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Solving Integral Equations via Hybrid Interpolative $\mathcal{RI}$-Type Contractions in $\mathfrak{b}$-Metric Spaces

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Abstract: Karapinar et al. established a more general class of contractions, namely, hybrid interpolative $\mathcal{R}$iech $\mathcal{I}$străstescue-type contractions, and presented some results on the platform of metric spaces. This research uses the domain of $\mathfrak{b}$-metric spaces to modify this class proficiently. Several interesting fixed-point results are presented by using this new class defined on $\mathfrak{b}$-metric space, where the symmetric condition is preserved in this study. Examples are provided for the authentication of proved results. Eventually, an application is also provided in order to comprehend our extensive effort in a better way.

Keywords: $\mathfrak{b}$-metric space ($\mathfrak{b}$MS); hybrid interpolative $\mathcal{R}$iech $\mathcal{I}$străstescue $\mathcal{RI}$-type contractions; picard operator; π-admissible mapping; Fredholm integral equations

MSC: 47H10; 54H25

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1. Introduction and Preliminaries

Today, fixed-point theory is a rapidly expanding and intriguing topic of mathematics with significant applications in a variety of disciplines. One of the powerful applications is in the context of integral equations, where recent numerical approaches have achieved significant scientific advancements. The objectives of this study are to propose some useful methods for predicting the solution of a family of integral equations by using certain modified results from fixed-point theory.

In 1922, Banach [1] established the famous Banach contraction principle (BCP), which used contraction mapping on complete metric spaces. It was later considered an effective technique for finding unique fixed points. It was also useful in nonlinear analysis, which resulted in a slew of developments in all directions. There are several generalizations of the Banach contraction principle in the literature on metric fixed-point theory. Three ingredients are very crucial for establishing fixed-point results. These are the underlying space, the completeness property, and an equality that is strong enough to guarantee the existence of a fixed point. It is suggested that readers look at some of the latest extensions established by [2–7]. In addition, during the past several decades, fixed-point theory has played a key role in solving many problems arising in nonlinear analysis and optimization [8,9].

In this context, Czerwik [10] and Bakhtin [11] proposed an interesting notion of metric spaces, termed $\mathfrak{b}$-metric space in 1993, by keeping the symmetric condition and changing the triangular inequality of the metric spaces. Furthermore, they presented some valuable fixed-point results to demonstrate the validity of this extension. This space was used by several authors for establishing new directions to prove the existence of fixed-points [12–24]. Recently, Karapinar et al. [25] presented a novel family of contraction mappings that combined some linear and non-linear contractions in metric spaces. These results are the generalization of the theorems of [26–28].

This research provided some interesting results in the context of $\mathfrak{b}$MS by using the contraction condition introduced by Karapinar et al. [25]. The fundamental advantage of these new conditions is that they allow us to include contractivity conditions involving a large number of terms, and we can also use either addition or multiplication at the same time. We suggested some significant convergence criteria and obtained fixed-point results in this regard. Finally, one of our results is applied to establish a novel existence condition for the solution of a class of the integral equations. Our results extend many existing results, including those given in [25].

Throughout the article, $\mathcal{M}$ refers to a non-empty set, $\mathbb{N}$ represents the set of natural numbers, $\mathbb{R}$ corresponds to the collection of real numbers, and $Fix_S(\mathcal{M})$ denotes set of all fixed points of a mapping S on $\mathcal{M}$.

Let us have a look at some core concepts that will be helpful for the proof of our main results.

Definition 1. *Let $\mathcal{M} \neq \phi$ and $\mathfrak{b} \geq 1$ be any real number, a map $d_{\mathfrak{b}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ satisfying the following properties on $\mathcal{M}$ is called a $\mathfrak{b}$ metric on $\mathcal{M}$:*
$\acute{d}b(1)$: $d_{\mathfrak{b}}(m,n) = 0 \iff m = n$;
$\acute{d}b(2)$: $d_{\mathfrak{b}}(m,n) = d_{\mathfrak{b}}(n,m)$;
$\acute{d}b(3)$: $d_{\mathfrak{b}}(m,n) \leq \mathfrak{b}\{d_{\mathfrak{b}}(m,k) + d_{\mathfrak{b}}(k,n)\}$,
for all $m,n,k \in \mathcal{M}$.
The pair $(\mathcal{M}, d_{\mathfrak{b}})$ is said to be a $\mathfrak{b}$-metric space ($\mathfrak{b}$MS).

Example 1. *Let $\mathcal{M} = \mathbb{R}^+$, define $d_{\mathfrak{b}} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ as $d_{\mathfrak{b}}(m,n) = \mid m-n \mid^q$ for all $m,n \in \mathcal{M}$, $q > 1$ be any constant; then, $(\mathcal{M}, d_{\mathfrak{b}})$ is a $\mathfrak{b}$MS with $\mathfrak{b} = 2^{q-1}$.*

Definition 2. *For a mapping $S : \mathcal{M} \to \mathcal{M}$ and $m \in \mathcal{M}$, the orbit of m with respect to S is defined as the following sequence of points:*

$$O_S(m) = \{m, Sm, S^2m, \ldots\}.$$

Definition 3. *Let $(\mathcal{M}, d_{\mathfrak{b}})$ be a $\mathfrak{b}$MS. A sequence $\{m_k\}$ defined by $m_k = S^k m_0$ for all $k \in \mathbb{N}$ is called the Picard sequence of S based on m_0, where $S^k = S \circ S \circ ....^{(k)} \circ S : \mathcal{M} \to \mathcal{M}$ is the k-th iterate of S and the mapping S is called a Picard operator if each Picard sequence of such an operator converges to one of its fixed points.*

