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Atomic Solution for Certain Gardner Equation

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Abstract: In this paper, a new technique using a tensor product is presented in order to provide exact solutions to some certain fractional differential equations. Particularly, the well-known third order Gardner's equation, which is also known in some contexts as KdV-mKdV, of the fractional type. This type of equations plays an important role in modeling many symmetric and asymmetric problems. Moreover, the existence of an atomic solution using a tensor product technique for certain second order equations has been proved.

Keywords: inverse problems; Gardner's equation; tensor product of Banach spaces

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1. Introduction

A large variety of biological, chemical and physical phenomena are governed by different kinds of nonlinear partial differential equations. Most of the exciting advances in theoretical physics and nonlinear science are the ones related to the development of methods to find the exact solutions of nonlinear partial differential equations. Such solutions play an important role in nonlinear science, since they can provide more insight of the physical aspects of the problem which can lead to further applications, see for instance, [1–9].

In recent years, several powerful methods for obtaining the exact solution of nonlinear evolution equations have been established, such as homogeneous balance method [2,7], Hirota's method [10,11], projective Riccati equation method [3,12], separation of variables [4,13,14] and Jacobi elliptic functions method [15].

Using a tensor product of two Banach spaces Zigan and others, [16,17], have presented a new method to find the exact solution of homogeneous and non-homogeneous first order abstract Cauchy problem. After that, different kinds of ordinary and fractional type differential equations have been solved using the tensor product methods [18–23], while atomic solution of certain inverse problems has been obtained [24,25].

It is known that the Gardner equation is one of those type of equations that plays an important role in applications in many different kinds of fields in science and engineering. For example, in fluid mechanics, plasma and physics, Gardner equation has the form

$$\frac{\partial u}{\partial s} + \lambda u \frac{\partial u}{\partial \omega} + \beta u^2 \frac{\partial u}{\partial \omega} + \gamma \frac{\partial^3 u}{\partial \omega^3} = 0,$$

which is also called the KdV-mKdV equation. By changing the value of λ, β and γ different types of equations can be obtained.

If β and γ are chosen to be 0 and 1, respectively, then the KdV equation has the form

$$\frac{\partial u}{\partial s} + \lambda u \frac{\partial u}{\partial \omega} + \frac{\partial^3 u}{\partial \omega^3} = 0, \quad (1)$$

where λ is any real number. Taking the value of $\lambda = \pm 1$ or $\lambda = \pm 6$, the KdV equation represent a large variety of phenomena such as diving waves in plasma and acoustic ion waves.

Moreover, if $\lambda = 0$ and $\gamma = 1$, the mKdV equation can be written as

$$\frac{\partial u}{\partial s} + \beta u^2 \frac{\partial u}{\partial \omega} + \frac{\partial^3 u}{\partial \omega^3} = 0, \tag{2}$$

that is completely integrable and can be obtained easily from the KdV using Miura transform [5].

In the last few years, differential equations of fractional order type received a great attention because they play a big aspect in the model life problems, symmetric and non-symmetric problems and their applications in physics, finance, and other branches of science and engineering, see e.g., [26–30].

Over the years, many different forms of fractional differential operators have been introduced, such as Riemann, Caputo, etc. [31,32]. Most of them use integral form and they are generally not equivalent. All definitions appear to satisfy linearity property and all of them do not satisfy the familiar rules of ordinary derivative. To overcome these difficulties and others, new simple interesting definition appeared and extended the definition of the usual derivation [33,34] as follows:

If g is a function from $[0, \infty)$ to the set of real numbers $\mathbb{R}$, the $\alpha-$ conformable derivative of the function g is defined as

$$D^\alpha(g)(s) = \lim_{\epsilon \to 0} \frac{g(s + \epsilon s^{1-\alpha}) - g(s)}{\epsilon},$$

for all $0 < \alpha \leq 1$ and all $s > 0$. If g has $\alpha-$conformable derivative in some interval $(0, b)$, $b > 0$ and $\lim_{s \to 0^+} g^\alpha(s)$ exists, then $g^\alpha(0) = \lim_{s \to 0^+} g^\alpha(s)$, where $g^\alpha(s)$ stands for $D^\alpha(g)(s)$ and $g^{(2\alpha)}(s)$ represents $D^\alpha D^\alpha(g)(s)$.

