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Computational Analysis for Fréchet Parameters of Life from Generalized Type-II Progressive Hybrid Censored Data with Applications in Physics and Engineering

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Abstract: Generalized progressive hybrid censored procedures are created to reduce test time and expenses. This paper investigates the issue of estimating the model parameters, reliability, and hazard rate functions of the Fréchet (Fr) distribution under generalized Type-II progressive hybrid censoring by making use of the Bayesian estimation and maximum likelihood methods. The appropriate estimated confidence intervals of unknown quantities are likewise built using the frequentist estimators' normal approximations. The Bayesian estimators are created using independent gamma conjugate priors under the symmetrical squared-error loss. The Bayesian estimators and the associated greatest posterior density intervals cannot be computed analytically since the joint likelihood function is obtained in complex form, but they may be assessed using Monte Carlo Markov chain (MCMC) techniques. Via extensive Monte Carlo simulations, the actual behavior of the proposed estimation methodologies is evaluated. Four optimality criteria are used to choose the best censoring scheme out of all the options. To demonstrate how the suggested approaches may be utilized in real scenarios, two real applications reflecting the thirty successive values of precipitation in Minneapolis–Saint Paul for the month of March as well as the number of vehicle fatalities for thirty-nine counties in South Carolina during 2012 are examined.

Keywords: Fréchet model; symmetric Bayes inference; MCMC techniques; maximum likelihood; reliability analysis; generalized Type-II progressive hybrid censoring

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1. Introduction

Reliability technology, as a measure of a system's capacity to properly perform its intended function under predetermined conditions for a specific period of time, is currently increasingly significant. In this regard, many research studies have been conducted, see for example, Chen et al. [1] and Luo et al. [2]. In the literature, progressive Type-II censoring (PC-T2) has received a lot of attention because it allows surviving subjects to be removed during an experiment at various stages other than the termination point, see Balakrishnan and Cramer [3]. To conduct this censoring, a researcher must first put n independent units into a test at time zero and determine the number of failures m and the progressive censoring $\underline{R} = (R_1, R_2, \ldots, R_m)$, where $n = \sum_{i=1}^{m} R_i + m$. At the moment of the first recorded failure (say $X_{1:m:n}$), the surviving units R_1 out of $n-1$ units are randomly chosen and removed from the test. Similarly, R_2 of $n - R_1 - 2$ are selected at random and removed from the test at the time of the second failure (say $X_{2:m:n}$) observed, and so on. All remaining survival units, $R_m = n - m - \sum_{j=1}^{m-1} R_j$, are withdrawn from the test at the moment of the mth failure (say $X_{m:m:n}$) observed, see Panahi [4]. The main disadvantage of this censoring is that it may take a longer time to complete the test when the experimental units are extremely trustworthy. To overcome this problem, the progressive Type-I hybrid censoring

(PHC-T1), which combines PC-T2 and traditional Type-I censoring, was presented by Kundu and Joarder [5]. However, PHC-T1 had the disadvantage that there are relatively few failures that may occur before time T, meaning that maximum likelihood estimators (MLEs) could not always be derived. To address this issue, Childs et al. [6] proposed the progressive Type-II hybrid censoring (PHC-T2) in which the experiment terminates at $T^* = \max\{X_{m:m:n}, T\}$, for details see Panahi [7]. On the other hand, to improve the efficiency of statistical inference, Ng et al. [8] proposed the adaptive progressively Type-II hybrid censoring, for further details see Panahi and Moradi [9].

Although the PHS-T2 guarantees an efficient number of observable failures, it may take a long time to collect the desired failures. Therefore, the generalized progressive Type-II hybrid censoring (GPHC-T2) was introduced by Lee et al. [10]. Assume that the two thresholds T_i, $i = 1, 2$ and the number m are preassigned such that $1 < m \leq n$ and $0 < T_1 < T_2$. The total number of failures up to periods T_1 and T_2 are shown as d_1 and d_2, respectively. Next, R_1 of $n-1$ are arbitrarily removed from the test at $X_{1:m:n}$; R_2 of $n - R_1 - 2$ are then removed at $X_{2:m:n}$, and so on. At $T^* = \max\{T_1, \min\{X_{m:m:n}, T_2\}\}$, the experiment is terminated and all remaining units are removed. If $X_{m:n} < T_1$, we continue to observe failures without any additional withdrawals up to time T_1 (Case-I); if $T_1 < X_{m:m:n} < T_2$, we end the test at $X_{m:m:n}$ (Case-II); otherwise, we end the test at time T_2 (Case-III). It is important to remember that the GPHC-T2 alters the PHC-T2 by ensuring that the test is finished at the designated time T_2. Thus, T_2 shows the maximum amount of time that the researcher is prepared to permit the experiment to run. As a result, the experimenter will see one of the following three data formats:

$$\{\underline{X}, \underline{R}\} = \begin{cases} \{(X_{1:m:n}, R_1), \ldots, (X_{m-1:m:n}, R_{m-1}), (X_{m:m:n}, 0), \ldots, (X_{d_1:n}, 0)\}; & \text{Case-I,} \\ \{(X_{1:m:n}, R_1), \ldots, (X_{d_1:n}, R_{d_1}), \ldots, (X_{m-1:m:n}, R_{m-1}), (X_{m:m:n}, R_m)\}; & \text{Case-II,} \\ \{(X_{1:m:n}, R_1), \ldots, (X_{d_1:n}, R_{d_1}), \ldots, (X_{d_2-1:n}, R_{d_2-1}), (X_{d_2:n}, R_{d_2})\}; & \text{Case-III.} \end{cases}$$

Assume that the variables $\underline{X}, \underline{R}$ stand for the respective lives in a distribution with cumulative distribution function (CDF) $F(\cdot)$ and probability density function (PDF) $f(\cdot)$. This leads to the following expression for the GPHC-T2 likelihood function as follows:

$$L_\rho(\theta|\underline{X}) = C_\rho \mathcal{R}_\rho(T_\tau; \theta) \prod_{j=1}^{D_\rho} f(x_{j:m:n}; \theta) \left[1 - F(x_{j:m:n}; \theta)\right]^{R_j}, \tag{1}$$

where Case-I, Case-II, and Case-III denoted by $\rho = 1, 2, 3$, correspondingly, $\tau = 1, 2$, and $\mathcal{R}_\rho(\cdot)$ is a composite form of reliability functions. From (1), the GPHC-T2 notations are listed in Table 1. Moreover, from (1), different censoring plans can be obtained as special cases, namely:

- PHC-T1 if $T_1 \to 0$.
- PHC-T2 if $T_2 \to \infty$.
- Hybrid-T1 if $T_1 \to 0$, $R_j = 0$, $j = 1, 2, \ldots, m-1$, $R_m = n - m$.
- Hybrid-T2 if $T_2 \to \infty$, $R_j = 0$, $j = 1, 2, \ldots, m-1$, $R_m = n - m$.
- Type-I censoring if $T_1 = 0$, $m = 1$, $R_j = 0$, $j = 1, 2, \ldots, m-1$, $R_m = n - m$.
- Type-II censoring if $T_1 = 0$, $T_2 \to \infty$, $R_j = 0$, $j = 1, 2, \ldots, m-1$, $R_m = n - m$.

Table 1. The GPHC-T2 notations.

ρ	C_ρ	D_ρ	$\mathcal{R}_\rho(T_\tau; \theta)$	$R^*_{d_\tau+1}$
1	$\Pi_{j=1}^{d_1} \sum_{i=j}^{m} (R_i + 1)$	d_1	$[1 - F(T_1)]^{R^*_{d_1+1}}$	$n - d_1 - \sum_{i=1}^{m-1} R_i$
2	$\Pi_{j=1}^{m} \sum_{i=j}^{m} (R_i + 1)$	m	1	0
3	$\Pi_{j=1}^{d_2} \sum_{i=j}^{m} (R_i + 1)$	d_2	$[1 - F(T_2)]^{R^*_{d_2+1}}$	$n - d_2 - \sum_{i=1}^{d_2} R_i$

On the basis of GPHC-T2, other research has also been carried out. For instance, the maximum likelihood and Bayes estimators of the Weibull parameters were produced by Ashour and Elshahhat [11]. The prediction problem of failure times from the Burr-XII distribution was studied by Ateya and Mohammed [12]. Seo [13] developed an objective Bayesian analysis with limited information about the Weibull distribution. The competing risks from exponential data were addressed by Cho and Lee [14], and more recently, Nagy et al. [15] looked at both the point and interval estimates of the Burr-XII parameters, and Wang et al. [16] addressed the estimation problem of the Kumaraswamy parameters using classical and Bayesian procedures.

The inverse Weibull (or Gumbel Type-II) distribution, commonly known as the two-parameter Fréchet (Fr) distribution, is well suited for data modeling with decreasing and upside-down bathtub hazard rates. To illustrate many environmental phenomena, including earthquakes, floods, wind speeds, rainfall, breakdown of insulating fluid, sea waves, etc., it has been widely employed. This model was originally proposed by Fréchet [17], and Kotz and Nadarajah [18] later discussed it. Suppose that X is a lifetime random variable that follows the Fr distribution, which is represented by the notation $\mathrm{Fr}(\delta, \theta)$, where $\delta > 0$ is the scale parameter and $\theta > 0$ is the shape parameter. Its PDF, CDF, and hazard rate function (HRF), denoted by $f(\cdot)$, $F(\cdot)$, and $h(\cdot)$, are provided by

$$f(x; \delta, \theta) = \delta\theta x^{-(\theta+1)} e^{-\delta x^{-\theta}}, \ x > 0; \tag{2}$$

$$F(x; \delta, \theta) = e^{-\delta x^{-\theta}}, \ x > 0; \tag{3}$$

and

$$h(t; \delta, \theta) = \frac{\delta\theta t^{-(\theta+1)}}{e^{\delta t^{-\theta}} - 1}, \ t > 0, \tag{4}$$

respectively, and its reliability function (RF), $R(\cdot)$, is given by $R(\cdot) = 1 - F(\cdot)$.

To our knowledge, no work has been done that estimates the Fr model parameters or survival characteristics in the presence of data from the generalized Type-II progressive hybrid censoring. Our goals in this study were the following in order to close this gap. The likelihood inference for the unknown Fr parameters and/or any function of them, such as $R(t)$, or $h(t)$, was first derived. The second goal was to create Bayes estimates for the same unknown parameters using independent gamma priors from the squared-error loss (SEL). Additionally, using the suggested estimating techniques, for all unknown parameters, the asymptotic confidence intervals (ACIs) and highest posterior density (HPD) interval estimators were obtained. The R programming language's "maxLik" and "coda" packages were used to calculate the acquired estimates because the theoretical results of δ and θ obtained by the proposed estimation methods cannot be expressed in closed form. These packages were proposed by Henningsen and Toomet [19] and Plummer et al. [20]. The final goal was to come up with the most effective progressive censoring scheme using four optimality criteria. A Monte Carlo simulation was used to examine the efficacy of the different estimators using various combinations of the total sample size, effective sample size, threshold times, and progressive censoring. All acquired estimators were compared using their simulated root-mean-square errors, mean relative absolute biases, average confidence lengths, and coverage percentages. To determine how well the suggested approaches worked in practice and choose the best censoring strategy, two different data sets the from physical and engineering domains were analyzed. The rest of the study is structured as follows: In Sections 2 and 3, the maximum likelihoods and Bayes inferences of the unknown parameters and reliability characteristics are obtained, respectively. In Section 4, the asymptotic and credible intervals are derived. The outcomes of the Monte Carlo simulation are detailed in Section 5. Section 6 investigates the methodology for determining the best progressive censoring strategy. In Section 7, two real applications are examined. Finally, in Section 8, some concluding remarks of the study are provided.

2. Likelihood Inference

Suppose $\underline{\mathrm{X}} = \{(X_{1:m:n}, R_1), .., (X_{d_1:n}, R_{d_1}), .., (X_{d_2:n}, R_{d_2})\}$ is a GPHC-T2 sample of size d_2 obtained from $\mathrm{Fr}(\delta, \theta)$. Thus, by inserting (2) and (3) into (1), where x_j is used in place of $x_{j:m:n}$, the likelihood function of GPHC-T2 may be expressed as

$$L_\rho(\delta, \theta|\underline{\mathrm{x}}) \propto \prod_{j=1}^{D_\rho} \delta\theta x_j^{-(\theta+1)} e^{-\delta x_j^{-\theta}} \left(1 - e^{-\delta x_j^{-\theta}}\right)^{R_j} \mathcal{R}_\rho(T_\tau; \delta, \theta), \tag{5}$$

where $\mathcal{R}_2(T_\tau; \delta, \theta) = 1$

$\mathcal{R}_1(T_1; \delta, \theta) = \left(1 - e^{-\delta T_1^{-\theta}}\right)^{R^*_{d_1+1}}$ and $\mathcal{R}_3(T_2; \delta, \theta) = \left(1 - e^{-\delta T_2^{-\theta}}\right)^{R^*_{d_2+1}}$.

