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On a New Subclass of q-Starlike Functions Defined in q-Symmetric Calculus

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Abstract: Geometric function theory combines geometric tools and their applications for information and communication analysis. It is also successfully used in the field of advanced signals, image processing, machine learning, speech and sound recognition. Various new subclasses of analytic functions have been defined using quantum calculus to investigate many interesting properties of these subclasses. This article defines a new class of q-starlike functions in the open symmetric unit disc ∇ using symmetric quantum calculus. Extreme points for this class as well as coefficient estimates and closure theorems have been investigated. By fixing several coefficients finitely, all results were generalized into families of analytic functions.

Keywords: univalent function; starlike function; uniformly convex function; q-starlike function; q-symmetric derivative operator

MSC: Primary 30C45; 30C50

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1. Introduction

Let $\nabla = \{w : w \in \mathbb{C}, |w| < 1\}$ be the open symmetric unit disc, and let the class of analytic functions be $\xi(w)$, which satisfies the normalization conditions $\xi(0) = 0$ and $\xi'(0) = 1$ in ∇. We generally represent functions of such a class $\mathcal{A}$ in the following series expansion form:

$$\xi(w) = w + \sum_{l=2}^{\infty} a_l w^l, \quad w \in \nabla. \tag{1}$$

The convolution of ξ and g are defined as:

$$\xi(w) * g(w) = w + \sum_{l=2}^{\infty} a_l b_l w^l, \quad \xi, g \in \mathcal{A},\ w \in \nabla,$$

where

$$g(w) = w + \sum_{l=2}^{\infty} b_l w^l.$$

Let $\mathcal{S} \subset \mathcal{A}$, which are univalent in ∇. The subclasses of starlike $\mathcal{ST}$ and convex functions $\mathcal{CV}$ of class $\mathcal{S}$ were defined by Goodman [1]. In [2], Ma and Minda generalized the convex and starlike functions by introducing the class of uniformly convex $\mathcal{UCV}$ and starlike $\mathcal{UST}$ functions, which were defined as:

$$\xi \in \mathcal{UCV} \Leftrightarrow \left| \frac{w\xi''(w)}{\xi'(w)} \right| \le \Re\left\{ 1 + \frac{w\xi''(w)}{\xi'(w)} \right\}$$

and

$$\xi \in \mathcal{ST} \Leftrightarrow \left| \frac{w\xi'(w)}{\xi(w)} - 1 \right| \leq \Re\left\{ \frac{w\xi'(w)}{\xi(w)} \right\}.$$

Note that:

$$\xi(w) \in \mathcal{UCV} \Leftrightarrow w\xi'(w) \in \mathcal{ST}.$$

Furthermore, Rønning [3] introduced the class $\mathcal{S}_p(\alpha)$ by using the parameter α, $-1 \leq \alpha < 1$,

$$\xi \in \mathcal{S}_p(\alpha) \Leftrightarrow \left| \frac{w\xi'(w)}{\xi(w)} - 1 \right| \leq \Re\left\{ \frac{w\xi'(w)}{\xi(w)} - \alpha \right\}.$$

Srivastava [4] studied such generalizations in detail within the context of univalent function theory.

The q-calculus operator theory has been used in many branches of mathematics, for example, in physics and mathematics; hypergeometric series; complex analysis; and particle physics, as well as in geometric function theory (GFT). Furthermore, its application has been studied on various differential and integral operators. The q-derivative operator (D_q) has a great importance in the study of a number of subclasses of analytic functions. Particularly, the theory of univalent functions are described by using the q-calculus operator theory. The q-analogue of the derivative and integral operator were defined by Jackson [5], in which he also discussed some of their applications in the field of geometric function theory. In [6], Ismail et al. discussed the idea of q-starlike functions. Arif et al. used the concepts of convolution to define the q-Noor integral operator [7] and then applied this operator to define new subclasses of analytic functions. Furthermore, in [8], the authors implemented the q-calculus operator theory to define the q-analogue of the differential operator and investigated a new subclass of analytic functions. Srivastava et al. [9] considered the q-derivative operator in order to define a class of k-symmetric harmonic functions. A number of mathematicians have been working in this field to define new subfamilies of analytic functions. Several subclasses of analytic functions have been defined by using the q-fractional integral and differential operators [10–17].