Definition 4. *A function $\pi : \mathcal{M} \times \mathcal{M} \to [0,\infty)$ in a $\mathfrak{b}$MS is called π-orbital admissible if for $m \in \mathcal{M}$ it satisfies*

$$\pi(m, Sm) \geq 1 \quad \Longrightarrow \quad \pi(Sm, S^2m) \geq 1.$$

Definition 5. *Let $(\mathcal{M}, d_{\mathfrak{b}})$ be a $\mathfrak{b}$MS and $\pi : \mathcal{M} \times \mathcal{M} \to [0,\infty)$ be a function. A mapping $S : \mathcal{M} \to \mathcal{M}$ is called a hybrid interpolative $\mathcal{RI}$-type contraction if there exists a $w \in [0,1)$ such that*

$$\pi(m,n) d_{\mathfrak{b}}(S^2m, S^2n) \leq w\mathbb{M}(m,n),$$

here

$$\mathbb{M}(m,n) = \begin{cases} \Big[c_1 d_{\mathfrak{b}}(m,n)^p + c_2 d_{\mathfrak{b}}(m,Sm)^p + c_3 d_{\mathfrak{b}}(n,Sn)^p & \\ +c_4 d_{\mathfrak{b}}(Sm,Sn)^p + c_5 d_{\mathfrak{b}}(Sm,S^2m)^p + q d_{\mathfrak{b}}(Sn,S^2n)^p\Big]^{\frac{1}{p}} & \sum_{i=1}^{5} c_i + q \leq 1 \\ & \text{if } p > 0 \\ d_{\mathfrak{b}}(m,n)^{c_1} . d_{\mathfrak{b}}(m,Sm)^{c_2} . d_{\mathfrak{b}}(n,Sn)^{c_3} . d_{\mathfrak{b}}(Sm,Sn)^{c_4} & \\ . d_{\mathfrak{b}}(Sm,S^2m)^{c_5} . d_{\mathfrak{b}}(Sn,S^2n)^{q} & \sum_{i=1}^{5} c_i + q = 1 \\ & \text{if } p = 0. \end{cases} \quad (1)$$

with $\{c_i : i = 1, 2, ..5 \geq 0\}$, $p \in \mathbb{R}$ *and* $q > 0$.

2. Main Results

In this section, we develop some fixed-point results for $\mathcal{RI}$-type contractions in the context of $\mathfrak{b}$MS. Furthermore, an example and an application are also presented for a deeper understanding of our results.

Proposition 1. *Given* $w \in [0, 1)$ *and* $\{m_k\} \subset \mathbb{R}^+$ *be any sequence such that*

$$m_{k+2} \leq w \max\{m_k, m_{k+1}\} \text{ for all } k \in \mathbb{N} \cup 0 \tag{2}$$

then,

$$m_{2k} \leq w^k \Lambda, \quad m_{2k+1} \leq w^k \Lambda \ \ \forall \ k \geq 1, \tag{3}$$

where $\Lambda = \max\{m_0, m_1\}$.

Proof. Setting $k = 0$ in (2), we obtain

$$m_2 \leq w \max\{m_0, m_1\} = w\Lambda,$$

for $k = 1$ we obtain

$$\begin{aligned} m_3 &\leq w \max\{m_1, m_2\} \\ &\leq w \max\{m_1, w \max\{m_0, m_1\}\} \\ &\leq w \max\{m_1, w\Lambda\} \\ &\leq w\Lambda. \end{aligned}$$

Suppose that (3) holds for some $k \in \mathbb{N}$; then,

$$\begin{aligned} m_{2k+2} &\leq w \max\{m_{2k}, m_{2k+1}\} \\ &\leq w \max\{w^k\Lambda, w^k\Lambda\} \\ &\leq w^{k+1}\Lambda, \end{aligned}$$

on the same lines we will obtain $m_{2k+3} \leq w^{k+1}\Lambda$.
This completes the proof by induction. □

Lemma 1. *Let* $\{r_k\}$ *be a sequence on a* $\mathfrak{b}$*MS, and there exists* $w \in [0, 1)$ *such that*

$$d_{\mathfrak{b}}(r_{k+2}, r_{k+3}) \leq w \max\{d_{\mathfrak{b}}(r_k, r_{k+1}), d_{\mathfrak{b}}(r_{k+1}, r_{k+2})\}, \ \ \forall k \in \mathbb{N}, \tag{4}$$

then, $\{r_k\}$ *is a Cauchy sequence in* $(\mathcal{M}, d_{\mathfrak{b}})$.

Proof. Let us consider a sequence $\{m_k\}$ in $\mathcal{M}$ defined as

$$m_k = d_{\mathfrak{b}}(r_k, r_{k+1})$$

for all $k \in \mathbb{N}$, this sequence guarantees the condition (3) ; then, for all $k \geq \mathbb{N}$ we have

$$d_{\mathfrak{b}}(r_{2k}, r_{2k+1}) = m_{2k} \leq w^k\Lambda,$$

also,

$$d_{\mathfrak{b}}(r_{2k+1}, r_{2k+2}) = m_{2k+1} \leq w^k\Lambda.$$

Adding both vertically

$$d_{\mathfrak{b}}(r_{2k}, r_{2k+1}) + d_{\mathfrak{b}}(r_{2k+1}, r_{2k+2}) \leq 2w^k\Lambda.$$

Notice that if $w = 0$ or $\Lambda = 0$, we will have a constant sequence which is Cauchy. So, let $w > 0$ and $\Lambda > 0$; let $\epsilon > 0$ be any arbitrary real number, so $\frac{\epsilon}{2\Lambda\mathfrak{b}^{n^*}} > 0$,; and for $w \in (0,1)$, there is a natural number $n^* > k_0 \geq 1$ such that

$$\sum_{k=k_0}^{+\infty} w^k < \frac{\epsilon}{2\Lambda\mathfrak{b}^{n^*}},$$

in particular,

$$2\mathfrak{b}^{n^*}\Lambda \sum_{k=k_0}^{n} w^k < 2\mathfrak{b}^{n^*}\Lambda \sum_{k=k_0}^{+\infty} w^k < \epsilon,$$

for all $n \in \mathbb{N}$ such that $n \geq k$.
Let $n^*, k, l \in \mathbb{N}$ such that $n^* > l > k \geq 2k_0$ and $n \geq k_0 + 1$ and $2n \geq l$; thus, we have

