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On Laplacian Energy of r-Uniform Hypergraphs

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Abstract: The matrix representations of hypergraphs have been defined via hypermatrices initially. In recent studies, the Laplacian matrix of hypergraphs, a generalization of the Laplacian matrix, has been introduced. In this article, based on this definition, we derive bounds depending pair-degree, maximum degree, and the first Zagreb index for the greatest Laplacian eigenvalue and Laplacian energy of r-uniform hypergraphs and r-uniform regular hypergraphs. As a result of these bounds, Nordhaus–Gaddum type bounds are obtained for the Laplacian energy.

Keywords: hypergraph; Laplacian matrix; graph energy

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1. Introduction and Preliminaries

Graph theory is based on the connections and arrangements of objects. Many complex structures, resembling a network in real life, are represented by hypergraphs, which are a generalization of graphs. Hypergraph theory has applications in chemistry and physics (see [1,2]).

Symmetry is a kind of invariant or a feature that a mathematical object remains the same under some operations or transformations. However, symmetry is a significant feature in hypergraph theory, especially in uniform hypergraph theory. For a recent study on the symmetric function-Lagrangian of linear 3-uniform hypergraphs, see [3].

Let $H = (V, E)$ be a hypergraph composed of a vertex set V and a hyperedge set E of cardinality n and m, respectively. Letting $\mathcal{P}(V)$ be the power set of V, then $E \subset \mathcal{P}(V) \setminus \{\emptyset\}$. Therefore, a hyperedge can connect multiple vertices. The vertices u and v are adjacent if there is a hyperedge that includes both u and v, represented by $u \sim v$. The number of hyperedges containing i is called its degree, d_i , and Δ, δ stands for the maximum and minimum vertex degrees. If $d_i = d$ for all $i \in V$, then H is called d-regular. H is said to be simple if all edges are distinct. If $|V|$ is finite, then H is called finite. Through this article, all hypergraphs are simple, finite, and connected.

If each hyperedge includes precisely r vertices, then H is called r-uniform hypergraph or r-graph. H will be the (ordinary) graph when $r = 2$. For any $u, v \in V$ $(u \neq v)$ if a hyperedge sequence of $e_1, e_2, \ldots, e_m$ exists such that $u \in e_1$, $v \in e_m$ and $e_k \cap e_{k+1} \neq \emptyset$ for all k $(1 \leq k \leq m-1)$; then, H is called connected. The complement $\overline{H}$ of the r-graph is defined to be an r-graph with $V(H) = V(\overline{H})$ and an $r-$subset of $V(H)$ is an edge of $\overline{H}$ iff it is not an edge of H.

Topological indices of graphs, which have a wide application area in chemical graph theory, are classified in two ways: degree-based and distance-based. The first Zagreb index of a graph G is defined as $Zg = Zg(G) = \sum_{i \in V} d_i^2$ in [4], for properties of the degree and distance-based indices which have been recently defined; see also [5,6].

Spectral graph theory is a remarkable theory that investigates the relationship between the eigenvalue features of the graph matrices and graph structure. Therefore, determining the characteristic polynomials of graph matrices is crucial. In [7], the authors establish criteria for a graph to be symmetric via the characteristic polynomials of block circulant matrices.

The matrices of hypergraphs are first studied via hypermatrices which are multidimensional arrays (see [8,9]). Recently, the matrix representations of hypergraphs have been defined. The adjacency matrix $\mathcal{A}(H) = \left[(\mathcal{A})_{ij}\right]$ of H, which is introduced in [10] as

$$(\mathcal{A})_{ij} = \begin{cases} \sum\limits_{e \in E_{ij}} \frac{1}{|e|-1} & \text{if } i \sim j \\ 0 & \text{otherwise,} \end{cases}$$

where $E_{ij} = \{e \in E : i, j \in e\}$. The number of hyperedges which include both of the vertices i, j is called the pair-degree, d_{ij}, of i, j. Thus, $d_{ij} = |E_{ij}|$.

The Laplacian matrix $\mathcal{L} = \mathcal{L}(H)$ is known as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = diag(d_1, d_2, \ldots, d_n)$ and has eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} > \rho_n = 0$. Notice that $\mathcal{L}$ is symmetric and positive semidefinite. The smallest eigenvalue of $\mathcal{L}$ is 0 with the corresponding eigenvector $(1, 1, \ldots, 1)^t$.