The log-likelihood function $\ell_\rho \propto L_\rho$ of (5) becomes

$$\ell_\rho(\delta, \theta|\underline{\mathrm{x}}) \propto D_\rho \ln(\delta\theta) - (\theta+1)\sum_{j=1}^{D_\rho} \ln\left(x_j\right) - \delta\sum_{j=1}^{D_\rho} x_j^{-\theta} + \sum_{j=1}^{D_\rho} R_j \ln\left(1 - e^{-\delta x_j^{-\theta}}\right) + \Upsilon_\rho(T_\tau; \delta, \theta), \tag{6}$$

where $\Upsilon_2(T_\tau; \delta, \theta) = 0$

$\Upsilon_1(T_1; \delta, \theta) = R^*_{d_1+1} \ln\left(1 - e^{-\delta T_1^{-\theta}}\right)$, and $\Upsilon_3(T_2; \delta, \theta) = R^*_{d_2+1} \ln\left(1 - e^{-\delta T_2^{-\theta}}\right)$.

The following two results are obtained by partly differentiating (6) with regard to δ and θ. To produce the MLEs $\hat{\delta}$ and $\hat{\theta}$, the following likelihood equations must be solved concurrently after being equated to zero as

$$\frac{\partial \ell_\rho}{\partial \delta} = \frac{D_\rho}{\delta} - \sum_{j-1}^{D_\rho} x_j^{-\theta} + \sum_{j=1}^{D_\rho} \frac{R_j x_j^{-\theta} e^{-\delta x_j^{-\theta}}}{\left(1 - e^{-\delta x_j^{-\theta}}\right)} + \frac{\partial \Upsilon_\rho(T_\tau; \delta, \theta)}{\partial \delta}, \tag{7}$$

and

$$\frac{\partial \ell_\rho}{\partial \theta} = \frac{D_\rho}{\theta} - \sum_{j=1}^{D_\rho} \ln(x_j) + \delta\sum_{j-1}^{D_\rho} x_j^{-\theta} \ln(x_j) - \sum_{j=1}^{D_\rho} \frac{R_j \delta x_j^{-\theta} \ln(x_j) e^{-\delta x_j^{-\theta}}}{\left(1 - e^{-\delta x_j^{-\theta}}\right)} - \frac{\partial \Upsilon_\rho(T_\tau; \delta, \theta)}{\partial \theta}, \tag{8}$$

where for $\rho = 1, 3$ and $\tau = 1, 2$, we have

$\frac{\partial \Upsilon_\rho(T_\tau;\delta,\theta)}{\partial \delta} = \frac{R^*_{d_\tau+1} T_\tau^{-\theta} e^{-\delta T_\tau^{-\theta}}}{\left(1-e^{-\delta T_\tau^{-\theta}}\right)}, \frac{\partial \Upsilon_\rho(T_\tau;\delta,\theta)}{\partial \theta} = \frac{R^*_{d_\tau+1} \delta T_\tau^{-\theta} \ln(T_\tau) e^{-\delta T_\tau^{-\theta}}}{\left(1-e^{-\delta T_\tau^{-\theta}}\right)}$.

As shown in (7) and (8), the MLEs of δ and θ must be obtained by solving a system of two nonlinear equations. Therefore, there is no closed-form analytical solution for $\hat{\delta}$ or $\hat{\theta}$. As a result, it may be calculated for every given GPHC-T2 data set using numerical methods such the Newton–Raphson iterative approach. The MLEs $\hat{R}(t)$ and $\hat{h}(t)$ can also be obtained by replacing δ and θ with $\hat{\delta}$ and $\hat{\theta}$, respectively.

3. Bayes Inference

The Bayes estimators of δ, θ, $R(t)$ and $h(t)$ and their corresponding HPD intervals are created in this section based on the SEL function. In order to do this, the Fr parameters δ and θ are taken to have independent gamma ($G(\cdot)$) priors with the form $G(\nu_1, \vartheta_1)$ and $G(\nu_2, \vartheta_2)$, respectively. Gamma priors should be taken into account for a number of reasons, including the fact that they (i) offer different shapes depending on parameter values, (ii) are adaptable, and (iii) are quite simple, brief, and might not produce a result with a difficult estimate problem. The joint prior density of δ and θ becomes

$$\pi(\delta, \theta) \propto \delta^{\nu_1 - 1} \theta^{\nu_2 - 1} e^{-(\delta\vartheta_1 + \theta\vartheta_2)}, \tag{9}$$

where $\nu_i > 0$ and $\vartheta_i > 0$ for $i = 1, 2$, are known. From (5) and (9), the joint posterior PDF of δ and θ is

$$\pi_\rho(\delta, \theta|\underline{x}) \propto \delta^{D_\rho+\nu_1-1}\theta^{D_\rho+\nu_2-1}e^{-(\delta\vartheta_1+\theta\vartheta_2)}\prod_{j=1}^{D_\rho} e^{-\delta x_j^{-\theta}} x_j^{-\theta}\left(1 - e^{-\delta x_j^{-\theta}}\right)^{R_j}\mathcal{R}_\rho(T_\tau; \delta, \theta). \quad (10)$$

There are many reasons to consider the SEL in a Bayesian analysis: (i) it is the commonly used symmetric loss; (ii) it is simple, clear, concise, and fairly easy; (iii) it assumes that the overestimation and underestimation are treated equally; and (iv) it develops the Bayes estimator directly by taking the posterior mean. However, under the SEL function, the posterior expectation of (10) yields the Bayes estimate of δ and θ (say $\tilde{\varphi}(\cdot)$) as

$$\tilde{\varphi}(\delta, \theta) = \int_0^\infty \int_0^\infty \varphi(\delta, \theta)\pi_\rho(\delta, \theta|\underline{x})d\delta d\theta.$$

It is obvious from (10), that the explicit expression of the marginal PDFs of δ and θ is not possible. Thus, to compute the acquired Bayes estimates and create their HPD intervals, we suggest generating samples from (10) using Bayesian MCMC techniques. Therefore, from (10), the conditional PDFs of δ and θ are provided, respectively, as

$$\pi_\rho^\delta(\delta|\theta, \underline{x}) \propto \delta^{D_\rho+\nu_1-1}e^{-\delta\vartheta_1}\prod_{j=1}^{D_\rho} e^{-\delta x_j^{-\theta}}\left(1 - e^{-\delta x_j^{-\theta}}\right)^{R_j}\mathcal{R}_\rho(T_\tau; \delta, \theta), \quad (11)$$

and

$$\pi_\rho^\theta(\theta|\delta, \underline{x}) \propto \theta^{D_\rho+\nu_2-1}e^{-\theta\left(\vartheta_2+\sum_{j=1}^{D_\rho}\ln(x_j)\right)}\prod_{j=1}^{D_\rho} e^{-\delta x_j^{-\theta}}\left(1 - e^{-\delta x_j^{-\theta}}\right)^{R_j}\mathcal{R}_\rho(T_\tau; \delta, \theta). \quad (12)$$

It is clear, from (11) and (12), that there is no analytical way to reduce the posterior PDFs of δ and θ, respectively, to any known distribution. Thus, the Metropolis–Hastings (M-H) method is seen to be the best option for solving this issue; for detail see Gelman et al. [21] and Lynch [22]. The M-H algorithm's sampling procedure based on the normal proposal distribution is carried out as follows:

Step-1: Set the starting values $\delta^{(0)} = \hat{\delta}$ and $\theta^{(0)} = \hat{\theta}$.

Step-2: Set $s = 1$.

Step-3: Create δ^* and θ^* from $N(\hat{\delta}, \hat{\sigma}^2_{\hat{\delta}})$ and $N(\hat{\theta}, \hat{\sigma}^2_{\hat{\theta}})$, respectively.

Step-4: Find $\xi_\delta = \min\left\{1, \frac{\pi_\rho^\delta(\delta^*|\theta^{(s-1)};\underline{x})}{\pi_\rho^\delta(\delta^{(s-1)}|\theta^{(s-1)};\underline{x})}\right\}$ and $\xi_\theta = \min\left\{1, \frac{\pi_\rho^\theta(\theta^*|\delta^{(s)};\underline{x})}{\pi_\rho^\theta(\theta^{(s-1)}|\delta^{(s)};\underline{x})}\right\}$.

Step-5: Create samples u_1 and u_2 using the uniform $U(0,1)$ distribution.

Step-6: If both u_1 and u_2 are less than ξ_δ and ξ_θ, respectively, then set $\delta^{(s)} = \delta^*$ and $\theta^{(s)} = \theta^*$, respectively. Otherwise, set $\delta^{(s)} = \delta^{(s-1)}$ and $\theta^{(s)} = \theta^{(s-1)}$, respectively.

Step-7: Set $s = s + 1$.

Step-8: Redo steps 3–7 $\mathcal{H}$ times to get $\delta^{(s)}$ and $\theta^{(s)}$ for $s = 1, 2, \ldots, \mathcal{H}$.

Step-9: Use $\delta^{(s)}$ and $\theta^{(s)}$, for $t > 0$, to compute the reliability $R(t)$ and hazard rate $h(t)$ parameters, respectively, as

$$R^{(s)}(t) = 1 - e^{-\delta^{(s)}t^{-\theta^{(s)}}} \quad \text{and} \quad h^{(s)}(t) = \frac{\delta^{(s)}\theta^{(s)}t^{-\theta^{(s)}-1}}{e^{\delta^{(s)}t^{-\theta^{(s)}}} - 1}.$$

In order to ensure the MCMC sampler's convergence and to eliminate the impact of initial guesses $\delta^{(0)}$ and $\theta^{(0)}$, the first simulated samples (say $\mathcal{H}_0$) are eliminated as burn-in.

The Bayesian estimates using the SEL function are therefore calculated using the remaining $\bar{\mathcal{H}} = \mathcal{H} - \mathcal{H}_0$ samples of δ, θ, $R(t)$, and $h(t)$, (say $\tilde{\varphi}(\cdot)$) as

$$\tilde{\varphi}(\delta,\theta) = \frac{1}{\bar{\mathcal{H}}} \sum_{s=\mathcal{H}_0+1}^{\mathcal{H}} \varphi^{(s)}(\delta,\theta).$$

Since the choice of symmetric (or asymmetric) loss is one of the main issues in the Bayes analysis, one may incorporated any other type of loss function instead of the SEL easily.

4. Interval Inference

In this section, the ACIs (based on observed Fisher information) and HPD intervals (based on MCMC simulated variates) of δ, θ, $R(t)$, and $h(t)$ are created.

4.1. Asymptotic Intervals

The asymptotic variance–covariance (AVC) matrix, which is created by inverting the Fisher information matrix, must first be computed in order to create the ACIs for δ and θ. The MLEs $(\hat{\delta},\hat{\theta})$ under some regularity criteria are normally distributed with mean (δ,θ) and variance $\mathbf{I}^{-1}(\delta,\theta)$. Following Lawless [23], we estimate $\mathbf{I}^{-1}(\delta,\theta)$ by $\mathbf{I}^{-1}(\hat{\delta},\hat{\theta})$ by substituting $\hat{\delta}$ and $\hat{\theta}$ in place of δ and θ as

$$\mathbf{I}^{-1}(\hat{\delta},\hat{\theta}) \cong \begin{bmatrix} -\mathcal{L}_{11} & -\mathcal{L}_{12} \\ -\mathcal{L}_{21} & -\mathcal{L}_{22} \end{bmatrix}^{-1}_{(\hat{\delta},\hat{\theta})} = \begin{bmatrix} \hat{\sigma}^2_{\hat{\delta}} & \hat{\sigma}_{\hat{\delta}\hat{\theta}} \\ \hat{\sigma}_{\hat{\theta}\hat{\delta}} & \hat{\sigma}^2_{\hat{\theta}} \end{bmatrix}, \tag{13}$$

where $\mathcal{L}_{ij}$ for $i,j = 1,2$ are

$$\mathcal{L}_{11} = -\frac{D_\rho}{\delta^2} - \sum_{j=1}^{D_\rho} \frac{R_j x_j^{-2\theta} e^{-\delta x_j^{-\theta}}}{\left(1-e^{-\delta x_j^{-\theta}}\right)^2} + \frac{\partial^2 \Upsilon_\rho(T_\tau;\delta,\theta)}{\partial\delta^2},$$

$$\mathcal{L}_{22} = -\frac{D_\rho}{\theta^2} - \delta\sum_{j-1}^{D_\rho} x_j^{-\theta}\ln^2(x_j) - \delta\sum_{j=1}^{D_\rho} \frac{R_j x_j^{-\theta}\ln^2(x_j) e^{-\delta x_j^{-\theta}}\left(e^{-\delta x_j^{-\theta}} + \delta x_j^{-\theta} - 1\right)}{\left(1-e^{-\delta x_j^{-\theta}}\right)^2} - \frac{\partial^2 \Upsilon_\rho(T_\tau;\delta,\theta)}{\partial\theta^2},$$

and

$$\mathcal{L}_{12} = \sum_{j-1}^{D_\rho} x_j^{-\theta}\ln(x_j) - \sum_{j=1}^{D_\rho} \frac{R_j x_j^{-\theta}\ln(x_j) e^{-\delta x_j^{-\theta}}\left(1-2e^{-\delta x_j^{-\theta}}\right)}{\left(1-e^{-\delta x_j^{-\theta}}\right)^2} - \frac{\partial^2 \Upsilon_\rho(T_\tau;\delta,\theta)}{\partial\theta\partial\delta},$$

$$\frac{\partial^2 \Upsilon_\rho(T_\tau;\delta,\theta)}{\partial\delta^2} = -\frac{R^*_{d_\tau+1} T_\tau^{-2\theta} e^{-\delta T_\tau^{-\theta}}}{\left(1-e^{-\delta T_\tau^{-\theta}}\right)^2},$$

$$\frac{\partial^2 \Upsilon_\rho(T_\tau;\delta,\theta)}{\partial\theta^2} = \frac{R^*_{d_\tau+1}\delta\ln^2(T_\tau) T_\tau^{-\theta} e^{-\delta T_\tau^{-\theta}}\left(e^{-\delta T_\tau^{-\theta}} + \delta T_\tau^{-\theta} - 1\right)}{\left(1-e^{-\delta T_\tau^{-\theta}}\right)^2},$$

and

$$\frac{\partial^2 \Upsilon_\rho(T_\tau;\delta,\theta)}{\partial\theta\partial\delta} = \frac{R^*_{d_\tau+1} T_\tau^{-\theta}\ln(T_\tau) e^{-\delta T_\tau^{-\theta}}\left(1-\delta T_\tau^{-\theta} - e^{-\delta T_\tau^{-\theta}}\right)}{\left(1-e^{-\delta T_\tau^{-\theta}}\right)^2}.$$

Thus, for δ and θ, respectively, the two-sided $100(1-\gamma)\%$ ACIs are provided by

$$\hat{\delta} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{\sigma}^2_{\hat{\delta}}} \quad \text{and} \quad \hat{\theta} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{\sigma}^2_{\hat{\theta}}},$$

where $Z_{\frac{\gamma}{2}}$ denotes the top $\frac{\gamma}{2}$ percentage points of the standard normal distribution.