$$\begin{aligned}
d_{\mathfrak{b}}(r_k, r_l) &\leq \mathfrak{b}[d_{\mathfrak{b}}(r_k, r_{k+1}) + d_{\mathfrak{b}}(r_{k+1}, r_l)] \\
&= \mathfrak{b}d_{\mathfrak{b}}(r_k, r_{k+1}) + \mathfrak{b}d_{\mathfrak{b}}(r_{k+1}, r_l) \\
&\leq \mathfrak{b}d_{\mathfrak{b}}(r_k, r_{k+1}) + \mathfrak{b}^2[d_{\mathfrak{b}}(r_{k+1}, r_{k+2}) + d_{\mathfrak{b}}(r_{k+2}, r_l)] \\
&\leq \mathfrak{b}d_{\mathfrak{b}}(r_k, r_{k+1}) + \mathfrak{b}^2 d_{\mathfrak{b}}(r_{k+1}, r_{k+2}) + \mathfrak{b}^3 d_{\mathfrak{b}}(r_{k+2}, r_{k+3}) + \ldots \\
&\quad .. + \mathfrak{b}^{l-k} d_{\mathfrak{b}}(r_{l-1}, r_l) \\
&= \sum_{j=k}^{l-1} \mathfrak{b}^{j-k+1} d_{\mathfrak{b}}(r_j, r_{j+1}) \\
&\leq \sum_{j=k}^{l-1} \mathfrak{b}^{n^*} d_{\mathfrak{b}}(r_j, r_{j+1}) \\
&\leq \sum_{j=2k_0}^{2n-1} \mathfrak{b}^{n^*} d_{\mathfrak{b}}(r_j, r_{j+1}) \\
&\leq \sum_{k=k_0}^{2n-1} \mathfrak{b}^{n^*} \{d_{\mathfrak{b}}(r_{2k}, r_{2k+1}) + d_{\mathfrak{b}}(r_{2k+1}, r_{2k+2})\} \\
&\leq \sum_{k=k_0}^{n-1} \mathfrak{b}^{n^*} 2w^k\Lambda \leq \sum_{k=k_0}^{n} \mathfrak{b}^{n^*} 2w^k\Lambda \\
&\leq \sum_{k=k_0}^{+\infty} \mathfrak{b}^{n^*} 2\Lambda w^k \\
&< \epsilon,
\end{aligned}$$

showing that $\{r_k\}$ is a Cauchy sequence in $\mathfrak{b}$MS. □

Corollary 1. *Let $\{r_k\}$ be a sequence in $\mathfrak{b}MS$, and $w \in [0,1)$ exists such that*

$$d_{\mathfrak{b}}(r_{k+2}, r_{k+3}) \leq w[d_{\mathfrak{b}}(r_k, r_{k+1})^{\beta_1}.d_{\mathfrak{b}}(r_{k+1}, r_{k+2})^{\beta_2}], \quad \forall\, k \in \mathbb{N}, \tag{5}$$

then, $\{r_k\}$ is a Cauchy sequence in $(\mathcal{M}, d_{\mathfrak{b}})$, where $\beta_1, \beta_2 \in [0,1]$ satisfy $\beta_1 + \beta_2 = 1$.

Proof. Consider

$$\begin{aligned}
d_{\mathfrak{b}}(r_{k+2}, r_{k+3}) &\leq w[d_{\mathfrak{b}}(r_k, r_{k+1})^{\beta_1}.d_{\mathfrak{b}}(r_{k+1}, r_{k+2})^{\beta_2}] \\
&\leq w[\max\{d_{\mathfrak{b}}(r_k, r_{k+1}), d_{\mathfrak{b}}(r_{k+1}, r_{k+2})\}^{\beta_1} \\
&\qquad .\max\{d_{\mathfrak{b}}(r_k, r_{k+1}), d_{\mathfrak{b}}(r_{k+1}, r_{k+2})\}^{\beta_2}] \\
&= w[\max\{d_{\mathfrak{b}}(r_k, r_{k+1}), d_{\mathfrak{b}}(r_{k+1}, r_{k+2})\}]^{\beta_1+\beta_2} \\
&= w[\max\{d_{\mathfrak{b}}(r_k, r_{k+1}), d_{\mathfrak{b}}(r_{k+1}, r_{k+2})\}],
\end{aligned}$$

then, the result follows form Lemma (1). □

Theorem 1. *Let $(\mathcal{M}, d_{\mathfrak{b}})$ be a complete $\mathfrak{b}$MS, $d_{\mathfrak{b}}$ be continuous, and S be a hybrid interpolative $\mathcal{RI}$-type contraction mapping. Assume that*

(1) S is continuous;
(2) S is a π orbital-admissible mapping;
(3) $r_0 \in \mathcal{M}$ exists such that $\pi(r_0, Sr_0) \geq 1$.

then S has a fixed point.

Proof. For $r_0 \in \mathcal{M}$ by assumption, $\pi(r_0, Sr_0) \geq 1$. Since S is an π orbital-admissible, we may write

$$\pi(Sr_0, S^2 r_0) \geq 1.$$

Continuing in the same manner, for r_0 we will obtain $\pi(S^k r_0, S^{k+1} r_0) \geq 1$ for any $k \in \mathbb{N}$, and for such a sequence $\{r_{k\}}$ we can write

$$r_1 = Sr_0, r_2 = Sr_1 = S^2 r_0, ..., r_k = Sr_{k-1} = S^k r_0.$$

If $r_k \in Sr_{k+1}$ this means that r_k is the fixed point of S trivially. Suppose $r_k \neq Sr_{k+1}$ for all $k \in \mathbb{N}$. Now consider

$$\begin{aligned} d_{\mathfrak{b}}(r_{k+2}, r_{k+3}) &\leq \pi(r_{k+2}, r_{k+3}) d_{\mathfrak{b}}(r_{k+2}, r_{k+3}) \\ &\leq \pi(r_{k+2}, r_{k+3}) d_{\mathfrak{b}}(S^2 r_k, S^2 r_{k+1}) \\ &\leq w\mathbb{M}(r_k, r_{k+1}) \end{aligned} \tag{6}$$

Now, we will discuss both possible cases of choice of p.

