, a_0), H(t_0, a_0))}{1 + G(a_n, a_0, a_0)} \in s(a_n, a_p).$$

Since, H has a g.l.b. property, then

$$G(a_n, a_p, a_p) \leq \lambda_1 G(a_n, a_0, a_0) + \frac{\lambda_2 G(a_n, H(t_n, a_n), H(t_n, a_n)) G(a_0, H(t_0, a_0), H(t_0, a_0))}{1 + G(a_n, a_0, a_0)} + \frac{\lambda_3 G(a_0, H(t_n, a_n), H(t_n, a_n)) G(a_n, H(t_0, a_0), H(t_0, a_0))}{1 + G(a_n, a_0, a_0)}.$$

which implies that

$$G(a_n, a_p, a_p) \leq \lambda_1 G(a_n, a_0, a_0) + \lambda_3 \frac{G(a_0, a_n, a_n) G(a_n, a_p, a_p)}{1 + G(a_n, a_0, a_0)}. \quad (11)$$

By using the following fact

$$\begin{aligned} G(a_n, a_0, a_0) &\leq G(a_n, a_0, a_n) \text{ where } a_0 \neq a_n. \\ G(a_n, a_0, a_n) &= G(a_0, a_n, a_n) \text{ using symmetric property.} \\ G(a_0, a_n, a_n) &< 1 + G(a_0, a_n, a_n). \end{aligned}$$

By using these equations, then the inequality 11 becomes

$$G(a_n, a_p, a_p) \leq \frac{\lambda_1}{1 - \lambda_3} G(a_n, a_0, a_0).$$

Furthermore, it is noted that, $G(a_0, a_p, a_p) \leq \tau G(a_0, a_n, a_n) + \tau G(a_n, a_n, a_p)$. Then, the above equality of becomes

$$G(a_0, a_p, a_p) \leq \tau G(a_0, a_n, a_n) + \frac{\tau \lambda_1}{1 - \lambda_3} G(a_n, a_0, a_0).$$

Using $n \rightarrow \infty$, we obtain

$$G(a_0, a_p, a_p) \leq \tau G(a_0, a_n, a_n) + \frac{\tau \lambda_1}{1 - \lambda_3} G(a_n, a_0, a_0) \rightarrow 0.$$

This implies that $a_p \rightarrow a_0 \in H(t_0, a_0)$, thus $a_0 \in \mathbb{M}$ means $(t_0, a_0) \in B$. using the definition of B , $(t, a) \leq (t_0, a_0) \in Z$, which provides (t_0, a_0) , is an upper bound of Z . By using Kuratowski–Zorn’s Lemma, Z has a maximal element (t_0, a_0) . Let us claim that $t_0 = 1$. However, suppose that $t_0 \leq 1$ and choose $0 < e \in \mathbb{R}$, $t_0 \leq t$ such that $\bar{B}(t_0, r) \subset G$. By condition (c), we have $g(t) - g(t_0) \in s(H(t, a), H(t_0, a_0))$, where $g(t) - g(t_0) \in (a_0, H(t, a))$ for all $a_0 \in H(t_0, a_0)$. Here, there exists $a \in H(t, a)$ such that $g(t) - g(t_0) \in s(G(a_0, a, a))$ and $G(a_0, a, a) \leq 2g(t) - 2g(t_0) < (1 - k)r$ for $r = \frac{2}{1-k}(2g(t) - 2g(t_0))$. It follows that $G(a, a_0, a_0) \leq (1 - k)r$. By condition (b), we deduced that the mapping $H(t, \cdot) : \bar{B}(a_0, r) \rightarrow CB(\mathbb{M})$ satisfies all the assertion of the Corollary. Therefore, for all $t \in [0, 1]$, there exist $a \in \bar{B}(a_0, r)$ such that $a \in H(t, a)$. Hence, $(a, t) \in B$. Since $G(a, a_0, a_0) \leq r = \frac{2}{1-k}(2g(t) - 2g(t_0))$, then we have $(t_0, a_0) \leq (t, a)$, which is a contradiction. So, $t_0 = 1$. This shows that $H(\cdot, 1)$ has a fixed point, then, on similar steps, one can prove that $H(0, \cdot)$ has a fixed point. $\square$

7. Conclusions

In this article, with the help of rational inequality, some results were acquired related to multi-valued functions over G –metric spaces. Some new definitions of open ball, open sets, and limit points of a set are introduced over G –metric spaces. Homotopy application are also provided. Some example provided to discuss and strengthen primary finding. We hope that the outcomes of this manuscript will be helpful to understand the literature of fixed point theory and its applications. In the future, we will try to explore these ideas in the directions of complex single-valued mapping and complex multi-valued mapping.

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