The ij-th entry of $\mathcal{L} = \mathcal{L}(H)$ is

$$\begin{cases} d_i & \text{if } i = j \\ -\sum\limits_{e \in E_{ij}} \frac{1}{|e|-1} & \text{if } i \sim j \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathcal{L} = \mathcal{L}(H)$ is a generalization of the Laplacian matrix of a graph. Clearly, the above definition coincides with the signless Laplacian matrix $\mathcal{Q} = \mathcal{Q}(H)$ whenever only its ij-th entry is positive, i.e., $\sum\limits_{e \in E_{ij}} \frac{1}{|e|-1}$ if $i \sim j$. Let $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$; $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ be the eigenvalues of $\mathcal{Q}$ and $\mathcal{A}$, respectively.

The adjacency, Laplacian, signless Laplacian matrices of hypergraphs and their energies have been introduced recently ([10–12]). Some of their spectral features and bounds for their eigenvalues are analyzed in the relevant references. It is naturally a matter of research for researchers to ask which results in spectral graph theory can be generalized to hypergraphs. It is shown that many of these results hold for hypergraph matrices.

The energy $E(H)$ of H is defined as

$$E(H) = \sum_{i=1}^{n} |\nu_i|,$$

see [11]. The incidence and signless Laplacian energy of uniform hypergraphs are studied in [13]. The Laplacian energy $EL(H)$ of a r-graph with n vertices and m hyperedges can be defined as

$$EL(H) = \sum_{i=1}^{n} \left| \rho_i - \frac{rm}{n} \right|, \tag{1}$$

note that $tr(\mathcal{L}(H)) = rm$.

In the present paper, we essentially deal with the bounds for the greatest Laplacian eigenvalue and Laplacian energy of r-uniform hypergraphs. Moreover, we derive Nordhaus–Gaddum type inequalities for the Laplacian energy by means of these bounds. Now, we can begin with the essential lemmas that will be used.

Lemma 1 ([14])**.** *Let H be a r-graph. Then,*

(i) $\sum\limits_{i=1}^{n} \rho_i = \sum\limits_{i \in V} d_i = rm,$

(ii) $\sum\limits_{i=1}^{n} \rho_i^2 = Zg + \sum\limits_{i,j \in V} \frac{d_{ij}^2}{(r-1)^2},$

(*iii*) $\rho_1 \geq \Delta + \frac{1}{r-1}$,
(*iv*) $\rho_1 \leq \theta_1$.

Lemma 2 ([15]). *Let $B = (b_{ij}) \in \mathbb{C}^{n\times n}$ and $E(B) = \{(i,j) : b_{ij} \neq 0, 1 \leq i \neq j \leq n\}$. If B is irreducible, then its each eigenvalue is included in the region*

$$\Lambda(B) = \bigcup_{(i,j)\in E(B)} \{z \in \mathbb{C} : |z - b_{ii}||z - b_{jj}| \leq s_i\acute{} s_j\acute{}\},$$

where $s_i\acute{}(B) = \sum_{i\neq j} |b_{ij}|$.

Lemma 3 ([16]). *Let $B = (B_{ij}) \in \mathbb{C}^{n\times n}$ with $b_{ii} > 0$ $(1 \leq i \leq n)$. Then, each oval region $Y_{ij} = \{z \in \mathbb{C} : |z - b_{ii}||z - b_{jj}| \leq s_i\acute{} s_j\acute{}, i \neq j\}$ of B is symmetrical about x axes. Moreover, the point of the intersection of the boundary of Y_{ij} and x axes at the most right side is $p_{ij} = (t(i,j), 0)$, where $t(i,j) = \frac{b_{ii}+b_{jj}+\sqrt{(b_{ii}-b_{jj})^2+4s_i\acute{} s_j\acute{}}}{2}$.*

Lemma 4 ([17]). *Let H be a graph and p be any polynomial, $s_i(p(Q))$ is the row sum of $p(Q)$ corresponding to the vertex $i \in V$, then*

$$\min_{i\in V} s_i(p(Q)) \leq p(\theta_1) \leq \max_{i\in V} s_i(p(Q)), \tag{2}$$

where Q is the signless Laplacian matrix.

2. Main Results

In this section, we first establish new bounds on ρ_1. Consequently, we present bounds on the Laplacian energy of r-graphs.

The following theorem presents an upper bound on ρ_1, consisting of degree and pair-degree. Let $s_i(\mathcal{R})$ denote the i-th row sum of any matrix $\mathcal{R}$.