Case-I :- If $p > 0$,

$$\begin{aligned} \mathbb{M}(r_k, r_{k+1}) = &\Big[c_1 d_{\mathfrak{b}}(r_k, r_{k+1})^p + c_2 d_{\mathfrak{b}}(r_k, Sr_k)^p + c_3 d_{\mathfrak{b}}(r_{k+1}, Sr_{k+1})^p \\ &+ c_4 d_{\mathfrak{b}}(Sr_k, Sr_{k+1})^p + c_5 d_{\mathfrak{b}}(Sr_k, S^2 r_k)^p + q d_{\mathfrak{b}}(Sr_{k+1}, S^2 r_{k+1})^p\Big]^{\frac{1}{p}} \\ = &\Big[c_1 d_{\mathfrak{b}}(r_k, r_{k+1})^p + c_2 d_{\mathfrak{b}}(r_k, r_{k+1})^p + c_3 d_{\mathfrak{b}}(r_{k+1}, r_{k+2})^p \\ &+ c_4 d_{\mathfrak{b}}(r_{k+1}, r_{k+2})^p + c_5 d_{\mathfrak{b}}(r_{k+1}, r_{k+2})^p + q d_{\mathfrak{b}}(r_{k+2}, r_{k+3})\Big]^{\frac{1}{p}} \\ = &\Big[(c_1 + c_2) d_{\mathfrak{b}}(r_k, r_{k+1})^p + (c_3 + c_4 + c_5) d_{\mathfrak{b}}(r_{k+1}, r_{k+2})^p \\ &+ q d_{\mathfrak{b}}(r_{k+2}, r_{k+3})^p\Big]^{\frac{1}{p}} \\ \leq &\Big[(c_1 + c_2 + c_3 + c_4 + c_5) \max\{d_{\mathfrak{b}}(r_k, r_{k+1})^p, d_{\mathfrak{b}}(r_{k+1}, r_{k+2})^p\} \\ &+ q d_{\mathfrak{b}}(r_{k+2}, r_{k+3})^p\Big]^{\frac{1}{p}}, \end{aligned} \tag{7}$$

taking power of p on both sides of (6), we have

$$\begin{aligned} d_{\mathfrak{b}}(r_{k+2}, r_{k+3})^p \leq &w^p (c_1 + c_2 + c_3 + c_4 + c_5) \max\{d_{\mathfrak{b}}(r_k, r_{k+1})^p, d_{\mathfrak{b}}(r_{k+1}, r_{k+2})^p\} \\ &+ w^p q d_{\mathfrak{b}}(r_{k+2}, r_{k+3})^p. \end{aligned} \tag{8}$$

Therefore,

$$(1-w^p q)d_{\mathfrak{b}}(r_{k+2},r_{k+3})^p \le w^p(c_1+c_2+c_3+c_4+c_5)\max\{d_{\mathfrak{b}}(r_k,r_{k+1})^p, d_{\mathfrak{b}}(r_{k+1},r_{k+2})^p\} \\ \le w^p(1-q)\max\{d_{\mathfrak{b}}(r_k,r_{k+1}), d_{\mathfrak{b}}(r_{k+1},r_{k+2})\}^p. \tag{9}$$

So for all $k \in \mathbb{N}$,

$$d_{\mathfrak{b}}(r_{k+2},r_{k+3})^p \le \frac{w^p(1-q)}{1-w^p q}\max\{d_{\mathfrak{b}}(r_k,r_{k+1}), d_{\mathfrak{b}}(r_{k+1},r_{k+2})\}^p, \tag{10}$$

or equally

$$d_{\mathfrak{b}}(r_{k+2},r_{k+3}) = \left(\frac{w^p(1-q)}{1-w^p q}\right)^{\frac{1}{p}}\max\{d_{\mathfrak{b}}(r_k,r_{k+1}), d_{\mathfrak{b}}(r_{k+1},r_{k+2})\} \\ = w\max\{d_{\mathfrak{b}}(r_k,r_{k+1}), d_{\mathfrak{b}}(r_{k+1},r_{k+2})\}. \tag{11}$$

Here, we used

$$w = \left(\frac{w^p(1-q)}{1-w^p q}\right)^{\frac{1}{p}}$$

and it is easy to verify that $w \in (0,1)$, so Lemma (1) concludes that $\{r_k\}$ is a Cauchy sequence.

Case-II:- If $p = 0$.

$$\mathbb{M}(r_k,r_{k+1}) = d_{\mathfrak{b}}(r_k,r_{k+1})^{c_1}.d_{\mathfrak{b}}(r_k,Sr_k)^{c_2}.d_{\mathfrak{b}}(r_{k+1},Sr_{k+1})^{c_3} \\ .d_{\mathfrak{b}}(Sr_k,Sr_{k+1})^{c_4}.d_{\mathfrak{b}}(Sr_k,S^2r_k)^{c_5}.d_{\mathfrak{b}}(Sr_{k+1},S^2r_{k+1})^q \\ = d_{\mathfrak{b}}(r_k,r_{k+1})^{c_1}.d_{\mathfrak{b}}(r_k,r_{k+1})^{c_2}.d_{\mathfrak{b}}(r_{k+1},r_{k+2})^{c_3} \\ .d_{\mathfrak{b}}(r_{k+1},r_{k+2})^{c_4}.d_{\mathfrak{b}}(r_{k+1},r_{k+2})^{c_5}.d_{\mathfrak{b}}(r_{k+2},r_{k+3})^q \\ = d_{\mathfrak{b}}(r_k,r_{k+1})^{c_1+c_2}.d_{\mathfrak{b}}(r_{k+1},r_{k+2})^{c_3+c_4+c_5} \\ .d_{\mathfrak{b}}(r_{k+2},r_{k+3})^q \tag{12}$$

(6) implies that

$$d_{\mathfrak{b}}(r_{k+2},r_{k+3}) \le wd_{\mathfrak{b}}(r_k,r_{k+1})^{c_1+c_2}.d_{\mathfrak{b}}(r_{k+1},r_{k+2})^{c_3+c_4+c_5} \\ .d_{\mathfrak{b}}(r_{k+2},r_{k+3})^q. \tag{13}$$

As by assumption

$$\sum_{i=1}^{5} c_i + q = 1,$$

so notice that if we set $q = 1$, then (13) will be a contradiction. Then, necessarily $q < 1$, so

$$\sum_{i=1}^{5} c_i = 1 - q > 0.$$

Let us consider,

$$\beta_1 = \frac{c_1+c_2}{1-q}, \quad \beta_2 = \frac{c_3+c_4+c_5}{1-q},$$

satisfying $\beta_1 + \beta_2 = 1$. Now, setting these in (13),

$$d_{\mathfrak{b}}(r_{k+2},r_{k+3})^{1-q} \le wd_{\mathfrak{b}}(r_k,r_{k+1})^{c_1+c_2}.d_{\mathfrak{b}}(r_{k+1},r_{k+2})^{c_3+c_4+c_5} \\ \implies d_{\mathfrak{b}}(r_{k+2},r_{k+3}) \le w^{\frac{1}{1-q}}d_{\mathfrak{b}}(r_k,r_{k+1})^{\beta_1}.d_{\mathfrak{b}}(r_{k+1},r_{k+2})^{\beta_2}.$$

