symmetry

MDPI

Article

Monstrous M-Theory †

Alessio Marrani [1,*], Mike Rios [2] and David Chester [3]

1 Instituto de Física Teorica, Departamento de Física, Universidad de Murcia, Campus de Espinardo, E-30100 Murcia, Spain
2 Dyonica ICMQG, Los Angeles, CA 90032, USA
3 Quantum Gravity Research, Los Angeles, CA 90290, USA
* Correspondence: jazzphyzz@gmail.com

† Dedicated to John H. Conway.

Abstract: In $26+1$ space–time dimensions, we introduce a gravity theory whose massless spectrum can be acted upon by the Monster group when reduced to $25+1$ dimensions. This theory generalizes M-theory in many respects, and we name it Monstrous M-theory, or M^2-theory. Upon Kaluza–Klein reduction to $25+1$ dimensions, the M^2-theory spectrum irreducibly splits as $\mathbf{1} \oplus \mathbf{196{,}883}$, where $\mathbf{1}$ is identified with the dilaton, and **196,883** is the dimension of the smallest non-trivial representation of the Monster. This provides a field theory explanation of the lowest instance of the Monstrous Moonshine, and it clarifies the definition of the Monster as the automorphism group of the Griess algebra by showing that such an algebra is not merely a sum of unrelated spaces, but descends from massless states for M^2-theory, which includes Horowitz and Susskind's bosonic M-theory as a subsector. Further evidence is provided by the decomposition of the coefficients of the partition function of Witten's extremal Monster SCFT in terms of representations of SO_{24}, the massless little group in $25+1$; the purely bosonic nature of the involved SO_{24}-representations may be traced back to the unique feature of 24 dimensions, which allow for a non-trivial generalization of the triality holding in 8 dimensions. Last but not least, a certain subsector of M^2-theory, when coupled to a Rarita–Schwinger massless field in $26+1$, exhibits the same number of bosonic and fermionic degrees of freedom; we cannot help but conjecture the existence of a would-be $\mathcal{N}=1$ supergravity theory in $26+1$ space–time dimensions.

Keywords: M-theory; Monster group; Monstrous Moonshine

Citation: Marrani, A.; Rios, M.; Chester, D. Monstrous M-Theory. *Symmetry* **2023**, *15*, 490. https://doi.org/10.3390/sym15020490

Academic Editors: Bobomurat Ahmedov, Ugur Camci and Ashfaque H. Bokhari

Received: 13 November 2022
Revised: 25 January 2023
Accepted: 10 February 2023
Published: 13 February 2023

1. Introduction

The Monster group $\mathbb{M}$, the largest of sporadic groups, was predicted to exist by Fischer and Griess back in the mid-1970s [1]. $\mathbb{M}$ is the automorphism group of the Griess algebra, as well as the automorphism group of the Monster vertex operator algebra (VOA) [2,3]. Conway and Norton defined *Monstrous Moonshine* as the observation that the Fourier coefficients of the j-function decompose into sums of dimensions of representations of $\mathbb{M}$ itself [4] and this was proven by Borcherds using generalized Kac–Moody algebras [5]. In the language of conformal field theory (CFT), Monstrous Moonshine is the statement that the states of an orbifold theory, which is the $D=25+1$ bosonic string theory on $(\mathbb{R}^{24}/\Lambda_{24})/\mathbb{Z}_2$ (where Λ_{24} is the Leech lattice [6–8]), are organized in representations of the Monster group, with a partition function equivalent to the j-function [9–11]. Witten also found the Monster group in three-dimensional pure gravity [12], for AdS_3, where the dual CFT is expected to be that of Frenkel, Lepowsky and Meurman (FLM) [3]. A Monster SCFT and fermionization of the Monster CFT were also defined and studied [9,13]

Eguchi, Ooguri and Tachikawa later noticed that the elliptic genus of the K3 surface has a natural decomposition in terms of dimensions of irreducible representations of the largest Mathieu group M_{24} [14], and this was named *Umbral Moonshine* [15,16], which generalizes the Moonshine correspondence for other sporadic groups [17].

With Witten's proposal [18] that M-theory unifies all of the ten-dimensional string theories with $\mathcal{N} = 1$ supergravity in $D = 10 + 1$ space–time dimensions, Horowitz and Susskind argued [19] that there exists a bosonic M-theory in $D = 26 + 1$ that reduces to the bosonic string in $25 + 1$ upon compactification. As the Monster group has a string theoretic interpretation in $D = 25 + 1$ [11,20,21], it is also natural to consider its action on fields from $D = 26 + 1$; support for this is found from bosonic M-theory's M2-brane near horizon geometry $AdS_4 \times S^{23}$, discussed by Horowitz and Susskind as an evidence for a dual $2 + 1$ CFT with global SO_{24} symmetry [19]. By observing that the automorphism group of the Leech lattice Λ_{24}, the Conway group Co_0 [6,8], is a maximal finite subgroup of SO_{24}, and its $\mathbb{Z}_2$ quotient $Co_1 \simeq Co_0/\mathbb{Z}_2$ is a maximal subgroup of the Monster [3,6], it is possible to realize some Monstrous symmetry as a finite subgroup of $\mathcal{R}$-symmetry in $26 + 1$ dimensions [22].

In the present paper, we introduce an Einstein gravity theory coupled to p-forms in $26 + 1$ space–time dimensions, which contains the aforementioned bosonic string theory [19] as a subsector. We name such a theory *Monstrous M-theory*, or shortly *M^2-theory*, because its massless spectrum (with gauge fields mod $\mathbb{Z}_2$) has the same dimension (196,884) as the Griess algebra and upon dimensional reduction can be acted upon by the Monster group $\mathbb{M}$ itself. When reducing to $25 + 1$, a web of gravito-dilatonic theories, named *Monstrous gravities*, is generated, in which the decomposition 196,884 = **196,883** $\oplus$ **1**, which first hinted at Monstrous Moonshine [4], entails the fact that the dilaton scalar field ϕ in $25 + 1$ is a singlet of $\mathbb{M}$ itself. As such, the irreducibility under $\mathbb{M}$ is crucially related to dilatonic gravity in $25 + 1$ space–time dimensions. The existence of a "weak" form of the SO_8-triality for SO_{24}, which we will name $\boldsymbol{\lambda}$-triality, gives rise to a p ($\geqslant$0)-parametrized tower of "weak" trialities involving p-form spinors in 24 dimensions, which we will regard as massless p-form spinor fields in $25 + 1$ space–time dimensions. Such "weak" trialities are instrumental to providing most of the Monstrous gravity theories with a fermionic (massless) spectrum such that the spectrum is still acted upon by the Monster $\mathbb{M}$.

All this gives an elegant description of the Monster's minimal non-trivial representation **196,883** in relation to the total number of massless degrees of freedom of Monstrous gravities in $D = 25 + 1$; as such, this also elucidates the definition of $\mathbb{M}$ as the automorphism group of the Griess algebra (the degree two piece of the Monster VOA), which was considered to be artificial in that it was thought to involve an algebra of two or more unrelated spaces [3,6,23].

The plan of the paper is as follows. We give motivation for Monstrous M-theory by lifting the M2-brane from $D = 10 + 1$ to $D = 26 + 1$ and breaking the Poincaré symmetry in its near-horizon geometry, which results in an SO_{24} $\mathcal{R}$-symmetry, that has the Conway group Co_0 as a maximal finite subgroup. We then reduce the near-horizon geometry of the M2-brane in $D = 26 + 1$ and relate the holography to Witten's BTZ black hole [12] with Monstrous symmetry. Next, in Section 3 we briefly review the *triality* among the 8-dimensional representations of the Lie algebra $\mathfrak{d}_4$, and then, in Section 3.1 we introduce some "weak" generalization for the Lie algebra $\mathfrak{d}_{12}$, which we will name $\boldsymbol{\lambda}$-triality, giving rise to the $\boldsymbol{\psi}$-triality, as discussed in Section 3.2. As it will be seen in the treatment below, the "weakness" of the aforementioned trialities relies on the *reducibility* of the bosonic representations involved. Then, in Section 4 we introduce and classify non-supersymmetric, gravito-dilatonic theories, named *Monstrous gravities*, in $25 + 1$ space–time dimensions, whose massless spectrum (also including fermions in most cases) has a dimension of 196,884, namely the same dimension as the Griess algebra [2,3]. A purely bosonic uplift to $26 + 1$ space–time dimensions is discussed in Section 5; in this framework, we introduce the *Monstrous M-theory*, also named *M^2-theory*, and we discuss its possible Lagrangian in Section 5.1. Moreover, Section 5.2 discusses a subsector of the M^2-theory which displays the same number of bosonic and fermionic massless degrees of freedom in $26 + 1$; in Section 5.2.1, this allows us to conjecture a Lagrangian and local supersymmetry transformations for the would-be $\mathcal{N} = 1$ Einstein supergravity theory in $26 + 1$ space–time dimensions. Then, Section 6 presents a cohomological construction of both the $\mathfrak{e}_8$ root lattice and the Leech lattice Λ_{24} (respectively determining optimal sphere packing in 8 and 24 dimensions [6]), and all this is again related to M-theory (i.e., $\mathcal{N}=1$ supergravity) in $D=10+1$ and to the aforementioned would-be $\mathcal{N}=1$ supergravity in $26+1$, respectively. Before concluding the paper, in order to provide further evidence for a consistent higher-dimensional field theory probed by $\mathbb{M}$, we decompose the first coefficients of the partition function of Monster CFT, firstly put forward by Witten [12], in terms of dimensions of representations of SO_{24}, namely of the massless little group in $25+1$ space–time dimensions; an interesting consequence of the aforementioned "weak" trialities characterizing SO_{24} is that the relevant SO_{24}-representations can be reduced to be *only* the p-form ones, $\wedge^p$, for suitable values of p and with non-trivial multiplicities. Final comments are then contained in the conclusive Section 8. An Appendix A, detailing the Chern–Simons Lagrangian terms for M^2-theory, concludes the paper.

2. Evidence for Monstrous M-Theory

2.1. Bosonic M-Theory in $D=26+1$

Horowitz and Susskind conjectured there exists a strong coupling limit of bosonic string theory that generalizes the relation between M-theory and superstring theory, called bosonic M-theory [19]. The main evidence for the existence of such a $D=26+1$ theory comes from the dilaton and its connection to the coupling constant, with the dilaton entering the action for the massless sector of bosonic string theory as

$$S=\int d^{26}x\sqrt{-g}e^{-2\phi}\left[R+4\nabla_\mu\phi\nabla^\mu\phi-\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right], \tag{1}$$

in a way similar to type IIA string theory, as if representing the compactification scale of a Kaluza–Klein reduction from $D=26+1$ space–time dimensions with $\mathbf{324}\to\mathbf{299}+\mathbf{24}+\mathbf{1}$ graviton decomposition. However, while in type IIA string theory, the existence of a vector boson in the string spectrum implies an S^1 compactification, in closed bosonic string theory, there is no massless vector. For this reason, an $S^1/\mathbb{Z}_2$ orbifold compactification of bosonic M-theory was proposed as its origin [19]. The bosonic string is then a stretched membrane across the interval; the orbifold breaks translation symmetry, and thus the massless vector does not appear. An orbifold construction was also used to eliminate the **24** vector in the Monster CFT partition function [3,9], which suggests a $D=26+1$ origin in light of bosonic M-theory [19]. In fact, the original FLM theory is bosonic string theory on $(\mathbb{R}^{24}/\Lambda_{24})/\mathbb{Z}_2$, and thus it can be regarded as certain compactification of bosonic M-theory. It is the sporadic SCFT constructions [9,12,17,24] with SO_{24} spinors and twisted sector states that require a generalization of bosonic M-theory with fermions.

Bosonic M-theory contains a three-form gauge field $C^{(3)}$ for its M2-brane, which, if one of its indices is reduced along the compact direction, becomes the familiar two-form $B^{(2)}$ of bosonic string theory [19]. If all components of $C^{(3)}$ are evaluated in the 26 dimensions, a (massless) 3-form (**2024**) results. In the present work, we will consider the case in which the $D=25+1$ massless 1-form (**24**) and 3-form (**2024**) persist, and actually they give rise to the so-called $\boldsymbol{\lambda}$-triality, which is the generalization of SO_8-triality up to SO_{24} in a "weaker", namely, reducible, way ("Weak" triality was suggested by Eric Weinstein in 2016, at the Advances in Quantum Gravity conference (San Francisco)), of the form

$$\underset{\wedge^1}{\mathbf{24}}\oplus\underset{\wedge^3}{\mathbf{2024}},\quad \underset{\boldsymbol{\lambda}}{\mathbf{2048}},\quad \underset{\boldsymbol{\lambda}_c}{\mathbf{2048}'}, \tag{2}$$

relating three 2048-dimensional representations of SO_{24} (with the subscript "c" denoting spinor conjugation).

2.2. Lifting the M2-Brane to $D=26+1$ and the Leech Lattice

In $D=10+1$, the presence of the M2-brane breaks the Poincaré symmetry from $SO_{10,1}$ to $SO_{2,1}\otimes SO_8$, with SO_8 being the $\mathcal{R}$-symmetry. The near-horizon geometry of the M2-brane is given by $AdS_4\otimes S^7$. From $D=26+1$ bosonic M-theory, the Poincaré symmetry breaks from $SO_{26,1}$ to $SO_{2,1} \otimes SO_{24}$, where the $\mathcal{R}$-symmetry is enhanced to SO_{24}, and the near-horizon geometry is $AdS_4 \otimes S^{23}$ [19]. By dimensional reduction, one can obtain a $AdS_3 \otimes S^{23}$ background (i.e., a generalized black string geometry), and make contact with Witten's three-dimensional BTZ black hole [12] by noting that the Conway group Co_0, the automorphism group of the *Leech lattice* Λ_{24}, is a maximal finite subgroup of SO_{24}. Geometrically, the 196,560 norm, four Leech vectors form a discrete S^{23} with symmetry given by the Conway group Co_0 [6,8]. (It is interesting to observe that 196,560 is not the dimension of a unique irreducible representation of Co_0, but rather it can be decomposed as a sum of dimensions of irreducible representations of Co_0 [25]. Remarkably, such a decomposition can be made purely in terms of irreducible representations of SO_{24} which all survive (and stay irreducible) under the maximal reduction $SO_{24} \to Co_0$, namely: $196{,}560 = 2 \times \mathbf{95{,}680} \oplus \mathbf{4576} \oplus 2 \times \mathbf{276} \oplus 3 \times \mathbf{24}$, which, by constraining the cardinality of $\mathbf{299}$ (massless graviton in $25+1$) not to exceed 1 (we will do this throughout the whole present paper), can also be rewritten as $196{,}560 = 2 \times \mathbf{95{,}680} \oplus \mathbf{4576} \oplus \mathbf{299} \oplus \mathbf{276} \oplus 2 \times \mathbf{24} \oplus \mathbf{1}$.) The quotient $Co_0/\mathbb{Z}_2$ yields the simple Conway group Co_1 [6], where $2^{1+24}.Co_1$ is a maximal subgroup of the Monster group $\mathbb{M}$ itself.

The set of norm four (i.e., minimal) Leech vectors is composed of three types of elements [3]:

$$\begin{aligned} \Lambda_4 &= \Lambda_4^1 \cup \Lambda_4^2 \cup \Lambda_4^3, && (3) \\ |\Lambda_4| &= |\Lambda_4^1| + |\Lambda_4^2| + |\Lambda_4^3| = 97{,}152 + 276 \times 4 + 98{,}304 = 196{,}560. && (4) \end{aligned}$$

Later, we show how to naturally recover these three types of elements from the field content of a Monstrous M-theory in $D = 26+1$. Moreover, the assignment permits a $\mathbb{Z}_2$ identification, that reduces 196,560 to 98,280, which can occur via an orbifold. Therefore, by carefully mapping $D = 25+1$ fields descending from Monstrous M-theory to the three types of norm four Leech vectors, and assigning the remaining fields to the degree two piece of the Monster VOA (the Griess algebra), the construction of the Moonshine module by FLM allows an action of the Monster $\mathbb{M}$ [3].

2.3. *Superalgebras and Central Extensions*

2.3.1. From 10 + 1 ...

Recalling some off-shell $SO_{10,1}$ representations and their Dynkin labels (In an odd number of dimensions (i.e., for $\mathfrak{b}_n$), the rank-2 symmetric bi-spinor is equivalent to the n-form representation (if this is interpreted as an n-brane, its Hodge dual is the $(n-3)$-brane). In $10+1$ dimensions $n = 5$, whereas in $26+1$ dimensions $n = 13$),

$$\begin{array}{rl} \underset{\wedge^1}{\mathbf{11}}: & (1,0^4); \\ \underset{\text{(0-form) spinor } \lambda}{\mathbf{32}}: & (0^4,1); \\ \underset{\wedge^2}{\mathbf{55}}: & (0,1,0^3); \\ \underset{\text{rank-2 symm. on spinor}}{\mathbf{462}} \simeq \underset{\wedge^5}{\binom{11}{5}}: & (0^4,2), \end{array} \tag{5}$$

the central charges that extend the $\mathcal{N} = 1$, $D = 10+1$ superalgebra (i.e., the M-theory superalgebra) can be computed from the anticommutator of the $\mathbf{2}^{\frac{11-1}{2}} \equiv \mathbf{32}$ Majorana spinor supercharge,

$$\begin{aligned} \underset{32\times 33/2=528}{\mathbf{32} \otimes_s \mathbf{32}} &= \underset{\text{1-form } P_\mu}{\mathbf{11}} \oplus \underset{\text{M2}}{\mathbf{55}} \oplus \underset{\text{M5}}{\mathbf{462}}, && (6) \\ \text{with Hodge duality} &: \underset{\text{M2}}{2} \to 4 \to 11-4 = 7 \to \underset{\text{M5}}{5}, && (7) \end{aligned}$$

thus yielding (here $\alpha, \beta = 1, \ldots, 32$, whereas μ-indices run $0, 1, \ldots, 10$)

$$\{Q_\alpha, Q_\beta\} = \left(\Gamma^\mu C^{-1}\right)_{\alpha\beta} P_\mu + \frac{1}{2}\left(\Gamma^{\mu_1\mu_2} C^{-1}\right)_{\alpha\beta} \underset{\text{M2}}{Z^{(2)}_{[\mu_1\mu_2]}} + \frac{1}{5!}\left(\Gamma^{\mu_1\cdots\mu_5} C^{-1}\right)_{\alpha\beta} \underset{\text{M5}}{Z^{(5)}_{[\mu_1\cdots\mu_5]}}. \tag{8}$$

The M-theory superalgebra has a higher dimensional origin. In fact, the central extensions in $D = 10 + 1$ in the right-hand side of (8) can be obtained by a Kaluza–Klein time-like reduction of the $(1, 0)$ minimal chiral superalgebra in $D = 10 + 2$, whose central extensions read as (cfr. (3.6) of [26] with $\mathbf{n} = 0$; $\hat{\mu}$-indices here run $\mathring{0}, 0, 1, \ldots, 10$)

$$\{Q_\alpha, Q_\beta\} = \frac{1}{2}\left(\Gamma^{\hat{\mu}_1\hat{\mu}_2}C^{-1}\right)_{\alpha\beta} Z^{(2)}_{[\hat{\mu}_1\hat{\mu}_2]} + \frac{1}{6!}\left(\Gamma^{\hat{\mu}_1\ldots\hat{\mu}_6}C^{-1}\right)_{\alpha\beta} Z^{(6)}_{[\hat{\mu}_1\ldots\hat{\mu}_6]}. \tag{9}$$

By splitting $\hat{\mu} = \mathring{0}, \mu$, one indeed obtains

$$Z^{(2)}_{\hat{\mu}_1\hat{\mu}_2} \rightarrow \begin{cases} Z^{(2)}_{\mu_1\mathring{0}} \sim P_\mu; \\ Z^{(2)}_{\mu_1\mu_2}; \end{cases} \tag{10}$$

$$Z^{(6)}_{\hat{\mu}_1\ldots\hat{\mu}_6} \rightarrow \begin{cases} Z^{(5)}_{\mu_1\ldots\mu_5\mathring{0}} \sim Z^{(5)}_{\mu_1\ldots\mu_5}; \\ Z^{(6)}_{\mu_1\ldots\mu_6} \rightarrow \epsilon_{\mu_1\ldots\mu_{11}} Z^{(6)}_{\nu_6\ldots\nu_{11}} \eta^{\mu_6\nu_6} \ldots \eta^{\mu_{11}\nu_{11}} \sim Z^{(5)}_{\mu_1\ldots\mu_5}, \end{cases} \tag{11}$$

and therefore (9) yields to (8).