In this paper, we focus on studying the atomic solution of the conformable fractional Gardner's equations:

$$\frac{\partial^\alpha u}{\partial s^\alpha} + \lambda u \frac{\partial^\alpha u}{\partial \omega^\alpha} + \frac{\partial^\alpha}{\partial \omega^\alpha}\left(\frac{\partial^\alpha}{\partial \omega^\alpha}\left(\frac{\partial^\alpha u}{\partial \omega^\alpha}\right)\right) = 0,$$

and

$$\frac{\partial^\alpha u}{\partial s^\alpha} + \beta u^2 \frac{\partial^\alpha u}{\partial \omega^\alpha} + \frac{\partial^\alpha}{\partial \omega^\alpha}\left(\frac{\partial^\alpha}{\partial \omega^\alpha}\left(\frac{\partial^\alpha u}{\partial \omega^\alpha}\right)\right) = 0,$$

using a tensor product technique.

2. Main Result

In the theory of tensor product of two Banach spaces A and B, a linear operator of the form $g = h \otimes j : A^* \to B$ is defined by $h \otimes j(\omega^*) = \omega^*(h)j$, where A^* is the dual of the Banach space A and $h \otimes j \in A \otimes B$, is called an atom.

It is known that, g is bounded as a linear operator and has norm equal to $||h||\ ||j||$. By $A \otimes B$ we denote the linear space spanned by the set $\{\omega \otimes y, (\omega, y) \in A \times B\}$. There are many norms that one can define on $A \otimes B$. One of most popular ones is the injective norm $||.||_\vee$, see [35]. For $\mathrm{Y} = \sum_{i=1}^{n} \omega_i \otimes y_i \in A \otimes B$,

$$||\mathrm{Y}||_\vee = \sup\{\sum_{i=1}^{n} |\langle \omega, \omega^* \rangle \langle y, y^* \rangle|, \omega^* \otimes y^* \in A^* \times B^*, ||\omega^*|| = ||y^*|| = 1\}.$$

The space $(A \otimes B, ||.||_\vee)$ need not to be complete. We let $A \overset{\vee}{\otimes} B$ denote the completion of $A \otimes B$ in the space of all bounded linear operators from A^* to B, which is denoted by

$L(A^*, B)$ with respect to the injective norm. It is known [35], that $C(I, A)$ (The space of all continuous function from I to A) is isometrically isomorphic to $C(I) \overset{\vee}{\otimes} A$. If $A = C(I)$, then $C(I) \overset{\vee}{\otimes} C(I) \simeq C(I, C(I))$ which is isomorphic to $C(I \times I)$, see [36]. That means any continuous function of two variables can be represented by an element in $C(I) \overset{\vee}{\otimes} C(I)$, the completion of $C(I) \otimes C(I)$, since there exists a one-one correspondence between the two spaces. If $u(x, y) \in C(I \times I)$, then $u(x, y)$ can be represented by $f \in C(I) \overset{\vee}{\otimes} C(I)$ which has the form $f = \sum_{i=1}^{\infty} P_i \otimes Q_i \; P_i, Q_i \in C(I), i = 1, 2, 3, \ldots.$

If sum is finite, then we say that $u(x, y)$ can be represented by a finite rank operator $f = \sum_{i=1}^{n} P_i \otimes Q_i$. If $n = 1$, then the function $u(x, y)$ can be represented by one atom. If the solution of the differential equation can be represented by an atom we say that it has an atomic solution.

In this section, we prove the existence of an atomic solution of the fractional type Gardner differential equation. To handle this, we need the following lemma.