The q-symmetric quantum calculus has found its applications in many fields of knowledge including quantum mechanics [18,19]. Some important properties of q-symmetric derivatives for the q-exponential function have been discussed in [19]. Such properties are generally not exhibited by the usual derivative. It is well known that the derivative of a differentiable function f can be approximated by the q-symmetric derivative. We believed that the q-symmetric derivative has, in general, better convergence properties than the h-derivative and the q-derivative, but this required additional investigation.

In the area of GFT, several academics have recently researched q-symmetric calculus. In order to create the symmetric operator of the q-derivative, Kanas et al. [20] implemented the fundamental ideas of q-symmetric calculus. They next examined the application of this operator by defining a new subclasses of analytic univalent functions in the open symmetric unit disc *nabla*. For the first time in the literature, Shahid et al. [21] studied q-symmetric calculus in the conic domain. In article [22], the authors generalized the conic domain by using the basic concepts of q-symmetric calculus. Khan et al. [23] also defined a novel subclass of multivalent q-starlike functions and looked into its relevant properties using a q-symmetric calculus operator.

To better explain our main results in this paper, we must first discuss basic definitions and concepts of the q-symmetric calculus. Therefore, we began with the idea of q-number:

Definition 1 ([24])**.** *Let $q \in (0,1)$ and $\eta \in \mathbb{C}$, so the q-number is defined as:*

$$[\eta]_q = \frac{1-q^\eta}{1-q}, \qquad [0]_q = 0,$$

$$[l]_q = 1+q+q^2+\ldots+q^{l-1}, \ \ \eta = l \in \mathbb{N}.$$

Definition 2. *For $l \in \mathbb{N}$, the q-symmetric number is defined as:*

$$\widetilde{[l]}_q = \frac{(q)^l-(q)^{-l}}{q-q^{-1}}, \ \ \widetilde{[0]}_q = 0.$$

Remark 1. *The q-symmetric number can be reduced into a symmetric number for $q \to 1-$, but it cannot be reduced into the q-number, as shown in [25].*

It is important to note that ordinary calculus is limiting in the case of the symmetric quantum calculus. A study of q-symmetric operators is expected to be of great importance in the development of q-function theory, which plays an important role in combinatorial analysis. This subject has received very little attention, particularly in relation to geometric function theory.

Definition 3. *For any non-negative integer l, the q-symmetric number shift factorial is defined as:*

$$\widetilde{[l]}_q! = \begin{cases} \widetilde{[l]}_q\widetilde{[l-1]}_q\widetilde{[l-2]}_q\widetilde{[2]}_q\widetilde{[1]}_q, & l \geq 1, \\ 1, & l = 0. \end{cases}$$

We have $\lim_{q\to 1-}\widetilde{[l]}_q! = l$.

Definition 4 ([5])**.** *The q-derivative operator or q-difference operator for $\xi \in \mathcal{A}$ is defined as:*

$$\partial_q\xi(w) = \frac{\xi(qw)-\xi(w)}{w(q-1)}, \quad w \in \nabla.$$

For $l \in \mathbb{N} := \{1,2,3,\ldots\}$ and $w \in \nabla$

$$\partial_q w^l = [l]_q w^{l-1}, \ \ \partial_q\left\{\sum_{l=1}^{\infty} a_l w^l\right\} = \sum_{l=1}^{\infty}[l]_q a_l w^{l-1}.$$

Definition 5. *As suggested in [26], the q-symmetric derivative operator for $\xi \in \mathcal{A}$ is defined as:*

$$\widetilde{\partial}_q\xi(w) = \frac{\xi(qw)-\xi(q^{-1}w)}{w(q-q^{-1})}, \quad w \in \nabla.$$