Additionally, we use the delta approach to first determine the estimated variance of $\hat{R}(t)$ and $\hat{h}(t)$ (see Greene [24]) before building the ACIs of $R(t)$ and $h(t)$ as

$$\hat{\sigma}^2_{\hat{R}(t)} = \nabla^{\mathsf{T}}_{\hat{R}} \boldsymbol{I}^{-1}(\hat{\delta},\hat{\theta})\nabla_{\hat{R}} \quad \text{and} \quad \hat{\sigma}^2_{\hat{h}(t)} = \nabla^{\mathsf{T}}_{\hat{h}} \boldsymbol{I}^{-1}(\hat{\delta},\hat{\theta})\nabla_{\hat{h}},$$

where $\nabla^{\mathsf{T}}_{\hat{R}} = \left[\frac{\partial R(t)}{\partial \delta}\ \frac{\partial R(t)}{\partial \theta}\right]_{(\hat{\delta},\hat{\theta})}$ and $\nabla^{\mathsf{T}}_{\hat{h}} = \left[\frac{\partial h(t)}{\partial \delta}\ \frac{\partial h(t)}{\partial \theta}\right]_{(\hat{\delta},\hat{\theta})}$.

Following that, the two-sided $100(1-\gamma)\%$ ACIs of $R(t)$ and $h(t)$ are provided, respectively, by

$$\hat{R}(t) \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{\sigma}^2_{\hat{R}(t)}} \quad \text{and} \quad \hat{h}(t) \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{\sigma}^2_{\hat{h}(t)}}.$$

4.2. HPD Intervals

Using the approach proposed by Chen and Shao [25], the $100(1-\gamma)\%$ HPD interval estimations of δ, θ, $R(t)$, or $h(t)$ are constructed. First, we rank the MCMC samples of $\varphi^{(s)}$ for $s = \mathcal{H}_0+1, \mathcal{H}_0+2, \ldots, \mathcal{H}$ as $\varphi_{(\mathcal{H}_0+1)}, \varphi_{(\mathcal{H}_0+2)}, \ldots, \varphi_{(\mathcal{H})}$. Hence, the $100(1-\gamma)\%$ two-sided HPD interval of φ is provided by

$$\varphi_{(j^*)}, \varphi_{(j^*+(1-\gamma)\bar{\mathcal{H}})},$$

where $j^* = \mathcal{H}_0+1, \mathcal{H}_0+2, \ldots, \mathcal{H}$ is selected so that

$$\varphi_{\left(j^*+[(1-\gamma)\left(\bar{\mathcal{H}}\right)]\right)} - \varphi_{(j^*)} = \min_{1\leqslant j\leqslant \gamma\bar{\mathcal{H}}}\left[\varphi_{(j+[(1-\gamma)\bar{\mathcal{H}}])} - \varphi_{(j)}\right].$$

5. Monte Carlo Simulation

To evaluate the true performance of the acquired point/interval estimators of δ, θ, $R(t)$, and $h(t)$, Monte Carlo simulations were conducted based on various combinations of $T_i, i = 1,2$ (threshold points), n (size of experimental items), m (size of effective sample), and $\underline{\mathrm{R}}$ (removal pattern). To establish this goal, for Fr(0.5, 1.5), we replicated the GPHC-T2 mechanism 1000 times. At $t = 0.3$, the true values of $R(t)$ and $h(t)$ were 0.9523 and 0.7620, respectively. Taking $(T_1, T_2) = (0.4,\ 0.8)$ and (0.8, 1.2), two different choices of n and m were used as n(=40, 80) and the choices of m were taken as failure percentages (FPs) of each n such as $\frac{m}{n}$(=50, 80)%. Additionally, for each (n, m), three progressive censoring plans $\underline{\mathrm{R}}$ were used, namely,

$$\begin{aligned}
&\text{Scheme-1}: \ \underline{\mathrm{R}} = (n-m, 0^*(m-1)),\\
&\text{Scheme-2}: \ \underline{\mathrm{R}} = \left(0^*\left(\frac{m}{2}-1\right), n-m, 0^*\left(\frac{m}{2}\right)\right),\\
&\text{Scheme-3}: \ \underline{\mathrm{R}} = (0^*(m-1), n-m),
\end{aligned}$$

where $\underline{\mathrm{R}} = (3,0,0,4)$ was used as $\underline{\mathrm{R}} = (3, 0^*2, 4)$

Once 1000 GPHC-T2 samples had been collected, via R 4.2.2 programming software by installing the "maxLik" package (by Henningsen and Toomet [19]), the MLEs and 95% ACI estimates of δ, θ, $R(t)$, and $h(t)$ were evaluated. Via the "coda" package (by Plummer et al. [20]) in R 4.2.2 programming software, to obtain the Bayes point estimates along with their HPD interval estimates of the same unknown parameters, we simulated 12,000 MCMC samples and ignored the first 2000 iterations as burn-in. According to the prior mean and prior variance criteria, two sets called Prior-I and -II of the hyperparameters (a_1, a_2, b_1, b_2) were considered as $(2.5, 7.5, 5, 5)$ and $(5, 15, 10, 10)$, respectively.

Specifically, the average point estimates (APEs) of δ, θ, $R(t)$, or $h(t)$ (say Ω) were given by

$$\overline{\check{\Omega}_\tau} = \frac{1}{\mathcal{B}}\sum_{i=1}^{\mathcal{B}} \check{\Omega}_\tau^{(i)},\ \tau = 1,2,3,4,$$

where $\mathcal{B}$ is the number of replications, $\check{\Omega}^{(i)}$ is the estimate of Ω at the ith sample, $\Omega_1 = \delta$, $\Omega_2 = \theta$, $\Omega_3 = R(t)$, and $\Omega_4 = h(t)$.

A comparison between point estimates of Ω was made based on their root-mean-square errors (RMSEs) and mean relative absolute biases (MRABs), respectively, as

$$\text{RMSE}(\check{\Omega}_\tau) = \sqrt{\frac{1}{\mathcal{B}}\sum_{i=1}^{\mathcal{B}} \left(\check{\Omega}_\tau^{(i)} - \Omega_\tau\right)^2},\ \tau = 1,2,3,4,$$

and

$$\text{MRAB}(\check{\Omega}_\tau) = \frac{1}{\mathcal{B}}\sum_{i=1}^{\mathcal{B}} \frac{1}{\Omega_\tau}\left|\check{\Omega}_\tau^{(i)} - \Omega_\tau\right|,\ \tau = 1,2,3,4.$$

On the other hand, the comparison between the interval estimates of Ω was made based on their average confidence lengths (ACLs) and coverage percentages (CPs) as

$$\text{ACL}_{(1-\gamma)\%}(\Omega_\tau) = \frac{1}{\mathcal{B}}\sum_{i=1}^{\mathcal{B}} \left(\mathcal{U}_{\check{\Omega}_\tau^{(i)}} - \mathcal{L}_{\check{\Omega}_\tau^{(i)}}\right),\ \tau = 1,2,3,4,$$

and

$$\text{CP}_{(1-\gamma)\%}(\Omega_\tau) = \frac{1}{\mathcal{B}}\sum_{i=1}^{\mathcal{B}} \mathbf{1}_{\left(\mathcal{L}_{\check{\Omega}_\tau^{(i)}};\mathcal{U}_{\check{\Omega}_\tau^{(i)}}\right)}(\Omega_\tau),\ \tau = 1,2,3,4,$$

respectively, where $\mathbf{1}(\cdot)$ is the indicator function, $(\mathcal{L}(\cdot), \mathcal{U}(\cdot))$ denote the (lower, upper) bounds of $(1-\gamma)\%$ ACI (or HPD) interval of Ω_τ.

Via a heatmap data visualization in R version 4.2.2 programming software (available in https://cran.r-project.org/bin/windows/base (accessed on 23 January 2023), the simulated RMSEs, MRABs, ACLs, and CPs of δ, θ, $R(t)$, or $h(t)$ are shown in Figures 1–4, respectively, while their numerical tables are available as Supplementary Materials. For specialization, some notations of the proposed methods have been defined on the "x-axis" line in Figures 1–3 such that (for Prior-I (say P1) as an example) the Bayes estimates is referred to as "BE-P1" and the HPD interval estimates is denoted as "HPD-P1".

From Figures 1–4, in terms of the lowest RMSE, MRAB, and ACL values as well as the highest CP values, useful observations were found and can be easily reported as:

- The main general point is that the proposed estimates of δ, θ, $R(t)$, or $h(t)$ provided good performance.
- As n(or m) increased, all estimates of μ, $R(t)$, and $h(t)$ performed satisfactory. A similar result was obtained when $\sum_{i=1}^{m} R_i$ decreased.
- As (T_1, T_2) increased, in most situations, the RMSEs, MRABs, and ACLs of all unknown parameters decreased while their CPs increased.
- The Bayes estimates of δ, θ, $R(t)$, or $h(t)$, due to the gamma information, behaved better compared to the other estimates as expected. A similar comment could also be made in the case of HPD credible intervals.
- Since the variance of Prior-II was smaller than the variance of Prior-I, the MCMC calculations under Prior-II provided more accurate estimates than others for all unknown parameters.
- Comparing the proposed schemes 1, 2, and 3, in most cases, it was noted that the proposed estimates of δ, θ, $R(t)$, and $h(t)$ behaved better using scheme 3 than the others.

- As a result, the Bayes M-H algorithm sampler is recommended to estimate the Fr parameters or its reliability characteristics in the presence of data obtained from generalized Type-II progressive hybrid censoring.

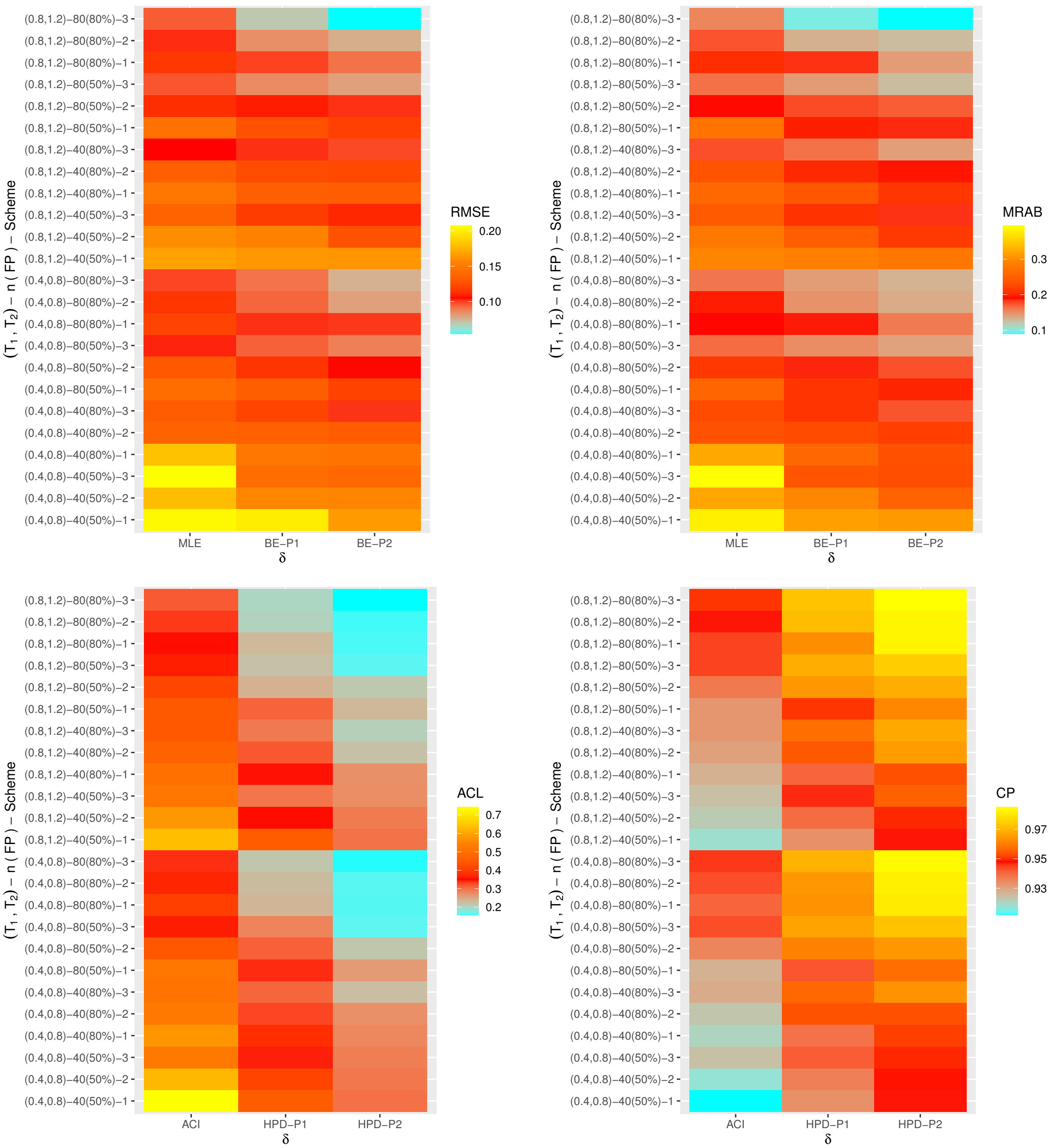

Figure 1. Heatmap plots for the simulation outputs of δ.