Theorem 1. *Let H be an r-graph. Then,*

$$\rho_1 \leq \sqrt{2}\max_{i\in V}\left\{d_i^2 + \frac{1}{r-1}\sum_{j:\, j\sim i} d_{ij}d_j\right\}. \tag{3}$$

Proof. As $s_i(\mathcal{A}) = \frac{1}{r-1}\sum_{j:\, j\sim i} d_{ij}$ and $s_i(\mathcal{A}^2) = \frac{1}{r-1}\sum_{j:\, j\sim i} d_{ij}d_j$ and $s_i(Q) = 2d_i$, $s_i(\mathcal{A}\mathcal{D}) = s_i(\mathcal{A}^2) = \frac{1}{r-1}\sum_{j:\, j\sim i} d_{ij}d_j$ for any $i \in V$, we obtain

$$\begin{aligned} s_i\left(Q^2\right) &= s_i(\mathcal{D}(\mathcal{D}+\mathcal{A})) + s_i(\mathcal{D}\mathcal{A}+\mathcal{A}^2) \\ &= d_i s_i(Q) + 2s_i(\mathcal{D}\mathcal{A}) \\ &= 2d_i^2 + \frac{2}{r-1}\sum_{j:\, j\sim i} d_{ij}d_j. \end{aligned}$$

Using Lemmas 1 and 4 leads to

$$\rho_1 \leq \theta_1 \leq \sqrt{2}\max_{i\in V}\left\{d_i^2 + \frac{1}{r-1}\sum_{j:\, j\sim i} d_{ij}d_j\right\},$$

which completes the proof. □

Now, we can express an upper bound including a new parameter m_i, the average of the degrees of the vertices adjacent to i.

Theorem 2. *Let H be a connected r-graph. Then,*

$$\rho_1 \leq \frac{1}{2}\max_{i\sim j}\left\{d_i + d_j + \sqrt{\left(d_i - d_j\right)^2 + \frac{4}{(r-1)^2}\sqrt{m_i \sum_{j:j\sim i} d_{ij}^2}\sqrt{m_j \sum_{k:k\sim j} d_{jk}^2}}\right\}. \tag{4}$$

Proof. Consider the matrix $\mathcal{B} = \left(b_{ij}\right)$ with $\mathcal{B} = \mathcal{D}^{-1/2}\,\mathcal{Q}\mathcal{D}^{1/2}$, where $\mathcal{D}^{1/2} = diag\left(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n}\right)$. Then,

$$b_{ij} = \begin{cases} d_i & \text{if } i = j \\ \frac{\sqrt{d_j}d_{ij}}{(r-1)\sqrt{d_i}} & \text{if } i \sim j \\ 0 & \text{elsewhere.} \end{cases}$$

Let $s_i'(\mathcal{B}) = s_i(\mathcal{B}) - d_i$. Applying the Cauchy–Schwarz inequality yields

$$\begin{aligned} [s_i'(\mathcal{B})]^2 = &\left(\sum_{j:i\sim j} \frac{\sqrt{d_j}d_{ij}}{(r-1)\sqrt{d_i}}\right)^2 \leq \sum_{j:i\sim j}\frac{d_j}{d_i}\sum_{j:i\sim j}\frac{d_{ij}^2}{(r-1)^2} \\ =&m_i \sum_{j:i\sim j}\frac{d_{ij}^2}{(r-1)^2} \\ =&\frac{m_i}{(r-1)^2}\sum_{j:i\sim j} d_{ij}^2, \end{aligned}$$

where $m_i = \frac{1}{d_i}\sum_{j:i\sim j} d_j$ is the average of the degrees of the vertices adjacent to i. As $\mathcal{B}$ is an irreducible, nonnegative matrix, from Lemma 2, there exists at least ij edge such that the largest eigenvalue $\nu = \nu(\mathcal{B})$ is included in the following oval region Y_{ij}, that is,

$$\begin{aligned} |\nu - d_i|\left|\nu - d_j\right| \leq& s_i'(\mathcal{B})s_j'(\mathcal{B}) \\ \leq& \frac{1}{(r-1)^2}\sqrt{m_i \sum_{j:i\sim j} d_{ij}^2}\sqrt{m_j \sum_{k:k\sim j} d_{jk}^2}, \end{aligned}$$

is verified. From Lemma 1

$$\theta_1 \geq \rho_1 \geq \Delta + \frac{1}{r-1}.$$

Then, $\nu = \theta_1 \geq \max\{d_i : i \in V\} + \frac{1}{r-1} > \max\left\{d_i, d_j\right\}$. Thus, solving (5) by using Lemma 3 yields

$$\rho_1 \leq \frac{d_i + d_j + \sqrt{\left(d_i - d_j\right)^2 + \frac{4}{(r-1)^2}\sqrt{m_i \sum_{j:j\sim i} d_{ij}^2}\sqrt{m_j \sum_{k:k\sim j} d_{jk}^2}}}{2}.$$

As $\rho_1 \leq \theta_1 = \nu(\mathcal{B})$, the proof is completed. □

Eventually, one can obtain the following bound on ρ_1 for d-regular r-graphs.