As $w \in (0,1)$,; therefore,

$$0 < 1 - q \leq 1 \implies 1 \leq \frac{1}{1-q} \implies w^{\frac{1}{1-q}} \leq w < 1,$$

so $\{r_k\}$ is a Cauchy sequence by Corollary (1). In both cases, we have shown that this Picard sequence is a Cauchy sequence. As $\mathcal{M}$ is complete, so there exists a point $r^* \in \mathcal{M}$ such that

$$d_{\mathfrak{b}}(r^*, Sr^*) = \lim_{k\to\infty} d_{\mathfrak{b}}(r_{k+1}, Sr^*) = \lim_{k\to\infty} d_{\mathfrak{b}}(Sr_k, Sr^*) = 0,$$

so $Sr^* = r^*$, that is, r^* is the fixed point of S. □

Corollary 2. *Let $(\mathcal{M}, d_{\mathfrak{b}})$ be a complete $\mathfrak{b}$MS and $d_{\mathfrak{b}}$ be continuous; also, $S : \mathcal{M} \to \mathcal{M}$ is a continuous map. Suppose that $c_1, c_2 \in (0,1)$ exists satisfying $c_1 + c_2 < 1$ such that*

$$d_{\mathfrak{b}}(S^2m, S^2n) \leq c_1 d_{\mathfrak{b}}(m,n) + c_1 d_{\mathfrak{b}}(Sm, Sn),$$

then S has a fixed point.

Theorem 2. *Let $(\mathcal{M}, d_{\mathfrak{b}})$ be a complete $\mathfrak{b}$MS, $d_{\mathfrak{b}}$ be continuous and S be a hybrid interpolative $\mathcal{RI}$-type contraction mapping. Assume that*

(1) S^2 is continuous;
(2) S is a π orbital-admissible mapping;
(3) $m_0 \in \mathcal{M}$ exists such that $\pi(m_0, Sm_0) \geq 1$,
(4) $\pi(m, Sm) \geq 1$ for all $m \in Fix_{S^2}(\mathcal{M})$

then S has a fixed point.

Proof. Let $\{r_k\}$ be the Picard sequence of S based on m_0 defined by $m_k = S^k m_0$. By completeness of $\mathcal{M}$, we have m^* such that

$$d_{\mathfrak{b}}(m^*, S^2m^*) = \lim_{k\to\infty} d_{\mathfrak{b}}(m_{k+1}, S^2m^*) = \lim_{k\to\infty} d_{\mathfrak{b}}(S^2m_k, S^2m^*) = 0,$$

showing that m^* is fixed point of S^2. This shows that the set $Fix_{S^2}(\mathcal{M})$ is non empty. Next, we verify that m^* is also a fixed point of S. On the contrary, assume that $Sm^* \neq m^*$ then $m^* \notin Fix_S(\mathcal{M})$.

$$\begin{aligned} 0 \leq d_{\mathfrak{b}}(m^*, Sm^*) &\leq \pi(m^*, Sm^*) d_{\mathfrak{b}}(m^*, Sm^*) \\ &\leq \pi(m^*, Sm^*) d_{\mathfrak{b}}(S^2m^*, S^2m^*) \\ &\leq w\mathbb{M}(m^*, Sm^*) \end{aligned} \tag{14}$$

Now, we will discuss both possible cases of choice of p.

Case-I:- If $p > 0$,

$$
\begin{aligned}
\mathbb{M}(m^*, Sm^*) = &\Big[c_1 d_{\mathfrak{b}}(m^*, Sm^*)^p + c_2 d_{\mathfrak{b}}(m^*, Sm^*)^p + c_3 d_{\mathfrak{b}}(Sm^*, S^2m^*)^p \\
&+ c_4 d_{\mathfrak{b}}(Sm^*, S^2m^*)^p + c_5 d_{\mathfrak{b}}(Sm^*, S^2m^*)^p + q d_{\mathfrak{b}}(S^2m^*, S^3m^*)^p\Big]^{\frac{1}{p}} \\
= &\Big[c_1 d_{\mathfrak{b}}(m^*, Sm^*)^p + c_2 d_{\mathfrak{b}}(m^*, Sm^*)^p + c_3 d_{\mathfrak{b}}(Sm^*, m^*)^p \\
&+ c_4 d_{\mathfrak{b}}(Sm^*, m^*)^p + c_5 d_{\mathfrak{b}}(Sm^*, m^*)^p + q d_{\mathfrak{b}}(m^*, Sm^*)^p\Big]^{\frac{1}{p}} \\
= &\Big[(c_1 + c_2 + c_3 + c_4 + c_5 + q) d_{\mathfrak{b}}(m^*, Sm^*)^p\Big]^{\frac{1}{p}} \\
\leq &\Big[d_{\mathfrak{b}}(m^*, Sm^*)^p\Big]^{\frac{1}{p}} \\
= &\, d_{\mathfrak{b}}, (m^*, Sm^*),
\end{aligned}
$$

which contradicts (14).