Lemma 1. *[21] Let $u_1 \otimes v_1$ and $u_2 \otimes v_2$ be two non zero atoms in $C(I) \overset{\vee}{\otimes} C(I)$. Then the following are equivalent:*

(1) $u_1 \otimes v_1 + u_2 \otimes v_2 = u_3 \otimes v_3$, a non zero atom.
(2) u_1, u_2 or v_1, v_2 are linearly dependent.

In the sequel, we state our main results of this paper, where we prove the existence of an atomic solution of the non-linear KdV-mKdV equations of conformable fractional type using a tensor product technique, a simple new method to find the exact solution of such non-linear fractional partial differential equations that cannot be solved by the well known separation of variables method, and in addition, is more accurate than the numerical approximate methods.

Theorem 1. *Let $u \in C(I \times I)$, where $I = [0, 1]$ or $[0, \infty)$ with values in the Banach space A. If u is an $\alpha-$differentiable in some $\alpha \in (0, 1]$, then the fractional partial differential equation*

$$u_s^{(\alpha)} + \lambda u u_\omega^{(\alpha)} + u_\omega^{(3\alpha)} = 0, \tag{3}$$

has an atomic solution.

Proof. Let $u(\omega, s) = X \otimes T$, where X and T are functions of ω and s, respectively. Then $u_\omega^{(\alpha)} = X^{(\alpha)} \otimes T$, $u_s^{(\alpha)} = X \otimes T^{(\alpha)}$and $u_\omega^{(3\alpha)} = X^{(3\alpha)} \otimes T$. This implies that Equation (3) becomes

$$X \otimes T^{(\alpha)} + \lambda [X \otimes T][X^{(\alpha)} \otimes T] = -X^{(3\alpha)} \otimes T,$$

which implies

$$X \otimes T^{(\alpha)} + \lambda X X^{(\alpha)} \otimes T^2 = -X^{(3\alpha)} \otimes T. \tag{4}$$

Using Equation (4) and Lemma 1, we have either $X = \lambda X X^\alpha$ or $T^{(\alpha)} = T^2$.
Case 1: If $T^{(\alpha)} = T^2$, then

$$\begin{aligned} T^{-2} T^{(\alpha)} &= 1 \\ \int T^{-2} d_\alpha T &= \int d_\alpha s \\ \frac{-1}{T} &= (\frac{1}{\alpha}) s^\alpha \\ T(s) &= -\alpha s^{-\alpha}. \end{aligned}$$

Hence,
$$X \otimes T^{(\alpha)} + \lambda X X^{(\alpha)} \otimes T^2 = -X^{(3\alpha)} \otimes T,$$
and
$$[X + \lambda X X^{(\alpha)}] \otimes T^2 = -X^{(3\alpha)} \otimes T.$$
Therefore, $T^2 = T$ and $X + \lambda X X^{(\alpha)} = -X^{(3\alpha)}$. This is a contradiction since in this case $T^2 \neq T$.
Case 2: If $X = \lambda X X^{(\alpha)}$, then $X - \lambda X X^{(\alpha)} = 0$ which implies $X[1 - \lambda X^{(\alpha)}] = 0$, and hence $X = 0$ or $1 - \lambda X^{(\alpha)} = 0$.

(i) If $X = 0$, then it is a zero solution.