Corollary 1. *Let H be a connected d-regular r-graph. Then,*

$$\rho_1 \leq \frac{1}{2}\max_{i\sim j}\left\{2d + \sqrt{\frac{4}{(r-1)^2}\sqrt{\sum_{j:j\sim i} d_{ij}^2 \sum_{k:k\sim j} d_{jk}^2}}\right\}. \tag{5}$$

Proof. If H is d-regular, then $m_i = 1$ and $d_i = d$ for all $i \in V$. Considering these facts in (4) leads (5). □

The lower bound can be presented by using the Rayleigh quotient for $\mathcal{L}^2$.

Theorem 3. *Let H be a connected r-graph. Then,*

$$\rho_1 \geq \left[\frac{1}{n} \sum_{i=1}^{n} \left(d_i^2 + \sum_{i,j \in V} \frac{d_{ij}^2}{(r-1)^2} - \frac{(d_i + d_j) d_{ij}}{r-1} + \frac{1}{(r-1)^2} \sum_{i,j,k \in V} d_{ik} d_{kj} \right)^2 \right]^{1/4}. \tag{6}$$

Proof. By using the Rayleigh quotient for a real symmetric matrix, $\mathcal{C}$ with $x = (1, 1, \ldots, 1)^t$ leads to

$$\rho_1^2 \geq \frac{x^t \mathcal{C}^2 x}{n} = \frac{(x\mathcal{C})^t (x\mathcal{C})}{n} = \sum_{i=1}^{n} s_i^2(\mathcal{C}) \Big/ n. \tag{7}$$

Let us apply (7) to $\mathcal{C} = \mathcal{L}^2$. We have $\left(\mathcal{L}^2\right)_{ii} = d_i^2 + \sum_{i,j \in V} \frac{d_{ij}^2}{(r-1)^2}$, and for $i \neq j$

$$\begin{aligned}
\left(\mathcal{L}^2\right)_{ij} &= \sum_{j=1}^{n} l_{ij} l_{ji} = l_{ii} l_{ij} + l_{ij} l_{jj} + \sum_{k \in V} l_{ik} l_{kj} \\
&= (d_i + d_j) l_{ij} + \sum_{k \in V} \left(\frac{-d_{ik}}{r-1} \right) \left(\frac{-d_{kj}}{r-1} \right) \\
&= -\, (d_i + d_j) \left(\frac{d_{ij}}{r-1} \right) + \sum_{k \in V} \left(\frac{d_{ik}}{r-1} \right) \left(\frac{d_{kj}}{r-1} \right) \\
&= -\, \frac{(d_i + d_j) d_{ij}}{r-1} + \frac{1}{(r-1)^2} \sum_{k \in V} d_{ik} d_{kj}.
\end{aligned}$$

Therefore, $s_i(\mathcal{L}^2) = d_i^2 + \sum_{i,j \in V} \frac{d_{ij}^2}{(r-1)^2} - \frac{(d_i + d_j) d_{ij}}{r-1} + \frac{1}{(r-1)^2} \sum_{k \in V} d_{ik} d_{kj}$. Applying (7) to $\mathcal{C} = \mathcal{L}^2$ yields

$$\rho_1 \geq \left[\frac{1}{n} \sum_{i=1}^{n} \left(d_i^2 + \sum_{i,j \in V} \frac{d_{ij}^2}{(r-1)^2} - \frac{(d_i + d_j) d_{ij}}{r-1} + \frac{1}{(r-1)^2} \sum_{k \in V} d_{ik} d_{kj} \right)^2 \right]^{1/4} - .$$

Thus, we obtain (6). □

Clearly, from (6), we achieve the bound (8) for regular r-graphs.

Corollary 2. *Let H be a connected d-regular r-graph. Then,*

$$\rho_1 \geq \left[\frac{1}{n} \sum_{i=1}^{n} \left(d^2 + \sum_{i,j \in V} \frac{d_{ij}^2}{(r-1)^2} - \frac{2d d_{ij}}{r-1} + \frac{1}{(r-1)^2} \sum_{i,j,k \in V} d_{ik} d_{kj} \right)^2 \right]^{1/4}. \tag{8}$$

We can now begin to be concerned with the bounds for the Laplacian energy of r-graphs. Let us give the following lemma.