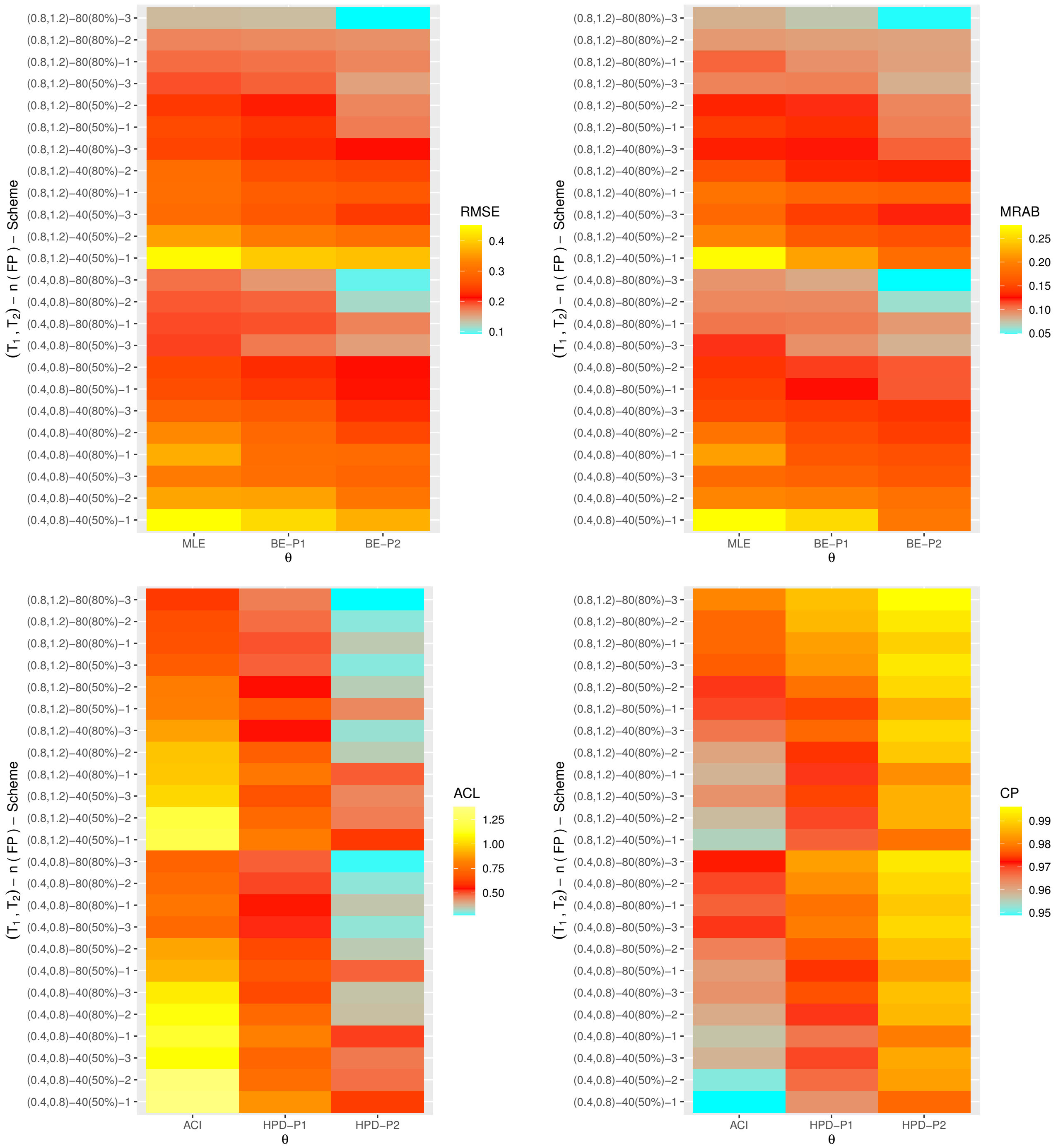

Figure 2. Heatmap plots for the simulation outputs of θ.

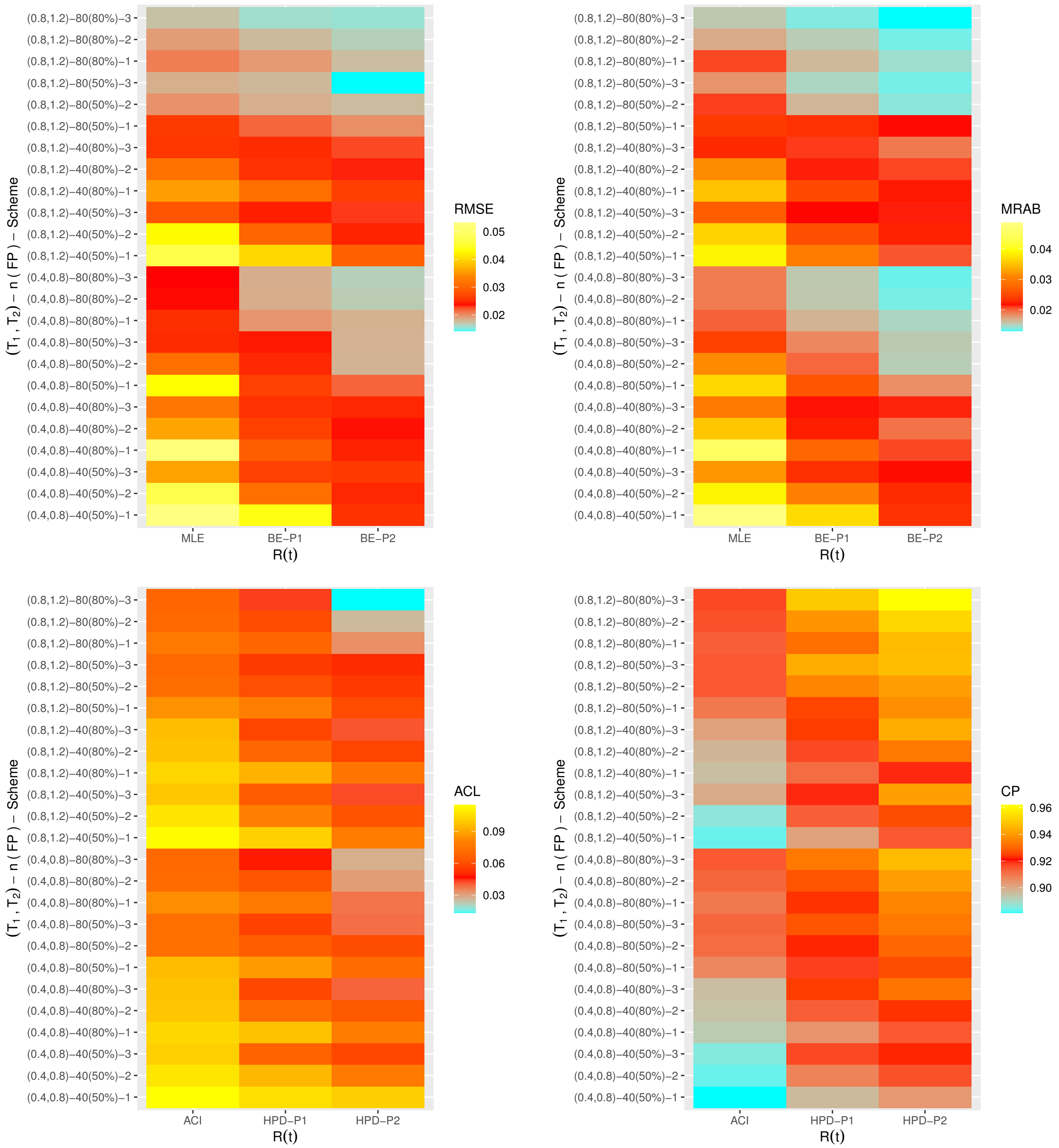

Figure 3. Heatmap plots for the simulation outputs of $R(t)$.

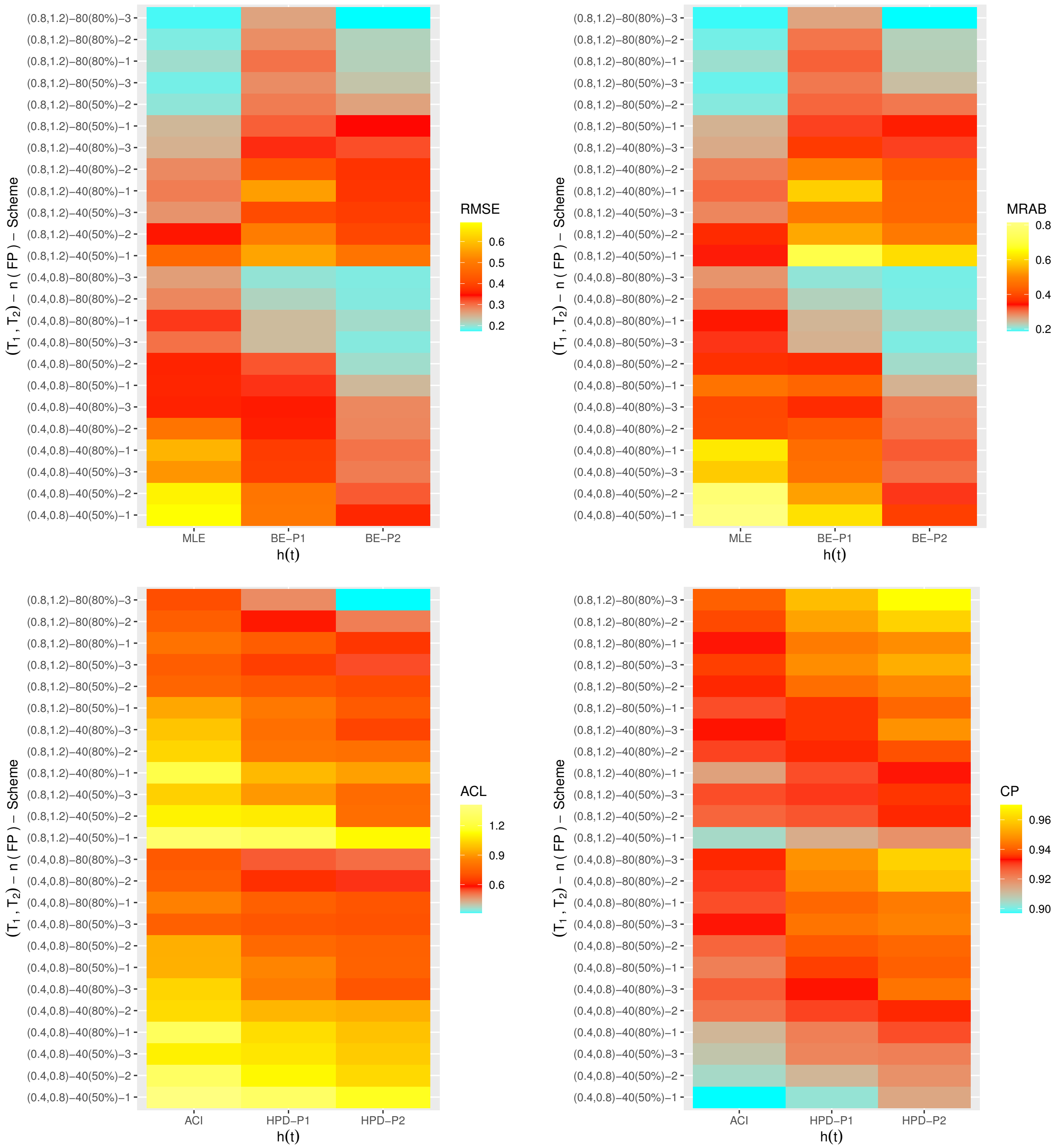

Figure 4. Heatmap plots for the simulation outputs of $h(t)$.

6. Optimal PC-T2 Designs

The experimenter may want to choose the "best" censoring scheme from a collection of all accessible censoring schemes in order to offer the most information about the unknown parameters under research, especially in the context of dependability. First, Balakrishnan and Aggarwala [26] looked at the issue of selecting the best censoring approach in various situations. However, several optimality criteria have been put forth, and many conclusions on the best censoring schemes have been examined. The ideal censoring design $\underline{R} = (R_1, R_2, \ldots, R_m)$ such that $n - m = \sum_{i=1}^{m} R_i$ can be proposed, and the precise values of n (size of test units), m (effective sample), and T_i, $i = 1, 2$ (ideal test thresholds) are chosen beforehand based on the availability of the units, experimental facilities, and cost considerations; for details, see Ng et al. [27].