From the right-hand side of (8), in terms of on-shell SO_9 representations,

$$\begin{array}{ccc} \underset{g}{\mathbf{44}}: & (2, 0^3) & \text{bosons;} \\ \underset{\text{M2 (3-form pot. } \wedge^3)}{\mathbf{84}}: & (0^2, 1, 0) & \text{bosons;} \\ \underset{\textit{gravitino} \text{ (1-form spinor) } \boldsymbol{\psi}}{\mathbf{128}}: & (1, 0^2, 1) & \text{fermions,} \end{array} \tag{12}$$

one obtains the field content of the massless multiplet of M-theory (i.e., of $\mathcal{N} = 1$, $D = 10 + 1$ supergravity), having

$$\underset{44+84}{B} = \underset{128}{F}. \tag{13}$$

2.3.2. . . . to 26 + 1

Let us generalize this to $D = (10 + 16) + 1 = 26 + 1$ space–time dimensions. We start some off-shell $SO_{26,1}$ representations and their Dynkin labels,

$$\begin{array}{cc} \underset{\wedge^1}{\mathbf{27}}: & (1, 0^{12}); \\ \underset{\boldsymbol{\lambda}}{\mathbf{8192}}: & (0^{12}, 1); \\ \underset{\wedge^2}{\mathbf{351}}: & (0, 1, 0^{11}); \\ \underset{\wedge^5}{\mathbf{80{,}730}}: & (0^4, 1, 0^8); \\ \underset{\wedge^6}{\mathbf{296{,}010}}: & (0^5, 1, 0^7); \\ \underset{\wedge^9}{\mathbf{4{,}686{,}825}}: & (0^8, 1, 0^4); \\ \underset{\wedge^{10}}{\mathbf{8{,}436{,}285}}: & (0^9, 1, 0^3); \\ \underset{\text{rank-2 symm. on spinor}}{\mathbf{20{,}058{,}300}} = \underset{\wedge^{13}}{\binom{27}{13}}: & (0^{12}, 2). \end{array} \tag{14}$$

Thus, the central charges that extends the $\mathcal{N} = 1$, $D = 26 + 1$ superalgebra can be computed from the anticommutator of the $\mathbf{2}^{\frac{27-1}{2}} \equiv \mathbf{8192}$ Majorana spinor supercharge,

$$\underset{8192\times 8193/2=33{,}558{,}528}{\mathbf{8192}\otimes_s\mathbf{8192}} = \underset{P_\mu}{\mathbf{27}}\oplus\underset{\mathrm{M2}}{\mathbf{351}}\oplus\underset{\mathrm{M5}}{\mathbf{80{,}730}}\oplus\underset{\mathrm{M6}}{\mathbf{296{,}010}}\oplus\underset{\mathrm{M9}}{\mathbf{4{,}686{,}825}}\oplus\underset{\mathrm{M10}}{\mathbf{8{,}436{,}285}}\oplus\underset{\mathrm{M13}}{\mathbf{20{,}058{,}300}}; \quad (15)$$

$$\text{with Hodge duality}\quad : \quad \begin{cases} \underset{\mathrm{M2}}{2} \to 4 \to 27-4=23 \to \underset{\mathrm{M21}}{21}; \\ \underset{\mathrm{M5}}{5} \to 7 \to 27-7=20 \to \underset{\mathrm{M18}}{18}; \\ \underset{\mathrm{M6}}{6} \to 8 \to 27-8=19 \to \underset{\mathrm{M17}}{17}; \\ \underset{\mathrm{M9}}{9} \to 11 \to 27-11=16 \to \underset{\mathrm{M14}}{14}; \\ \underset{\mathrm{M10}}{10} \to 12 \to 27-12=15 \to \underset{\mathrm{M13}}{13}, \end{cases} \quad (16)$$

thus yielding (here $\alpha,\beta = 1,\ldots,8192$, whereas μ-indices run $0,1,\ldots,26$)

$$\begin{aligned}\{Q_\alpha, Q_\beta\} = {} & \left(\Gamma^\mu C^{-1}\right)_{\alpha\beta} P_\mu + \frac{1}{2}\left(\Gamma^{\mu_1\mu_2} C^{-1}\right)_{\alpha\beta} \underset{\mathrm{M2}}{Z^{(2)}_{[\mu_1\mu_2]}} + \frac{1}{5!}\left(\Gamma^{\mu_1\cdots\mu_5} C^{-1}\right)_{\alpha\beta} \underset{\mathrm{M5}}{Z^{(5)}_{[\mu_1\cdots\mu_5]}} \\ & + \frac{1}{6!}\left(\Gamma^{\mu_1\cdots\mu_6} C^{-1}\right)_{\alpha\beta} \underset{\mathrm{M6}}{Z^{(6)}_{[\mu_1\cdots\mu_6]}} + \frac{1}{9!}\left(\Gamma^{\mu_1\cdots\mu_9} C^{-1}\right)_{\alpha\beta} \underset{\mathrm{M9}}{Z^{(9)}_{[\mu_1\cdots\mu_9]}} \\ & + \frac{1}{10!}\left(\Gamma^{\mu_1\cdots\mu_{10}} C^{-1}\right)_{\alpha\beta} \underset{\mathrm{M10}}{Z^{(10)}_{\mu_1\cdots\mu_{10}}} + \frac{1}{13!}\left(\Gamma^{\mu_1\cdots\mu_{13}} C^{-1}\right)_{\alpha\beta} \underset{\mathrm{M13}}{Z^{(13)}_{\mu_1\cdots\mu_{13}}}. \end{aligned} \quad (17)$$

Additionally, the $\mathcal{N}=1$, $D=26+1$ superalgebra has a higher dimensional origin. In fact, the central extensions in $D=26+1$ in the right-hand side of (17) can be obtained by a Kaluza–Klein time-like reduction from the $(1,0)$ minimal chiral superalgebra in $D=26+2$, whose central extensions read as (cfr. (3.6) of [26] with $\mathbf{n}=2$; $\hat{\mu}$-indices here run $\mathring{0}$, $0,1,\ldots,26$)

$$\begin{aligned}\{Q_\alpha, Q_\beta\} = {} & \frac{1}{2}\left(\Gamma^{\hat{\mu}_1\hat{\mu}_2} C^{-1}\right)_{\alpha\beta} Z^{(2)}_{[\hat{\mu}_1\hat{\mu}_2]} + \frac{1}{6!}\left(\Gamma^{\hat{\mu}_1\cdots\hat{\mu}_6} C^{-1}\right)_{\alpha\beta} Z^{(6)}_{[\hat{\mu}_1\cdots\hat{\mu}_6]} \\ & + \frac{1}{10!}\left(\Gamma^{\hat{\mu}_1\cdots\hat{\mu}_{10}} C^{-1}\right)_{\alpha\beta} Z^{(10)}_{[\hat{\mu}_1\cdots\hat{\mu}_{10}]} + \frac{1}{14!}\left(\Gamma^{\hat{\mu}_1\cdots\hat{\mu}_{14}} C^{-1}\right)_{\alpha\beta} Z^{(14)}_{[\hat{\mu}_1\cdots\hat{\mu}_{14}]}. \end{aligned} \quad (18)$$

By splitting $\hat{\mu} = \mathring{0}, \mu$, one indeed obtains

$$Z^{(2)}_{\hat{\mu}_1\hat{\mu}_2} \to \begin{cases} Z^{(2)}_{\mu_1\mathring{0}} \sim P_\mu; \\ Z^{(2)}_{\mu_1\mu_2}; \end{cases} \quad (19)$$

$$Z^{(6)}_{\hat{\mu}_1\cdots\hat{\mu}_6} \to \begin{cases} Z^{(5)}_{\mu_1\cdots\mu_5\mathring{0}} \sim Z^{(5)}_{\mu_1\cdots\mu_5}; \\ Z^{(6)}_{\mu_1\cdots\mu_6}; \end{cases} \quad (20)$$

$$Z^{(10)}_{\hat{\mu}_1\cdots\hat{\mu}_{10}} \to \begin{cases} Z^{(10)}_{\mu_1\cdots\mu_9\mathring{0}} \sim Z^{(9)}_{\mu_1\cdots\mu_9}; \\ Z^{(10)}_{\mu_1\cdots\mu_{10}}, \end{cases} \quad (21)$$

$$Z^{(14)}_{\hat{\mu}_1\cdots\hat{\mu}_{14}} \to \begin{cases} Z^{(14)}_{\mu_1\cdots\mu_{13}\mathring{0}} \sim Z^{(13)}_{\mu_1\cdots\mu_{13}}; \\ Z^{(14)}_{\mu_1\cdots\mu_{14}} \to \epsilon_{\mu_1\cdots\mu_{27}} Z^{(14)}_{\nu_{14}\cdots\nu_{27}} \eta^{\mu_{14}\nu_{14}} \ldots \eta^{\mu_{27}\nu_{27}} \sim Z^{(13)}_{\mu_1\cdots\mu_{13}}, \end{cases} \quad (22)$$

and therefore (18) yields to (17).

We will elaborate on the possible existence of local supersymmetry in $26+1$ further below. For the time being, we confine ourselves to observe that Susskind and Horowitz identified a subset of the above (central, p-brane) charges for bosonic M-theory [19], whereas the most general set of central extensions is provided by the right-hand side of (17). We note that the automorphic form of the fake Monster Lie algebra satisfies functional equations generated by transformations in the group $\mathrm{Aut}(II_{26,2})^+$ [27], a discrete subgroup of $O_{26,2}$ which can transform fields in $D=26+2$, $D=26+1$ and $D=25+1$. Thus, the signature

$D = 26 + 2$ has proven essential in the proof of Monstrous Moonshine, and it gives further evidence for an M-theoretical origin. We can anticipate that Monstrous M-theory, in fact, has its most natural origin in $D = 29 + 1$ or $D = 28 + 2$ with purely bosonic massless states descending from 5-form and dual 23-form gauge fields, respectively of a 4-brane and 22-brane. By dimensional reduction to $D = 26 + 1$, such higher 5-form and 23-form gauge fields break up non-trivially, providing a rich structure to possibly realize a would-be supergravity with a **98,304** Rarita–Schwinger field, as we will see in Section 5.2.

2.4. M-Branes, Horava–Witten and the Monster SCFT

The presence of an M10-brane breaks $SO_{26,1}$ Poincaré symmetry to $SO_{10,1} \otimes SO_{16}$, giving $D = 10 + 1$ Poincaré symmetry on its world volume [22]. The **8192** spinor then factorizes as $(\mathbf{32}, \mathbf{128}) \oplus (\mathbf{32}, \mathbf{128}')$, thus isolating a hidden $\mathbf{128}^{(\prime)}$ spinor, which can be used to form $\mathfrak{e}_8 = \mathfrak{so}_{16} \oplus \mathbf{128}^{(\prime)}$. Intriguingly, this may suggest an origin for Horava–Witten theory [28,29], which requires an eleven-manifold M^{11} with boundary, whose boundary points are the $\mathbb{Z}_2$ fixed points in M^{11}; in this theory, a M2-brane stretched between these fixed points yields the strongly coupled heterotic string [28,29]. On the other hand, in the presence of the broken **8192** spinor, we see a possible reason for the E_8 symmetry that arises at the fixed points, as the hidden spinor fermions may contribute to anomalies induced via the orbifold of the M10-brane worldvolume.

If instead we reduce directly from $D = 26 + 1$ on an orbifold $S^1/\mathbb{Z}_2$, we break half the supersymmetry and remove the **24** vector, while the **8192** spinor projects down to **4096**. This is in agreement with the Monster SCFT [9], where the fixed points of the orbifold contain 4096 twisted states. This differs from the orbifold reduction of bosonic M-theory, where the fixed points have no extra degrees of freedom due to the absence of chiral bosons and fermions [19]. A $D = 26 + 1$ M-theory with $\mathbf{24} \cdot \mathbf{4096} = \mathbf{98{,}304}$ Rarita–Schwinger field would have fermionic anomalies at each orbifold fixed point that must be canceled by vector multiplets as in the $D = 10 + 1$ M-theory case [28,29]. One would expect a generalization of E_8 symmetry at each fixed point that contains at least $\mathbf{24} \cdot \mathbf{2^{12}} = \mathbf{98{,}304}$ vector multiplets for RNS twisted sector states. The Griess algebra provides such minimal degrees of freedom, and thus could possibly be used to cancel anomalies at the fixed points. Another possibility is the *Leech algebra*, which we introduce in a later section.

It was shown that bosonic M-theory can reduce to the bosonic string in $D = 25 + 1$ by reduction along $S^1/\mathbb{Z}_2$ [19], and thus the Monster CFT in its relation to $D = 25 + 1$ bosonic string theory on $(\mathbb{R}^{24}/\Lambda_{24})/\mathbb{Z}_2$ can trace its origin back to $26 + 1$ space–time dimensions. This suggests that the Monster CFT describes states on the boundary of $AdS_3 \otimes S^{23}$, originating from the M2-brane near-horizon geometry $AdS_4 \otimes S^{23}$, where the transverse directions are discretized and given by the Leech lattice Λ_{24}. In going from $D = 26 + 1$ to $D = 25 + 1$, the **324** graviton breaks as $\mathbf{324} = \mathbf{299} \oplus \mathbf{24} \oplus \mathbf{1}$, where the orbifold removes the translation symmetry, and hence eliminates the 1-form **24** [19] from the closed string spectrum [19].

Recall, that a holomorphic CFT for the Leech lattice Λ_{24} has partition function

$$Z_{\text{Leech}}(q) = \frac{\Theta_{\Lambda_{24}}}{\eta^{24}} = \frac{1}{q} + 24 + 196{,}884q + 21{,}493{,}760q^2 + 864{,}299{,}970q^3 + \mathcal{O}(q^4) \quad (23)$$

$$= J(q) + 24, \quad (24)$$

where $J(q) = j(q) - 744$, and $j(q)$ is the *j-function* [9]. FLM used a $\mathbb{Z}_2$-twisted version of the Leech theory to remove the unwanted **24** states that contribute to the constant term in the partition function and to obtain the appropriate finite group structure [3,9]. From a modern perspective, this can be accomplished by a $S^1/\mathbb{Z}_2$ orbifold reduction of $D = 26 + 1$ bosonic M-theory, which has been shown to reduce to the $D = 25 + 1$ bosonic string [19].

A superconformal field theory ("the Beauty and the Beast") description of the FLM model was given by Dixon, Ginsparg and Harvey [9]. The supersymmetric extension of the Virasoro algebra introduces moments G_n that satisfy the relations

$$[L_m, G_n] = \left(\frac{m}{2} - n\right) G_{m+n} \tag{25}$$

and

$$\{G_m, G_n\} = 2L_{m+n} + \frac{\hat{c}}{2}(m^2 - \frac{1}{4})\delta_{m+n,0}. \tag{26}$$

For integer moding of G_n ($n \in \mathbb{Z}$), the supersymmetric extension is named the Ramond (R) algebra, while for half-integer moding ($n \in \mathbb{Z} + \frac{1}{2}$) it is named the Neveu–Schwarz (NS) algebra [9]. The $\mathbf{2}^{12} \equiv \mathbf{4096}$ twisted states of the FLM model are half-integer moded [3,9], thus suggesting a fermionic origin. This can arise from projecting half the degrees of freedom of the **8192** spinor from $D = 26 + 1$, which is expected from an orbifold reduction, analogous to the case of $D = 10 + 1$ M-theory where the **32** spinor is projected to a **16** [28,29]. A $S^1/\mathbb{Z}_2$ orbifold reduction reduces the **8192** spinor to **4096** spinor, where under SO_{24} one has **4096** = **2048** $\oplus$ **2048**$'$. The **2048** spinors can yield worldsheet fermions in $D = 25 + 1$. Such SO_{24} spinors are seen in Duncan's SCFT with Conway group symmetry [24]. These spinors can be used to build RNS states in $D = 25 + 1$, that generalize the gravitino and dilatino states of type IIA in $D = 9 + 1$ from 128 to 98,304 degrees of freedom.

The untwisted states of the FLM model include the 24 Ramond ground states and the 196,560/2 = 98,280 Leech lattice states, which G_0 pairs with 24 $\times$ 2^{12} = 98,304 dimension 2 Ramond fields as 98,280 + 24 = 98,304 [3,9]. In $D = 26 + 1$, the massless Rarita–Schwinger (1-form spinor) field has **98,304** degrees of freedom, and thus is a candidate for the origin of the dimension 2 Ramond fields in a SCFT. The remaining 98,280 states come from a discretized transverse space, where in the $AdS_4 \otimes S^{23}$ near-horizon geometry of the M2-brane in $D = 26 + 1$ the 23-sphere is discretized by the 196,560 norm four Leech lattice vectors. This is consistent with the Conway group Co_0 being a maximal finite subgroup of the $\mathcal{R}$-symmetry group SO_{24}. The $S^1/\mathbb{Z}_2$ orbifold reduces the 196,560 vectors to 98,280, while also reducing $AdS_4 \otimes S^{23}$ to $AdS_3 \otimes S^{23}$, and breaking the discrete $\mathcal{R}$-symmetry group Co_0 down to the simple Conway group (It is interesting to observe that **98,280** is not the dimension of a unique irreducible representation of Co_1, but rather it can be decomposed as a sum of irreducible representation of Co_1 [30]. Remarkably, one finds a decomposition only in terms of irreducible representations of Co_0 which all survive (and stay irreducible) under the maximal reduction $Co_0 \to Co_1$, namely **98,280** $=$ **80,730** $\oplus$ **17,250** $\oplus$ **299** $\oplus$ **1**.) $Co_1 \simeq Co_0/\mathbb{Z}_2$ [6], thus making contact with Witten's holographic interpretation of the Monster [12] with Co_1 as a discrete R-symmetry.