Lemma 5. *Let α $(1 \leq \alpha \leq n-1)$ be the greatest integer such that $\rho_\alpha \geq \frac{rm}{n}$. Then,*

$$EL(H) = 2T_\alpha(H) - \frac{2\alpha r m}{n}, \tag{9}$$

where $T_\alpha = T_\alpha(H) = \sum\limits_{i=1}^{\alpha} \rho_i$.

Proof. By Lemma 1, $\sum\limits_{i=1}^{n} \rho_i = rm$. Then, by (1), we have

$$\begin{aligned} EL(H) =& \sum_{i=1}^{\alpha}\left(\rho_i - \frac{rm}{n}\right) + \sum_{i=\alpha+1}^{n}\left(\frac{rm}{n} - \rho_i\right) \\ =& \sum_{i=1}^{\alpha}\rho_i - \frac{\alpha rm}{n} + \frac{(n-\alpha)rm}{n} - \left(\sum_{i=1}^{n}\rho_i - \sum_{i=1}^{\alpha}\rho_i\right) \\ =& 2T_\alpha - \frac{2\alpha rm}{n}. \end{aligned}$$

□

Using (1), we obtain a lower bound for $EL(H)$ below.

Theorem 4. *Let H be a connected r-graph. Then,*

$$EL(H) \geq 2\left(\Delta + \frac{1}{r-1} - \frac{rm}{n}\right).$$

Proof. $EL(H) = \max\limits_{1\leq i\leq n-1}\left\{2T_i - \frac{2rmi}{n}\right\}$, which can be concluded from [18] (see Theorem 3.1). Then, using Lemma 1 yields

$$\begin{aligned} EL(H) =& \max_{1\leq i\leq n-1}\left\{2T_i - \frac{2rmi}{n}\right\} \geq 2T_1 - \frac{2rm}{n} \\ =& 2\left(\rho_1 - \frac{rm}{n}\right) \\ \geq& 2\left(\Delta + \frac{1}{r-1} - \frac{rm}{n}\right), \end{aligned}$$

which is the expected result. □

The Laplacian spread of a hypergraph is also defined to be $\rho_1 - \rho_{n-1}$. The following lemma expresses an upper bound for the Laplacian spread of r-graphs.

Lemma 6. *Let H be a connected r-graph of order $n(\geq 3)$. Then,*

$$\rho_1 - \rho_{n-1} \leq \sqrt{\frac{2}{n-1}}\sqrt{(n-1)\left(Zg + \sum_{i,j\in V}\frac{d_{ij}^2}{(r-1)^2}\right) - r^2m^2}. \tag{10}$$

Proof. An analogous proof can be followed from [19] (see Theorem 2.1), by considering Lemma 1. □

Theorem 5. *Let H be a connected r-graph of order $n(\geq 3)$. Then,*

$$EL(H) \geq \sqrt{\frac{2}{n-1}}\sqrt{(n-1)\left(Zg + \sum_{i,j\in V}\frac{d_{ij}^2}{(r-1)^2}\right) - r^2m^2}. \tag{11}$$

Proof. Let x_i $(1 \leq i \leq n)$ are real numbers provided that there are $m, M \in \mathbb{R}$ such that $-\infty < m \leq x_i \leq M < +\infty$, for each $1 \leq i \leq n$. Then, for any nonnegative q_i, $(1 \leq i \leq n)$ satisfying $\sum_{i=1}^{n} q_i = 1$ implies

$$0 \leq \sum_{i=1}^{n} q_i x_i^2 - \left(\sum_{i=1}^{n} q_i x_i\right)^2 \leq \frac{1}{2}(M-m)\sum_{i=1}^{n} q_i \left| x_i - \sum_{i=1}^{n} q_i x_i \right|, \tag{12}$$

see [20]. Setting $q_i := \frac{1}{n-1}$, $x_i := \rho_i$ for $i = 1, 2, \ldots, n-1$ and $m := \rho_{n-1}$, $M := \rho_1$, (12) then becomes

$$\frac{1}{n-1}\sum_{i=1}^{n-1} \rho_i^2 - \frac{1}{(n-1)^2}\left(\sum_{i=1}^{n-1} \rho_i\right)^2 \leq \frac{\rho_1 - \rho_{n-1}}{2(n-1)}\sum_{i=1}^{n-1}\left| \rho_i - \frac{1}{n-1}\sum_{i=1}^{n-1} \rho_i \right|.$$