(ii) if $1 - \lambda X^{(\alpha)} = 0$, then $X^{(\alpha)} = \frac{1}{\lambda}$, which leads to
$$\frac{d^\alpha X}{d\omega^\alpha} = \frac{1}{\lambda},$$
by integrating both sides, we obtain
$$X = \int \frac{1}{\lambda} d_\alpha \omega,$$
and hence,
$$X(\omega) = \frac{1}{\alpha\lambda}\omega^\alpha.$$
Now, we have $X(\omega) = \frac{1}{\alpha\lambda}\omega^\alpha$, $X^{(\alpha)}(\omega) = \frac{1}{\lambda}$, $X^{(2\alpha)}(\omega) = X^{(3\alpha)}(\omega) = 0$. Hence,
$$X \otimes T^{(\alpha)} + \lambda X X^{(\alpha)} \otimes T^2 = -X^{(3\alpha)} \otimes T,$$
which implies,
$$X \otimes T^{(\alpha)} + X \otimes T^2 = 0 \otimes T,$$
and hence,
$$X \otimes [T^{(\alpha)} + T^2] = 0.$$
Since here $X \neq 0$, we have $T^{(\alpha)} + T^2 = 0$, which implies
$$\begin{aligned} T^{(\alpha)} &= -T^2, \\ T^{-2}T^\alpha &= -1, \\ \int T^{-2} d_\alpha T &= \int -1 d_\alpha s, \\ \frac{-1}{T} &= (\frac{-1}{\alpha})s^\alpha, \\ T(s) &= \alpha s^{-\alpha}. \end{aligned}$$
To verify (4), set $u = X \otimes T$, $u_\omega^{(\alpha)} = X^{(\alpha)} \otimes T$, $u_s^{(\alpha)} = X \otimes T^{(\alpha)}$ and $u_\omega^{(3\alpha)} = X^{(3\alpha)} \otimes T$, where $X(\omega) = \frac{1}{\alpha\lambda}\omega^\alpha$, $X^{(\alpha)}(\omega) = \frac{1}{\lambda}$, $X^{(2\alpha)}(\omega) = X^{(3\alpha)}(\omega) = 0$, and $T(s) = \alpha s^{-\alpha}$ $T^{(\alpha)}(s) = -\alpha^2 s^{-2\alpha} = -T^2(s)$, then $X^{(3\alpha)} \otimes T = 0$.
Now,
$$\begin{aligned} X \otimes T^{(\alpha)} + \lambda X X^{(\alpha)} \otimes -T^{(\alpha)} &= X \otimes T^{(\alpha)} + X \otimes -T^{(\alpha)} \\ &= 0 \\ &= X^{(3\alpha)} \otimes T. \end{aligned}$$
This implies that $u = X \otimes T$ is an atomic solution of Equation (3), where $X(\omega) = \frac{1}{\alpha\lambda}\omega^\alpha$ and $T(s) = \alpha s^{-\alpha}$. $\square$

Theorem 2. *Let $u \in C(I \times I)$, where $I = [0,1]$ or $[0,\infty)$ with values in the Banach space A. If u is an $\alpha-$differentiable in some $\alpha \in (0,1]$, then the fractional partial differential equation*

$$u_s^{(\alpha)} + \beta u^2 u_\omega^{(\alpha)} + u_\omega^{(3\alpha)} = 0, \tag{5}$$

has an atomic solution.

Proof. Let $u(\omega, s) = X \otimes T$, where X and T are functions of ω and s, respectively. Then, $u^2 = [X \otimes T][X \otimes T] = X^2 \otimes T^2$, $u_\omega^{(\alpha)} = X^{(\alpha)} \otimes T$, $u_s^{(\alpha)} = X \otimes T^{(\alpha)}$ and $u_\omega^{(3\alpha)} = X^{(3\alpha)} \otimes T$. This implies that Equation (5) becomes

$$\begin{aligned} X \otimes T^{(\alpha)} + \beta[X^2 \otimes T^2][X^{(\alpha)} \otimes T] + X^{(3\alpha)} \otimes T &= 0, \\ X \otimes T^{(\alpha)} + \beta X^2 X^{(\alpha)} \otimes T^3 + X^{(3\alpha)} \otimes T &= 0, \end{aligned}$$

so,

$$\beta X^2 X^{(\alpha)} \otimes T^3 + X^{(3\alpha)} \otimes T = -X \otimes T^{(\alpha)}. \tag{6}$$