Finally, it is here worth mentioning that the $\mathbb{Z}_{2A}$-fermionization of the Monster CFT [13] reveals representations of the Baby Monster finite group $\mathbb{BM}$ in the NS and R sectors,

$$Z^{\mathcal{F}}_{NS}(\tau) = \frac{1}{q} + \frac{1}{\sqrt{q}} + 4372\sqrt{q} + 100{,}628q + \mathcal{O}(q^{3/2}), \tag{27}$$

and

$$Z^{\mathcal{F}}_{R}(\tau) = 192{,}512q + 21{,}397{,}504q^2 + \mathcal{O}(q^3), \tag{28}$$

where 4372 + 192,512 = 196,884. In terms of SO_{24} irreducible representations, we note that 4372 = **276** $\oplus$ **2048** $\oplus$ **2048**$'$, where worldsheet fermions are suggested. This implies a $D = 25 + 1$ string theory with **2048** $\oplus$ **2048**$'$ worldsheet fermions that generalizes the $D = 9 + 1$ superstring with SO_8 spinors. In the treatment given below, we propose a $D = 26 + 1$ origin for such a string theory, supported by the fermionization of the Monster CFT [13], which suggests a $(2 + 1)$-dimensional fermionic gravitational Chern–Simons term that can live on the boundary of AdS_4. Once again, given an $S^1/\mathbb{Z}_2$ orbifold reduction of $D = 26 + 1$ M-theory with fermions, one does expect anomalies, and to cancel such anomalies may necessitate the use of the Leech lattice Λ_{24} at each fixed hyperplane. The resulting $D = 25 + 1$ closed string theory is then very similar to the Bimonster string theory introduced by Harvey et al. in [11].

3. "Weak" Trialities in 24 Dimensions

By *triality*, denoted by $\mathbb{T}$, in this paper, we refer to a property of the Lie algebra $\mathfrak{d}_4$ (see [31]), namely a map among its three 8-dimensional irreducible representations

$$\mathfrak{d}_4 : \begin{cases} \wedge^1 \equiv \mathbf{8}_v := (1,0,0,0) \text{ (1-form)}; \\ \\ \boldsymbol{\lambda} \equiv \mathbf{8}_s := (0,0,0,1) \text{ (semispinor)}; \\ \\ \boldsymbol{\lambda}' \equiv \boldsymbol{\lambda}_c \equiv \mathbf{8}'_s \equiv \mathbf{8}_c := (0,0,1,0) \text{ (conjugate semispinor)} \end{cases} \tag{29}$$

among themselves:

$$\mathbb{T} : \begin{array}{ccc} \wedge^1 & & \\ \uparrow\downarrow & \searrow\nwarrow & \\ \boldsymbol{\lambda}' & \rightleftarrows & \boldsymbol{\lambda} \end{array} . \tag{30}$$

The origin of $\mathbb{T}$ can be traced back to the three-fold structural symmetry of the Dynkin diagram of $\mathfrak{d}_4$, and to the existence of an outer automorphism of $\mathfrak{d}_4$ which interchanges $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$; in fact, the outer automorphism group of $\mathfrak{d}_4$ (or, more precisely, of the corresponding spin group $Spin_8$, the double cover of the Lie group SO_8) is isomorphic to the symmetric group S_3 that permutes such three representations.

Thence, through suitably iterated tensor products of representations $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$, $\mathbb{T}$ affects higher-dimensional representations, as well. For instance, $\mathbb{T}$ maps also the three 56-dimensional irreducible representations of $\mathfrak{d}_4$:

$$\mathfrak{d}_4 : \begin{cases} \wedge^3 \equiv \mathbf{56}_v := (0,0,1,1) \ \big(\text{3-form } \wedge^3\big); \\ \\ \boldsymbol{\psi} \equiv \mathbf{56}_s := (1,0,0,1) \text{ (1-form spinor, } \textit{aka} \text{ gravitino)}; \\ \\ \boldsymbol{\psi}_c \equiv \boldsymbol{\psi}' \equiv \mathbf{56}'_s \equiv \mathbf{56}_c := (1,0,1,0) \text{ (conjugate gravitino)} \end{cases} \tag{31}$$

among themselves

$$\mathbb{T} : \begin{array}{ccc} \wedge^3 & & \\ \uparrow\downarrow & \searrow\nwarrow & \\ \boldsymbol{\psi} & \rightleftarrows & \boldsymbol{\psi}' \end{array} . \tag{32}$$

By *gravitino*, we mean the gamma-traceless 1-form spinor; indeed, in order to correspond to an irreducible representation, the spinor-vector ψ^α_μ must be gamma-traceless:

$$\Gamma^\mu_{\alpha\beta}\psi^\beta_\mu = 0, \tag{33}$$

where μ and α are the vector resp. spinor indices, and $\Gamma^\mu_{\alpha\beta} \equiv (\Gamma^\mu)_{\alpha\beta}$ denote the gamma matrices of $\mathfrak{d}_4$. ψ is a Rarita–Schwinger (RS) field of spin/helicity $\frac{3}{2}$, and, in the context of supersymmetric theories, it is named *gravitino* (being the spartner of the graviton $g_{\mu\nu}$). As (30) denotes the action of triality $\mathbb{T}$ on (semi)spinors, (32) expresses the triality $\mathbb{T}$ acting on RS fields. $\mathbb{T}$ plays an important role in type II string theory in $9+1$ space–time dimensions, in which $\mathfrak{so}_8$ (compact real form of $\mathfrak{d}_4$) is the algebra of the massless little group (cfr. e.g., [32]).

3.1. $\boldsymbol{\lambda}$-Triality

In certain dimensions, there may be a "weaker" variant of $\mathbb{T}$, in which $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ have the same dimension of a *reducible* (bosonic) representation, namely of a sum of *irreducible* (bosonic) representations, of $\mathfrak{d}_n$. In fact, for $n = 12$ (i.e., in $\mathfrak{d}_{12}$) something remarkable takes place; in $\mathfrak{d}_{12}$, the following three representations all have the same dimension 2048:

$$\mathfrak{d}_{12}:\begin{cases}\boldsymbol{\lambda}\equiv\mathbf{2}^{11}=\mathbf{2048}:=(0^{11},1);\\ \\ \boldsymbol{\lambda}'\equiv\left(\mathbf{2}^{11}\right)'=\mathbf{2048}':=(0^{10},1,0);\\ \\ \wedge^1\oplus\wedge^3=\mathbf{24}\oplus\mathbf{2024}=(1,0^{11})\oplus(0^2,1,0^9).\end{cases}\tag{34}$$

In other words, in $\mathfrak{d}_{12}$ the *reducible* bosonic representation given by the sum of the vector (1-form) representation $\wedge^1$ and of the 3-form representation $\wedge^3$ has the same dimension of each of the (semi)spinors $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$. Analogously to the aforementioned case of $\mathfrak{d}_4$, one can then define a "triality-like" map, named $\boldsymbol{\lambda}$*-triality* and denoted by $\tilde{\mathbb{T}}_{\lambda}$, between the corresponding representation vector representation spaces,

$$\tilde{\mathbb{T}}_{\lambda}:\begin{array}{ccc}\left(\wedge^1\oplus\wedge^3\right) & & \\ \uparrow\downarrow & \searrow\nwarrow & \\ \boldsymbol{\lambda} & \rightleftarrows & \boldsymbol{\lambda}'\end{array}\tag{35}$$

It is immediately realized that a crucial difference with (30) relies on the *reducibility* of the bosonic sector of the map, which we will henceforth associate with the "weakness" of $\tilde{\mathbb{T}}_{\lambda}$. However, since no other Dynkin diagram (besides $\mathfrak{d}_4$) has an automorphism group of order greater than 2, one can also conclude that (34) and (35) cannot be realized as an automorphism of $\mathfrak{d}_{12}$, nor can it be traced back to some structural symmetry of the Dynkin diagram of $\mathfrak{d}_{12}$ itself.

3.2. $\boldsymbol{\psi}$-Triality

As triality $\mathbb{T}$ of $\mathfrak{d}_4$ (30) affects all tensor products stemming from $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$, implying in particular (32), so the $\boldsymbol{\lambda}$-triality $\tilde{\mathbb{T}}_{\lambda}$ of $\mathfrak{d}_{12}$ (35) affects all tensor products stemming from $\wedge^1\oplus\wedge^3$, $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$; in particular, in $\mathfrak{d}_{12}$, the following three representations have the same dimension 47,104:

$$\mathfrak{d}_{12}:\begin{cases}\boldsymbol{\psi}\equiv\mathbf{47{,}104}:=(1,0^{10},1);\\ \\ \boldsymbol{\psi}'\equiv\mathbf{47{,}104}':=(1,0^{9},1,0);\\ \\ 2\times\left(2\times\wedge^4\oplus\wedge^3\oplus\wedge^2\right)=2\times\left(2\times\mathbf{10{,}626}\oplus\mathbf{2024}\oplus\mathbf{276}\right)\\ =2\times\left(2\times(0^3,1,0^8)\oplus(0^2,1,0^9)\oplus(0,1,0^{10})\right).\end{cases}\tag{36}$$

In other words, in $\mathfrak{d}_{12}$ the *reducible* bosonic representation given by the sum of the 4-form $\wedge^4$, 3-form $\wedge^3$ and 2-form $\wedge^2$ representations (with multiplicity 4, 2 and 2, respectively) has the same dimension of each of the RS field representations $\boldsymbol{\psi}$ and $\boldsymbol{\psi}'$. Analogously to the aforementioned case of $\mathfrak{d}_4$, one can then define a "triality-like" map, named $\boldsymbol{\psi}$*-triality* and denoted by $\tilde{\mathbb{T}}_{\psi}$, between the corresponding representation vector spaces,

$$\tilde{\mathbb{T}}_{\psi}:\begin{array}{ccc}2\cdot\left(2\cdot\wedge^4\oplus\wedge^3\oplus\wedge^2\right) & & \\ \uparrow\downarrow & \searrow\nwarrow & \\ \boldsymbol{\psi} & \rightleftarrows & \boldsymbol{\psi}'\end{array}.\tag{37}$$

Again, (34) and (35) cannot be realized as an automorphism of $\mathfrak{d}_{12}$, nor can it be traced back to some structural symmetry of the Dynkin diagram of $\mathfrak{d}_{12}$ itself.

3.3. Iso-Dimensionality among (Sets of) p-Forms: An Example

Representations with the same dimensions can also be only bosonic. Still, $\mathfrak{d}_{12}$ provides the following example of such a phenomenon (Another example is provided by the iso-dimensionality map $\wedge^1\oplus\wedge^2\leftrightarrow S_0^2\oplus\mathbf{1}$, holding for any orthogonal Lie algebra. However, since we will fix the number of graviton fields to be 1, we will not make use of such an iso-dimensionality map): the following two sets of representations have the same dimension 42,504:

$$\mathfrak{d}_{12}:\begin{cases}\wedge^5\equiv\mathbf{42{,}504}:=(0^4,1,0^7);\\ \\ 4\times\wedge^4=4\times\mathbf{10{,}626}=4\times(0^3,1,0^8).\end{cases}\tag{38}$$

In other words, in $\mathfrak{d}_{12}$ the 5-form representation $\wedge^5$ has the same dimension, namely 42,504, of four copies of the 4-form representation $\wedge^4$. Again, one can then define a map, denoted by $\mathcal{B}$, between the corresponding representation vector spaces(Of course, all instances of iso-dimensionality among representations given by (34) and (35), (36) and (37) and (38) and (39), hold up to Poincaré/Hodge duality (in the bosonic sector); cfr. (46) further below. Note that other iso-dimensionality maps besides (39) may exist, but we will not make use of them in the present paper):

$$\mathcal{B}:\wedge^5\leftrightarrow 4\cdot\wedge^4.\tag{39}$$

4. Monstrous Dilatonic Gravity in 25 + 1

In the previous Section 3, we introduced some maps among fermionic and bosonic representations of $\mathfrak{d}_{12}$, having the same dimension but different Dynkin labels:

- The $\boldsymbol{\lambda}$-triality $\tilde{\mathbb{T}}_{\lambda}$ (34) and (35), generalizing the triality $\mathbb{T}$ (30) of $\mathfrak{d}_4$ to $\mathfrak{d}_{12}$;
- he $\boldsymbol{\psi}$-triality $\tilde{\mathbb{T}}_{\psi}$ (36) and (37), extending the weak triality of $\mathfrak{d}_{12}$ to its Rarita–Schwinger sector;
- The iso-dimensionality map $\mathcal{B}$ (38) and (39) among certain sets of bosonic (p-form) representations of $\mathfrak{d}_{12}$.

As triality $\mathbb{T}$ (30) of $\mathfrak{d}_4$ plays a role in the type II string theories (which all have $\mathfrak{so}_8$ as the algebra of the massless little group), one might ask whether (35), (37) and (39) have some relevance in relation to bosonic string theory [19], or in relation to more general field theories in $D=25+1$ space–time dimensions, in which $\mathfrak{so}_{24}$ is the algebra of the massless little group. Below, we will show that this is actually the case for a quite large class of *non-supersymmetric* dilatonic (Einstein) gravity theories in $25+1$, named *Monstrous gravities*, which we are now going to introduce.

To this aim, we start and display various massless fields in $D=25+1$ space–time dimensions. As mentioned, each massless field fits into the following irreducible representation (A priori, one could also consider $\wedge^6\equiv\mathbf{134{,}596}$ (because $134{,}596<196{,}884$—see below), but it actually does not enter in any way in the treatment of this section) $\mathbf{R}$ of the massless little group SO_{24} (recall that $g\equiv S_0^2$ and $\phi\equiv\mathbf{1}$ throughout):

$$\begin{array}{ccc}\text{field} & \mathbf{R} & \text{Dynkin labels}\\ g: & \mathbf{299} & (2,0^{11})\\ \psi: & \mathbf{47{,}104} & (1,0^{10},1)\\ \psi': & \mathbf{47{,}104}' & (1,0^9,1,0)\\ \wedge^1 & \mathbf{24} & (1,0^{11})\\ \lambda: & \mathbf{2048} & (0^{11},1)\\ \lambda': & \mathbf{2048}' & (0^{10},1,0)\\ \phi: & \mathbf{1} & (0^{12})\\ \wedge^5: & \mathbf{42{,}504} & (0^4,1,0^7)\\ \wedge^4: & \mathbf{10{,}626} & (0^3,1,0^8)\\ \wedge^3: & \mathbf{2024} & (0^2,1,0^9)\\ \wedge^2: & \mathbf{276} & (0,1,0^{10})\end{array}\tag{40}$$

We are now going to classify field theories in $25+1$ space–time dimensions which share the following features:

a They all contain gravity (in terms of *one* 26-bein, then yielding *one* metric tensor $g_{\mu\nu}$) and *one* dilaton scalar field ϕ; thus, the Lagrangian density of their gravito-dilatonic sector reads as follows (Throughout our analysis, we rely on the conventions and treatment given in Secs. 22 and 23 of [33]):

$$\mathcal{L} = e^{-2\phi}\left(R - 4\partial_\mu\phi\partial^\mu\phi\right). \tag{41}$$

b The relations among all such theories are due to the $\boldsymbol{\lambda}$-triality $\tilde{\mathbb{T}}_{\boldsymbol{\lambda}}$ (34) and (35), the *weak* $\boldsymbol{\psi}$-triality $\tilde{\mathbb{T}}_{\boldsymbol{\psi}}$ (36) and (37), as well as the bosonic map $\mathcal{B}$ (38) and (39) of $\mathfrak{so}_{24}$ (real compact form of $\mathfrak{d}_{12}$), which is the Lie algebra of the massless little group.

c By constraining the theories to contain *only one* graviton and *only one* dilaton, the total number of degrees of freedom of the massless spectrum must sum up to

$$\begin{aligned}
& 1 + 299 + 47{,}104 \cdot (\#\psi) + 24 \cdot \left(\#\wedge^1\right) + 2048 \cdot (\#\lambda) \\
& +42{,}504 \cdot \left(\#\wedge^5\right) + 10{,}626 \cdot \left(\#\wedge^4\right) + 2024 \cdot \left(\#\wedge^3\right) + 276 \cdot \left(\#\wedge^2\right) \\
= \; & 196{,}884.
\end{aligned} \tag{42}$$

Consequently, the whole set of massless degrees of freedom of such theories may be acted upon by the *Monster group* $\mathbb{M}$, the largest sporadic group, because **196,883** is the dimension of its smallest non-trivial representation [1]. For this reason, the gravito-dilatonic theories under consideration will all be named *Monstrous gravities*. They will be characterized by the following split:

$$196{,}884 = \mathbf{196{,}883} \oplus \mathbf{1}, \tag{43}$$

which is at the origin of the so-called Monstrous Moonshine [4,5]. The dilaton ϕ, which is a singlet of $\mathbb{M}$, coincides with the vacuum state $|\Omega\rangle$ of the chiral Monster SCFT discussed in [9,34,35]. Thus, Monstrous gravities in $25+1$ space–time dimensions, and the presence of a unique ϕ, are intimately related to the 196,883-dimensional representation of $\mathbb{M}$, and thus, *they may provide an explanation of the (observation of who firstly ignited the) Monstrous Moonshine in terms of (higher-dimensional, gravitational) field theory*.

In the context of Witten's three-dimensional gravity [12], this suggests that the 196,883 primary operators that create black holes are carrying dilatonic gravity field content. As in [12], it is enlightening to compare the number 196,883 of primaries with the Bekenstein–Hawking entropy of the corresponding black hole: an exact quantum degeneracy of 196,883 yields an entropy of Witten's BTZ black hole given by $\ln(196{,}883) \simeq 12.19$, whereas the Bekenstein–Hawking entropy-area formula yields to $4\pi \simeq 12.57$. Of course, one should not expect a perfect agreement between such two quantities, because the Bekenstein–Hawking entropy-area formula holds in the semi-classical regime and not in the exact quantum one. As given in (42), 196,883 comes from gauge fields (potentials), graviton, etc., albeit without dilaton; in this sense, *the quantum entropy* $\ln(196{,}883) \simeq 12.19$ *has a manifest higher-dimensional interpretation* since the BTZ black hole degrees of freedom can be expressed in terms of massless degrees of freedom of fields in $25+1$ space–time dimensions.

4.1. Classification

All Monstrous gravities will be classified by using two sets of numbers:

- $\mathbf{s}_1$, a length-5 string, providing the number of independent "helicity"-h massless fields, with $h = 2, \frac{3}{2}, 1, \frac{1}{2}, 0$, respectively denoted by g (graviton), ψ (Rarita–Schwinger field), $\wedge^1$ (1-form potential), λ [spinor field (The spinor field gets named *gaugino* (or *dilatino*) in the presence of supersymmetry.)], and ϕ (dilaton); as pointed out above, we fix $\#g = \#\phi = 1$ throughout (additionally, note that any theory with $\#\wedge^1 \geqslant 1$ is a Maxwell–Einstein–dilaton theory in $25+1$ space–time dimensions):

$$\mathbf{s}_1 := \left(\#g, \#\psi, \#\wedge^1, \#\lambda, \#\phi\right) = \left(1, \#\psi, \#\wedge^1, \#\lambda, 1\right); \tag{44}$$

- $\mathbf{s}_2$, a length-4 string, providing the number of independent p-form brane potentials, for the smallest values of p, namely for $p = 5, 4, 3, 2$,

$$\mathbf{s}_2 := \left(\#\wedge^5, \#\wedge^4, \#\wedge^3, \#\wedge^2\right). \tag{45}$$

Before starting, we should point out that the classification below is unique up to Poincaré/Hodge duality $*$ for the p-form potentials, namely, for $p = 1, \ldots 5$:

$$\begin{array}{ccc} p\text{-form pot.} & \overset{*}{\longleftrightarrow} & p'\text{-form pot.} \\ \wedge^1 & & \wedge^{23} \\ \wedge^2 & & \wedge^{22} \\ \wedge^3 & & \wedge^{21} \\ \wedge^4 & & \wedge^{20} \\ \wedge^5 & & \wedge^{19} \end{array} \tag{46}$$

as well as up to chiral/non-chiral arrangements in the fermionic sector,

$$\begin{array}{cc} \#\psi & \text{chiral/non-chiral arr.s} \\ 2 & (2,0), (0,2), (1,1) \\ 4 & (4,0), (0,4), (3,1), (1,3), (2,2); \end{array} \tag{47}$$

$$\begin{array}{cc} \#\lambda & \text{chiral/non-chiral arr.s} \\ 1 & (1,0), (0,1) \\ 2 & (2,0), (0,2), (1,1) \\ 3 & (3,0), (0,3), (2,1), (1,2). \end{array} \tag{48}$$

Clearly, (46)–(48) are particularly relevant if the (local) supersymmetry in $25+1$ is considered; however, in this paper, we will not be dealing with such an interesting topic, and we will confine ourselves to make some comments further below (in $26+1$).