By Lemma 1,

$$\begin{aligned}(n-1)\left(Zg + \sum_{i,j\in V} \frac{d_{ij}^2}{(r-1)^2}\right) - r^2 m^2 \leq& \frac{(n-1)}{2}(\rho_1 - \rho_{n-1})\sum_{i=1}^{n-1}\left| \rho_i - \frac{rm}{n-1} \right| \\ \leq& \frac{(n-1)}{2}(\rho_1 - \rho_{n-1}) EL(H),\end{aligned}$$

as $\sum_{i=1}^{n-1} \left| \rho_i - \frac{rm}{n-1} \right| \leq \sum_{i=1}^{n} \left| \rho_i - \frac{rm}{n} \right|$. Then, regarding (10) leads to (11). □

In [21], the energy of a graph is redefined as the sum of the singular values $\sigma_i(A)$ $(1 \leq i \leq n)$ of its adjacency matrix A. Based on this definition, we can define the energy of a hypergraph H as

$$E(H) = \sum_{i=1}^{n} \sigma_i(\mathcal{A}). \tag{13}$$

Let $\mathcal{I}_n$ be the the identity matrix. As $\mathcal{L} - \frac{rm}{n}\mathcal{I}_n$ is a real symmetric matrix, from (1), the Laplacian energy of a r-graph can also be expressed in terms of singular values of the matrix $\mathcal{L} - \frac{rm}{n}\mathcal{I}_n$ as

$$EL(H) = \sum_{i=1}^{n} \sigma_i\left(\mathcal{L} - \frac{rm}{n}\mathcal{I}_n\right).$$

Lemma 7 ([22])**.** *Let $A, B \in \mathbb{C}^{n\times n}$ be symmetric matrices. Then,*

$$\sigma_i(A+B) \leq \sigma_i(A) + \sigma_i(B).$$

The following upper bound including the energy of a r-graph for $EL(H)$ can be given.

Theorem 6. *Let H be a r-graph. Then,*

$$EL(H) \leq E(H) + \sum_{i\in V}\left| d_i - \frac{rm}{n} \right|. \tag{14}$$

Proof. Clearly, $\mathcal{L} - \frac{rm}{n}\mathcal{I}_n = \mathcal{D} - \mathcal{A} - \frac{rm}{n}\mathcal{I}_n = (-\mathcal{A}) + \left(\mathcal{D} - \frac{rm}{n}\mathcal{I}_n\right)$. Applying Lemmas 7 and (13) leads to

$$\begin{aligned}EL(H) =& \sum_{i=1}^{n}\sigma_i\left[(-\mathcal{A})+\left(\mathcal{D}-\frac{rm}{n}\mathcal{I}_n\right)\right]\\ \leq& \sum_{i=1}^{n}\sigma_i(-\mathcal{A})+\sum_{i=1}^{n}\sigma_i\left(\mathcal{D}-\frac{rm}{n}\mathcal{I}_n\right)\\ =& \sum_{i=1}\sigma_i(\mathcal{A})+\sum_{i=1}^{n}\sigma_i\left(\mathcal{D}-\frac{rm}{n}\mathcal{I}_n\right)\\ =& E(H)+\sum_{i\in V}\left|d_i-\frac{rm}{n}\right|,\end{aligned}$$

which completes the proof. □

Consequently, from (14), we can establish an upper bound as follows:

Theorem 7. *Let H be a r-graph. Then,*

$$EL(H) \leq nr\Delta + \sqrt{nZg - r^2m^2}. \tag{15}$$

Proof. Using the Cauchy–Schwarz inequality and Lemma 1 gives

$$\begin{aligned}\sum_{i\in V}\left|d_i-\frac{rm}{n}\right| \leq& \sqrt{n\sum_{i\in V}\left(d_i-\frac{rm}{n}\right)^2}\\ \leq& \sqrt{n\sum_{i\in V}\left(d_i^2-2d_i\frac{rm}{n}+\frac{r^2m^2}{n^2}\right)}\\ =& \sqrt{n\left(Zg-\frac{2rm}{n}\sum_{i\in V}d_i+\sum_{i\in V}\frac{r^2m^2}{n^2}\right)}\\ =& \sqrt{n\left(Zg-\frac{2(rm)^2}{n}+\frac{r^2m^2}{n}\right)}\\ =& \sqrt{nZg-r^2m^2}.\end{aligned}$$