Note that:

$$\widetilde{\partial}_q w^l = \widetilde{[l]}_q w^{l-1}, \ \ \widetilde{\partial}_q\left\{\sum_{l=1}^{\infty} a_l w^l\right\} = \sum_{l=1}^{\infty}\widetilde{[l]}_q a_l w^{l-1}.$$

It is easy to see that following properties hold the following:

$$\begin{aligned}
\widetilde{\partial}_q(\xi(w)+g(w)) &= \left(\widetilde{\partial}_q\xi\right)(w)+\left(\widetilde{\partial}_q g\right)(w),\\
\widetilde{\partial}_q(\xi(w)g(w)) &= \xi(qw)\left(\widetilde{\partial}_q g\right)(w)+g(q^{-1}w)\left(\widetilde{\partial}_q\xi\right)(w),\\
\widetilde{\partial}_q\left(\frac{\xi(w)}{g(w)}\right) &= \frac{\xi(qw)\left(\widetilde{\partial}_q g\right)(w)-g(q^{-1}w)\left(\widetilde{\partial}_q\xi\right)(w)}{g(q^{-1}w)g(qw)},\\
\widetilde{\partial}_q\xi(w) &= \widetilde{\partial}_{q^2}\xi(q^{-1}w).
\end{aligned}$$

Throughout in this paper, we assumed that:

$$0<q<1,\ -1\le\alpha<1,\quad \text{and}\quad k\ge 0.$$

Heat transfer and other issues in cylindrical and spherical coordinates have been solved using the q-symmetric operator theory, which has been linked to a wide range of issues in significant areas of mathematical physics and engineering. Using q-differential operators, various new subclasses of convex and starlike functions have been defined to discover many interesting properties of new subclasses of analytic functions. The study of certain subclasses of starlike functions and its generalization is a classical focus in the field of geometric function theory. In this paper, we investigated several properties of specific subclasses using a well-known q-operator.

For example, we considered the q-symmetric derivative operator and introduced the following new subclasses of q-starlike functions.

Definition 6. *Let $\xi\in\mathcal{S}_{(\alpha,k,q)}$ of the form (1) if it satisfies the following condition:*

$$\Re\left\{\frac{w\widetilde{\partial}_q\xi(w)}{\xi(w))}-\alpha\right\}>k\left|\frac{w\widetilde{\partial}_q\xi(w)}{\xi(w)}-1\right|.$$

Let $\mathcal{TS}_{(\alpha,k,q)}=\mathcal{S}_{(\alpha,k,q)}\cap\mathcal{T}$ and $\mathcal{T}\subset\mathcal{S}$, consisting of functions of the form

$$\xi(w)=w-\sum_{l=2}^{\infty}a_lw^l,\quad a_l\ge 0,\ \textit{for all } l\ge 2. \tag{2}$$

Remark 2. *(i) If $q\to 1-$, then the subclass $\mathcal{S}_{(\alpha,k,q)}=\mathcal{S}_{(\alpha,k)}$, which is a subclass of starlike functions.*

(ii) If $q\to 1-$ and $k=0$, then $\mathcal{S}_{(\alpha,k,q)}=\mathcal{S}^*(\alpha)$ $(0\le\alpha<1)$, which is a class of starlike functions of order α.

(iii) If $q\to 1-$, $\alpha=0$, and $k=0$, then $\mathcal{S}_{(\alpha,k,q)}=\mathcal{S}^*(\alpha)$and $(0\le\alpha<1)$, which is a class of starlike functions, respective to the origin.