We will split the Monstrous gravity theories, sharing the features **a–c** listed above, in five groups, labeled with Latin numbers: 0, 1, 2, 3, 4, specifying the number $\#\psi$ of $h = 3/2$ RS fields. The $\boldsymbol{\psi}$-triality $\tilde{\mathbb{T}}_{\boldsymbol{\psi}}$ (36) and (37) of $\mathfrak{so}_{24}$ maps such five groups among themselves. Then, each of these groups will be split into four subgroups, labeled with Greek letters: α, β, γ and δ, respectively characterized by the following values of $\#\wedge^1$ and $\#\lambda$:

$$\left(\#\wedge^1, \#\lambda\right) = \underset{\alpha}{(3,0)}, \quad \underset{\beta}{(2,1)}, \quad \underset{\gamma}{(1,2)}, \quad \underset{\delta}{(0,3)} \ . \tag{49}$$

The $\boldsymbol{\lambda}$-triality $\tilde{\mathbb{T}}_{\boldsymbol{\lambda}}$ (34) and (35) of $\mathfrak{so}_{24}$ allows to move among such four subgroups (within the same group). The theories belonging to each of such four subgroups will share the same split of the massless degrees of freedom into bosonic and fermionic ones, respectively specified, as above, by the numbers B and F. Each of such four subgroups is a set of a varying number of theories, which will be labeled in lowercase Latin letters: *i*, *ii*, *iii*, etc. Such theories will be connected by the action of the bosonic map $\mathcal{B}$ (38) and (39) of $\mathfrak{so}_{24}$, and thus they will differ for the content of 5-form $\wedge^5$ and 4-form $\wedge^4$ (potential) fields.

Modulo all possibilities arising from the combinations of (46) and (48), the classification of *Monstrous gravity* theories in $25+1$ space–time dimensions is as follows.

0 Group 0 (ψ-less theories):

$$\underset{\substack{(B|F)=(196{,}884|0),\ \lambda\text{-less}\\ \textbf{purely bosonic}}}{\alpha} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,0,3,0,1) & (0,16,12,8) & \text{bosonic}, \wedge^5\text{-less} \\ ii & '' & (1,12,12,8) & \text{bosonic} \\ iii & '' & (2,8,12,8) & \text{bosonic} \\ iv & '' & (3,4,12,8) & \text{bosonic} \\ v & '' & (4,0,12,8) & \text{bosonic}, \wedge^4\text{-less} \end{bmatrix} \tag{50}$$

$$\underset{(B|F)=(194{,}836|2048)}{\beta} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,0,2,1,1) & (0,16,11,8) & \wedge^5\text{-less} \\ ii & '' & (1,12,11,8) & - \\ iii & '' & (2,8,11,8) & - \\ iv & '' & (3,4,11,8) & - \\ v & '' & (4,0,11,8) & \wedge^4\text{-less} \end{bmatrix} \tag{51}$$

$$\underset{(B|F)=(192{,}788|4096)}{\gamma} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,0,1,2,1) & (0,16,10,8) & \wedge^5\text{-less} \\ ii & '' & (1,12,10,8) & - \\ iii & '' & (2,8,10,8) & - \\ iv & '' & (3,4,10,8) & - \\ v & '' & (4,0,10,8) & \wedge^4\text{-less} \end{bmatrix} \tag{52}$$

$$\underset{(B|F)=(190{,}740|6144),\ \wedge^1\text{-less}}{\delta} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,0,0,3,1) & (0,16,9,8) & \wedge^5\text{-less} \\ ii & '' & (1,12,9,8) & - \\ iii & '' & (2,8,9,8) & - \\ iv & '' & (3,4,9,8) & - \\ v & '' & (4,0,9,8) & \wedge^4\text{-less} \end{bmatrix} \tag{53}$$

1 Group 1 ($\#\psi = 1$ theories):

$$\underset{(B|F)=(149{,}780|47{,}104),\ \lambda\text{-less}}{\alpha} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,1,3,0,1) & (0,12,10,6) & \wedge^5\text{-less} \\ ii & '' & (1,8,10,6) & - \\ iii & '' & (2,4,10,6) & - \\ iv & '' & (3,0,10,6) & \wedge^4\text{-less} \end{bmatrix} \tag{54}$$

$$\underset{(B|F)=(147{,}732|49{,}152)}{\beta} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,1,2,1,1) & (0,12,9,6) & \wedge^5\text{-less} \\ ii & '' & (1,8,9,6) & - \\ iii & '' & (2,4,9,6) & - \\ iv & '' & (3,0,9,6) & \wedge^4\text{-less} \end{bmatrix} \tag{55}$$

$$\underset{(B|F)=(145{,}684|51{,}200)}{\gamma} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,1,1,2,1) & (0,12,8,6) & \wedge^5\text{-less} \\ ii & '' & (1,8,8,6) & - \\ iii & '' & (2,4,8,6) & - \\ iv & '' & (3,0,8,6) & \wedge^4\text{-less} \end{bmatrix} \tag{56}$$

$$\underset{(B|F)=(143{,}636|53{,}248),\ \wedge^1\text{-less}}{\delta} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,1,0,3,1) & (0,12,7,6) & \wedge^5\text{-less} \\ ii & '' & (1,8,7,6) & - \\ iii & '' & (2,4,7,6) & - \\ iv & '' & (3,0,7,6) & \wedge^4\text{-less} \end{bmatrix} \tag{57}$$

2 Group 2 ($\#\psi = 2$ theories):

$$\underset{(B|F)=(102{,}676|94{,}208),\ \lambda\text{-less}}{\alpha} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,2,3,0,1) & (0,8,8,4) & \wedge^5\text{-less} \\ ii & '' & (1,4,8,4) & - \\ iii & '' & (2,0,8,4) & \wedge^4\text{-less} \end{bmatrix} \tag{58}$$

$$\underset{(B|F)=(100{,}628|96{,}256)}{\beta} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,2,2,1,1) & (0,8,7,4) & \wedge^5\text{-less} \\ ii & '' & (1,4,7,4) & - \\ iii & '' & (2,0,7,4) & \wedge^4\text{-less} \end{bmatrix} \tag{59}$$

$$\underset{(B|F)=(98{,}580|98{,}304)}{\gamma} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,2,1,2,1) & (0,8,6,4) & \wedge^5\text{-less} \\ ii & '' & (1,4,6,4) & - \\ iii & '' & (2,0,6,4) & \wedge^4\text{-less} \end{bmatrix} \tag{60}$$

$$\underset{(B|F)=(96{,}532|100{,}352),\ \wedge^1\text{-less}}{\delta} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,2,0,3,1) & (0,8,5,4) & \wedge^5\text{-less} \\ ii & '' & (1,4,5,4) & - \\ iii & '' & (2,0,5,4) & \wedge^4\text{-less} \end{bmatrix} \tag{61}$$

3 Group 3 ($\#\psi = 3$ theories):

$$\underset{(B|F)=(55{,}572|141{,}312),\ \lambda\text{-less}}{\alpha} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,3,3,0,1) & (0,4,6,2) & \wedge^5\text{-less} \\ ii & '' & (1,0,6,2) & \wedge^4\text{-less} \end{bmatrix} \tag{62}$$

$$\underset{(B|F)=(53{,}524|143{,}360)}{\beta} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,3,2,1,1) & (0,4,5,2) & \wedge^5\text{-less} \\ ii & '' & (1,0,5,2) & \wedge^4\text{-less} \end{bmatrix} \tag{63}$$

$$\underset{(B|F)=(51{,}476|145{,}408)}{\gamma} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,3,1,2,1) & (0,4,4,2) & \wedge^5\text{-less} \\ ii & '' & (1,0,4,2) & \wedge^4\text{-less} \end{bmatrix} \tag{64}$$

$$\underset{(B|F)=(49{,}428|147{,}456),\ \wedge^1\text{-less}}{\delta} : \begin{bmatrix} & \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ i & (1,3,0,3,1) & (0,4,3,2) & \wedge^5\text{-less} \\ ii & '' & (1,0,3,2) & \wedge^4\text{-less} \end{bmatrix} \tag{65}$$

4 Group 4 ($\#\psi = 4$ theories):

$$\underset{(B|F)=(8{,}468|188{,}416),\ \lambda\text{-less}}{\alpha} : \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ (1,4,3,0,1) & (0,0,4,0) & \wedge^5,\wedge^4,\wedge^2\text{-less} \end{bmatrix} \tag{66}$$

$$\underset{(B|F)=(6420|190{,}464)}{\beta} : \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ (1,4,2,1,1) & (0,0,3,0) & \wedge^5,\wedge^4,\wedge^2\text{-less} \end{bmatrix} \tag{67}$$

$$\underset{(B|F)=(4372|192{,}512)}{\gamma} : \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ (1,4,1,2,1) & (0,0,2,0) & \wedge^5,\wedge^4,\wedge^2\text{-less} \end{bmatrix} \tag{68}$$

$$\underset{(B|F)=(2324|194{,}560)}{\delta} : \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \textbf{features} \\ (1,4,0,3,1) & (0,0,1,0) & \wedge^5,\wedge^4,\wedge^2,\wedge^1\text{-less} \end{bmatrix} \tag{69}$$

The above classification contains 60 Monstrous gravity theories, from the purely bosonic, $\wedge^5$-less, $0.\alpha.i$ theory (50) to the theory with the highest F, i.e., the $4.\delta$ theory (69). Note that, since we have imposed $\#g = \#\phi = 1$, no purely fermionic Monstrous gravity can exist. Moreover, as far as linear realizations of (local) supersymmetry are concerned, Monstrous gravity theories are *not* supersymmetric, as it is evident from $B \neq F$ in all cases. It is also worth remarking that all such theories (but the ones of the group **4** (66) and (69)) contain bosonic string theory, whose (massless, closed string) field content is $\#g = \#\phi = \#\wedge^2 = 1$ (see e.g., [19]), as a subsector.

5. Monstrous M-Theory in 26 + 1

At this point, the natural question arises as to whether the Monstrous gravities classified above can be uplifted (The possibility of an uplift/oxidation to $26+1$ is far from being trivial, and when possible, it uniquely fixes the content of the higher dimensional (massless) spectrum) to $26+1$ space–time dimensions, in which the massless little group is SO_{25}.

At least in one case, namely for the purely bosonic Monstrous gravity labeled by $0.\alpha.iii$, the answer to this question is positive. The field content of such a theory is specified by the following: $\mathbf{s}_1$ and $\mathbf{s}_2$, as from (50):

$$\underset{(B|F)=(196{,}884|0)}{0.\alpha.iii}:\left\{\begin{array}{l}\mathbf{s}_1=(1,0,3,0,1);\\ \\ \mathbf{s}_2=(2,8,12,8),\end{array}\right. \tag{70}$$

or equivalently (absent fields are not reported):

$$\begin{array}{ccc} \text{field} & \mathbf{R}\text{ of }\mathfrak{so}_{24} & \# \\ g: & \mathbf{299} & 1 \\ \wedge^1 & \mathbf{24} & 3 \\ \phi: & \mathbf{1} & 1 \\ \wedge^5: & \mathbf{42{,}504} & 2 \\ \wedge^4: & \mathbf{10{,}626} & 8 \\ \wedge^3: & \mathbf{2024} & 12 \\ \wedge^2: & \mathbf{276} & 8 \end{array} \tag{71}$$

One can indeed realize that all such bosonic massless (SO_{24}-covariant) fields in $25+1$ can be obtained by a KK reduction of the following set of (SO_{25}-covariant) bosonic massless fields fields in $26+1$:

$$\begin{array}{ccccc} \mathfrak{so}_{25} & & \mathfrak{so}_{24} & \text{fields} & \\ \underset{\mathbf{324}}{g} & \longrightarrow & \underset{\mathbf{299}}{g}, & \underset{\mathbf{24}}{\wedge^1}, & \underset{\mathbf{1}}{\phi} \\ \underset{\mathbf{53{,}130}}{\wedge^5} & \longrightarrow & \underset{\mathbf{42{,}504}}{\wedge^5}, & \underset{\mathbf{10{,}626}}{\wedge^4} & \\ \underset{\mathbf{12{,}650}}{\wedge^4} & \longrightarrow & \underset{\mathbf{10{,}626}}{\wedge^4}, & \underset{\mathbf{2024}}{\wedge^3} & \\ \underset{\mathbf{2300}}{\wedge^3} & \longrightarrow & \underset{\mathbf{2024}}{\wedge^3}, & \underset{\mathbf{276}}{\wedge^2} & \\ \underset{\mathbf{300}}{\wedge^2} & \longrightarrow & \underset{\mathbf{276}}{\wedge^2}, & \underset{\mathbf{24}}{\wedge^1} & \end{array} \tag{72}$$

In other words, the (massless) field content (71) of the Monstrous gravity $0.\alpha.iii$ in $25+1$ can be obtained by the S^1 reduction of the following (massless) field content in $26+1$:

$$\begin{array}{ccc} \text{field} & \mathbf{R}\text{ of }\mathfrak{so}_{25} & \# \\ g: & \mathbf{324} & 1 \\ \wedge^5: & \mathbf{53{,}130} & 2 \\ \wedge^4: & \mathbf{12{,}650} & 6 \\ \wedge^3: & \mathbf{2300} & 6 \\ \wedge^2: & \mathbf{300} & 2 \end{array} \tag{73}$$

Therefore, we picked an Einstein gravity theory coupled to p-forms, with $p = 2, 3, 4, 5$, in $26+1$ space–time dimensions (that can be coupled to a **98,304** Rarita–Schwinger field), whose massless spectrum contains $196,884$ degrees of freedom that may be acted upon by the Monster group $\mathbb{M}$, *at least* after reduction to $D = 25+1$, and after suitable assignment. The assignment is as follows in $D = 25+1$: 98,280 = (**42,504** + 4 × **10,626** + 6 × **2024**) + 4 × **276** + **24** to the norm four (i.e., minimal) Leech vectors modulo $\mathbb{Z}_2$, and hence 196,884 = 1 + 299 + 98,280 + 98,304 which corresponds to the Griess algebra, namely to the sum of the two smallest representations of $\mathbb{M}$, namely the trivial (singlet) **1** and the smallest non-trivial one **196,883**. Such a theory will be henceforth named *Monstrous M-theory*, or simply *M^2-theory*. Note that the disentangling of the 196,884 degrees of freedom into **196,883** $\oplus$ **1** occurs only when reducing the theory to $25+1$, in which case the dilaton ϕ is identified with the singlet of $\mathbb{M}$: in other words, *the (observation which firstly hinted the) Monstrous Moonshine* [4] *is crucially related to the S^1 compactification of M^2-theory down to $25+1$ space–time dimensions*.

5.1. Lagrangian(s) for Bosonic Monstrous M-Theory

A priori, the purely bosonic 196,884-dimensional degrees of freedom of the massless spectrum of M^2-theory can be realized in various ways at the Lagrangian level. Here, within the framework defined above, we will attempt to write down a general Lagrangian for the bosonic part of M^2-theory.

We start by labeling the massless fields of M^2-theory, given by (73), as follows (It is amusing to note that the p-form (potentials) content of M^2-theory follows from a pair of 5-form (potentials) of SO_{28}, which is the massless little group in 30 dimensions. Thus, the bosonic non-gravitational content of M^2-theory descends from a pair of massless 4-branes in $D=s+t=30$,or better from a self-dual pair of massless p-form potentials in $D=30$, namely from a 5-form and its dual 23-form potentials, respectively related to *massless* 4-brane and its dual 22-brane in $D=30$):

$$\begin{array}{ccc} \text{field} & \text{label} & \# \\ g: & g_{\mu\nu} & 1 \\ \wedge^5: & C^{(5)A}_{\lambda\mu\nu\rho\sigma} & 2 \\ \wedge^4: & C^{(4)i}_{\lambda\mu\nu\rho} & 6 \\ \wedge^3: & C^{(3)i}_{\lambda\mu\nu} & 6 \\ \wedge^2: & C^{(2)A}_{\lambda\mu} & 2 \end{array} \tag{74}$$

The uppercase Latin indices take values $1,2$, whereas the lowercase Latin indices run $1,2,\ldots,6$. A general definition of the field strengths reads

$$\begin{aligned} G^{(3)A} &:= dC^{(2)A} + \mathbf{A}^A_j C^{(3)j}; \\ G^{(4)i} &:= dC^{(3)i} + \mathbf{B}^i_{(AB)} C^{(2)A} \wedge C^{(2)B} + \mathbf{C}_{ij} C^{(4)j}; \\ G^{(5)i} &:= dC^{(4)i} + \mathbf{D}^i_{Aj} C^{(2)A} \wedge C^{(3)j} + \mathbf{E}^i_A C^{(5)A}; \\ G^{(6)A} &:= dC^{(5)A} + \mathbf{F}^A_{(BCD)} C^{(2)B} \wedge C^{(2)C} \wedge C^{(2)D} + \mathbf{G}^A_{ij} C^{(3)i} \wedge C^{(3)j} + \mathbf{H}^A_{Bi} C^{(4)i} \wedge C^{(2)B}, \end{aligned} \tag{75}$$

where the uppercase bold Latin tensors are constant (all (uppercase and calligraphic) Latin tensors introduced in (76) and (78) are constant because there is no scalar field in the (massless) spectrum of the theory), and they are possibly given by suitable representation theoretic projectors (Here, we will not analyze possible characterizations of such tensors as (invariant) projectors. We confine ourselves to remark that, in a very simple choice of covariance (namely, $A=1,2$ and $i=1,2,\ldots,6$ running over the spin-$1/2$ and spin-$5/2$ representations **2** and **6** of $\mathfrak{sl}_2$), most of them vanish).