By (14),

$$EL(H) \leq E(H) + \sqrt{nZg - r^2m^2}. \tag{16}$$

We may have

$$E(H) = \sum_{i=1}^{n}|\nu_i| \leq \sum_{i=1}^{n}|\nu_1| = n|\nu_1|.$$

Thus,

$$E(H) \leq nr\Delta, \tag{17}$$

as $|\nu_1| \leq r\Delta$ (see Corollary 14, [23]). Putting (17) in (16) implies

$$EL(H) \leq nr\Delta + \sqrt{nZg - r^2m^2},$$

which is the desired result. □

Let us set $\varepsilon_i := \rho_i - \frac{rm}{n}$ in (1). Then, $EL(H)$ can also be expressed as follows:

$$EL(H) = \sum_{i=1}^{n}|\varepsilon_i|. \tag{18}$$

Lemma 8. *Let H be a r-graph. Then, ε_i verifies the following assertions:*

(i) $\sum\limits_{i=1}^{n}\varepsilon_i = 0$,

(ii) $\sum\limits_{i=1}^{n}\varepsilon_i^2 = Zg + \sum\limits_{i,j\in V}\frac{d_{ij}^2}{(r-1)^2} - \frac{r^2m^2}{n}$.

Proof. Clearly, by Lemma 1 we have

$$\sum_{i=1}^{n}\varepsilon_i = \sum_{i=1}^{n}\rho_i - \sum_{i=1}^{n}\frac{rm}{n} = rm - rm = 0,$$

$$\begin{aligned}\sum_{i=1}^{n}\varepsilon_i^2 =& \sum_{i=1}^{n}\left(\rho_i - \frac{rm}{n}\right)^2\\ =& \sum_{i=1}^{n}\rho_i^2 - \frac{2rm}{n}\sum_{i=1}^{n}\rho_i - \frac{r^2m^2}{n}\\ =& Zg + \sum_{i,j\in V}\frac{d_{ij}^2}{(r-1)^2} - \frac{r^2m^2}{n}.\end{aligned}$$

□

Several bounds may be derived by applying mathematical inequalities and considering Lemma 8. We obtain three lower bounds on $EL(H)$:

Theorem 8. *Let H be a r-graph. Then,*

$$EL(H) \geq \sqrt{Zg + \sum_{i,j\in V}\frac{d_{ij}^2}{(r-1)^2} - \frac{r^2m^2}{n}}.$$

Proof. Applying the Radon inequality in (18) leads

$$EL(H) = \sum_{i=1}^{n}|\varepsilon_i| = \sum_{i=1}^{n}\frac{|\varepsilon_i|^2}{|\varepsilon_i|} \geq \frac{\sum\limits_{i=1}^{n}|\varepsilon_i|^2}{\sum\limits_{i=1}^{n}|\varepsilon_i|},$$

using Lemma 8 yields the result. □

The following lower bound includes ρ_1, Zg, and pair-degree of vertices.

Theorem 9. *Let H be a r-graph. Then,*

$$EL(H) \geq \frac{2\left(Zg + \sum\limits_{i,j\in V}\frac{d_{ij}^2}{(r-1)^2} - \frac{r^2m^2}{n}\right)}{\rho_1}.$$

Proof. Let (r_i),(q_i) $(1 \leq i \leq n)$ be real numbers with $\sum\limits_{i=1}^{n}|r_i| = 1$ and $\sum\limits_{i=1}^{n}r_i = 0$. Then,

$$\left|\sum_{i=1}^{n}r_iq_i\right| \leq \frac{1}{2}\left(\max_{1\leq i\leq n}q_i - \min_{1\leq i\leq n}q_i\right), \tag{19}$$

holds ([24], p. 346). Setting $r_i = \sum\limits_{i=1}^{n} \frac{\varepsilon_i}{|\varepsilon_i|}$ and $q_i = \varepsilon_i$ in (19) implies

$$\begin{aligned}\left| \frac{\sum\limits_{i=1}^{n} \varepsilon_i^2}{\sum\limits_{i=1}^{n} |\varepsilon_i|} \right| \leq & \frac{1}{2}\left(\max_{1\leq i\leq n} \varepsilon_i - \min_{1\leq i\leq n} \varepsilon_i \right) \\ = & \frac{1}{2}(\rho_1 - rm - (0 - rm)),\end{aligned}$$

from Lemma 8 and (18), we obtain

$$EL(H) \geq \frac{2\left(Zg + \sum\limits_{i,j\in V} \frac{d_{ij}^2}{(r-1)^2} - \frac{r^2m^2}{n} \right)}{\rho_1}.$$

□

The below result can be given by using the previous theorem and (3).