In this case, either $\beta X^2 X^{(\alpha)} = X^{(3\alpha)}$ or $T^3 = T$.
Case 1: If $\beta X^2 X^{(\alpha)} = X^{(3\alpha)}$, then substitute it in Equation (6) to obtain

$$\begin{aligned} X^{(3\alpha)} \otimes T^3 + X^{(3\alpha)} \otimes T &= -X \otimes T^{(\alpha)} \\ X^{(3\alpha)} \otimes [T^3 + T] &= X \otimes -T^{(\alpha)}, \end{aligned}$$

and therefore, $X^{(3\alpha)} = X$ and $T^3 + T = -T^{(\alpha)}$. This is a contradiction, since $X = X^{(3\alpha)} = \beta X^2 X^{(\alpha)}$ implies $X = 0$ or $X = \beta X^2 X^{(\alpha)}$.
(i) $X = 0 \Longrightarrow X^{(\alpha)} = X^{(3\alpha)} = 0$, which is a zero solution.
(ii) $X = \beta X^2 X^{(\alpha)} \Longrightarrow 1 = \beta X X^{(\alpha)} \Longrightarrow X = \sqrt{\frac{2}{\alpha\beta}}\omega^{\frac{\alpha}{2}} \neq X^{(3\alpha)}$, a contradiction.
Case 2: If $T^3 = T$, then $T = 0$ and $T^2 = 1$.
(i) If $T(s) = 0$, then it is a zero solution (trivial solution).
(ii) If $T^2 = 1 \Rightarrow T(s) = \pm 1$ and $T^{(\alpha)}(s) = 0$, then Equation (6) becomes

$$\begin{aligned} \beta X^2 X^{(\alpha)} \otimes T + X^{(3\alpha)} \otimes T &= -X \otimes T^{(\alpha)}, \\ [\beta X^2 X^{(\alpha)} + X^{(3\alpha)}] \otimes T &= 0, \end{aligned}$$

and since $T(s) \neq 0$, we have $\beta X^2 X^{(\alpha)} + X^{(3\alpha)} = 0$, and hence

$$\begin{aligned} X^{(3\alpha)} &= -\beta X^2 X^{(\alpha)}, \\ \int X^{(3\alpha)} d\omega^\alpha &= -\beta \int X^2 X^{(\alpha)} d_\alpha \omega, \\ X^{(2\alpha)} &= \frac{-\beta}{3} X^3. \end{aligned}$$

Set $\eta = \frac{-\beta}{3}$, then $X^{(2\alpha)} = \eta X^3$. Put $X^{(2\alpha)} = y^{(\alpha)}$ by chain rule we have $y^{(\alpha)} = y'(X) d_\alpha X = y' y$, so

$$\begin{aligned} y^{(\alpha)} &= y' y = \eta X^3, \\ \int y dy &= \int \eta X^3 dX, \\ \frac{y^2}{2} &= \frac{\eta}{4} X^4, \\ y &= \sqrt{\frac{\eta}{2}} X^2. \end{aligned}$$

Then,

$$\begin{aligned} X^{(\alpha)} &= \sqrt{\frac{\eta}{2}}X^2, \\ \frac{d^\alpha X}{d\omega^\alpha} &= \sqrt{\frac{\eta}{2}}X^2, \\ X(\omega) &= \frac{-\alpha}{\sqrt{\frac{\eta}{2}}}\omega^{-\alpha}. \end{aligned}$$

To verify (6), set $u = X \otimes T$, $u_\omega^{(\alpha)} = X^{(\alpha)} \otimes T$, $u_s^{(\alpha)} = X \otimes T^{(\alpha)}$ and $u_\omega^{(3\alpha)} = X^{(3\alpha)} \otimes T$, where $X(\omega) = \frac{-\alpha}{\sqrt{\frac{\eta}{2}}}\omega^{-\alpha}$, then

$$\begin{gathered} X^{(\alpha)}(\omega) = \tfrac{\alpha^2}{\sqrt{\frac{\eta}{2}}}\omega^{-2\alpha} = \sqrt{\tfrac{\eta}{2}}X^2, \\ X^{(2\alpha)}(\omega) = \sqrt{\tfrac{\eta}{2}}X^{2+\alpha}, \text{ and} \\ X^{(3\alpha)}(\omega) = -\beta X^2 X^{(\alpha)}, \end{gathered}$$