Then, a general Lagrangian density can be written as

$$\begin{aligned} \mathcal{L} &= R - \frac{1}{2\cdot 3!}\mathcal{A}_{AB} G^{(3)A}\cdot G^{(3)B} - \frac{1}{2\cdot 4!}\mathcal{B}_{ij} G^{(4)i}\cdot G^{(4)j} \\ &\quad - \frac{1}{2\cdot 5!}\mathcal{C}_{ij} G^{(5)i}\cdot G^{(5)j} - \frac{1}{2\cdot 6!}\mathcal{D}_{AB} G^{(6)A}\cdot G^{(6)B} + \mathcal{L}_{CS\text{-like}}, \end{aligned} \tag{76}$$

where the calligraphic Latin constant tensors are (symmetric and) positive definite in order for all kinetic terms of p-forms to be consistent. A minimal, Maxwell-like choice is $\mathcal{A}_{AB}=\mathcal{D}_{AB}=\delta_{AB}$ and $\mathcal{B}_{ij}=\mathcal{C}_{ij}=\delta_{ij}$, such that (76) simplifies down to

$$\begin{aligned}\mathcal{L} &= R - \frac{1}{2\cdot 3!}\sum_{A=1}^{2} G^{(3)A}_{\mu\nu\rho} G^{(3)A|\mu\nu\rho} - \frac{1}{2\cdot 4!}\sum_{i=1}^{6} G^{(4)i}_{\lambda\mu\nu\rho} G^{(4)i|\lambda\mu\nu\rho} \\ &\quad - \frac{1}{2\cdot 5!}\sum_{i=1}^{6} G^{(5)i}_{\lambda\mu\nu\rho\sigma} G^{(5)i|\lambda\mu\nu\rho\sigma} - \frac{1}{2\cdot 6!}\sum_{A=1}^{2} G^{(6)A}_{\lambda\mu\nu\rho\sigma\tau} G^{(6)A|\lambda\mu\nu\rho\sigma\tau} + \mathcal{L}_{CS\text{-like}}. \end{aligned} \tag{77}$$

The "topological", "Chern–Simons-like" Lagrangian occurring in (76) and (77) is composed by a number of a priori non-vanishing terms, such as, for instance,

$$\begin{aligned}\sqrt{|g|}\mathcal{L}_{CS} &= \epsilon\mathcal{E}_3^{ABCDi} G^{(6)}_A G^{(6)}_B G^{(6)}_C G^{(6)}_D C^{(3)}_i \\ &\quad + \epsilon\mathcal{I}_2^{ijklmA} G^{(5)}_i G^{(5)}_j G^{(5)}_k G^{(5)}_l G^{(5)}_m C^{(2)}_A + \dots \\ &\quad + \epsilon\mathcal{S}_3^{ijklmnp} G^{(4)}_i G^{(4)}_j G^{(4)}_k G^{(4)}_l G^{(4)}_m G^{(4)}_n C^{(3)}_p + \dots \\ &\quad + \epsilon\mathcal{W}_3^{ABCDEFGHi} G^{(3)}_A G^{(3)}_B G^{(3)}_C G^{(3)j}_D G^{(3)}_E G^{(3)}_F G^{(3)}_G G^{(3)}_H C^{(3)}_i + \dots \end{aligned} \tag{78}$$

where ϵ denotes the Ricci–Levi–Civita tensor in $26+1$, and the full Lagrangian is shown in Appendix A. We leave the study of the constant tensors $\mathbf{A}, \dots, \mathbf{H}$, $\mathcal{A}, \dots, \mathcal{D}$, and $\mathcal{E}, \dots, \mathcal{W}$ respectively in (75)–(78) (as well as others occurring in Appendix A) for further future work.

It is immediately realized that M^2-theory includes Horowitz and Susskind's bosonic M-theory [19] as a truncation; indeed, by setting

$$\begin{aligned} C^{(2)A} &= 0; \\ C^{(3)i} &= \delta^{i1} C; \\ C^{(4)i} &= 0; \\ C^{(5)A} &= 0, \end{aligned} \tag{79}$$

one obtains ($F = dC$)

$$\mathcal{L} = R - \frac{1}{2\cdot 4!} F^2, \tag{80}$$

which is the Lagrangian of the bosonic string theory discussed by Susskind and Horowitz in [19].

Finally, we observe that a Scherk–Schwarz reduction of the Lagrangian (76) to $25+1$ would provide a quite general Lagrangian for the $0.\alpha.iii$ Monster (dilatonic, Einstein) gravity; we leave this task for further future work.

5.2. $B = F$ in $26+1$

Remarkably, a certain subsector of M^2-theory, when coupled to an $h = 3/2$ Rarita–Schwinger field, exhibits $B = F$, which is a necessary condition for (linearly realized, conventional) supersymmetry to hold. Such a subsector is given by the following (Analogously to what observed in Section 5.1, it is amusing to observe that the bosonic content of the $B = F$ sector of $26+1$ M^2-theory (which we are tempted to conjecture to be $\mathcal{N} = 1$, $D = 26+1$ supergravity; see further below) derives from a single 5-form potential, corresponding to a *massless* 4-brane, in $D = 30$, complemented by a "transmutation" of the 2-form potential **300** of $\mathfrak{so}_{25}$ into the rank-2 symmetric traceless tensor (graviton) **324** of $\mathfrak{so}_{25}$, namely by the replacement of a massless string (1-brane) with a massless graviton in $D = 26+1$):

field	**R** of $\mathfrak{so}_{25}$	#
g :	**324**	1
$\wedge^5$:	**53,130**	1
$\wedge^4$:	**12,650**	3
$\wedge^3$:	**2300**	3
$\wedge^2$:	**300**	0

(81)

Thus, when coupled to a an $h = 3/2$ RS field ψ (fitting the **98,304** irreducible representation of $\mathfrak{so}_{25}$, with Dynkin labels $(1, 0^{10}, 1)$), the resulting theory has (Again, bosonic M-theory [19] trivially is a subsector of (the purely bosonic sector of) such a theory in $26+1$)

$$B = F = 98{,}304. \tag{82}$$

By recalling (72) and observing that the massless RS field branches from $26+1$ to $25+1$ as

$$\underbrace{\underset{\psi}{\mathbf{98{,}304}}}_{\mathfrak{so}_{25}\text{ repr.}} = \underbrace{\underset{\psi}{\mathbf{47{,}104}} \oplus \underset{\psi'}{\mathbf{47{,}104}'} \oplus \underset{\lambda}{\mathbf{2048}} \oplus \underset{\lambda'}{\mathbf{2048}'}}_{\mathfrak{so}_{24}\text{ reprs.}}, \tag{83}$$

the subsector of M^2-theory with $B = F = 98{,}304$ gives rise to the following massless spectrum, when reduced to $25+1$:

$$\begin{array}{ccc}
\text{field} & \mathbf{R} \text{ of } \mathfrak{so}_{24} & \# \\
g: & \mathbf{299} & 1 \\
\psi: & \mathbf{47{,}104} & 2 \equiv (\psi \oplus \psi') \\
\wedge^1: & \mathbf{24} & 1 \\
\lambda: & \mathbf{2048} & 2 \equiv (\lambda \oplus \lambda') \\
\varphi: & \mathbf{1} & 1 \\
\wedge^5: & \mathbf{42{,}504} & 1 \\
\wedge^4: & \mathbf{10{,}626} & 4 \\
\wedge^3: & \mathbf{2024} & 6 \\
\wedge^2: & \mathbf{276} & 3
\end{array} \tag{84}$$

By recalling the treatment of Section 4, one can recognize (84) as a subsector (in which (82) holds) of the Monstrous gravity $2.\gamma.ii$ in (60), simply obtained by decreasing $\#\wedge^2$ from 4 to 3.

Other subsectors of Monstrous gravity theories in $25+1$ exist such that $B = F$. Below, we list some of them:

$$\begin{array}{cccccc}
 & & \mathbf{s}_1 & \mathbf{s}_2 & B=F \\
0. & \gamma.i-iv & (0,0,1,1,0) & (0,0,1,0) & 2048 \\
2. & \left\{\begin{array}{l} \alpha.i-ii \\ \beta.i-ii \\ \gamma.i-ii \\ \delta.i-ii \end{array}\right. & (0,2,0,0,0) & (1,4,4,4) & 94{,}208 \\
2. & \left\{\begin{array}{l} \gamma.ii \\ \delta.ii \end{array}\right. & \begin{array}{c}(0,2,1,1,0)\\(1,2,0,1,1)\end{array} & \begin{array}{c}(1,4,5,4)\\(1,4,5,3)\end{array} & 96{,}256
\end{array} \tag{85}$$

Note that, among the $B = F$ subsectors in $25+1$ reported above, only (84) and the second in the last line of (85) (i.e., the subsector of the $2.\delta.ii$ Monstrous gravity) contain gravity.

5.2.1. $\mathcal{N} = 1$ Supergravity in $26+1$?

As pointed out, $B = F$ is a necessary but not sufficient condition for (linearly realized, local, conventional) supersymmetry to hold. It is thus tantalizing to conjecture that the theory in $26+1$ with massless spectrum (81) and one Rarita–Schwinger field ψ is actually a $\mathcal{N} = 1$ supergravity theory.

Inspired by M-theory (throughout our treatment, we refer to the conventions used in Section 22 of [33]) (i.e., $\mathcal{N} = 1$ supergravity) in $10+1$, and exploiting a truncation of the purely bosonic Lagrangians discussed in Section 5.1 (the capped lowercase Latin indices run $\hat{\imath} = 1, 2, 3$ throughout), one can write down a tentative Lagrangian for the would-be $\mathcal{N} = 1$ supergravity in $26 + 1$:

$$\begin{aligned}
\mathcal{L} \quad = \quad & R - \frac{1}{2 \cdot 4!} \sum_{\hat{\imath}=1}^{3} G^{(4)\hat{\imath}} \cdot G^{(4)\hat{\imath}} - \frac{1}{2 \cdot 5!} \sum_{\hat{\imath}=1}^{3} G^{(5)\hat{\imath}} \cdot G^{(5)\hat{\imath}} - \frac{1}{2 \cdot 6!} G^{(6)} \cdot G^{(6)} + \mathcal{L}_{CS\text{-like}} \\
& - \mathbf{a} \frac{i}{2} \overline{\psi}_\mu \Gamma^{\mu\nu\rho} \nabla_\nu \left(\frac{\omega + \tilde{\omega}}{2} \right) \psi_\rho \\
& + \sum_{\hat{\imath}=1}^{3} \mathbf{b}_{\hat{\imath}} \overline{\psi}_\mu \Gamma^{[\mu} \Gamma^{(4)} \Gamma^{\nu]} \psi_\nu \cdot \left(G^{(4)\hat{\imath}} + \tilde{G}^{(4)\hat{\imath}} \right) \\
& + \sum_{\hat{\imath}=1}^{3} \mathbf{c}_{\hat{\imath}} \overline{\psi}_\mu \Gamma^{[\mu} \Gamma^{(5)} \Gamma^{\nu]} \psi_\nu \left(G^{(5)\hat{\imath}} + \tilde{G}^{(5)\hat{\imath}} \right) \\
& + \mathbf{d} \overline{\psi}_\mu \Gamma^{[\mu} \Gamma^{(6)} \Gamma^{\nu]} \psi_\nu \cdot \left(G^{(6)} + \tilde{G}^{(6)} \right),
\end{aligned} \tag{86}$$

where

$$\Gamma^{(4)} \cdot G^{(4)\hat{\imath}} = \Gamma_{\alpha\beta\gamma\delta} G^{(4)\hat{\imath}|\alpha\beta\gamma\delta}, \text{ etc.} \tag{87}$$

and, upon truncation of (75) respectively (78),

$$\begin{aligned}
& G^{(4)\hat{\imath}} := dC^{(3)\hat{\imath}} + \mathbf{C}_{\hat{\imath}\hat{\jmath}} C^{(4)\hat{\jmath}}; \\
& G^{(5)\hat{\imath}} := dC^{(4)\hat{\imath}} + \mathbf{E}^{\hat{\imath}} C^{(5)}; \\
& G^{(6)} := dC^{(5)} + \mathbf{G}_{\hat{\imath}\hat{\jmath}} C^{(3)\hat{\imath}} \wedge C^{(3)\hat{\jmath}};
\end{aligned} \tag{88}$$

$$\begin{aligned}
\sqrt{|g|} \mathcal{L}_{CS\text{-like}} \quad = \quad & \epsilon \mathcal{E}_{\hat{\imath}} G^{(6)} G^{(6)} G^{(6)} G^{(6)} C^{(3)\hat{\imath}} \\
& + \epsilon \mathcal{G}_{\hat{\imath}\hat{\jmath}\hat{k}\hat{l}\hat{m}\hat{n}\hat{p}} G^{(4)\hat{\imath}} G^{(4)\hat{\jmath}} G^{(4)\hat{k}} G^{(4)\hat{l}} G^{(4)\hat{m}} G^{(4)\hat{n}} C^{(3)\hat{p}} \\
& + \epsilon \mathcal{H}_{\hat{\imath}\hat{\jmath}\hat{k}\hat{l}} G^{(6)} G^{(6)} G^{(4)\hat{\imath}} G^{(4)\hat{\jmath}} G^{(4)\hat{k}} C^{(3)\hat{l}} \\
& + \epsilon \mathcal{I}_{\hat{\imath}\hat{\jmath}\hat{k}\hat{l}\hat{m}} G^{(6)} G^{(5)\hat{\imath}} G^{(5)\hat{\jmath}} G^{(4)\hat{k}} G^{(4)\hat{l}} C^{(3)\hat{m}} \\
& + \epsilon \mathcal{J}_{\hat{\imath}} G^{(6)} G^{(6)} G^{(6)} G^{(5)\hat{\imath}} C^{(4)\hat{\jmath}} \\
& + \epsilon \mathcal{K}_{\hat{\imath}} G^{(6)} G^{(6)} G^{(6)} G^{(4)\hat{\imath}} C^{(5)} \\
& + \epsilon \mathcal{L}_{\hat{\imath}\hat{\jmath}\hat{k}\hat{l}\hat{m}} G^{(6)} G^{(5)\hat{\imath}} G^{(4)\hat{\jmath}} G^{(4)\hat{k}} G^{(4)\hat{l}} C^{(4)\hat{m}} \\
& + \epsilon \mathcal{M}_{\hat{\imath}\hat{\jmath}} G^{(6)} G^{(6)} G^{(5)\hat{\imath}} G^{(5)\hat{\jmath}} C^{(5)} \\
& + \epsilon \mathcal{N}_{\hat{\imath}\hat{\jmath}\hat{k}\hat{l}} G^{(6)} G^{(4)\hat{\imath}} G^{(4)\hat{\jmath}} G^{(4)\hat{k}} G^{(4)\hat{l}} C^{(5)} \\
& + \epsilon \mathcal{O}_{\hat{\imath}\hat{\jmath}\hat{k}\hat{l}\hat{m}\hat{n}} G^{(5)\hat{\imath}} G^{(5)\hat{\jmath}} G^{(5)\hat{k}} G^{(5)\hat{l}} G^{(4)\hat{m}} C^{(3)\hat{n}} \\
& + \epsilon \mathcal{P}_{\hat{\imath}\hat{\jmath}\hat{k}\hat{l}\hat{m}\hat{n}} G^{(5)\hat{\imath}} G^{(5)\hat{\jmath}} G^{(5)\hat{k}} G^{(4)\hat{l}} G^{(4)\hat{m}} C^{(4)\hat{n}}.
\end{aligned} \tag{89}$$

Moreover,

$$\begin{aligned}
& \tilde{G}^{(4)\hat{\imath}} := G^{(4)\hat{\imath}} + \mathbf{e}^{\hat{\imath}} \overline{\psi} \Gamma^{(2)} \psi; \\
& \tilde{G}^{(5)\hat{\imath}} := G^{(5)\hat{\imath}} + \mathbf{f}^{\hat{\imath}} \overline{\psi} \Gamma^{(3)} \psi; \\
& \tilde{G}^{(6)} := G^{(6)} + \mathbf{g} \overline{\psi} \Gamma^{(4)} \psi
\end{aligned} \tag{90}$$

are the would-be supercovariant field strengths, and

$$\nabla_\mu(\omega) \psi_\nu := \partial_\mu \psi_\nu + \mathbf{h} \omega_\mu^{ab} \Gamma_{ab} \psi_\nu \tag{91}$$

is the covariant derivative with

$$\begin{aligned}\tilde{\omega}_\mu^{ab} &:= \omega_\mu^{ab} + i\mathbf{l}\overline{\psi}_\alpha \Gamma_\mu^{ab\alpha\beta}\psi_\beta; \qquad (92)\\ \omega_\mu^{ab} &:= \omega_\mu^{ab}(e) + K_\mu^{ab}; \qquad (93)\\ K_\mu^{ab} &:= i\Big[\mathbf{m}\overline{\psi}_\alpha \Gamma_\mu^{ab\alpha\beta}\psi_\beta + \mathbf{n}\Big(\overline{\psi}_\mu \Gamma^b \psi^a - \overline{\psi}_\mu \Gamma^a \psi^b + \overline{\psi}^b \Gamma_\mu \psi^a\Big)\Big]. \qquad (94)\end{aligned}$$

We can therefore formulate the following.

Conjecture

The Lagrangian (86) should be invariant under the following local supersymmetry transformations with parameter ε (a Majorana spinor):

$$\begin{aligned}\delta_\varepsilon e_\mu^a &= -\frac{i}{2}\bar{\varepsilon}\Gamma^a \psi_\mu; \qquad (95)\\ \delta_\varepsilon \psi_\mu &= \mathbf{p}\nabla_\mu(\tilde{\omega})\varepsilon + \sum_{\hat{\imath}=1}^{3}\mathbf{q}_{\hat{\imath}}\Big(\Gamma^{\alpha\beta\gamma\delta}{}_\mu + \mathbf{r}_{\hat{\imath}}\Gamma^{\beta\gamma\delta}\delta_\mu^\alpha\Big)\varepsilon \tilde{G}^{(4)\hat{\imath}}_{\alpha\beta\gamma\delta}\\ &\quad + \sum_{\hat{\imath}=1}^{3}\mathbf{s}_{\hat{\imath}}\Big(\Gamma^{\alpha\beta\gamma\delta\rho}{}_\mu + \mathbf{t}_{\hat{\imath}}\Gamma^{\beta\gamma\delta\rho}\delta_\mu^\alpha\Big)\varepsilon \tilde{G}^{(5)\hat{\imath}}_{\alpha\beta\gamma\delta\rho} + \mathbf{u}\Big(\Gamma^{\alpha\beta\gamma\delta\rho\sigma}{}_\mu + \mathbf{v}\Gamma^{\beta\gamma\delta\rho\sigma}\delta_\mu^\alpha\Big)\varepsilon \tilde{G}^{(6)}_{\alpha\beta\gamma\delta\rho\sigma}; \qquad (96)\\ \delta_\varepsilon C^{(3)\hat{\imath}}_{\mu\nu\rho} &= \mathbf{w}^{\hat{\imath}}\bar{\varepsilon}\Gamma_{[\mu\nu}\psi_{\rho]}; \qquad (97)\\ \delta_\varepsilon C^{(4)\hat{\imath}}_{\mu\nu\rho\sigma} &= \mathbf{x}^{\hat{\imath}}\bar{\varepsilon}\Gamma_{[\mu\nu\rho}\psi_{\sigma]}; \qquad (98)\\ \delta_\varepsilon C^{(5)}_{\mu\nu\rho\sigma\tau} &= \mathbf{y}\bar{\varepsilon}\Gamma_{[\mu\nu\rho\rho}\psi_{\tau]}. \qquad (99)\end{aligned}$$

To prove (or disprove) the invariance of the Lagrangian (86) (with definitions (87) and (94)) under the local supersymmetry transformations (95) and (99), and thus fixing the real parameters $\mathbf{a}, \ldots, \mathbf{y}$ as well as the tensors $\mathbf{C}$, $\mathbf{E}$, $\mathbf{G}$ and $\mathcal{E}$, $\mathcal{G}$, seems a formidable task, which deserves to be pursued in a separate paper.

Under dimensional reduction to $25+1$, one would then obtain a would-be type IIA $\mathcal{N} = (1,1)$ supergravity theory, with massless spectrum (84); as observed above, this would correspond to a suitable truncation of the Monstrous gravity 2.γ.*ii* in (60), in which $\#\wedge^2$ decreases from 4 to 3. Again, we leave the investigation of this interesting task for further future work.