The issue of contrasting two (or more) alternative censoring strategies has been addressed in a number of publications in the literature (for examples, see Sen et al. [28], Elshahhat and Abu El Azm [29], Elshahhat et al. [30], among others). In our situation, Table 2 provides a selection of frequently used metrics to assist us in selecting the ideal censoring approach, $\mathcal{C}_i$.

Table 2. Useful criteria for the best PC-T2 plan.

Criterion	Target
$\mathcal{C}_1$	Maximize trace$(\boldsymbol{I}(\hat{\delta},\hat{\theta}))$
$\mathcal{C}_2$	Minimize trace$(\boldsymbol{I}^{-1}(\hat{\delta},\hat{\theta}))$
$\mathcal{C}_3$	Minimize det$(\boldsymbol{I}^{-1}(\hat{\delta},\hat{\theta}))$
$\mathcal{C}_4$	Minimize $\widehat{var}(\log(\hat{\mathcal{T}}_\varrho))$

It is recommended to maximize the observed Fisher information, the $\boldsymbol{I}^{-1}(\cdot)$ values for $\mathcal{C}_1$. In addition, for criteria $\mathcal{C}_2$ and $\mathcal{C}_3$, we want to minimize the determinant and trace of $\boldsymbol{I}^{-1}(\cdot)$. For multiparameter distributions, the ideal censoring approach may be chosen using scale-invariant criteria. While comparing the two Fisher information matrices is more difficult when dealing with unknown multiparameter distributions, scale-invariant criteria can be utilized to compare numerous criteria when dealing with single-parameter distributions $\mathcal{C}_4$. Criterion $\mathcal{C}_4$ tends to minimize the variance of the logarithmic MLE of the ϱth quantile, $\mathcal{T}_\varrho$. Thus, from (3), the logarithmic of the Fr distribution $\mathcal{T}_\varrho$ is given by

$$\log(\hat{\mathcal{T}}_\varrho) = \left[-\frac{\log(\varrho)}{\delta}\right]^{-\frac{1}{\theta}}_{(\hat{\delta},\hat{\theta})},\ 0 < \varrho < 1. \tag{14}$$

Applying the delta approach to (14), the approximation of the variance for the Fr distribution's $\log(\hat{\mathcal{T}}_\varrho)$ is obtained as

$$\widehat{var}(\log(\hat{\mathcal{T}}_\varrho)) = \Sigma^{\mathsf{T}}_{\log(\hat{\mathcal{T}}_\varrho)}\ \mathbf{I}^{-1}(\hat{\delta},\hat{\theta})\ \Sigma_{\log(\hat{\mathcal{T}}_\varrho)},$$

where

$$\Sigma^{\mathsf{T}}_{\log(\hat{\mathcal{T}}_\varrho)} = \left[\frac{\partial}{\partial\delta}\log(\hat{\mathcal{T}}_\varrho), \frac{\partial}{\partial\theta}\log(\hat{\mathcal{T}}_\varrho)\right]_{(\hat{\delta},\hat{\theta})}.$$

The highest value of the criteria $\mathcal{C}_1$ and the minimum value of the criterion $\mathcal{C}_i$ for $i = 2,3,4$ correspond to the optimum censoring. Contrarily, the highest value of the criterion $\mathcal{C}_1$ and the lowest value of other criteria correspond to the optimal censoring.

7. Real-Life Applications

To highlight the utility of the proposed estimation procedures and the applicability of the study objectives to actual situations, this section presents two different applications by analyzing two sets of useful real data taken from the physical and engineering fields. These applications show that the proposed inferential approaches work satisfactorily under real-life data using the proposed censoring plan.

7.1. March Precipitation

In this application, we considered a data set representing thirty successive values (in inches) of precipitation in Minneapolis–Saint Paul for the month of March, see Table 3. This data set was provided by Hinckley [31] and recently reanalyzed by Elshahhat et al. [32]. To examine whether March precipitation data fit the Fr distribution or not, the Kolmogorov–Smirnov (KS) statistic and its p-value were calculated. To establish this goal, from Table 3, the MLEs (with their standard errors (SEs)) of δ and θ were 1.0252 (0.1978) and 1.5496

(0.2027), respectively, meanwhile the KS (p-value) was 0.1524 (0.489). It means that the Fr lifetime model fit the March precipitation data well. Using a graphic visualization, based on the complete March precipitation data, Figure 5 displays (i) the estimated and empirical RFs and (ii) the contour of the log-likelihood function with respect to various choices of δ and θ. It supported the same findings as the KS test and showed that the MLEs $\hat{\delta} \cong 1.025$ and $\hat{\theta} \cong 1.550$ existed and were unique.

Table 3. Successive values of March precipitation.

0.77	1.74	0.81	1.20	1.95	1.20	0.47	1.43	3.37	2.20
3.00	3.09	1.51	2.10	0.52	1.62	1.31	0.32	0.59	0.81
2.81	1.87	1.18	1.35	4.75	2.48	0.96	1.89	0.90	2.05

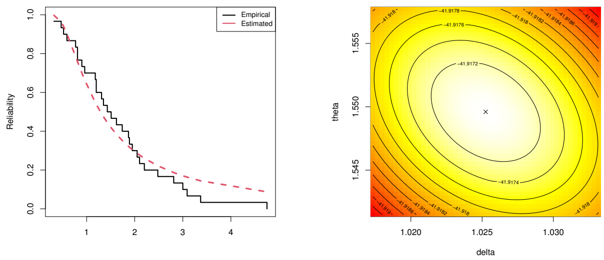

Figure 5. Empirical/fitted RFs (**left**); contour (**right**) plots from March precipitation data.

For the explanation of the proposed estimation methodologies, from the complete March precipitation data, three GPHC-T2 samples with $m = 10$ and various thresholds $T_i, i = 1, 2$ were generated and are reported in Table 4. Moreover, in Table 4, different censoring plans $\underline{R}$ were utilized, namely, $\mathcal{S}_1 : (2^*10)$, $\mathcal{S}_2 : (5^*2, 0^*6, 5^*2)$, $\mathcal{S}_3 : (6^*3, 0^*6, 2)$, and $\mathcal{S}_4 : (2, 0^*6, 6^*3)$. From Table 4, the maximum likelihood estimates (along with their SEs) as well as the ACI estimates (along with their widths) of δ, θ, $R(t)$, and $h(t)$ (at $t = 1$) were computed and are listed in Table 5. Since there was no prior information about the unknown Fr parameters δ and θ from the given data set, by repeating the MCMC sampler 50,000 times and ignoring the first 10,000 times as burn-in, the Bayes estimates (with their SEs) as well as the HPD interval estimates (with their widths) were evaluated using improper gamma priors and are provided in Table 5 as well. For the computational logic, the unknown hyperparameters were set to 0.001. It is clear, from Table 5, that the MCMC estimates of δ, θ, $R(t)$, and $h(t)$ behaved better than the others in terms of the smallest standard error and interval width values.

Table 4. GPHC-T2 samples from March precipitation data.

Scheme	Sample	$T_1(d_1)$	$T_2(d_2)$	Generated Data	R^*	T^*
$\mathcal{S}_1$	1	3.40(11)	5.00(11)	0.32, 0.59, 0.81, 1.18, 1.31, 1.51, 1.87, 2.05, 2.48, 3.09, 3.37	1	3.40
	2	2.00(7)	3.25(10)	0.32, 0.59, 0.81, 1.18, 1.31, 1.51, 1.87, 2.05, 2.48, 3.09	0	3.09
	3	2.00(7)	2.50(9)	0.32, 0.59, 0.81, 1.18, 1.31, 1.51, 1.87, 2.05, 2.48	3	2.50
$\mathcal{S}_2$	1	2.95(11)	3.05(11)	0.32, 0.81, 1.31, 1.35, 1.43, 1.51, 1.62, 1.74, 1.87, 2.48, 2.81	4	2.95
	2	2.25(9)	2.50(10)	0.32, 0.81, 1.31, 1.35, 1.43, 1.51, 1.62, 1.74, 1.87, 2.48	0	2.48
	3	1.50(5)	2.00(9)	0.32, 0.81, 1.31, 1.35, 1.43, 1.51, 1.62, 1.74, 1.87	6	2.00
$\mathcal{S}_3$	1	3.50(11)	4.80(11)	0.32, 0.90, 1.43, 2.05, 2.10, 2.20, 2.48, 2.81, 3.00, 3.09, 3.37	1	3.50
	2	2.75(7)	3.25(10)	0.32, 0.90, 1.43, 2.05, 2.10, 2.20, 2.48, 2.81, 3.00, 3.09	0	3.09
	3	2.25(6)	3.05(9)	0.32, 0.90, 1.43, 2.05, 2.10, 2.20, 2.48, 2.81, 3.00	3	3.05
$\mathcal{S}_4$	1	2.60(11)	4.80(11)	0.32, 0.59, 0.77, 0.81, 0.81, 0.90, 0.96, 1.18, 1.62, 2.20, 2.48	5	2.60
	2	1.10(7)	2.50(10)	0.32, 0.59, 0.77, 0.81, 0.81, 0.90, 0.96, 1.18, 1.62, 2.20	0	2.20
	3	1.50(8)	2.10(9)	0.32, 0.59, 0.77, 0.81, 0.81, 0.90, 0.96, 1.18, 1.62	7	2.10

Table 5. Point and 95% interval estimates of δ, θ, $R(t)$, and $h(t)$ from March precipitation data.

Scheme	Sample	Par.	MLE		MCMC		ACI			HPD		
			Est.	SE	Est.	SE	Lower	Upper	Width	Lower	Upper	Width
$\mathcal{S}_1$	1	δ	1.8514	0.3703	1.6588	0.2364	1.1256	2.5772	1.4517	1.3969	1.9224	0.5255
		θ	0.9547	0.1974	0.8078	0.1878	0.5677	1.3417	0.7739	0.5750	1.0277	0.4527
		$R(1)$	0.8430	0.0581	0.8079	0.0439	0.7290	0.9570	0.2279	0.7526	0.8538	0.1011
		$h(1)$	0.3292	0.1084	0.3162	0.0557	0.1168	0.5416	0.4248	0.2183	0.4275	0.2093
	2	δ	1.8803	0.3765	1.6879	0.2337	1.1424	2.6182	1.4758	1.4424	1.9576	0.5152
		θ	0.9039	0.1991	0.7523	0.1922	0.5136	1.2941	0.7804	0.5353	0.9912	0.4559
		$R(1)$	0.8475	0.0574	0.8135	0.0421	0.7349	0.9600	0.2251	0.7637	0.8588	0.0952
		$h(1)$	0.3059	0.1046	0.2889	0.0545	0.1010	0.5108	0.4098	0.1962	0.3969	0.2007
	3	δ	1.9040	0.3823	1.7049	0.2416	1.1547	2.6534	1.4988	1.4514	1.9965	0.5451
		θ	0.8633	0.2033	0.7049	0.1974	0.4648	1.2618	0.7970	0.4664	0.9247	0.4583
		$R(1)$	0.8510	0.0570	0.8165	0.0428	0.7394	0.9627	0.2233	0.7685	0.8662	0.0977
		$h(1)$	0.2877	0.1028	0.2679	0.0545	0.0862	0.4892	0.4030	0.1634	0.3608	0.1974
$\mathcal{S}_2$	1	δ	2.3307	0.4210	2.1201	0.2523	1.5056	3.1559	1.6503	1.8228	2.3704	0.5476
		θ	0.8229	0.1722	0.6788	0.1806	0.4853	1.1605	0.6752	0.4675	0.8874	0.4199
		$R(1)$	0.9028	0.0409	0.8788	0.0294	0.8226	0.9830	0.1604	0.8466	0.9129	0.0662
		$h(1)$	0.2066	0.0680	0.1969	0.0374	0.0733	0.3398	0.2666	0.1295	0.2686	0.1391
	2	δ	1.8799	0.3918	1.6843	0.2392	1.1120	2.6479	1.5359	1.4255	1.9528	0.5273
		θ	1.0014	0.2118	0.8466	0.1956	0.5863	1.4164	0.8301	0.6217	1.0810	0.4594
		$R(1)$	0.8474	0.0598	0.8127	0.0432	0.7302	0.9646	0.2344	0.7596	0.8581	0.0985
		$h(1)$	0.3390	0.1155	0.3261	0.0565	0.1126	0.5654	0.4528	0.2244	0.4378	0.2134
	3	δ	1.8845	0.3952	1.6894	0.2366	1.1099	2.6591	1.5493	1.4467	1.9610	0.5144
		θ	0.9891	0.2206	0.8256	0.2046	0.5568	1.4214	0.8647	0.6076	1.0796	0.4720
		$R(1)$	0.8481	0.0600	0.8137	0.0425	0.7304	0.9658	0.2353	0.7646	0.8593	0.0946
		$h(1)$	0.3339	0.1184	0.3168	0.0570	0.1018	0.5660	0.4642	0.2180	0.4256	0.2075
$\mathcal{S}_3$	1	δ	2.3735	0.4667	2.1580	0.2556	1.4589	3.2882	1.8294	1.8927	2.4194	0.5267
		θ	0.8882	0.1730	0.7458	0.1808	0.5490	1.2273	0.6783	0.5500	0.9770	0.4269
		$R(1)$	0.9069	0.0435	0.8834	0.0284	0.8216	0.9921	0.1704	0.8529	0.9141	0.0611
		$h(1)$	0.2165	0.0766	0.2109	0.0366	0.0664	0.3667	0.3002	0.1397	0.2787	0.1389
	2	δ	2.0295	0.4390	1.8276	0.2421	1.1691	2.8898	1.7207	1.5706	2.0924	0.5218
		θ	1.0130	0.2028	0.8574	0.1962	0.6155	1.4105	0.7950	0.6199	1.0857	0.4658
		$R(1)$	0.8686	0.0577	0.8378	0.0376	0.7555	0.9816	0.2261	0.7953	0.8787	0.0834
		$h(1)$	0.3110	0.1121	0.3012	0.0508	0.0914	0.5307	0.4393	0.2087	0.4009	0.1921
	3	δ	2.0708	0.4490	1.8624	0.2513	1.1908	2.9508	1.7600	1.5918	2.1203	0.5286
		θ	0.9542	0.2040	0.7971	0.1973	0.5545	1.3540	0.7996	0.5466	1.0094	0.4627
		$R(1)$	0.8739	0.0566	0.8432	0.0378	0.7630	0.9849	0.2219	0.7990	0.8817	0.0827
		$h(1)$	0.2851	0.1065	0.2737	0.0485	0.0764	0.4938	0.4173	0.1826	0.3615	0.1788
$\mathcal{S}_4$	1	δ	1.7138	0.3237	1.5320	0.2232	1.0794	2.3482	1.2688	1.2781	1.7873	0.5092
		θ	0.8887	0.2006	0.7435	0.1850	0.4955	1.2820	0.7865	0.5247	0.9780	0.4532
		$R(1)$	0.8198	0.0583	0.7821	0.0471	0.7055	0.9341	0.2286	0.7277	0.8376	0.1099
		$h(1)$	0.3347	0.1087	0.3149	0.0590	0.1216	0.5478	0.4262	0.2034	0.4170	0.2136
	2	δ	1.6400	0.3231	1.4614	0.2202	1.0067	2.2733	1.2666	1.2305	1.7284	0.4979
		θ	0.9122	0.2171	0.7602	0.1941	0.4867	1.3377	0.8510	0.5090	0.9807	0.4717
		$R(1)$	0.8060	0.0627	0.7662	0.0500	0.6832	0.9289	0.2457	0.7115	0.8248	0.1133
		$h(1)$	0.3600	0.1228	0.3365	0.0655	0.1193	0.6007	0.4814	0.2122	0.4520	0.2398
	3	δ	1.6816	0.3330	1.4968	0.2297	1.0289	2.3343	1.3055	1.2307	1.7600	0.5293
		θ	0.8508	0.2181	0.6970	0.1939	0.4232	1.2783	0.8551	0.4756	0.9421	0.4665
		$R(1)$	0.8139	0.0620	0.7740	0.0506	0.6925	0.9354	0.2429	0.7130	0.8319	0.1189
		$h(1)$	0.3271	0.1185	0.3020	0.0639	0.0948	0.5594	0.4646	0.1882	0.4165	0.2283