Case-II:- If $p = 0$.

$$
\begin{aligned}
\mathbb{M}(m^*, Sm^*) &= d_{\mathfrak{b}}(m^*, Sm^*)^{c_1}.d_{\mathfrak{b}}(m^*, Sm^*)^{c_2}.d_{\mathfrak{b}}(Sm^*, S^2m^*)^{c_3} \\
&\quad .d_{\mathfrak{b}}(Sm^*, S^2m^*)^{c_4}.d_{\mathfrak{b}}(Sm^*, S^2m^*)^{c_5}.d_{\mathfrak{b}}(S^2m^*, S^3m^*)^{q} \\
&= d_{\mathfrak{b}}(m^*, Sm^*)^{c_1}.d_{\mathfrak{b}}(m^*, Sm^*)^{c_2}.d_{\mathfrak{b}}(Sm^*, m^*)^{c_3} \\
&\quad .d_{\mathfrak{b}}(Sm^*, m^*)^{c_4}.d_{\mathfrak{b}}(Sm^*, m^*)^{c_5}.d_{\mathfrak{b}}(Sm^*, m^*)^{q} \\
&= d_{\mathfrak{b}}(m^*, m^*)^{c_1+c_2+c_3+c_4+c_5+q} \\
&= d_{\mathfrak{b}}(m^*, Sm^*)
\end{aligned}
$$

which is again a contradiction to (14), thus m^* is fixed point of S □

Theorem 3. *Let* $(\mathcal{M}, d_{\mathfrak{b}})$ *be a complete* $\mathfrak{b}$*MS,* $d_{\mathfrak{b}}$ *be continuous. Let* $\pi : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ *be a function, and let* $S : \mathcal{M} \to \mathcal{M}$ *be a mapping such that*

(1) *Either* S *is* S^2 *is continuous;*
(2) S *is a* π *orbital-admissible mapping;*
(3) $m_0 \in \mathcal{M}$ *exists such that* $\pi(m_0, Sm_0) \geq 1$,
(4) *A constant* $w \in [0, 1)$ *exists such that at least one of the following conditions is fulfilled for all distinct* $m, n \in \mathcal{M} \backslash Fix_S(\mathcal{M})$:
   (a) $\pi(m, n) d_{\mathfrak{b}}(S^2m, S^2n) \leq w d_{\mathfrak{b}}(m, n)$
   (b) $\pi(m, n) d_{\mathfrak{b}}(S^2m, S^2n) \leq w d_{\mathfrak{b}}(m, Sm)$
   (c) $\pi(m, n) d_{\mathfrak{b}}(S^2m, S^2n) \leq w d_{\mathfrak{b}}(n, Sn)$
   (d) $\pi(m, n) d_{\mathfrak{b}}(S^2m, S^2n) \leq w d_{\mathfrak{b}}(Sm, Sn)$
   (e) $\pi(m, n) d_{\mathfrak{b}}(S^2m, S^2n) \leq w d_{\mathfrak{b}}(Sm, S^2m)$
   (f) $\pi(m, n) d_{\mathfrak{b}}(S^2m, S^2n) \leq w d_{\mathfrak{b}}(Sn, S^2n)$.

then S *has a fixed point.*

Proof. The proof of this result follows from the case of $p > 0$ from of theorem (1) and theorem (2), and it can be observed easily that the contractions condition (1) is satisfied by the following suitable choice of parameters

$$\begin{aligned} c_1 = 1,\ c_2 = 0,\ c_3 = 0,\ c4 = 0,\ c_5 = 0,\ q = 0 \\ c_1 = 0,\ c_2 = 1,\ c_3 = 0,\ c4 = 0,\ c_5 = 0,\ q = 0 \\ c_1 = 0,\ c_2 = 0,\ c_3 = 1,\ c4 = 0,\ c_5 = 0,\ q = 0 \\ c_1 = 0,\ c_2 = 0,\ c_3 = 0,\ c4 = 1,\ c_5 = 0,\ q = 0 \\ c_1 = 0,\ c_2 = 0,\ c_3 = 0,\ c4 = 0,\ c_5 = 1,\ q = 0 \\ c_1 = 0,\ c_2 = 0,\ c_3 = 0,\ c4 = 0,\ c_5 = 0,\ q = 1 \end{aligned} \tag{15}$$

□

Corollary 3. *Let $(\mathcal{M}, d_{\mathfrak{b}})$ be a complete $\mathfrak{b}$MS, $d_{\mathfrak{b}}$ be continuous, and $S : \mathcal{M} \to \mathcal{M}$ be a mapping such that either S is S^2 or is continuous . Suppose that there exists a constant $w \in (0,1)$ such that at least one of the following conditions is fulfilled for all distinct $m, n \in \mathcal{M}\ Fix_S(\mathcal{M})$:*

1. $d_{\mathfrak{b}}(S^2m, S^2n) \leq wd_{\mathfrak{b}}(m, n)$
2. $d_{\mathfrak{b}}(S^2m, S^2n) \leq wd_{\mathfrak{b}}(m, Sm)$
3. $d_{\mathfrak{b}}(S^2m, S^2n) \leq wd_{\mathfrak{b}}(n, Sn)$
4. $d_{\mathfrak{b}}(S^2m, S^2n) \leq wd_{\mathfrak{b}}(Sm, Sn)$
5. $d_{\mathfrak{b}}(S^2m, S^2n) \leq wd_{\mathfrak{b}}(Sm, S^2m)$
6. $d_{\mathfrak{b}}(S^2m, S^2n) \leq wd_{\mathfrak{b}}(Sn, S^2n)$

then S has a fixed point.

In particular, if we drop the condition of continuity of either S or S^2, then we have a special case of existence of the fixed point provided that $\delta = 0$. For this, observe the following example.

Example 2. *Let $(\mathcal{M}, d_{\mathfrak{b}})$ be a complete metric space where $\mathcal{M} = [-1, 1]$, and $d_{\mathfrak{b}} : \mathcal{M} \times \mathcal{M} \to (0, \infty)$ is defined as $d_{\mathfrak{b}}(m, n) = |m - n|^2$. Let $S : \mathcal{M} \to \mathcal{M}$ be defined by*

$$Sm = \begin{cases} \sqrt{1 - m^2} & \textit{if } -1 \leq m \leq 0 \\ \\ \frac{m^2}{2} & \textit{if } 0 \leq m \leq 1, \end{cases}$$

then

$$S^2m = \begin{cases} \frac{1-m^2}{2} & \textit{if } -1 \leq m \leq 0 \\ \\ \frac{m^4}{8} & \textit{if } 0 \leq m \leq 1. \end{cases}$$

Next, define $\pi : \mathcal{M} \times \mathcal{M} \to [0, \infty)$, which is defined as

$$\pi(m, n) = \begin{cases} \frac{3}{2} & \textit{if } 0 \leq m \leq 1 \\ 1 & \textit{if } n = 1,\ \ m = -1 \\ 0 & \textit{if otherwise} \end{cases}$$