6. Cohomological Construction of Lattices: From $\mathfrak{e}_8$ to the Leech Lattice

Let us consider the following (commutative) diagram, starting from the Lie algebra $\mathfrak{e}_8$,

$$\begin{tikzcd} & \mathfrak{a}_8 \oplus \wedge^3 \oplus \overline{\wedge^3} \arrow[dr] & \\ \mathfrak{e}_8 \arrow[ur] \arrow[dr] & & \mathfrak{b}_4 \oplus g \oplus \wedge^3 \oplus *\wedge^3 \\ & \mathfrak{d}_8 \oplus \boldsymbol{\lambda} \arrow[ur] & \end{tikzcd} \qquad (100)$$

where $g \equiv S_0^2$, as above, denotes the $D = 10+1$ graviton representation (which is related to "super-Ehlers" embeddings in [36]), and $*$ stands for the Hodge dual ($*\wedge^p := \wedge^{D-p}$). Thus, the number $\#\mathfrak{e}_8$ of roots of the $\mathfrak{e}_8$ root lattice reads

$$\underset{240}{\#\mathfrak{e}_8} = \underset{248}{\dim \mathfrak{e}_8} - 8 = (\underset{32}{\dim \mathfrak{b}_4} - 4) + (\underset{40}{\dim \mathbf{g}} - 4) + \dim(\underset{84\cdot 2}{\wedge^3 \oplus *\wedge^3}). \qquad (101)$$

Therefore, the number $\#\mathfrak{e}_8^+$ of positive roots of $\mathfrak{e}_8$ is

$$\#\mathfrak{e}_8^+ = \frac{1}{2}\Big((\underset{32}{\dim \mathfrak{b}_4} - 4) + (\underset{40}{\dim \mathbf{g}} - 4)\Big) + \underset{84}{\dim \wedge^3} = 120. \qquad (102)$$

Note that it also holds that

$$\frac{1}{2}\Big((\dim \underset{32}{\mathfrak{b}_4} - 4) + (\dim \underset{40}{\mathbf{g}} - 4) \Big) = \dim \mathfrak{b}_4. \tag{103}$$

It should be also remarked that $120 = \binom{10}{3}$, i.e., it matches the number of degrees of freedom of a massless 3-form potential in $11+1$ space-dimensions; indeed, a massless 3-form in $11+1$ (corresponding to $\wedge^3$ of the little group SO_{10}) gives rise to a massless 3-form and a massless 2-form in $10+1$ (corresponding to $\wedge^3 \oplus \wedge^2 \simeq \wedge^3 \oplus \mathfrak{b}_4$ of the little group SO_9).

The case of $\mathfrak{e}_8$ is peculiar, because the closure (as well as the commutativity) of the diagram (100) relies on the existence of the "anomalous" embedding:

$$\begin{gathered} \mathfrak{d}_8 \supset \mathfrak{b}_4; \\ \mathbf{16} = \mathbf{16} \equiv \boldsymbol{\lambda}, \end{gathered} \tag{104}$$

where $\boldsymbol{\lambda}$ is the spinor representation.

By replacing $\mathfrak{b}_4$ and $\wedge^3$, respectively, as follows,

$$\mathfrak{b}_4 \rightarrow \mathfrak{b}_{12}; \tag{105}$$

$$\wedge^3 \rightarrow \wedge^5 \oplus 3 \cdot \wedge^4 \oplus 3 \cdot \wedge^3, \tag{106}$$

one can define the "Leech algebra" $\mathfrak{L}_{24}$ in analogy with $\mathfrak{e}_8$ (albeit with $D = 26+1$ graviton g), through the following diagram:

$$\begin{array}{ccccc} & & \begin{array}{l} \mathfrak{a}_{24} \oplus (\wedge^5 \oplus 3 \cdot \wedge^4 \oplus 3 \cdot \wedge^3) \\ \oplus \left(\overline{\wedge^5} \oplus 3 \cdot \overline{\wedge^4} \oplus 3 \cdot \overline{\wedge^3}\right) \end{array} & & \\ & \nearrow & & \searrow & \\ \mathfrak{L}_{24} & & & & \begin{array}{l} \mathfrak{b}_{12} \oplus g \oplus (\wedge^5 \oplus 3 \cdot \wedge^4 \oplus 3 \cdot \wedge^3) \\ \oplus \left(* (\wedge^5 \oplus 3 \cdot \wedge^4 \oplus 3 \cdot \wedge^3) \right) \end{array} \\ & \searrow & & \nearrow & \\ & & ? & & \end{array} \tag{107}$$

The question mark in (107) occurs because *there is no analogue of the "anomalous" embedding (104) for* $\mathfrak{L}_{24}$. Thus, it holds that

$$\begin{aligned} \underset{196{,}560}{\#\mathfrak{L}_{24}} &= \dim \underset{196{,}584}{\mathfrak{L}_{24}} - 24 \\ &= (\dim \underset{288}{\mathfrak{b}_{12}} - 12) + (\dim \underset{312}{\mathbf{g}} - 12) + \underset{2\times(53{,}130+3\times 12{,}650+3\times 2300)}{\dim\Big(\wedge^5 \oplus 3 \cdot \wedge^4 \oplus 3 \cdot \wedge^3 + *\big(\wedge^5 \oplus 3 \cdot \wedge^4 \oplus 3 \cdot \wedge^3\big)\Big)} \\ &= 196{,}560, \end{aligned} \tag{108}$$

where $\#\mathfrak{L}_{24}$ denotes the number of minimal, non-trivial vectors (of norm 4) of the Leech lattice Λ_{24}. Therefore, the $\mathbb{Z}_2$-modded number of minimal, non-trivial vectors of Λ_{24} is

$$\#\mathfrak{L}_{24}^{+} = \frac{1}{2}\Big((\dim \underset{288}{\mathfrak{b}_{12}} - 12) + (\dim \underset{312}{\mathbf{g}} - 12) \Big) + \dim(\underset{53{,}130+3\times 12{,}650+3\times 2300}{\wedge^5 \oplus 3 \cdot \wedge^4 \oplus 3 \cdot \wedge^3}) = 98{,}280, \tag{109}$$

which is the number entering the construction of the smallest non-trivial representation of the Monster group $\mathbb{M}$ (cfr. [37]). Note that it also holds that

$$\frac{1}{2}\Big((\dim \underset{288}{\mathfrak{b}_{12}} - 12) + (\dim \underset{312}{\mathbf{g}} - 12) \Big) = \dim \mathfrak{b}_{12}. \tag{110}$$

It should moreover be also remarked that $98{,}280 = \binom{28}{5}$, i.e., it matches the number of degrees of freedom of a massless 5-form potential in $D = 29+1$ space-dimensions; indeed, it can be checked that a massless 5-form potential in $D = 29 + 1$ (corresponding to $\wedge^5$ of the little group SO_{28}) gives rise to 1 massless 5-form, 3 massless 4-forms, 3 massless 3-forms and 1 massless 2-form in $26 + 1$ (corresponding to $\left(\wedge^5 \oplus 3 \cdot \wedge^4 \oplus 3 \cdot \wedge^3\right) \oplus \wedge^2 \simeq \left(\wedge^5 \oplus 3 \cdot \wedge^4 \oplus 3 \cdot \wedge^3\right) \oplus \mathfrak{b}_{12}$ of the little group SO_{25}).

Equations (102) and (109) define a cohomological construction of the 8-dimensional $\mathfrak{e}_8$ root lattice and of the 24-dimensional Leech lattice Λ_{24}, respectively, based on the analogy between the following:

- M-theory in $10 + 1$ space–time dimensions, with SO_9 massless little group and massless spectrum given by $\mathbf{128}$ (gravitino $\boldsymbol{\psi}$) $=$ $\mathbf{84}$ (3-form potential $\wedge^3$) $\oplus$ $\mathbf{44}$ (graviton $g \simeq S^0_2$); this corresponds to $D0$-branes (supergravitons) in BFSS M(atrix) model, carrying $256 = 128(B) + 128(F)$ KK states [38].
- The would-be $\mathcal{N} = 1$ supergravity in $26 + 1$ space–time dimensions, with SO_{25} massless little group and massless spectrum given by **98,304** (would-be gravitino $\boldsymbol{\psi}$) $= 3 \times \mathbf{2300} \oplus 3 \times \mathbf{12{,}650} \oplus \mathbf{53{,}130}$ (set of massless p-forms which is the "$(26 + 1)$-dimensional analogue" of the 3-form in $10 + 1$) $\oplus$ $\mathbf{324}$ (graviton $g \simeq S^0_2$); this would correspond to $D0$-branes (i.e., the would-be "supergravitons") in the would-be BFSS-like M(atrix) model, carrying $196{,}608 = 98{,}304(B) + 98{,}304(F)$ KK states.

There are many analogies, but the big difference is (local) supersymmetry in $D = 26 + 1$ (and possibly in $D = 25 + 1$), whose nature is at present still conjectural.

The "Leech algebra" $\mathfrak{L}_{24}$ encodes $\dim \mathfrak{su}_{25} = 624 = 324 + 300$, and $2 \times 97{,}980 = 2 \times (3 \times 2300 + 3 \times 12{,}650 + 53{,}130) = 195{,}960$ to get 624 + 195,960 = 196,584. Removing the 12 + 12 = 24 Cartans gives 196,560, which is the number of minimal Leech vectors. It is thus tempting to conjecture "Monstrous supergravitons" as $D0$-branes, as $\mathfrak{L}_{24}$ "sees" 98, 304 of the bosonic KK states. On the other hand, the Monster $\mathbb{M}$ acts on almost all of these, albeit seeing only $299 + 1$ of the 324 graviton degrees of freedom from $324 + 300$, giving 299 + 1 + (300 + 97,980) = 299 + 1 + 98,280 of the Griess algebra [1,6].

Therefore, *the relation between the "Leech algebra" $\mathfrak{L}_{24}$ and the Griess algebra is realized in field theory by the relation between M^2-theory and its subsector (81) coupled to one RS field (the would-be gravitino) in $D = 26 + 1$*, discussed in Section 5.2.

6.1. $26 + 1 \longrightarrow 10 + 1$ *through Vinberg's T-Algebras*

How can one relate M-theory in $D = 10 + 1$ with M^2-theory in $D = 26 + 1$?

The dimensional reduction $26 + 1 \longrightarrow 10 + 1$ may have a non-trivial structure: one can proceed along a decomposition proved by Wilson [8], characterizing the aforementioned number of minimal Leech vectors as

$$196{,}560 = 3 \times 240 \times (1 + 16 + 256). \tag{111}$$

Therefore, we identify $1 + 16 + 256 = 273$ with the (Hermitian part of) Vinberg's T-algebra and 240 with E_8 fibers (in (112), the Greek subscripts discriminate among $\mathfrak{so}_{16}$-singlets) [26,39]:

$$T_3^{8,2} = \underbrace{\begin{pmatrix} \mathbf{1}_\alpha & \mathbf{16} & \mathbf{128} \\ * & \mathbf{1}_\beta & \mathbf{128}' \\ * & * & \mathbf{1}_\gamma \end{pmatrix}}_{\text{written in a } \mathfrak{so}_{16} \text{ covariant way}}, \tag{112}$$

with spin factor lightcone coordinates $\mathbf{1}_\alpha$ and $\mathbf{1}_\beta$ removed, thus yielding $128 + 128 + 16 + 1 = 273$ degrees of freedom. The spin factor $\mathbf{1}_\alpha \oplus \mathbf{1}_\beta \oplus \mathbf{16}$ of $T_3^{8,2}$ (112) enjoys an enhancement from $\mathfrak{so}_{16}$ (massless little algebra in $17 + 1$) to $\mathfrak{so}_{17,1}$ Lorentz algebra ($\mathfrak{so}_{17,1}$ would be the Lorentz symmetry of the 18-dimensional string theory suggested by Lorentzian Kac–Moody algebras [40]), and $\mathfrak{der}\left(T_3^{8,2}\right) = \mathrm{mcs}(\mathfrak{so}_{17,1}) = \mathfrak{so}_{17}$ [41]. Breaking the $\mathfrak{so}_{25}$ Lie algebra of massless little group in $26 + 1$ with respect to $\mathfrak{so}_{17}$, as well as its $\mathbf{4096}$ spinor (both encoded in the so-called "Exceptional Periodicity" algebra $\mathfrak{f}_4^3$ [42]), one obtains the decomposition

$$\mathfrak{f}_4^3 := \mathfrak{so}_{25} \oplus \mathbf{4096} = \mathfrak{so}_{17} \oplus \mathfrak{so}_8 \oplus (\mathbf{17}, \mathbf{8}_v) \oplus (\mathbf{256}, \mathbf{8}_s) \oplus (\mathbf{256}, \mathbf{8}_c). \tag{113}$$

As $\mathfrak{so}_8$ acts on S^7, one can take the 240 roots as forming a discrete 7-sphere, and the 273 is constructed as $17 + 256 = 273$ by picking one of the **256** spinors. This gives a discrete form of the maximal Hopf fibration:

$$S^7 \hookrightarrow S^{15} \to S^8, \tag{114}$$

and the three maps yield three charts of the form $196{,}560 = 3 \cdot 240 \cdot 273$ (cfr. (111)) in a discrete Cayley plane [7,22,26]. Through the super-Ehlers embedding [36]

$$\mathfrak{e}_{8(8)} = \mathfrak{sl}_9(\mathbb{R}) \oplus \mathbf{84} \oplus \mathbf{84}' = \mathfrak{so}_9 \oplus \mathbf{44} \oplus \mathbf{84} \oplus \mathbf{84}, \tag{115}$$

we can identify each discrete S^7 fiber of 240 E_8 roots with the M2- and M5-brane gauge fields of $D = 10 + 1$ M-theory, as well as with little group ($\mathfrak{so}_9$) and graviton (**44**) degrees of freedom, albeit with all $4 + 4$ Cartans removed. This is understood with $\mathfrak{so}_9 \subset \mathfrak{so}_{25}$ acting isometrically on the S^8 base. From this perspective, the reduction from $D = 26 + 1$ to $D = 10 + 1$ occurs first along three charts, and gauge and gravity data are encoded in discrete S^7 chart fibers therein.

This picture is further supported by noting that the Conway group Co_0 is a maximal finite subgroup of SO_{24}, and that Co_0 can be generated by unitary 3×3 octonionic matrices [8] of F_4 type [7]. In general, the stabilizer subgroup of 3×3 unitary matrices over the octonions $\mathbb{O}$ lies in $SO_9 \subset F_4$ through Peirce decomposition; since there are three independent primitive idempotents in the exceptional Jordan algebra $J_3^{\mathbb{O}}$, there are three such embedded copies of SO_9, providing three charts for the reduction $26 + 1 \longrightarrow 10 + 1$.

6.2. $10 + 1 \longrightarrow 3 + 1$ through S^7 Fiber

As it is well known, a remarkable class of M-theory compactifications is provided by G_2 compactifications to $D = 3 + 1$, where the internal manifold with G_2 holonomy is characterized by its invariant 3-form (which comes from an octonionic structure) [43]. In the $26 + 1$ framework under consideration, a compactification down to $3 + 1$ dimensions can involve a 23-sphere S^{23}, which in turn can be fibrated with an $\mathbb{OP}^2$ base and S^7 fibers. Since S^7 is the quintessential G_2 manifold [44], this provides a natural $26 + 1 \longrightarrow 10 + 1 \longrightarrow 3 + 1$ pattern of reduction along a G_2 manifold from Monstrous M-theory.

7. Further Evidence for M^2-Theory: Monster SCFT and Massless p-Forms in $25 + 1$

In order to conclude the present investigation of higher-dimensional gravity theories which can exhibit the Monster group as symmetry of their massless spectrum, we reconsider Witten's Monster $\mathcal{N} = 1$ SCFT dual to three-dimensional gravity [12]. We will show that the coefficients of its partition function enjoy rather simple interpretations as sums of degrees of freedom of massless fields in $D = 25 + 1$ space–time dimensions, namely as sums of dimensions of suitable representations of the corresponding massless little group SO_{24}. This fact provides further evidence of how a purely bosonic theory of gravity and massless p-forms in $25 + 1$ space–time dimensions can be probed by the Monster group $\mathbb{M}$ in terms of its lowest dimensional representations.

We start and recall the partition function of Witten's $\mathcal{N} = 1$ Monster SCFT (cfr. (3.35) of [12]):

$$\begin{aligned} K(q) &= q^{-1/2} + 276q^{1/2} + 2048q^1 + 11{,}202q^{3/2} + 49{,}152q^2 \\ &\quad +184{,}024q^{5/2} + 614{,}400q^3 + 1{,}881{,}471q^{7/2} + \mathcal{O}(q^4), \qquad (116) \\ &= q^{1/2} Z^{2B}(q), \qquad (117) \end{aligned}$$

where $Z^{2B}(q)$ is the 2B McKay–Thompson series (cfr. (C.1) of [13]). The coefficients of $K(q)$, which in [12] have been related to the (smallest) representations of the Monster group

$\mathbb{M}$ [30], also admit a rather simple (in generally not unique, especially for large coefficients) interpretation in terms of representations of SO_{24}, thus strengthening the evidence for the existence of a gravitational field theory probed by the lowest-dimensional, non-trivial representation(s) of $\mathbb{M}$ itself. Indeed, a tedious but straightforward computation yields the following result:

$$\begin{aligned}
276 &= \underset{\mathbf{276}}{\left|\wedge^2\right|}; \\
2048 &= \underset{\mathbf{2048}}{|\boldsymbol{\lambda}|}; \\
11{,}202 &= \underset{\mathbf{24}}{\left|\wedge^1\right|} + 2\underset{\mathbf{276}}{\left|\wedge^2\right|} + \underset{\mathbf{10{,}626}}{\left|\wedge^4\right|}; \\
49{,}152 &= \underset{\mathbf{2048}}{|\boldsymbol{\lambda}|} + \underset{\mathbf{47{,}104}}{|\boldsymbol{\psi}|}; \\
184{,}024 &= \underset{\mathbf{2048}}{|\boldsymbol{\lambda}|} + \underset{\mathbf{47{,}104}}{|\boldsymbol{\psi}|} + \underset{\mathbf{276}}{\left|\wedge^2\right|} + \underset{\mathbf{134{,}596}}{\left|\wedge^6\right|}; \\
614{,}400 &= 2\underset{\mathbf{2048}}{|\boldsymbol{\lambda}|} + 2\underset{\mathbf{47{,}104}}{|\boldsymbol{\psi}|} + \underset{\mathbf{516{,}096}}{\left|\boldsymbol{\psi}^{(2)}\right|}; \\
1{,}881{,}471 &= 23\underset{\mathbf{1}}{|\phi|} + 2\underset{\mathbf{24}}{\left|\wedge^1\right|} + 4\underset{\mathbf{276}}{\left|\wedge^2\right|} + \underset{\mathbf{2024}}{\left|\wedge^3\right|} + \underset{\mathbf{10{,}626}}{\left|\wedge^4\right|} + 3\underset{\mathbf{42{,}504}}{\left|\wedge^5\right|} + 2\underset{\mathbf{134{,}596}}{\left|\wedge^6\right|} + 2\underset{\mathbf{735{,}471}}{\left|\wedge^8\right|},
\end{aligned} \tag{118}$$

where $\boldsymbol{\psi}^{(p)}$ denotes the p-form spinor representation of SO_{24}, and we have used the notation $\boldsymbol{\psi}^{(0)} \equiv \boldsymbol{\lambda}$, $\boldsymbol{\psi}^{(1)} \equiv \boldsymbol{\psi}$ (cfr. Section 3).