Various properties, namely, the mean, mode, median, first/third quartiles, standard deviation (St.D), and skewness (Skew.) from 40,000 MCMC variates of δ, θ, $R(t)$, and $h(t)$ were obtained and are presented in Table 6. To highlight the convergence of the MCMC iterations, from each generated sample by $\mathcal{S}_1$ (for example), Figure 6 displays both the density and trace plots of δ, θ, $R(t)$, and $h(t)$. For discrimination, the solid (—) line represents the Bayes estimate while the dashed (- - -) lines represent the HPD interval bounds. It is clear, from Figure 6, that the MCMC technique converged favorably, and the suggested size of the burn-in sample was sufficient to eliminate the influence of the suggested initial values. Moreover, for each sample, Figure 6 shows that the calculated estimates of δ, θ, and $h(t)$ were fairly symmetrical while those associated with $R(t)$ were negatively skewed.

Table 6. Summary of MCMC draws of δ, θ, $R(t)$, and $h(t)$ from March precipitation data.

Scheme	Sample	Par.	Mean	Mode	1st Quart.	Median	3rd Quart.	St.D	Skew.
$\mathcal{S}_1$	1	δ	1.65884	1.44240	1.56456	1.66014	1.75286	0.13707	0.06177
		θ	0.80782	0.78396	0.72868	0.80569	0.88953	0.11712	0.03572
		$R(1)$	0.80785	0.76364	0.79082	0.80988	0.82672	0.02632	−0.30505
		$h(1)$	0.31615	0.34999	0.27967	0.31448	0.35220	0.05417	0.21963
	2	δ	1.68791	1.33249	1.59475	1.68677	1.77693	0.13267	0.04592
		θ	0.75233	0.75994	0.67512	0.74995	0.83252	0.11832	0.03766
		$R(1)$	0.81346	0.73618	0.79704	0.81488	0.83084	0.02478	−0.35833
		$h(1)$	0.28894	0.36288	0.25335	0.28748	0.32189	0.05183	0.24360
	3	δ	1.70488	1.39610	1.61328	1.70442	1.79652	0.13674	−0.00785
		θ	0.70490	0.56174	0.62146	0.70547	0.78358	0.11780	0.06035
		$R(1)$	0.81649	0.75244	0.80076	0.81812	0.83412	0.02523	−0.44389
		$h(1)$	0.26785	0.25802	0.23289	0.26549	0.30269	0.05073	0.26045
$\mathcal{S}_2$	1	δ	2.12012	1.71551	2.02876	2.11993	2.21242	0.13895	−0.03581
		θ	0.67875	0.71366	0.60156	0.67956	0.75084	0.10878	0.07255
		$R(1)$	0.87881	0.82013	0.86851	0.87996	0.89056	0.01698	−0.48792
		$h(1)$	0.19689	0.26852	0.17194	0.19544	0.22075	0.03612	0.27805
	2	δ	1.68425	1.47094	1.58825	1.68506	1.77899	0.13749	0.07098
		θ	0.84661	0.83066	0.76259	0.84304	0.93005	0.11962	0.04259
		$R(1)$	0.81266	0.77029	0.79572	0.81457	0.83119	0.02572	−0.29661
		$h(1)$	0.32609	0.36437	0.28879	0.32443	0.36352	0.05503	0.21025
	3	δ	1.68935	1.47765	1.59572	1.68609	1.77855	0.13377	0.07314
		θ	0.82559	0.63312	0.74114	0.82271	0.90966	0.12298	0.07918
		$R(1)$	0.81371	0.77183	0.79724	0.81476	0.83112	0.02491	−0.32491
		$h(1)$	0.31675	0.27657	0.27736	0.31461	0.35079	0.05436	0.28526
$\mathcal{S}_3$	1	δ	2.15798	1.86332	2.06271	2.15792	2.25137	0.13729	0.14454
		θ	0.74578	0.79158	0.67012	0.74546	0.82224	0.11142	0.11278
		$R(1)$	0.88336	0.84484	0.87289	0.88444	0.89475	0.01591	−0.22485
		$h(1)$	0.21087	0.27088	0.18488	0.20956	0.23429	0.03621	0.32873
	2	δ	1.82763	1.69518	1.73442	1.82638	1.91578	0.13367	0.12385
		θ	0.85736	0.76249	0.77643	0.85412	0.93867	0.11953	0.05221
		$R(1)$	0.83777	0.81643	0.82349	0.83901	0.85277	0.02159	−0.26012
		$h(1)$	0.30121	0.29062	0.26634	0.29833	0.33361	0.04988	0.29036
	3	δ	1.86243	1.63762	1.76459	1.86529	1.95677	0.14061	0.12317
		θ	0.79707	0.61631	0.72013	0.80131	0.87467	0.11919	0.03519
		$R(1)$	0.84317	0.80556	0.82874	0.84515	0.85869	0.02193	−0.23488
		$h(1)$	0.27368	0.24361	0.24158	0.27169	0.30419	0.04713	0.25244
		δ	1.53202	1.32668	1.44701	1.53007	1.61764	0.12947	0.04637
$\mathcal{S}_4$	1	θ	0.74345	0.60097	0.66593	0.74052	0.82041	0.11459	0.07596
		$R(1)$	0.78209	0.73464	0.76473	0.78348	0.80163	0.02825	−0.34439
		$h(1)$	0.31494	0.28799	0.27636	0.31184	0.35151	0.05561	0.25228
	2	δ	1.46145	1.28186	1.37635	1.45918	1.54745	0.12887	0.04011
		θ	0.76024	0.68648	0.68097	0.75701	0.84119	0.12069	0.05982
		$R(1)$	0.76617	0.72248	0.74751	0.76757	0.78721	0.03021	−0.37683
		$h(1)$	0.33652	0.33802	0.29477	0.33514	0.37521	0.06116	0.26036
	3	δ	1.49676	1.11375	1.40627	1.49904	1.58976	0.13635	−0.05557
		θ	0.69695	0.63129	0.61659	0.69281	0.77564	0.11814	0.15571
		$R(1)$	0.77405	0.67168	0.75494	0.77666	0.79603	0.03108	−0.47954
		$h(1)$	0.30203	0.34369	0.26199	0.29914	0.34097	0.05874	0.29132

According to the optimum criteria $\mathcal{C}_i,\ i = 1,2,3,4$ presented in Section 6, utilizing the generated samples in Table 4, the best PC-T2 plan was also proposed; see Table 7. It is evident that

- Via criterion $\mathcal{C}_1$, the schemes $\mathcal{S}_2$ (in sample 1) and $\mathcal{S}_1$ (in samples 2 and 3) were the optimum plans.
- Via criteria $\mathcal{C}_i,\ i = 2,3,4$, the scheme $\mathcal{S}_4$ (in samples 1, 2, and 3) was the optimum plan.
- The ideal PC-T2 plans suggested here confirmed the findings listed in Section 5.

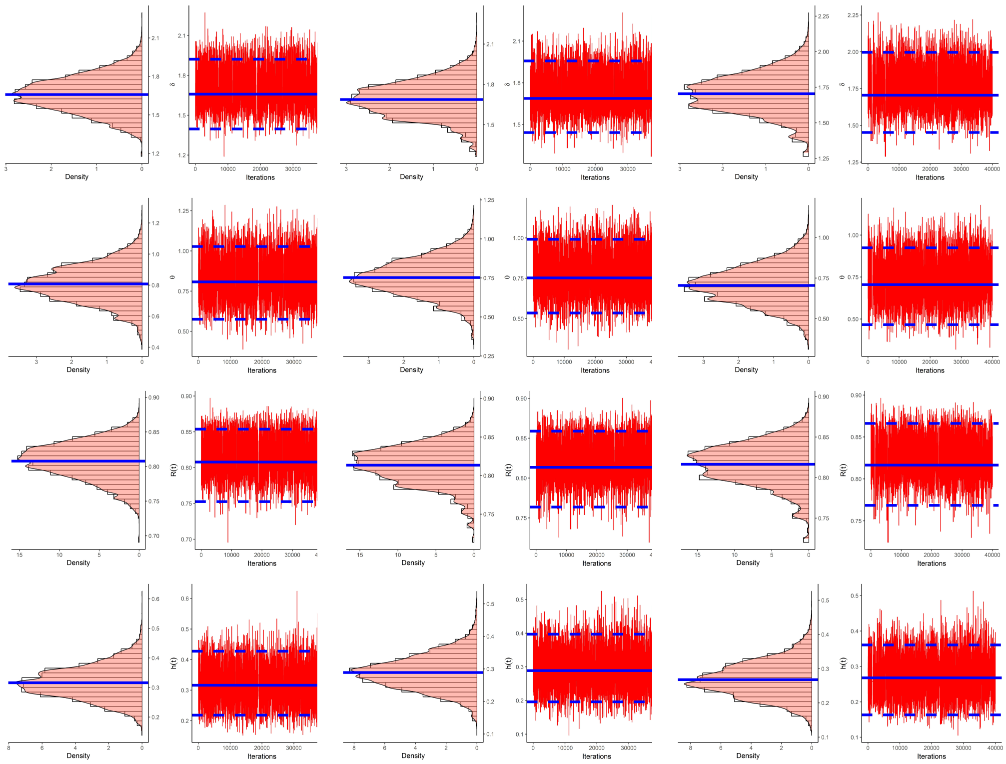

Figure 6. Density (**left**) and trace (**right**) plots of δ, θ, $R(t)$, and $h(t)$ from March precipitation data.

Table 7. Optimum PC-T2 plans from March precipitation data.