Corollary 3. *Let H be a r-graph. Then,*

$$EL(H) \geq \frac{\sqrt{2}\left(Zg + \sum\limits_{i,j\in V} \frac{d_{ij}^2}{(r-1)^2} - \frac{r^2m^2}{n} \right)}{\max\limits_{i\in V}\left\{ d_i^2 + \frac{1}{r-1} \sum\limits_{j:\, j\sim i} d_{ij}d_j \right\}}.$$

Now, we establish Nordhaus–Gaddum type inequalities (see [25]) on the Laplacian energy of r-graphs, namely $EL(H) + EL(\overline{H})$:

Theorem 10. *Let H be a connected r-graph. Then,*

$$EL(H) + EL(\overline{H}) \geq 2\left(\binom{n-1}{r-1} + \Delta - \delta + \frac{2}{r-1} - \frac{r\binom{n}{r}}{n} \right).$$

Proof. From Theorem 4, we have

$$EL(H) + EL(\overline{H}) \geq 2\left(\Delta + \overline{\Delta} + \frac{2}{r-1} - \frac{r(m+\overline{m})}{n} \right).$$

As $\overline{m} = \binom{n}{r} - m$ and $\overline{\Delta} = \binom{n-1}{r-1} - \delta$ (see [14]), the proof is obvious. □

For d-regular r-graphs, we have $\Delta = \delta = d$. Therefore, we can present the result below.

Corollary 4. *Let H be a connected d-regular r-graph. Then,*

$$EL(H) + EL(\overline{H}) \geq 2\left(\binom{n-1}{r-1} + \frac{2}{r-1} - \frac{r\binom{n}{r}}{n} \right).$$

Finally, the following bound is obtained by utilizing Theorem 7.

Theorem 11. *Let H be a r-graph. Then,*

$$\begin{aligned}EL(H) + EL(\overline{H}) \leq & nr\left(\Delta + \binom{n-1}{r-1} - \delta \right) + \sqrt{nZg(H) - r^2m^2} \\ & + \sqrt{nZg(\overline{H}) - r^2\left[\binom{n}{r} - m\right]^2},\end{aligned}$$

where $Zg(\overline{H}) = n\binom{n-1}{r-1}^2 - 2rm\binom{n-1}{r-1} + Zg(H)$.

Proof. We have $\overline{d_i} = \binom{n-1}{r-1} - d_i$ (see [14]). By using (15), we obtain

$$\begin{aligned} EL(H) + EL(\overline{H}) \leq & nr(\Delta + \overline{\Delta}) + \sqrt{nZg(H) - r^2m^2} + \sqrt{nZg(\overline{H}) - r^2\overline{m}^2} \\ = & nr\left(\Delta + \binom{n-1}{r-1} - \delta\right) + \sqrt{nZg(H) - r^2m^2} \\ & + \sqrt{nZg(\overline{H}) - r^2\left[\binom{n}{r} - m\right]^2}. \end{aligned} \tag{20}$$

In addition, by Lemma 1,

$$\begin{aligned} Zg(\overline{H}) = & \sum_{i\in V}\left(\overline{d_i}\right)^2 = \sum_{i\in V}\left[\binom{n-1}{r-1} - d_i\right]^2 \\ = & \sum_{i\in V}\binom{n-1}{r-1}^2 - 2\binom{n-1}{r-1}\sum_{i\in V} d_i + \sum_{i\in V} d_i^2 \\ = & n\binom{n-1}{r-1}^2 - 2rm\binom{n-1}{r-1} + Zg(H). \end{aligned} \tag{21}$$

The proof is clear by writing (21) in (20). □

3. Conclusions

The definition of the (ordinary) graph energy [26] has led the authors to define other types of energy in time, and various studies have carried out in spectral graph theory. Initially, the hypergraph matrices have been defined via hypermatrices. The energies of hypergraphs started to work within the definition of matrix representations of hypergraphs in recent years. These new definitions have allowed many techniques and features in spectral graph theory to be investigated in terms of hypergraphs.

In this paper, we propose upper and lower bounds for the greatest Laplacian eigenvalue and Laplacian energy of the uniform hypergraphs and regular uniform hypergraphs, depending on many hypergraph invariants such as the degree, pair-degree, maximum degree, and the first Zagreb index.

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