Remarkably, *the degrees of freedom of p-form spinors $\boldsymbol{\psi}^{(p)}$ can always be expressed only in terms of the degrees of freedom of p-form fields*: for the first cases, i.e., for $p = 0$, 1 and 2, by recalling $\boldsymbol{\lambda}$-triality (34) (which in turn implies $\boldsymbol{\psi}$-triality (36)), it holds that

$$\underset{(\boldsymbol{\lambda}\text{-triality (34)})}{p=0} \quad : \quad \underset{\mathbf{2048}}{\left|\boldsymbol{\psi}^{(0)}\right|} = \underset{\mathbf{24}}{\left|\wedge^1\right|} + \underset{\mathbf{2024}}{\left|\wedge^3\right|}; \tag{119}$$

$$\underset{(\boldsymbol{\psi}\text{-triality (36)})}{p=1} \quad : \quad \underset{\mathbf{47{,}104}}{\left|\boldsymbol{\psi}^{(1)}\right|} = 2\underset{\mathbf{276}}{\left|\wedge^2\right|} + 2\underset{\mathbf{2024}}{\left|\wedge^3\right|} + 4\underset{\mathbf{10{,}626}}{\left|\wedge^4\right|} = 2\underset{\mathbf{276}}{\left|\wedge^2\right|} + 2\underset{\mathbf{2024}}{\left|\wedge^3\right|} + \underset{\mathbf{42{,}504}}{\left|\wedge^5\right|}; \tag{120}$$

$$\underset{(\boldsymbol{\psi}^{(2)}\text{-triality})}{p=2} \quad : \quad \underset{\mathbf{516{,}096}}{\left|\boldsymbol{\psi}^{(2)}\right|} = 14\underset{\mathbf{1}}{|\phi|} + 8\underset{\mathbf{24}}{\left|\wedge^1\right|} + \underset{\mathbf{276}}{\left|\wedge^2\right|} + 5\underset{\mathbf{2024}}{\left|\wedge^3\right|} + 3\underset{\mathbf{10{,}626}}{\left|\wedge^4\right|} + 3\underset{\mathbf{42{,}504}}{\left|\wedge^5\right|} + \underset{\mathbf{346{,}104}}{\left|\wedge^7\right|}. \tag{121}$$

Thus, by using (119) and (121), the sums on the right-hand sides of (118) can be expressed only in terms of p-form bosonic fields, as follows:

$$\begin{aligned}
276 &= \underset{\mathbf{276}}{\left|\wedge^2\right|}; \\
2048 &= \underset{\mathbf{24}}{\left|\wedge^1\right|} + \underset{\mathbf{2024}}{\left|\wedge^3\right|}; \\
11{,}202 &= \underset{\mathbf{24}}{\left|\wedge^1\right|} + 2\underset{\mathbf{276}}{\left|\wedge^2\right|} + \underset{\mathbf{10{,}626}}{\left|\wedge^4\right|}; \\
49{,}152 &= \underset{\mathbf{24}}{\left|\wedge^1\right|} + 2\underset{\mathbf{276}}{\left|\wedge^2\right|} + 3\underset{\mathbf{2024}}{\left|\wedge^3\right|} + \underset{\mathbf{42{,}504}}{\left|\wedge^5\right|}; \\
184{,}024 &= \underset{\mathbf{24}}{\left|\wedge^1\right|} + 3\underset{\mathbf{276}}{\left|\wedge^2\right|} + 3\underset{\mathbf{2024}}{\left|\wedge^3\right|} + \underset{\mathbf{42{,}504}}{\left|\wedge^5\right|} + \underset{\mathbf{134{,}596}}{\left|\wedge^6\right|}; \\
614{,}400 &= 16\underset{\mathbf{1}}{|\phi|} + 8\underset{\mathbf{24}}{\left|\wedge^1\right|} + 7\underset{\mathbf{276}}{\left|\wedge^2\right|} + 2\underset{\mathbf{2024}}{\left|\wedge^3\right|} + 3\underset{\mathbf{42{,}504}}{\left|\wedge^5\right|} + \underset{\mathbf{134{,}596}}{\left|\wedge^6\right|} + \underset{\mathbf{346{,}104}}{\left|\wedge^7\right|}; \\
1{,}881{,}471 &= 23\underset{\mathbf{1}}{|\phi|} + 2\underset{\mathbf{24}}{\left|\wedge^1\right|} + 4\underset{\mathbf{276}}{\left|\wedge^2\right|} + \underset{\mathbf{2024}}{\left|\wedge^3\right|} + \underset{\mathbf{10{,}626}}{\left|\wedge^4\right|} + 3\underset{\mathbf{42{,}504}}{\left|\wedge^5\right|} + 2\underset{\mathbf{134{,}596}}{\left|\wedge^6\right|} + 2\underset{\mathbf{735{,}471}}{\left|\wedge^8\right|}.
\end{aligned} \tag{122}$$

Thus, the first coefficients of the partition function (116) and (117) of the $\mathcal{N} = 1$ Monster SCFT [12] can be decomposed as sums of the dimensions of purely bosonic, p-form representations of SO_{24}; since this latter is the massless little group in $D = 25 + 1$ space–time dimensions, the above results imply that, *at least* for the first coefficients, *the coefficients of the partition function of $\mathcal{N} = 1$ Monster SCFT can be expressed in terms of degrees of freedom of massless, purely bosonic, p-form fields in* $25 + 1$ *space–time dimensions*.

The purely bosonic nature of such degrees of freedom is ultimately due to the $\boldsymbol{\lambda}$-triality (34) (or, equivalently, (119)), which is the generalization of the triality $\mathbb{T}$, discussed at the start of Section 3 from 8 to 24 dimensions. To the best of our knowledge, no other examples of such a generalized, "weak" triality are known in other dimensions, so 24 stands out as a very peculiar number in this respect.

Note how all the purely bosonic decompositions (122) share a common feature: for each $p \geqslant 0$, the decompositions (122) exhibit the lowest possible multiplicity of p-form fields, constrained to correspond to a number of degrees of freedom which is strictly smaller than the dimensions of the subsequent $(p+1)$-form field: namely, the condition

$$\# \wedge^{p} \cdot |\wedge^{p}| \leqslant \left|\wedge^{p+1}\right| \tag{123}$$

holds in (122) for all p's appearing.

8. Final Remarks

Monstrous M-theory, Monstrous dilatonic gravities and Monstrous Moonshine

We have shown that in $26+1$ space–time dimensions, there exists a Monstrous M-theory, or simply M^2-theory, whose massless spectrum (73) contains 196,884 degrees of freedom that may be acted upon by the Monster group $\mathbb{M}$ after reduction to $D = 25+1$, because it corresponds to the sum of the two smallest representations of $\mathbb{M}$, namely the trivial (singlet) **1** and the non-trivial one **196,883**. A subsector of M^2-theory yields Horowitz and Susskind's bosonic M-theory [19]. Crucially, the disentangling of the 196,884 degrees of freedom into $\mathbf{196{,}883} \oplus \mathbf{1}$ occurs only when reducing M^2-theory down to $25+1$, obtaining the massless spectrum (71), in which the dilaton ϕ is identified with the singlet of $\mathbb{M}$: in other words, *the (initial observation giving rise to) Monstrous Moonshine* [4] *is crucially related to the KK compactification of* M^2*-theory down to a certain Monstrous dilatonic gravity (namely, the theory* $0.\alpha.iii$ *within the classification carried out in Section 4.1) in* $25+1$ *space–time dimensions*.

Remarkably, such a Monstrous dilatonic theory in $25+1$ contains a subsector given by the massless excitations of the closed and open bosonic string in $25+1$, namely a graviton, an antisymmetric rank-2 field, a dilaton, and a 1-form potential. Actually, by generalizing the *triality* $\mathbb{T}$ of SO_8 (massless little group of string theory in $9+1$) to SO_{24} (massless little group of bosonic string theory in $25+1$), such a dilatonic (Einstein) gravity theory can be shown to be part of a web of some 60 gravito-dilatonic theories, collectively named *Monstrous gravity theories*, whose coarse-grained classification is given in Section 4.1.

The relation between SO_8 and SO_{24} (which at present is the unique dimension enjoying a kind of generalization of $\mathbb{T}$) can be interpreted in terms of the Conway group (The Conway group Co_0 is the full automorphism of the Leech lattice Λ_{24}; however, it is not a simple group, nor is it contained in the Monster. In fact, its quotient by its center $\mathbb{Z}_2$, namely the Conway simple group $Co_1 \sim Co_0/\mathbb{Z}_2$ is contained in $\mathbb{M}$. This means the Monster's maximal finite subgroup Co_1 has the $\mathbb{Z}_2$ action built in, which acts on only half the minimal Leech vectors $196,560/2 = 98,280$.) Co_0, which is a maximal finite subgroup of SO_{24} itself; as shown by Wilson [8], Co_0 is generated by unitary 3×3 octonion matrices, namely by F_4 matrices [7]. Interestingly, SO_9 can be maximally embedded into F_4 in three possible ways, each one providing the manifestly $\mathbb{T}$-invariant breaking

$$\mathfrak{f}_4 \rightarrow \mathfrak{so}_9 = \mathfrak{so}_8 \oplus \mathbf{8}_v \oplus \mathbf{8}_s \oplus \mathbf{8}_c; \tag{124}$$

in this sense, no triality is needed for $\mathfrak{so}_{24}$, but rather just the threefold nature of the (symmetric) embedding $SO_9 \subset F_4$. In turn, the "anomalous" embedding [45]

$$\mathfrak{f}_4 \oplus \mathbf{273} \hookrightarrow \mathfrak{so}_{26} \tag{125}$$

allows one to reduce from $26+1$ to lower dimensions in a non-trivial way, namely along the chain $26+1 \rightarrow 25+1 \rightarrow 10+1 \rightarrow 3+1$.This, as remarked in [22], confirms and strengthens Ramond and Sati's argument that $D = 10 + 1$ M-theory has hidden Cayley plane $\mathbb{OP}^2$ fibers [46].

The Moonshine decomposition (43),

$$196{,}884 = \mathbf{196{,}883} \oplus \mathbf{1} \tag{126}$$

always holds in Monstrous gravities, due to the very existence of the dilatonic scalar field ϕ in their spectrum. In particular, the dilaton ϕ is a singlet of $\mathbb{M}$. Monstrous gravities in $25 + 1$ space–time dimensions, and the presence of a unique ϕ, are intimately related to the representation **196,883** of $\mathbb{M}$, and thus *they may provide an explanation of the (initial observation giving rise to) Monstrous Moonshine in terms of (higher-dimensional, gravitational) field theory*.

Black hole entropy in 2 + 1

Along the lines of Witten's investigation of three-dimensional gravity [12], the present paper suggests that *the quantum entropy* $\ln(196{,}883) \simeq 12.19$ *has a manifest higher-dimensional interpretation* since the BTZ black hole degrees of freedom can be expressed in terms of massless degrees of freedom of fields in $25 + 1$ space–time dimensions.

Local SUSY in 26 + 1?

Remarkably, a certain subsector of the spectrum of M^2-theory, given by (81), when coupled to one massless Rarita–Schwinger field ψ in $26 + 1$, gives rise to a theory which has the same number of bosonic and fermionic massless degrees of freedom, namely

$$B = F = 98{,}304, \tag{127}$$

for a total of 196,608 degrees of freedom. We have been therefore tempted to ask ourselves to ask whether this subsector of M^2-theory, when coupled to a RS field ψ, may actually enjoy (local) supersymmetry in $26 + 1$ space–time dimensions, thus giving rise to a would-be $\mathcal{N} = 1$, $D = 26 + 1$ supergravity theory. In this line of reasoning, we have conjectured a "M-theory-inspired" Lagrangian density, as well as the corresponding local supersymmetry transformations in $26 + 1$. The invariance of such a Lagrangian under those supersymmetry transformations is still conjectural, and to prove (or disprove) it seems quite a formidable, though absolutely worthy, task, and we leave it for further future work.

At any rate, the reduction of the bosonic sector (81) of such a would-be $\mathcal{N} = 1$ supergravity from $26 + 1$ to $25 + 1$ yields a suitable subsector of the Monstrous gravity labeled by $2.\gamma.ii$ in the classification of Section 4.1, simply obtained by letting $\#\wedge^2 : 4 \longrightarrow 3$. In light of this, we cannot help but point out a certain mismatch, essentially amounting to the **276** degrees of freedom of a massless 2-form in $25 + 1$, between the total (bosonic + fermionic) degrees of freedom of the would-be $\mathcal{N} = 1$ supergravity in $26 + 1$ (98,304 + 98,304 = 196,608) and the (purely bosonic) 196,884 degrees of freedom of M^2-theory: 196,884 − 196,608 = 276. In this sense, "monstrousity" and (would-be) "supersymmetry" in $26 + 1$ (as well as, predictably, in $25 + 1$) space–time dimensions exhibit a slight disalignment, though being tightly related.

Leech lattice and Griess algebra

All this suggests that the Monster group $\mathbb{M}$ has its origin in a gravity theory in $26 + 1$ dimensions, as its definition as the automorphism of the Griess algebra [1,3,23] is clarified by showing that such an algebra is not merely a sum of unrelated spaces, but related to the massless spectrum of Monstrous gravities in $25 + 1$, which in *at least* one case (namely, the $0.\alpha.iii$ theory, whose massless spectrum is given by (70) and (71)) oxidates up to M^2-theory in $26 + 1$. The spectrum of M^2-theory dimensionally reduced to $25 + 1$ contains a subsector given by the massless excitations of the closed and open bosonic string in $25 + 1$, namely a graviton, an antisymmetric rank-2 field, a dilaton, and a 1-form potential. Therefore, *the relation between the "Leech algebra" $\mathfrak{L}_{24}$ and the Griess algebra is realized in field theory by the relation between M^2-theory and its subsector (81) coupled to one RS field (the would-be gravitino) in* $26 + 1$, discussed in Section 5.2.

On the other hand, the discussion of the analogies between the $\mathfrak{e}_8$ root lattice and the Leech lattice Λ_{24} seems to suggest that M-theory in $10 + 1$ and the would-be $\mathcal{N} = 1$

supergravity in $26+1$ are tightly related to the lattices $\mathfrak{e}_8$ respectively Λ_{24}, which determine the optimal lattice packings in $D=8$ respectively 24.

Developments

Many directions for further future developments stem from the present work, which is a preliminary investigation of higher-dimensional structures in space–time, which reflect themselves in large-dimensional, yet finite, group theoretical structures. Below, we list a few possible developments.

- It would be interesting to explore the implications of the characterization of the $\mathbb{M}$ as acting on the whole massless spectrum of M^2-theory in $26+1$ space–time dimensions.
- One could further study the maps discussed in Section 3; as pointed out above, no other Dynkin diagram (besides $\mathfrak{d}_4$) has an automorphism group of order greater than 2, and thus, such maps cannot be realized as an automorphism of $\mathfrak{d}_{12}$, nor they can be traced back to some structural symmetry of the Dynkin diagram of $\mathfrak{d}_{12}$ itself.
- Additionally, one could study the Lagrangian structure of M^2-theory, as well as of its Scherk–Schwarz reduction to $25+1$.
- Further evidence may be gained by investigating whether the dimensions of representations of finite groups, such as the Baby Monster group $\mathbb{BM}$, the Conway group Co_0 and the simple Conway group $Co_1 \simeq Co_0/\mathbb{Z}_2$, can all be rather simply interpreted as sums of dimensions of representation of SO_{24} or SO_{25} itself, and study the decomposition of the (smallest) coefficients of the partition functions of the SCFT derived from the Monster SCFT.
- Further study may concern the double copy structure of Monster dilatonic gravities in $25+1$, as well as of M^2-theory, and its possibly supersymmetric subsector, in $26+1$.
- The investigation on the existence of local SUSY in $26+1$, and the determination of the corresponding Lagrangian and SUSY transformations is of the utmost relevance, of course.
- Last but not least, it would be interesting to study the massive spectrum of (massive variants of) Monstrous gravities and of M^2-theory.

We would like to conclude with a sentence by John H. Conway, to whom this paper is dedicated, on the Monster group [47]: "*There's never been any kind of explanation of why it's there, and it's obviously not there just by coincidence. It's got too many intriguing properties for it all to be just an accident*."

Author Contributions: Conceptualization, A.M., M.R. and D.C.; methodology, A.M., M.R. and D.C.; validation, A.M., M.R. and D.C.; formal analysis, A.M., M.R. and D.C.; investigation, A.M., M.R. and D.C.; writing—original draft preparation, A.M., M.R. and D.C.; writing—review and editing, A.M., M.R. and D.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: We would like to acknowledge an informal yet inspiring discussion with Eric Weinstein during the "Advances in Quantum Gravity" Conference, hosted by Laura Deming at Topos House, San Francisco, CA, on July 2016. Since that occasion, we started thinking about the idea of a "weak" triality for $\mathfrak{d}_{12}$. We also thank Richard Borcherds for insights on the domain of the fake Monster automorphic form in signature $D=26+2$.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Chern–Simons Lagrangian Terms for Monstrous M-Theory