In this paper, we used the q-symmetric derivative operator for $\xi\in\mathcal{A}$, so the new subclasses $\mathcal{S}_{(\alpha,k,q)}$ and $\mathcal{TS}_{(\alpha,k,q)}$ of q-starlike functions were defined. Then, in Theorems 1 and 2, the necessary and sufficient conditions were proved for the classes $\mathcal{S}_{(\alpha,k,q)}$ and $\mathcal{TS}_{(\alpha,k,q)}$. By fixing the second coefficient in $\mathcal{TS}_{(\alpha,k,q)}$, the new subclass $\mathcal{TS}_{c(\alpha,k,q)}$ was introduced, and for this class, we found the necessary and sufficient conditions in Theorems 3 and 4. Therefore, we also proved that the class $\mathcal{TS}_{c(\alpha,k,q)}$ was closed using a convex combination. Next, we provided several theorems that included conditions for the functions in class $\mathcal{TS}_{c(\alpha,k,q)}$. In the final step, by fixing several coefficients finitely, we introduced the new subclass $\mathcal{TS}_{c_l,i(\alpha,k,q)}$ and investigated extreme points for the class $\mathcal{TS}_{c_l,i(\alpha,k,q)}$ in Theorem 7.

2. Main Results

Theorem 1. *The function $\xi(w)$ of the form (1) is in $\mathcal{S}_{(\alpha,k,q)}$ if*

$$\sum_{l=2}^{\infty}\left\{[\widetilde{l}]_q(1+k)-(\alpha+k)\right\}|a_l| \leq 1-\alpha. \tag{3}$$

Proof. It is sufficient to show that

$$k\left|\frac{w\widetilde{\partial}_q\xi(w)}{\xi(w))}-1\right|-\Re\left\{\frac{w\widetilde{\partial}_q\xi(w)}{\xi(w))}-1\right\} \leq 1-\alpha$$

in order to find

$$\begin{aligned} & k\left|\frac{w\widetilde{\partial}_q\xi(w)}{\xi(w))}-1\right|-\Re\left\{\frac{w\widetilde{\partial}_q\xi(w)}{\xi(w))}-1\right\} \\ \leq \quad & (1+k)\left|\frac{w\widetilde{\partial}_q\xi(w)}{\xi(w))}-1\right| \\ \leq \quad & \frac{(1+k)\sum_{l=2}^{\infty}([\widetilde{l}]_q-1)|a_l|}{1-\sum_{l=2}^{\infty}|a_l|}. \end{aligned}$$

The above inequality should be bounded with $(1-\alpha)$ if

$$\sum_{l=2}^{\infty}\left\{[\widetilde{l}]_q(1+k)-(\alpha+k)\right\}|a_l| \leq 1-\alpha.$$

Hence, the proof is complete. □

Theorem 2. *Let the function $\xi(w)$ of the form (2) belong to the class $\mathcal{TS}_{(\alpha,k,q)}$ if*

$$\sum_{l=2}^{\infty}\left\{[\widetilde{l}]_q(1+k)-(\alpha+k)\right\}a_l \leq 1-\alpha. \tag{4}$$

Proof. Now, in this theorem, we had to prove the necessary conditions. If $\xi(w) \in \mathcal{TS}_{(\alpha,k,p,q)}$ and w is real, then

$$\begin{aligned} & \frac{1-\sum_{l=2}^{\infty}[\widetilde{l}]_q a_l w^{l-1}}{1-\sum_{l=2}^{\infty}a_l w^{l-1}}-\alpha \\ \geq \quad & k\left|\frac{\sum_{l=2}^{\infty}\left\{[\widetilde{l}]_q-1\right\}w^{l-1}}{1-\sum_{l=2}^{\infty}a_l w^{l-1}}\right|. \end{aligned}$$

If $w \to 1$ along the real axis, we achieve the desired result, as follows:

$$\sum_{l=2}^{\infty}\left\{[\widetilde{l}]_q(1+k)-(\alpha+k)\right\}a_l \leq 1-\alpha.$$

□

Corollary 1. *If the function $\xi(w)$ is defined by (2) and $\xi(w) \in \mathcal{TS}_{(\alpha,k,q)}$, then*

$$a_l \leq \frac{1-\alpha}{[\widetilde{l}]_q(1+k)-(\alpha+k)}, \quad l \geq 2.$$

Corollary 2. *If the function* $\xi(w)$ *is defined by* (2) *and* $\xi(w) \in \mathcal{TS}_{(\alpha,k,q)}$, *then*

$$a_2 = \frac{1-\alpha}{\widetilde{[2]}_q(1+k)-(\alpha+k)}. \tag{5}$$

The Class $TS_{c(\alpha,k,q)}$

By fixing the second coefficient in $TS_{(\alpha,k,q)}$**, we introduced a new subclass,** $TS_{c(\alpha,k,q)}$ **as follows:**