Now to check the contractivity condition, for $0 \leq m \leq 1$

$$\begin{aligned}
\pi(m,n)d_{\mathfrak{b}}(S^2m,S^2n) &= \frac{3}{(2)(8)}|m^4-n^4| \\
&= \frac{3}{(2)(8)}|(m^2-n^2)(m^2+n^2)| \\
&\leq \frac{3}{8}|m^2-n^2| \\
&= \frac{3}{8}|\sqrt{|m^2-n^2|}\sqrt{|m^2-n^2|} \\
&= \frac{3}{8}\sqrt{|m-n|.|m+n|}\sqrt{2\frac{|m^2-n^2|}{2}} \\
&\leq \frac{3}{4}d_{\mathfrak{b}}(m,n)^{\frac{1}{4}}d_{\mathfrak{b}}(Sm,Sn)^{\frac{1}{4}}.
\end{aligned}$$

For $m=-1,\ n=1$, we have

$$\begin{aligned}
\pi(m,n)d_{\mathfrak{b}}(S^2m,S^2n) &= \frac{1}{8} \\
&< \frac{3}{4} \\
&= \frac{3}{4}d_{\mathfrak{b}}(m,n)^{\frac{1}{4}}d_{\mathfrak{b}}(Sm,Sn)^{\frac{1}{4}}.
\end{aligned}$$

All of the conditions of theorem (1) except the continuity of S are satisfied with $\delta=0$. Hence, S has a fixed point.

3. Application

Now, we apply our main result to find a solution to an integral equation of the Fredholm type.

Suppose $I=[0,1]$ and $\mathcal{M}=\mathcal{C}(I,\mathbb{R}^2)$ are the space of all continuous functions defined from I to $\mathbb{R}^2$, endowed with the usual sup-norm.

We define a $\mathfrak{b}$ metric on $\mathcal{M}$ as

$$d_{\mathfrak{b}}(\phi,\psi)=\parallel \phi-\psi \parallel_{\infty}=\sup_{m\in I}\{|\phi(m)-\psi(m)|^{q^*}\} \quad q^*>1,$$

for all $\phi,\psi\in\mathcal{M}$. It is easy to verify that $(\mathcal{M},d_{\mathfrak{b}})$ is a complete $\mathfrak{b}$MS.

Consider a Fredholm integral equation

$$\phi(\zeta)=f(\zeta)+\int_0^1 k_{\phi}(\zeta,x^*,\phi(x^*))dx^*. \tag{16}$$

Define a mapping $S:\mathcal{M}\to\mathcal{M}$, as

$$S(\phi^*(\zeta))=f(\zeta)+\int_0^1 k_{\phi}(\zeta,x^*,\phi(x^*))dx^* \tag{17}$$

Theorem 4. *Suppose that the following conditions hold:*

(1) Let $k_{\phi}:I\times I\times\mathbb{R}^2\to\mathbb{R}^2$ and $f:I\to\mathbb{R}^2$ be continuous;
(2) $\phi_0\in\mathcal{M}$ exists such that $\phi_k=S\phi_{k-1}$;
(3) A continuous function $\mathfrak{f}:I\times I\to I$ exists such that

$$|k_{\phi}(m,m^*,\phi(m^*))-k_{\psi}(m,m^*,\psi(m^*))|^{q^*}\leq|\mathfrak{f}(\phi(m^*),\psi(m^*))||\phi(m^*)-\psi(m^*)|^{q^*},$$

for each $m,m^*\in I$ and $|\mathfrak{f}(\phi(m^*),\psi(m^*))|\leq\frac{1}{\nu}$ where $\nu>0$.

Then, the integral Equation (16) has a solution.

Proof. Let $(\mathcal{M}, d_{\mathfrak{b}})$ be a complete $\mathfrak{b}$MS and $\pi(\phi,\psi) = 1$. Let $p^* > 1$ such that $\frac{1}{p^*} + \frac{1}{q^*} = 1$; then, for $\phi^* \in S(\phi)$ we have

$$\begin{aligned}
d_{\mathfrak{b}}(S\phi^*(m), S(\psi^*(m))) &= \sup_{m\in I} |S\phi^*(m) - S\psi^*(m)|^{q^*} \\
&= \sup_{m\in I} \left| \int_0^1 k_\phi(m, m^*, \phi(m^*)) - k_\psi(m, m^*, \psi(m^*)) \right|^{q^*} dm^* \\
&\leq \sup_{m\in I} \Bigg[\left(\int_0^1 |1|^{p^*} dm^* \right)^{\frac{1}{p^*}} \\
&\quad \int_0^1 \left(|k_\phi(m, m^*, \phi(m^*)) - k_\psi(m, m^*, \psi(m^*))|^{q^*} \right)^{\frac{1}{q^*}} \Bigg]^{q^*} dm^* \\
&= \sup_{m\in I} \int_0^1 |k_\phi(m, m^*, \phi(m^*)) - k_\psi(m, m^*, \psi(m^*))|^{q^*} dm^* \\
&\leq \sup_{m\in I} \int_0^1 |\mathfrak{f}(\phi(m^*), \psi(m^*))| \left|\phi(m^*) - \psi(m^*)\right|^{q^*} dm^* \\
&\leq (\frac{1}{\nu}) \parallel \phi(m^*) - \psi(m^*) \parallel_\infty \\
&= (\frac{1}{\nu}) d_{\mathfrak{b}}(\phi(m^*), \psi(m^*))
\end{aligned}$$

Similarly, one can easily obtain

$$\begin{aligned}
d_{\mathfrak{b}}(S^2\phi^*(m), S^2\psi^*(m)) &= \sup_{m\in I} |S^2\phi^*(m) - S^2\psi^*(m)|^{q^*} \\
&\leq (\frac{1}{\nu})^2 d_{\mathfrak{b}}(\phi(m^*), \psi(m^*)) \\
&= (\frac{1}{\nu})^2 \mathbb{M} d_{\mathfrak{b}}(\phi(m^*), \psi(m^*)).
\end{aligned}$$

All of the conditions of the theorem (1) are satisfied by choice of $w = (\frac{1}{\nu})^2 \in (0,1)$ and

$$c_1 = 1\,, c_2 = c_3 = c_4 = c_5 = \delta = 0,$$

hence the integral Equation (16) has a solution. □

4. Conclusions and Future Work

Several results from the literature may be considered as a special cases of our developed extension, which illustrates the degree of the validity of our results. If $\mathfrak{b} = 1$ then the main results of Karapinar et al. [25] are just a subcase of our results. The proposed idea also yields some consequences of work done by Istrăţescu ([26,27]).