and $T(s) = \pm 1 = T^3(s), T^{(\alpha)}(s) = 0$, then $X \otimes T^{(\alpha)} = \beta X^2 X^{(\alpha)} = 0$.
Now,

$$\begin{aligned} X \otimes T^{(\alpha)} + \beta X^2 X^{(\alpha)} \otimes T^3 + X^{(3\alpha)} \otimes T &= \beta X^2 X^{(\alpha)} \otimes T^3 + X^{(3\alpha)} \otimes T, \\ &= [\beta X^2 X^{(\alpha)} + X^{(3\alpha)}] \otimes T, \\ &= \beta X^2 X^{(\alpha)} + -\beta X^2 X^{(\alpha)} \\ &= 0 \\ &= \beta X^2 X^{(\alpha)} \\ &= X \otimes T^{(\alpha)}. \end{aligned}$$

Therefore, $u = X \otimes T$ is a solution of (5), where $X(\omega) = \frac{-\alpha}{\sqrt{\frac{\eta}{2}}}\omega^{-\alpha}$ and $T(s) = \pm 1$. □

3. A Further Result

In this section, we give an atomic solution of a conformable fractional version of a second order Gardner's equation type of the form

$$\frac{\partial u}{\partial s} + \lambda u \frac{\partial u}{\partial \omega} + \frac{\partial^2 u}{\partial \omega^2} = 0.$$

Theorem 3. *Let $u \in C(I \times I)$, where $I = [0,1]$ or $[0,\infty)$ with values in the Banach space A. If u is an $\alpha-$differentiable in some $\alpha \in (0,1]$, then the fractional partial diffrential equation*

$$u_s^{(\alpha)} + \lambda u u_\omega^{(\alpha)} + u_\omega^{(2\alpha)} = 0, \tag{7}$$

has an atomic solution.

Proof. Let $u(\omega, s) = X \otimes T$, where X is a function of ω and T is a function of s. Then

$$u_\omega^{(\alpha)} = X^{(\alpha)} \otimes T,\ u_s^{(\alpha)} = X \otimes T^{(\alpha)} \text{and } u_\omega^{(2\alpha)} = X^{(2\alpha)} \otimes T.$$

This implies that Equation (7) becomes

$$X \otimes T^{(\alpha)} + \lambda [X \otimes T][X^{(\alpha)} \otimes T] = -X^{(2\alpha)} \otimes T,$$

then we have

$$X \otimes T^{(\alpha)} + \lambda X X^{(\alpha)} \otimes T^2 = -X^{(2\alpha)} \otimes T. \tag{8}$$

This is the form where the sum of two atoms equal an atom. Hence, by Lemma 1 either $X = \lambda X X^{\alpha}$ or $T^{\alpha} = T^2$.
Case 1: If $T^{(\alpha)} = T^2$, then

$$\begin{aligned} T^{-2}T^{\alpha} &= 1, \\ \int T^{-2} d_{\alpha} T &= \int d_{\alpha} s, \\ \frac{-1}{T} &= (\frac{1}{\alpha}) s^{\alpha}, \\ T(s) &= -\alpha s^{-\alpha}. \end{aligned}$$

Hence,

$$\begin{aligned} X \otimes T^{(\alpha)} + \lambda X X^{(\alpha)} \otimes T^2 &= -X^{(2\alpha)} \otimes T, \\ [X + \lambda X X^{(\alpha)}] \otimes T^2 &= -X^{(2\alpha)} \otimes T. \end{aligned}$$

Therefore, $T^2 = T$ and $X + \lambda X X^{(\alpha)} = -X^{(2\alpha)}$. This is a contradiction since $T = -\alpha s^{-\alpha} \neq \alpha s^{-2\alpha} = T^2$.
Case 2: If $X = \lambda X X^{(\alpha)}$, then $X - \lambda X X^{(\alpha)} = 0$, and so $X[1 - \lambda X^{(\alpha)}] = 0$, which implies $X = 0$ or $1 - \lambda X^{(\alpha)} = 0$.