Sample	**Scheme**	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$		
$\varrho \rightarrow$					**0.3**	**0.6**	**0.9**
1	$\mathcal{S}_1$	33.1788	0.17613	0.00531	0.13763	2.38870	311.913
	$\mathcal{S}_2$	40.1670	0.20691	0.00515	0.32813	6.92420	1148.77
	$\mathcal{S}_3$	38.4794	0.24774	0.00644	0.29390	4.63153	535.519
	$\mathcal{S}_4$	34.9765	0.14503	0.00415	0.12876	1.95256	182.945
2	$\mathcal{S}_1$	32.4967	0.18138	0.00558	0.17374	2.88478	333.724
	$\mathcal{S}_2$	28.9297	0.19837	0.00686	0.13316	2.11421	253.737
	$\mathcal{S}_3$	29.5870	0.23382	0.00790	0.16429	1.92208	137.821
	$\mathcal{S}_4$	31.9176	0.15153	0.00475	0.11751	1.70550	135.791
3	$\mathcal{S}_1$	31.2816	0.18753	0.00599	0.21677	4.17175	580.510
	$\mathcal{S}_2$	27.1783	0.20486	0.00754	0.16105	3.54841	556.485
	$\mathcal{S}_3$	29.0505	0.24319	0.00837	0.21440	2.93514	262.146
	$\mathcal{S}_4$	31.2035	0.15850	0.00508	0.14613	2.00214	168.034

7.2. Vehicle Fatalities

For this application, we analyzed a real data set representing the number of vehicle fatalities for thirty-nine counties in South Carolina during 2012. These data were obtained from the National Highway Traffic Safety Administration (www-fars.nhtsa.dot.gov/States) and reported first by Mann [33]; see Table 8. First, to check the fit status, the KS statistics (with its p-value) and MLEs (with their SEs) based on all of vehicle fatalities data were computed. From Table 8, the MLEs (SEs) of δ and θ were 7.8474 (1.8243) and 0.9719 (0.1068), respectively, meanwhile the KS (p-value) was 0.1648 (0.240). It showed that the Fr distribution was a suitable life model for the vehicle fatalities data. Additionally, Figure 7 corroborated the same goodness-of-fit results and suggested taking the estimates $\hat{\delta} \cong 7.8474$ and $\hat{\theta} \cong 0.9719$ (that is, $\hat{\theta}$ existed and was unique) as initial guesses to run any proposed numerical evaluations.

Table 8. Motor vehicle deaths in South Carolina for 2012.

22	26	17	4	48	9	9	31	27	20
12	6	5	14	9	16	3	33	9	20
68	13	51	13	2	4	17	16	6	52
50	48	23	12	13	10	15	8	1	

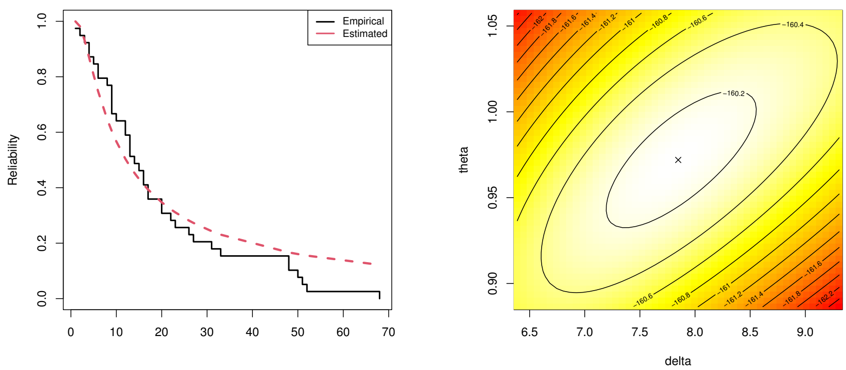

Figure 7. Empirical/fitted RFs (**left**); contour (**right**) plots from vehicle fatalities data.

To evaluate our acquired estimators, different artificial GPHC-T2 samples (when $m = 20$) based on different choices of $\underline{R}$ and T_i, $i = 1, 2$ were obtained from the vehicle fatalities data and are presented in Table 9. The censoring mechanisms used here were designed as $\mathcal{S}_1 : (1^*19, 0)$, $\mathcal{S}_2 : (3^*3, 0^*13, 1, 3^*3)$, $\mathcal{S}_3 : (6^*3, 0^*16, 1)$, and $\mathcal{S}_4 : (1, 0^*16, 6^*3)$.

From Table 9, the point and interval estimates obtained via the maximum likelihood and Bayes estimation approaches of δ, θ, $R(t)$, and $h(t)$ (at $t = 5$) were determined and are provided in Table 10. The Bayesian results were carried out under the noninformative priors. As a result, from Table 10, the point estimates of the unknown parameters had the same behavior, as they appeared to be near each other. A similar behavior was also observed in the case of interval estimates. This was an expected result due to the lack of additional historical information that could be used, which in turn made no significant difference between the proposed frequentist and Bayesian estimates.

Table 9. GPHC-T2 samples from vehicle fatalities data.

Scheme	Sample	$T_1(d_1)$	$T_2(d_2)$	Generated Data	R^*	T^*
$\mathcal{S}_1$	1	70(21)	75(21)	1, 2, 4, 5, 6, 9, 9, 10, 12, 13, 14, 16, 17, 20, 22, 26, 31, 48, 50, 52, 68	0	70
	2	35(17)	60(20)	1, 2, 4, 5, 6, 9, 9, 10, 12, 13, 14, 16, 17, 20, 22, 26, 31, 48, 50, 52	0	52
	3	25(15)	49(18)	1, 2, 4, 5, 6, 9, 9, 10, 12, 13, 14, 16, 17, 20, 22, 26, 31, 48	4	49
$\mathcal{S}_2$	1	55(22)	70(22)	1, 4, 8, 9, 10, 12, 12, 13, 13, 13, 14, 15, 16, 16, 17, 17, 20, 22, 31, 50, 51, 52	1	55
	2	18(16)	60(20)	1, 4, 8, 9, 10, 12, 12, 13, 13, 13, 14, 15, 16, 16, 17, 17, 20, 22, 31, 50	0	50
	3	21(17)	40(19)	1, 4, 8, 9, 10, 12, 12, 13, 13, 13, 14, 15, 16, 16, 17, 17, 20, 22, 31	4	40
$\mathcal{S}_3$	1	70(21)	75(21)	1, 6, 12, 16, 16, 17, 17, 20, 20, 22, 23, 26, 27, 31, 33, 48, 48, 50, 51, 52, 68	0	70
	2	40(15)	70(20)	1, 6, 12, 16, 16, 17, 17, 20, 20, 22, 23, 26, 27, 31, 33, 48, 48, 50, 51, 52	0	52
	3	19(7)	49(17)	1, 6, 12, 16, 16, 17, 17, 20, 20, 22, 23, 26, 27, 31, 33, 48, 48	4	49
$\mathcal{S}_4$	1	49(22)	70(22)	1, 3, 4, 4, 5, 6, 6, 8, 9, 9, 9, 9, 10, 12, 12, 13, 13, 13, 20, 33, 48, 48	4	49
	2	30(19)	50(20)	1, 3, 4, 4, 5, 6, 6, 8, 9, 9, 9, 9, 10, 12, 12, 13, 13, 13, 20, 33	0	33
	3	19(18)	32(19)	1, 3, 4, 4, 5, 6, 6, 8, 9, 9, 9, 9, 10, 12, 12, 13, 13, 13, 20	6	30

Table 10. Point and 95% interval estimates of δ, θ, $R(t)$, and $h(t)$ from vehicle fatalities data.

Scheme	Sample	Par.	MLE		MCMC		ACI			HPD		
			Est.	SE	Est.	SE	Lower	Upper	Width	Lower	Upper	Width
∞_2	1	δ	6.4352	1.6264	6.2790	0.1963	3.2476	9.6229	6.3753	6.0529	6.5206	0.4677
		θ	0.6926	0.1043	0.6494	0.0729	0.4881	0.8971	0.4090	0.5393	0.7655	0.2262
		$R(5)$	0.8789	0.0430	0.8887	0.0252	0.7945	0.9632	0.1687	0.8425	0.9308	0.0883
		$h(5)$	0.0403	0.0106	0.0359	0.0093	0.0195	0.0611	0.0415	0.0215	0.0527	0.0313
	2	δ	6.2622	1.5457	6.0989	0.2048	3.2327	9.2917	6.0590	5.8370	6.3190	0.4820
		θ	0.6751	0.1031	0.6313	0.0734	0.4729	0.8772	0.4043	0.5138	0.7441	0.2303
		$R(5)$	0.8791	0.0424	0.8887	0.0252	0.7961	0.9621	0.1660	0.8417	0.9307	0.0889
		$h(5)$	0.0392	0.0104	0.0349	0.0092	0.0189	0.0596	0.0407	0.0201	0.0511	0.0310
	3	δ	5.8676	1.4876	5.6906	0.2217	2.9520	8.7833	5.8313	5.3873	5.9128	0.5255
		θ	0.6320	0.1048	0.5869	0.0731	0.4265	0.8375	0.4109	0.4742	0.6977	0.2236
		$R(5)$	0.8802	0.0426	0.8893	0.0243	0.7966	0.9638	0.1672	0.8429	0.9296	0.0866
		$h(5)$	0.0365	0.0099	0.0324	0.0086	0.0172	0.0558	0.0386	0.0188	0.0477	0.0288
$\mathcal{S}_2$	1	δ	7.5035	2.0715	7.2561	0.2921	3.4434	11.564	8.1202	6.9272	7.5138	0.5866
		θ	0.6528	0.0995	0.6161	0.0627	0.4577	0.8479	0.3902	0.5127	0.7128	0.2001
		$R(5)$	0.9275	0.0326	0.9312	0.0155	0.8636	0.9914	0.1278	0.9007	0.9579	0.0572
		$h(5)$	0.0268	0.0078	0.0245	0.0062	0.0115	0.0421	0.0305	0.0141	0.0359	0.0218
	2	δ	7.7362	2.0966	7.4992	0.2785	3.6270	11.846	8.2186	7.2111	7.7641	0.5530
		θ	0.7523	0.1103	0.7077	0.0744	0.5362	0.9684	0.4323	0.5860	0.8211	0.2352
		$R(5)$	0.9003	0.0401	0.9079	0.0225	0.8216	0.9789	0.1573	0.8667	0.9483	0.0816
		$h(5)$	0.0384	0.0109	0.0344	0.0092	0.0170	0.0598	0.0428	0.0188	0.0507	0.0318
	3	δ	7.6067	2.0985	7.3867	0.2595	3.4937	11.720	8.2262	7.1073	7.6411	0.5337
		θ	0.7420	0.1124	0.6938	0.0783	0.5217	0.9623	0.4406	0.5712	0.8133	0.2421
		$R(5)$	0.9002	0.0404	0.9094	0.0236	0.8210	0.9794	0.1584	0.8676	0.9493	0.0817
		$h(5)$	0.0379	0.0108	0.0334	0.0095	0.0167	0.0592	0.0425	0.0178	0.0495	0.0317
$\mathcal{S}_3$	1	δ	9.2039	2.7553	8.9694	0.2811	3.8036	14.604	10.801	8.6897	9.2987	0.6090
		θ	0.7815	0.1100	0.7387	0.0745	0.5658	0.9971	0.4313	0.6238	0.8582	0.2344
		$R(5)$	0.9270	0.0378	0.9333	0.0191	0.8529	0.9978	0.1449	0.8984	0.9652	0.0668
		$h(5)$	0.0322	0.0112	0.0287	0.0085	0.0102	0.0543	0.0441	0.0149	0.0440	0.0291
	2	δ	8.9795	2.6017	8.7588	0.2627	3.8802	14.079	10.199	8.5049	9.0478	0.5430
		θ	0.7661	0.1083	0.7230	0.0748	0.5539	0.9784	0.4246	0.5989	0.8415	0.2425
		$R(5)$	0.9270	0.0368	0.9336	0.0190	0.8548	0.9991	0.1444	0.8984	0.9662	0.0678
		$h(5)$	0.0316	0.0109	0.0280	0.0084	0.0102	0.0530	0.0427	0.0138	0.0428	0.0291
	3	δ	8.1794	2.3471	7.9443	0.2780	3.5792	12.780	9.2004	7.6748	8.2128	0.5381
		θ	0.7061	0.1083	0.6605	0.0774	0.4938	0.9184	0.4246	0.5440	0.7805	0.2366
		$R(5)$	0.9276	0.0361	0.9341	0.0194	0.8569	0.9983	0.1414	0.8988	0.9659	0.0671
		$h(5)$	0.0289	0.0100	0.0255	0.0081	0.0094	0.0485	0.0391	0.0124	0.0395	0.0271

Table 10. *Cont.*

Scheme	Sample	Par.	MLE		MCMC		ACI			HPD		
			Est.	SE	Est.	SE	Lower	Upper	Width	Lower	Upper	Width
$\mathcal{S}_4$	1	δ	6.2440	1.5894	6.0212	0.2627	3.1289	9.3591	6.2302	5.7671	6.2754	0.5083
		θ	0.7337	0.1116	0.6827	0.0818	0.5149	0.9524	0.4375	0.5578	0.8022	0.2444
		$R(5)$	0.8530	0.0461	0.8641	0.0299	0.7625	0.9434	0.1809	0.8112	0.9156	0.1044
		$h(5)$	0.0485	0.0118	0.0431	0.0111	0.0254	0.0715	0.0461	0.0251	0.0616	0.0365
	2	δ	6.0989	1.6122	5.8599	0.2795	2.9389	9.2588	6.3199	5.6177	6.1841	0.5664
		θ	0.7190	0.1167	0.6667	0.0816	0.4902	0.9479	0.4577	0.5402	0.7844	0.2443
		$R(5)$	0.8530	0.0466	0.8637	0.0296	0.7615	0.9444	0.1829	0.8109	0.9164	0.1055
		$h(5)$	0.0475	0.0118	0.0422	0.0109	0.0245	0.0706	0.0461	0.0242	0.0600	0.0359
	3	δ	6.0137	1.5585	5.7737	0.2813	2.9591	9.0684	6.1093	5.4699	6.0277	0.5578
		θ	0.7203	0.1177	0.6635	0.0880	0.4896	0.9510	0.4614	0.5395	0.8025	0.2631
		$R(5)$	0.8484	0.0471	0.8609	0.0321	0.7561	0.9408	0.1847	0.7997	0.9141	0.1143
		$h(5)$	0.0486	0.0122	0.0426	0.0118	0.0247	0.0724	0.0478	0.0243	0.0630	0.0387

The vital statistics of δ, θ, $R(t)$, and $h(t)$, obtained based on 40,000 MCMC variates, namely, the mean, mode, median, first/third quartiles, St.D, and skewness were calculated and are listed in Table 11. Moreover, using the data sets generated by $\mathcal{S}_1$ as an example, the density and trace plots of δ, θ, $R(t)$, and $h(t)$ were plotted and are displayed in Figure 8. They demonstrated that the MCMC method converged effectively. It is also clear that the MCMC iterations of δ and θ were fairly symmetrical while those associated with $R(t)$ and $h(t)$ were negatively and positively skewed, respectively.