Further work should be considered on how to redevelop the contractivity condition in order to formulate these results in more general abstract spaces, for example, controlled, as well as double-controlled, metric spaces; fuzzy metric spaces; fuzzy $\mathfrak{b}$-metric spaces; and generalized fuzzy metric spaces .

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References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrals. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Ali, M.U.; Kamran, T.; Din, F.; Anwar, M. Fixed and Common Fixed Point Theorems for Wardowski Type Mappings in Uniform Spaces. *UPB Sci. Bull. Ser. A* **2018**, *80*, 1–12.
3. Anwar, M.; Shehwar, D.; Ali, R.; Hussain, N. Wardowski Type α-F-Contractive Approach for Nonself Multivalued Mappings. *UPB Sci. Bull. Ser. A* **2020**, *82*, 69–77.
4. Boyd, D.W.; Wong, J.S. On nonlinear contractions. *Proc. Am. Math. Soc.* **1969**, *20*, 458–464.
5. Caristi, J. Fixed point theorems for mappings satisfying inwardness conditions. *Trans. Am. Math. Soc.* **1976**, *215*, 241–251. [CrossRef]
6. Chatterjea, S. Fixed-point theorems. *Dokl. Bolg. Akad. Nauk.* **1972**, *25*, 727.
7. Hardy, G.E.; Rogers, T.D. A generalization of a fixed point theorem of Reich. *Can. Math. Bull.* **1973**, *16*, 201–206. [CrossRef]
8. He, L.; Cui, Y.-L.; Ceng, L.-C.; Zhao, T.-Y.; Wang, D.-Q.; Hu, H.-Y. Strong convergence for monotone bilevel equilibria with constraints of variational inequalities and fixed points using subgradient extragradient implicit rule. *J. Inequalities Appl.* **2021**, *2021*, 146. [CrossRef]
9. Ceng, L.-C.; Wen, C.-F.; Liou, Y.-C.; Yao, J.-C. A General Class of Differential Hemivariational Inequalities Systems in Reflexive Banach Spaces. *Mathematics* **2021**, *9*, 3173. [CrossRef]
10. Czerwik, S. Contraction mappings in b-metric spaces. *Acta Math. Inform. Univ. Ostrav.* **1993**, *1*, 5–11.
11. Bakhtin, I. The contraction mapping principle in quasi metric spaces. *Funct. Anal.* **1989**, *30*, 26–37.
12. Adiguzel, R.S.; Aksoy, U.; Karapınar, E.; Erhan, I.M. On the solutions of fractional differential equations via Geraghty type hybrid contractions. *Appl. Comput. Math.* **2021**, *20*, 313–333.
13. Anwar, M.; Shehwar, D.; Ali, R. Fixed Point Theorems on α-F-contractive Mapping in Extended $\mathfrak{b}$-Metric Spaces. *J. Math. Anal.* **2020**, *11*, 43–51. ISSN 2217-3412.
14. Batul, S.; Sagheer, D.; Anwar, M.; Aydi, H.; Parvaneh, V. Fuzzy fixed-point results of Fuzzy Mappings on b-metric Spaces via ($\alpha*$-F)-contractions. *Adv. Math. Phys.* **2022**, *2022*, 4511632. [CrossRef]
15. Bernide, V.; Pacurar, M. The early developments in fixed-point theory on $\mathfrak{b}$-metric spaces. *Carpathian J. Math.* **2022**, *38*, 523–538.
16. Bota, M.F.; Micula, S. Ulam–Hyers stability via fixed-point results for special contractions in $\mathfrak{b}$-metric spaces. *Symmetry* **2022**, *14*, 2461. [CrossRef]
17. Kirk, M.; Kiziltunc, H. On some well known fixed point theorems in b-metric spaces. *Turk. J. Anal. Number Theory* **2013**, *1*, 13–16.
18. Shatanawi, W.; Pitea, A.; Lazović, R. Contraction conditions using comparison functions on b-metric spaces. *Fixed-Point Theory Appl.* **2014**, *2014*, 135. [CrossRef]
19. Suzuki, T. Basic inequality on a $\mathfrak{b}$-metric space and its applications. *J. Inequalities Appl.* **2017**, *256*, 1–11.
20. Younis, M.; Singh, D.; Abdou, A.A. A fixed point approach for tuning circuit problem in dislocated $\mathfrak{b}$-metric spaces. *Math. Methods Appl. Sci.* **2022**, *45*, 2234–2253. [CrossRef]
21. Bhat, I.A.; Mishra, L.N. Numerical Solutions of Volterra Integral Equations of Third Kind and Its Convergence Analysis. *Symmetry* **2022**, *14*, 2600. [CrossRef]
22. Pathak, V.K.; Mishra, L.N. Application of Fixed Point Theorem to Solvability for Non-Linear Fractional Hadamard Functional Integral Equations. *Mathematics* **2022**, *10*, 2400. [CrossRef]
23. Al-Rawashdeh, A.; Hassen, A.; Abdelbasset, F.; Sehmim, S.; Shatanawi, W. On common fixed points for $\alpha - F$-contractions and applications. *J. Nonlinear Sci. Appl.* **2016**, *9*, 3445–3458.
24. Mlaiki, N.; Özgür, N.i.Y.; Mukheimer, A.; Taş, N. A new extension of the M_b-metric spaces. *J. Math. Anal.* **2018**, *9*, 118–133.
25. Karapinar, E.; Fulga, A.; Shahzad, N.; de Hierro, A.F.R.L. Solving Integral Equations by Means of fixed-point theory. *J. Funct. Spaces* **2022**, *2022*, 7667499. [CrossRef]
26. Istraţescu, V.I. Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters. I. *Ann. Mat. Pura Appl.* **1982**, *130*, 89–104. [CrossRef]
27. Istraţescu, V.I. Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters, II. *Ann. Mat. Pura Appl.* **1983**, *134*, 327–362. [CrossRef]
28. Karapinar, E. Revisiting the Kannan type contractions via interpolation. *Adv. Theory Nonlinear Anal. Its Appl.* **2018**, *2*, 85–87. [CrossRef]