(i) If $X = 0$, then it is a zero solution.

(ii) if $1 - \lambda X^{(\alpha)} = 0$, then $X^{(\alpha)} = \frac{1}{\lambda} \Longrightarrow$

$$\begin{aligned} \frac{d^{\alpha} X}{d\omega^{\alpha}} &= \frac{1}{\lambda}, \\ X &= \int \frac{1}{\lambda} d_{\alpha} \omega, \\ X(\omega) &= \frac{1}{\alpha\lambda} \omega^{\alpha}. \end{aligned}$$

Now, we have $X(\omega) = \frac{1}{\alpha\lambda}\omega^{\alpha}$, $X^{(\alpha)}(\omega) = \frac{1}{\lambda}$, $X^{(2\alpha)}(\omega) = 0$.
Hence,

$$\begin{aligned} X \otimes T^{(\alpha)} + \lambda X X^{(\alpha)} \otimes T^2 &= -X^{(2\alpha)} \otimes T, \\ X \otimes T^{(\alpha)} + X \otimes T^2 &= 0 \otimes T, \\ X \otimes [T^{(\alpha)} + T^2] &= 0. \end{aligned}$$

Since here $X \neq 0$, we have $T^{\alpha} + T^2 = 0$ which implies

$$\begin{aligned} T^{\alpha} &= -T^2, \\ T^{-2}T^{\alpha} &= -1, \\ \int T^{-2} d_{\alpha} T &= \int -1 d_{\alpha} s, \\ -T^{-1} &= (\frac{-1}{\alpha}) s^{\alpha}, \\ T(s) &= \alpha s^{-\alpha}. \end{aligned}$$

To verify (8), set $u = X \otimes T$, $u_{\omega}^{(\alpha)} = X^{(\alpha)} \otimes T$, $u_s^{(\alpha)} = X \otimes T^{(\alpha)}$and $u_{\omega}^{(2\alpha)} = X^{(2\alpha)} \otimes T$, where

$$X(\omega) = \tfrac{1}{\alpha\lambda}\omega^{\alpha},$$
$$X^{(\alpha)}(\omega) = \tfrac{1}{\lambda}, \lambda X X^{(\alpha)} = \lambda \tfrac{1}{\alpha\lambda}\omega^{\alpha} \tfrac{1}{\lambda} = \tfrac{1}{\alpha\lambda}\omega^{\alpha} = X,$$
$$X^{(2\alpha)}(\omega) = 0,$$

and

$$T(s) = \alpha s^{-\alpha}, T^{(\alpha)}(s) = -\alpha^2 s^{-2\alpha} = -T^2(s).$$

Then, $X^{(2\alpha)} \otimes T = 0$,

$$\begin{aligned} X \otimes T^{(\alpha)} + \lambda X X^{(\alpha)} \otimes -T^{(\alpha)} &= X \otimes T^{(\alpha)} + \lambda X X^{(\alpha)} \otimes -T^{(\alpha)} \\ &= X \otimes T^{(\alpha)} + X \otimes -T^{(\alpha)} \\ &= X \otimes T^{(\alpha)} - X \otimes T^{(\alpha)} \\ &= 0 \\ &= X^{(2\alpha)} \otimes T. \end{aligned}$$

This implies that $u = X \otimes T$ is an atomic solution of equation (7), where $X(\omega) = \frac{1}{\alpha\lambda}\omega^{\alpha}$ and $T(s) = \alpha s^{-\alpha}$. $\square$

4. Conclusions

In this paper, we presented exact solutions to some certain fractional differential equations, specifically, the well-known second and third order Gardner's equations of fractional type using a tensor product technique, a simple and new method to solve such non-linear equations that gives an exact and accurate solution of well-known problems appear in many applications in science and engineering rather than other methods that are known to be approximate methods and include some kind of error estimate or some kind of conclusion and computational difficulties.

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