Table 11. MCMC properties of δ, θ, $R(t)$, and $h(t)$ from vehicle fatalities data.

Scheme	Sample	Par.	Mean	Mode	1st Quart.	Median	3rd Quart.	St.D	Skew.
$\mathcal{S}_1$	1	δ	6.27902	6.07238	6.19440	6.27871	6.35796	0.11885	0.12109
		θ	0.64937	0.60836	0.60836	0.64920	0.68829	0.05868	0.10751
		$R(5)$	0.88874	0.89782	0.87410	0.89025	0.90534	0.02314	−0.45401
		$h(5)$	0.03592	0.03158	0.02987	0.03539	0.04103	0.00819	0.47887
	2	δ	6.09886	5.83703	6.01562	6.10033	6.17950	0.12355	−0.00218
		θ	0.63130	0.59457	0.59062	0.63021	0.66983	0.05894	0.16323
		$R(5)$	0.88871	0.89373	0.87396	0.89032	0.90551	0.02330	−0.50969
		$h(5)$	0.03494	0.03169	0.02915	0.03425	0.03991	0.00811	0.54586
	3	δ	6.09886	5.83703	6.01562	6.10033	6.17950	0.12355	−0.00218
		θ	0.63130	0.59457	0.59062	0.63021	0.66983	0.05894	0.16323
		$R(5)$	0.88871	0.89373	0.87396	0.89032	0.90551	0.02330	−0.50969
		$h(5)$	0.03494	0.03169	0.02915	0.03425	0.03991	0.00811	0.54586
$\mathcal{S}_2$	1	δ	7.25614	6.92722	7.14857	7.26059	7.36376	0.15535	−0.05403
		θ	0.61609	0.58145	0.58145	0.61666	0.65115	0.05087	0.09922
		$R(5)$	0.93118	0.93395	0.92209	0.93217	0.94205	0.01510	−0.47310
		$h(5)$	0.02451	0.02235	0.02046	0.02411	0.02811	0.00574	0.50290
	2	δ	7.49919	7.29232	7.39787	7.49533	7.59669	0.14611	0.14063
		θ	0.70769	0.71028	0.66841	0.70805	0.74481	0.05960	0.06435
		$R(5)$	0.90789	0.90220	0.89555	0.90828	0.92348	0.02120	−0.45521
		$h(5)$	0.03443	0.03580	0.02853	0.03409	0.03933	0.00825	0.47964
	3	δ	7.38668	7.10877	7.28818	7.38192	7.47749	0.13747	0.20688
		θ	0.69382	0.73684	0.65019	0.69430	0.73573	0.06176	0.08418
		$R(5)$	0.90938	0.88600	0.89566	0.91127	0.92470	0.02170	−0.53495
		$h(5)$	0.03341	0.04118	0.02753	0.03274	0.03856	0.00838	0.53320

Table 11. *Cont.*

Scheme	Sample	Par.	Mean	Mode	1st Quart.	Median	3rd Quart.	St.D	Skew.
$\mathcal{S}_3$	1	δ	8.96935	8.58414	8.87126	8.97489	9.07186	0.15489	−0.11977
		θ	0.73871	0.79551	0.69502	0.73939	0.78131	0.06101	0.07219
		$R(5)$	0.93335	0.90800	0.92240	0.93476	0.94659	0.01795	−0.57091
		$h(5)$	0.02873	0.03846	0.02306	0.02822	0.03361	0.00776	0.54583
	2	δ	8.75879	8.52297	8.65921	8.75582	8.85316	0.14244	0.13097
		θ	0.72303	0.66433	0.68138	0.72298	0.76373	0.06108	0.10960
		$R(5)$	0.93364	0.94638	0.92273	0.93515	0.94626	0.01782	−0.63479
		$h(5)$	0.02805	0.02203	0.02266	0.02752	0.03260	0.00762	0.61696
	3	δ	7.94430	7.67479	7.83967	7.94385	8.04518	0.14838	0.20050
		θ	0.66047	0.71265	0.61505	0.65708	0.70477	0.06251	0.13608
		$R(5)$	0.93411	0.91262	0.92193	0.93677	0.94782	0.01824	−0.63194
		$h(5)$	0.02554	0.03326	0.01996	0.02442	0.03041	0.00734	0.62253
		δ	6.02117	5.85999	5.91772	6.01872	6.11777	0.13903	0.23533
$\mathcal{S}_4$	1	θ	0.68273	0.61467	0.63882	0.68367	0.72573	0.06396	0.05950
		$R(5)$	0.86408	0.88685	0.84640	0.86541	0.88432	0.02776	−0.37107
		$h(5)$	0.04313	0.03418	0.03578	0.04260	0.04920	0.00973	0.39941
	2	δ	5.85988	5.53493	5.76293	5.85735	5.95667	0.14492	0.07564
		θ	0.66665	0.68793	0.62290	0.66654	0.70781	0.06254	0.07949
		$R(5)$	0.86374	0.83947	0.84488	0.86631	0.88260	0.02761	−0.37640
		$h(5)$	0.04218	0.04813	0.03531	0.04143	0.04813	0.00947	0.41417
	3	δ	5.77368	5.47975	5.67229	5.76992	5.87478	0.14664	0.09816
		θ	0.66351	0.63075	0.61691	0.66383	0.70766	0.06721	0.06895
		$R(5)$	0.86094	0.86270	0.84218	0.86270	0.88203	0.02955	−0.42338
		$h(5)$	0.04258	0.03986	0.03520	0.04198	0.04890	0.01013	0.45365

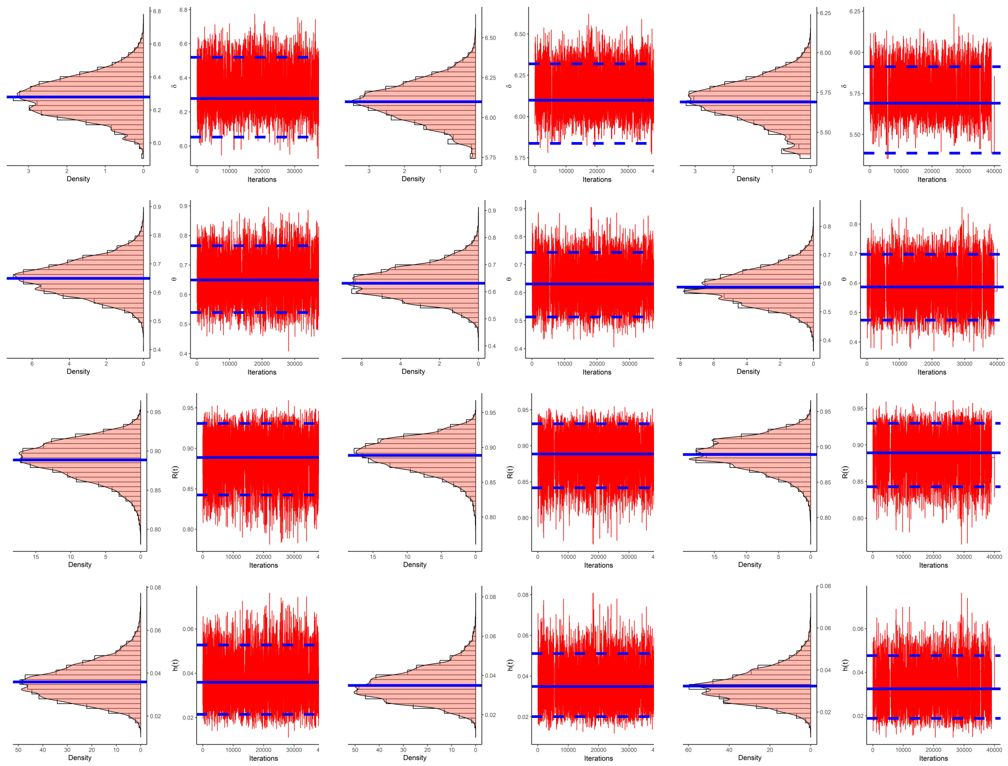

Figure 8. Density (**left**) and trace (**right**) plots of δ, θ, $R(t)$, and $h(t)$ from vehicle fatalities data.

Again, from Table 9, the problem of selecting an optimum PC-T2 plan is also illustrated based on vehicle fatalities data; see Table 12. It shows that:

- Via criterion $\mathcal{C}_1$, the schemes $\mathcal{S}_2$ (in samples 1 and 2) and $\mathcal{S}_3$ (in sample 3) were the optimum plans.
- Via criteria $\mathcal{C}_i,\ i = 2,3$, the scheme $\mathcal{S}_1$ (in samples 1, 2, and 3) was the optimum plan.
- Via criterion $\mathcal{C}_4$; the scheme $\mathcal{S}_4$ (in samples 1, 2, and 3) was the optimum plan.
- The ideal PC-T2 plans provided here supported our findings from Section 5 as well.

Finally, based on both physical and engineering scenarios, we can draw the conclusion that the investigated approaches provided an adequate interpretation of the Fréchet lifetime model when a sample was generated from the generalized Type-II progressive hybrid censoring mechanism.

Table 12. Optimum PC-T2 plans from vehicle fatalities data.

Sample	**Scheme**	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$		
$\varrho \rightarrow$					**0.3**	**0.6**	**0.9**
1	$\mathcal{S}_1$	224.0199	2.395783	0.010695	9.631151	256.7524	92,617.13
	$\mathcal{S}_2$	308.4324	4.30110	0.013945	17.72368	517.4279	227,824.3
	$\mathcal{S}_3$	205.2758	7.603747	0.037042	11.68548	179.4937	29,616.80
	$\mathcal{S}_4$	196.9198	2.538555	0.012891	4.878394	104.0239	26,231.06
2	$\mathcal{S}_1$	200.1599	2.492961	0.012455	8.04740	196.9935	60,769.58
	$\mathcal{S}_2$	206.0574	4.407945	0.021392	8.279247	157.4971	35,747.67
	$\mathcal{S}_3$	205.3611	6.780726	0.033019	12.5469	208.1168	37,439.37
	$\mathcal{S}_4$	190.6955	2.612971	0.01370	5.328222	125.0294	33,173.61
3	$\mathcal{S}_1$	201.6662	2.360953	0.011707	10.68031	328.4031	145,470.3
	$\mathcal{S}_2$	203.6842	4.416535	0.021683	8.808981	176.8395	40,252.86
	$\mathcal{S}_3$	207.2369	5.52049	0.026639	17.9720	393.9506	106,057.5
	$\mathcal{S}_4$	180.4068	2.442885	0.013541	5.187278	123.4222	34,735.87

8. Concluding Remarks

This work considered the generalized Type-II progressive hybrid censoring-based Fréchet model's reliability analysis of the unknown parameters, reliability and hazard rate functions. The Newton–Raphson iterative approach was used to calculate the frequentist estimates with their asymptotic confidence intervals for the unknown parameters and any function of them using the R programming language's "maxLik" package. The posterior density function was derived in nonlinear form since the likelihood function was generated in complex form. Therefore, using the Metropolis–Hastings method and taking into account the squared-error loss, the Bayesian estimates and the corresponding HPD intervals were constructed. Numerous simulation experiments based on various selections of total test units, observed failure data, threshold times, and progressive censoring plans were carried out to compare the behavior of the acquired estimates, and they demonstrated that the Bayes MCMC approach outperformed the frequentist approach quite satisfactorily. It is advised to use the Bayesian MCMC paradigm to estimate the Fréchet distribution's parameters, reliability, and hazard functions under generalized Type-II progressive hybrid censoring. To show how the suggested methods could be applied in practical situations, two applications representing the successive values (in inches) of precipitation in Minneapolis–Saint Paul and the number of vehicle fatalities in South Carolina were examined. We anticipate reliability practitioners will find the findings and methodology presented here useful and that they will be applied to further filtering strategies.

Supplementary Materials: The following supporting information can be downloaded at: https://www.mdpi.com/article/10.3390/sym15020348/s1, Table S1: The APEs (1st column), RMSEs (2nd column) and MRABs (3rd column) of δ; Table S2: The APEs (1st column), RMSEs (2nd column) and MRABs (3rd column) of θ; Table S3: The APEs (1st column), RMSEs (2nd column) and MRABs (3rd column) of $R(t)$; Table S4: The APEs (1st column), RMSEs (2nd column) and MRABs (3rd column) of $h(t)$; Table S5: The ACLs (1st column) and CPs (2nd column) of 95% ACI/HPD credible intervals of δ; Table S6: The ACLs (1st column) and CPs (2nd column) of 95% ACI/HPD credible intervals of θ;

Table S7: The ACLs (1st column) and CPs (2nd column) of 95% ACI/HPD credible intervals of $R(t)$;
Table S8: The ACLs (1st column) and CPs (2nd column) of 95% ACI/HPD credible intervals of $h(t)$.

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