The full Lagrangian from Equation (78) is given by

$$
\begin{aligned}
\sqrt{|g|}\mathcal{L}_{CS} = {} & \epsilon\mathcal{E}_2^{ABCiDE} G_A^{(6)} G_B^{(6)} G_C^{(6)} G_i^{(4)} G_D^{(3)} C_E^{(2)} + \epsilon\mathcal{F}_2^{ABijCD} G_A^{(6)} G_B^{(6)} G_i^{(5)} G_j^{(5)} G_C^{(3)} C_D^{(2)} \\
& +\epsilon\mathcal{G}_2^{ABijkC} G_A^{(6)} G_B^{(6)} G_i^{(5)} G_j^{(4)} G_k^{(4)} C_C^{(2)} + \epsilon\mathcal{H}_2^{AijklB} G_A^{(6)} G_i^{(5)} G_j^{(5)} G_k^{(5)} G_l^{(4)} C_B^{(2)} \\
& +\epsilon\mathcal{I}_2^{ijklmA} G_i^{(5)} G_j^{(5)} G_k^{(5)} G_l^{(5)} G_m^{(5)} C_A^{(2)} + \epsilon\mathcal{J}_2^{ABiCDEF} G_A^{(6)} G_B^{(6)} G_i^{(4)} G_C^{(3)} G_D^{(3)} G_E^{(3)} C_F^{(2)} \\
& +\epsilon\mathcal{K}_2^{AijBCDE} G_A^{(6)} G_i^{(5)} G_j^{(5)} G_B^{(3)} G_C^{(3)} G_D^{(3)} C_E^{(2)} + \epsilon\mathcal{L}_2^{AijkBCD} G_A^{(6)} G_i^{(5)} G_j^{(4)} G_k^{(4)} G_B^{(3)} G_C^{(3)} C_D^{(2)} \\
& +\epsilon\mathcal{M}_2^{ijklABC} G_i^{(5)} G_j^{(5)} G_k^{(5)} G_l^{(4)} G_A^{(3)} G_B^{(3)} C_C^{(2)} + \epsilon\mathcal{N}_2^{AijklBC} G_A^{(6)} G_i^{(4)} G_j^{(4)} G_k^{(4)} G_l^{(4)} G_B^{(3)} C_C^{(2)} \\
& +\epsilon\mathcal{P}_2^{ijklmAB} G_i^{(5)} G_j^{(5)} G_k^{(4)} G_l^{(4)} G_m^{(4)} G_A^{(3)} C_B^{(2)} + \epsilon\mathcal{W}_2^{ijklmnA} G_i^{(5)} G_j^{(4)} G_k^{(4)} G_l^{(4)} G_m^{(4)} G_n^{(4)} C_A^{(2)} \\
& +\epsilon\mathcal{Q}_2^{AiBCDEFG} G_A^{(6)} G_i^{(4)} G_B^{(3)} G_C^{(3)} G_D^{(3)} G_E^{(3)} G_F^{(3)} C_G^{(2)} + \epsilon\mathcal{R}_2^{ijABCDEF} G_i^{(5)} G_j^{(5)} G_A^{(3)} G_B^{(3)} G_C^{(3)} G_D^{(3)} G_E^{(3)} C_F^{(2)} \\
& +\epsilon\mathcal{S}_2^{ijkABCDE} G_i^{(5)} G_j^{(4)} G_k^{(4)} G_A^{(3)} G_B^{(3)} G_C^{(3)} G_D^{(3)} C_E^{(2)} + \epsilon\mathcal{T}_2^{ijklABCD} G_i^{(4)} G_j^{(4)} G_k^{(4)} G_l^{(4)} G_A^{(3)} G_B^{(3)} G_C^{(3)} C_D^{(2)} \\
& +\epsilon\mathcal{U}_2^{iABCDEFGH} G_i^{(4)} G_A^{(3)} G_B^{(3)} G_C^{(3)} G_D^{(3)} G_E^{(3)} G_F^{(3)} G_G^{(3)} C_H^{(2)} + \epsilon\mathcal{E}_3^{ABCDi} G_A^{(6)} G_B^{(6)} G_C^{(6)} G_D^{(6)} C_i^{(3)} \\
& +\epsilon\mathcal{F}_3^{ABCDEi} G_A^{(6)} G_B^{(6)} G_C^{(6)} G_D^{(3)} G_E^{(3)} C_i^{(3)} + \epsilon\mathcal{G}_3^{ABijCk} G_A^{(6)} G_B^{(6)} G_i^{(5)} G_j^{(4)} G_C^{(3)} C_k^{(3)} \\
& +\epsilon\mathcal{H}_3^{AijkBl} G_A^{(6)} G_i^{(5)} G_j^{(5)} G_k^{(5)} G_B^{(3)} C_l^{(3)} + \epsilon\mathcal{I}_3^{ABijkl} G_A^{(6)} G_B^{(6)} G_i^{(4)} G_j^{(4)} G_k^{(4)} C_l^{(3)} \\
& +\epsilon\mathcal{J}_3^{Aijklm} G_A^{(6)} G_i^{(5)} G_j^{(5)} G_k^{(4)} G_l^{(4)} C_m^{(3)} + \epsilon\mathcal{K}_3^{ijklmn} G_i^{(5)} G_j^{(5)} G_k^{(5)} G_l^{(5)} G_m^{(4)} C_n^{(3)} \\
& +\epsilon\mathcal{L}_3^{ABCDEFi} G_A^{(6)} G_B^{(6)} G_C^{(3)} G_D^{(3)} G_E^{(3)} G_F^{(3)} C_i^{(3)} + \epsilon\mathcal{M}_3^{AijBCDk} G_A^{(6)} G_i^{(5)} G_j^{(4)} G_B^{(3)} G_C^{(3)} G_D^{(3)} C_k^{(3)} \\
& +\epsilon\mathcal{N}_3^{ijkABCl} G_i^{(5)} G_j^{(5)} G_k^{(5)} G_A^{(3)} G_B^{(3)} G_C^{(3)} C_l^{(3)} + \epsilon\mathcal{O}_3^{AijkBCl} G_A^{(6)} G_i^{(4)} G_j^{(4)} G_k^{(4)} G_B^{(3)} G_C^{(3)} C_l^{(3)} \\
& +\epsilon\mathcal{P}_3^{ijklABm} G_i^{(5)} G_j^{(5)} G_k^{(4)} G_l^{(4)} G_A^{(3)} G_B^{(3)} C_m^{(3)} + \epsilon\mathcal{R}_3^{ijklmAn} G_i^{(5)} G_j^{(4)} G_k^{(4)} G_l^{(4)} G_m^{(4)} G_A^{(3)} C_n^{(3)} \\
& +\epsilon\mathcal{S}_3^{ijklmno} G_i^{(4)} G_j^{(4)} G_k^{(4)} G_l^{(4)} G_m^{(4)} G_n^{(4)} C_o^{(3)} + \epsilon\mathcal{T}_3^{ABCDEFGi} G_A^{(6)} G_B^{(3)} G_C^{(3)} G_D^{(3)} G_E^{(3)} G_F^{(3)} G_G^{(3)} C_i^{(3)} \\
& +\epsilon\mathcal{U}_3^{ijABCDEk} G_i^{(5)} G_j^{(4)} G_A^{(3)} G_B^{(3)} G_C^{(3)} G_D^{(3)} G_E^{(3)} C_k^{(3)} + \epsilon\mathcal{V}_3^{ijkABCDl} G_i^{(4)} G_j^{(4)} G_k^{(4)} G_A^{(3)} G_B^{(3)} G_C^{(3)} G_D^{(3)} C_l^{(3)} \\
& +\epsilon\mathcal{W}_3^{ABCDEFGHi} G_A^{(3)} G_B^{(3)} G_C^{(3)} G_D^{(3)} G_E^{(3)} G_F^{(3)} G_G^{(3)} G_H^{(3)} C_i^{(3)} + \epsilon\mathcal{E}_4^{ABCij} G_A^{(6)} G_B^{(6)} G_C^{(6)} G_i^{(5)} C_j^{(4)} \\
& +\epsilon\mathcal{F}_4^{ABiCDj} G_A^{(6)} G_B^{(6)} G_i^{(5)} G_C^{(3)} G_D^{(3)} C_j^{(4)} + \epsilon\mathcal{G}_4^{ABijCk} G_A^{(6)} G_B^{(6)} G_i^{(4)} G_j^{(4)} G_C^{(3)} C_k^{(4)} \\
& +\epsilon\mathcal{H}_4^{AijkBl} G_A^{(6)} G_i^{(5)} G_j^{(5)} G_k^{(4)} G_B^{(3)} C_l^{(4)} + \epsilon\mathcal{I}_4^{ijklAm} G_i^{(5)} G_j^{(5)} G_k^{(5)} G_l^{(5)} G_A^{(3)} C_m^{(4)} \\
& +\epsilon\mathcal{J}_4^{Aijklm} G_A^{(6)} G_i^{(5)} G_j^{(4)} G_k^{(4)} G_l^{(4)} C_m^{(4)} + \epsilon\mathcal{K}_4^{ijklmn} G_i^{(5)} G_j^{(5)} G_k^{(5)} G_l^{(4)} G_m^{(4)} C_n^{(4)} \\
& +\epsilon\mathcal{L}_4^{AiBCDEj} G_A^{(6)} G_i^{(5)} G_B^{(3)} G_C^{(3)} G_D^{(3)} G_E^{(3)} C_j^{(4)} + \epsilon\mathcal{M}_4^{AijBCDk} G_A^{(6)} G_i^{(4)} G_j^{(4)} G_B^{(3)} G_C^{(3)} G_D^{(3)} C_k^{(4)} \\
& +\epsilon\mathcal{N}_4^{ijkABCl} G_i^{(5)} G_j^{(5)} G_k^{(4)} G_A^{(3)} G_B^{(3)} G_C^{(3)} C_l^{(4)} + \epsilon\mathcal{O}_4^{ijklABm} G_i^{(5)} G_j^{(4)} G_k^{(4)} G_l^{(4)} G_A^{(3)} G_B^{(3)} C_m^{(4)} \\
& +\epsilon\mathcal{P}_4^{ijklmAn} G_i^{(4)} G_j^{(4)} G_k^{(4)} G_l^{(4)} G_m^{(4)} G_A^{(3)} C_n^{(4)} + \epsilon\mathcal{Q}_4^{iABCDEFj} G_i^{(5)} G_A^{(3)} G_B^{(3)} G_C^{(3)} G_D^{(3)} G_E^{(3)} G_F^{(3)} C_j^{(4)} \\
& +\epsilon\mathcal{R}_4^{ijABCDEk} G_i^{(4)} G_j^{(4)} G_A^{(3)} G_B^{(3)} G_C^{(3)} G_D^{(3)} G_E^{(3)} C_k^{(4)} + \epsilon\mathcal{E}_5^{ABCiD} G_A^{(6)} G_B^{(6)} G_C^{(6)} G_i^{(4)} C_D^{(5)} \\
& +\epsilon\mathcal{F}_5^{ABijC} G_A^{(6)} G_B^{(6)} G_i^{(5)} G_j^{(5)} C_C^{(5)} + \epsilon\mathcal{G}_5^{ABiCDE} G_A^{(6)} G_B^{(6)} G_i^{(4)} G_C^{(3)} G_D^{(3)} C_E^{(5)} \\
& +\epsilon\mathcal{H}_5^{AijBCD} G_A^{(6)} G_i^{(5)} G_j^{(5)} G_B^{(3)} G_C^{(3)} C_D^{(5)} + \epsilon\mathcal{I}_5^{AijkBC} G_A^{(6)} G_i^{(5)} G_j^{(4)} G_k^{(4)} G_B^{(3)} C_C^{(5)} \\
& +\epsilon\mathcal{J}_5^{ijklAB} G_i^{(5)} G_j^{(5)} G_k^{(5)} G_l^{(4)} G_A^{(3)} C_B^{(5)} + \epsilon\mathcal{K}_5^{AijklB} G_A^{(6)} G_i^{(4)} G_j^{(4)} G_k^{(4)} G_l^{(4)} C_B^{(5)} \\
& +\epsilon\mathcal{L}_5^{ijklmA} G_i^{(5)} G_j^{(5)} G_k^{(4)} G_l^{(4)} G_m^{(4)} C_A^{(5)} + \epsilon\mathcal{M}_5^{AiBCDEF} G_A^{(6)} G_i^{(4)} G_B^{(3)} G_C^{(3)} G_D^{(3)} G_E^{(3)} C_F^{(5)} \\
& +\epsilon\mathcal{N}_5^{ijABCDE} G_i^{(5)} G_j^{(5)} G_A^{(3)} G_B^{(3)} G_C^{(3)} G_D^{(3)} C_E^{(5)} + \epsilon\mathcal{O}_5^{ijkABCD} G_i^{(5)} G_j^{(4)} G_k^{(4)} G_A^{(3)} G_B^{(3)} G_C^{(3)} C_D^{(5)} \\
& +\epsilon\mathcal{P}_5^{ijklABC} G_i^{(4)} G_j^{(4)} G_k^{(4)} G_l^{(4)} G_A^{(3)} G_B^{(3)} C_C^{(5)} + \epsilon\mathcal{Q}_5^{iABCDEFG} G_i^{(4)} G_A^{(3)} G_B^{(3)} G_C^{(3)} G_D^{(3)} G_E^{(3)} G_F^{(3)} C_G^{(5)} .
\end{aligned}
\tag{A1}
$$

References

1. Griess, R.L. The structure of the monster simple group. In *Proceedings of the Conference on Finite Groups*; Academic Press: Boston, MA, USA, 1976; pp. 113–118.
2. Conway, J.H. A simple construction for the Fischer-Griess monster group. *Invent. Math.* **1985**, *79*, 513–540.
3. Frenkel, I.B.; Lepowsky, J.; Meurman, A. *Vertex Operator Algebras and the Monster*; Academic Press: Boston, MA, USA, 1988.
4. Conway, J.H.; Norton, S. Monstrous moonshine. *Bull. Lond. Math. Soc.* **1979**, *11*, 308–339. [CrossRef]
5. Borcherds, R.E. Monstrous moonshine and monstrous lie superalgebras. *Invent. Math.* **1992**, *109*, 405–444. [CrossRef]
6. Conway, J.H.; Sloane, N.J.A. *Sphere Packings, Lattices and Groups*; Grundlehren der mathematischen Wissenschaften v. 290; Springer: Berlin, Germany, 2013.
7. Rios, M. U-Duality and the Leech Lattice. *arXiv* **2013**, arXiv:1307.1554.
8. Wilson, R.A. Octonions and the Leech lattice. *J. Algebra* **2009**, *322*, 2186–2190.
9. Dixon, L.; Ginsparg, P.; Harvey, J. Beauty and the beast: Superconformal symmetry in a Monster module. *Commun. Math. Phys.* **1988**, *119*, 221–241. [CrossRef]
10. Paquette, N.M.; Persson, D.; Volpato, R. Monstrous BPS-Algebras and the Superstring Origin of Moonshine. *arXiv* **2016**, arXiv:1601.05412.
11. Craps, B.; Gaberdiel, M.R.; Harvey, J.A. Monstrous branes. *arXiv* **2002**, arXiv:hep-th/0202074.
12. Witten, E. Three-Dimensional Gravity Revisited. *arXiv* **2007**, arXiv:0706.3359.
13. Lin, Y.; Shao, S. Duality Defect of the Monster CFT. *arXiv* **2019**, arXiv:1911.00042.
14. Eguchi, T.; Ooguri, H.; Tachikawa, Y. Notes on the K3 Surface and the Mathieu group M_{24}. *Exp. Math.* **2011**, *20*, 91–96.
15. Cheng, M.C.N.; Duncan, J.F.R.; Harvey, J.A. Umbral Moonshine. *Commun. Number Theory Phys.* **2014**, *8*, 101–242. [CrossRef]
16. Duncan, J.F.R.; Griffin, M.J.; Ono, K. Proof of the Umbral Moonshine Conjecture. *Res. Math. Sci.* **2015**, *2*, 26. [CrossRef]
17. Bae, J.B.; Harvey, J.A.; Lee, K.; Lee, S.; Rayhaun, B.C. Conformal Field Theories with Sporadic Group Symmetry. *arXiv* **2021**, arXiv:2002.02970.
18. Witten, E. String Theory Dynamics In Various Dimensions. *Nucl. Phys. B* **1995**, *443*, 85–126. [CrossRef]
19. Horowitz, G.T.; Susskind, L. Bosonic M-theory. *J. Math. Phys.* **2001**, *42*, 3152. [CrossRef]
20. Tuite, M.P. On the Relationship between the Uniqueness of the Moonshine Module and Monstrous Moonshine. *Commun. Math. Phys.* **1995**, *166*, 495. [CrossRef]
21. Cummins, C.J.; Gannon, T. Modular equations and the genus zero property of moonshine functions. *Invent. Math.* **1997**, *129*, 413–443. [CrossRef]
22. Rios, M.; Marrani, A.; Chester, D. Exceptional Super Yang-Mills in $D = 27 + 3$ and Worldvolume M-Theory. *Phys. Lett.* **2020**, *808*, 135674. [CrossRef]
23. Borcherds, R.E. What is the monster? *arXiv* **2002**, arXiv:math/0209328.
24. Duncan, J.F. Super-moonshine for Conway's largest sporadic group. *Duke Math. J.* **2007**, *139*, 255–315. [CrossRef]
25. Conway, J.H. A perfect group of order 8,315,553,613,086,720,000 and the sporadic simple groups. *Proc. Natl. Acad. Sci. USA* **1968**, *61*, 398–400. [CrossRef] [PubMed]
26. Rios, M.; Marrani, A.; Chester, D. The Geometry of Exceptional Super Yang-Mills Theories. *Phys. Rev. D* **2019**, *99*, 046004. [CrossRef]
27. Borcherds, R.E. Automorphic forms and Lie algebras. *Curr. Dev. Math.* **1996**, *1996*, 1–36. [CrossRef]
28. Horava, P.; Witten, E. Heterotic and Type I String Dynamics from Eleven Dimensions. *Nucl. Phys. B* **1996**, *460*, 506–524. [CrossRef]
29. Horava, P.; Witten, E. Eleven-Dimensional Supergravity on a Manifold with Boundary. *Nucl. Phys. B* **1996**, *475*: 94–114.
30. Conway, J.H.; Curtis, R.T.; Norton, S.P.; Parker, R.A.; Wilson, R.A. *Atlas of Finite Groups*; Oxford University Press: Oxford, UK, 1985.
31. Porteous, I. *Clifford Algebras and the Classical Groups*; Cambridge Studies in Advanced Mathematics 50; Cambridge University Press: Cambridge, UK, 1995.
32. Adams, J.F. $Spin(8)$, Triality, F_4 and all that. In *Superspace and Supergravity*; Hawking, S., Roček, M., Eds.; Cambridge University Press: Cambridge, UK, 1981.
33. Ortín, T. *Gravity and Strings*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2015.
34. Dolan, L.; Goddard, P.; Montague, P. Conformal field theory, triality and the Monster group. *Phys. Lett. B* **1990**, *236*, 165–172. [CrossRef]
35. Dolan, L.; Goddard, P.; Montague, P. Conformal field theory of twisted vertex operators. *Nucl. Phys. B* **1990**, *338*, 529–601. [CrossRef]
36. Ferrara, S.; Marrani, A.; Trigiante, M. Super-Ehlers in Any Dimension. *J. High Energy Phys.* **2012**, *11*, 068.
37. Conway, J.H. The $\sqrt{Monster\ construction}$. In *Groups, Difference Sets, and the Monster*; Arasu, K.T., Dillon, J.F., Harada, K., Sehgal, S., Solomon, R., Eds.; De Gruyter: Berlin, Germany, 1996; pp. 345–348.
38. Banks, T.; Fischler, W.; Shenker, S.H.; Susskind, L. M Theory as a Matrix Model: A Conjecture. *Phys. Rev. D* **1997**, *55*, 5112–5128. [CrossRef]
39. Vinberg, E.B. The theory of Convex Homogeneous Cones. In *Transactions of the Moscow Mathematical Society for the Year 1963*; American Mathematical Society: Providence, RI, USA, 1965; pp. 340–403.
40. Gaberdiel, M.R.; Olive, D.I.; West, P.C. A class of Lorentzian Kac-Moody algebras. *Nucl. Phys. B* **2002**, *645*, 403–437. [CrossRef]

41. Truini, P.; De Graaf, W.; Marrani, A. Exceptional Periodicity and Magic Star Algebras. III: The Algebra $\mathbf{f}(n)^4$ and the Derivations of HT-Algebras. In Proceedings of the 24th International Colloquium on Group Theoretical Methods in Physics (GROUP 24), Paris, France, 15–20 July 2002.
42. Truini, P.; Rios, M.; Marrani, A. The Magic Star of Exceptional Periodicity. *Contemp. Math.* **2019**, *721*, 277–297.
43. Gukov, S. M-theory on manifolds with exceptional holonomy. *Fortschr. Phys.* **2003**, *51*, 719–731. [CrossRef]
44. Joyce, D. *Compact Manifolds with Special Holonomy*; Oxford University Press: Oxford, UK, 2000.
45. Ramond, P. Exceptional groups and physics. In Proceedings of the 24th International Colloquium on Group Theoretical Methods in Physics (GROUP 24), Paris, France, 15–20 July 2002; Institute of Physics Conference Series No. 173; IOP Publishing: London, UK, 2003.
46. Sati, H. $\mathbb{OP}^2$ bundles in M-theory. *Commun. Num. Theor. Phys.* **2009**, *3*, 495–530. [CrossRef]
47. Haran, B. Life, Death and the Monster (John Conway), Videoclip. 2014. Available online: https://www.numberphile.com/videos/life-death-and-the-monster-john-conway (accessed on 12 November 2022).