Definition 7. *Let* $\xi(w) \in \mathcal{TS}_{(\alpha,k,q)}$, *so then* $\xi(w) \in \mathcal{TS}_{c(\alpha,k,q)}$ $(0 < c \leq 1)$, *if it can be represented as:*

$$\xi(w) = w - \frac{c(1-\alpha)}{\widetilde{[2]}_q(1+k)-(\alpha+k)}w^2 - \sum_{l=3}^{\infty} a_l w^l. \tag{6}$$

Theorem 3. *Let the function* $\xi(w)$ *be defined by* (6), *so then* $\xi(w) \in \mathcal{TS}_{c(\alpha,k,q)}$, *if*

$$\sum_{l=3}^{\infty}\left\{\widetilde{[l]}_q(1+k)-(\alpha+k)\right\}a_l \leq (1-c)(1-\alpha). \tag{7}$$

Proof. We substituted

$$a_2 = \frac{c(1-\alpha)}{\widetilde{[2]}_q(1+k)-(\alpha+k)}$$

in (4), and after additional calculations, we achieved the required result. □

Corollary 3. *If the function* $\xi(w)$ *is defined by* (6) *and* $\xi(w) \in \mathcal{TS}_{c(\alpha,k,p,q)}$, *then*

$$a_l \leq \frac{(1-c)(1-\alpha)}{\widetilde{[l]}_q(1+k)-(\alpha+k)}, \quad l \geq 3. \tag{8}$$

Theorem 4. *Using a convex linear combination, the class* $\mathcal{TS}_{c(\alpha,k,q)}$ *was closed.*

Proof. Let $\xi(w)$ and $g(w) \in \mathcal{TS}_{c(\alpha,k,q)}$. Then suppose $\xi(w)$ is provided by (6) and

$$g(w) = w - \frac{(1-\alpha)}{\widetilde{[l]}_q(1+k)-(\alpha+k)}w^2 - \sum_{l=3}^{\infty} d_l w^l, \tag{9}$$

where

$$d_l \geq 0$$

This then proves the following function

$$\mathcal{H}(w) = \lambda\xi(w) + (1-\lambda)g(w), \;\; 0 \leq \lambda \leq 1 \tag{10}$$

is in $\mathcal{TS}_{c(\alpha,k,q)}$. Using (6) and (9) on (10), we found

$$\begin{aligned} \mathcal{H}(w) \;\;=\;\; & w - \frac{c(1-\alpha)}{\left\{\widetilde{[2]}_q(1+k)-(\alpha+k)\right\}}w^2 \\ & -\sum_{l=3}^{\infty}\{\lambda a_l + (1-\lambda)d_l\}w^l. \end{aligned} \tag{11}$$

With the knowledge that $\xi(w)$ and $g(w)$ belong to $\mathcal{TS}_{c(\alpha,k,q)}$, as well as $0 \leq \lambda \leq 1$, we used Theorem 3 to obtain

$$\begin{aligned} & \sum_{l=3}^{\infty}\left\{\widetilde{[l]}_q(1+k)-(\alpha+k)\right\}\{\lambda a_l+(1-\lambda)d_l\} \\ \leq \quad & (1-c)(1-\alpha). \end{aligned} \tag{12}$$

Once again, according to Theorem 3 and inequality (12), we found $\mathcal{H}(w) \in \mathcal{TS}_{c(\alpha,k,q)}$. □

Theorem 5. *If every j is $(j = 1, 2, 3, \ldots, m)$ and*

$$\xi_j(w) = w - \frac{c(1-\alpha)}{\left\{\widetilde{[2]}_q(1+k)-(\alpha+k)\right\}}w^2 - \sum_{l=3}^{\infty} a_{l,j}w^l, \;\; a_{l,j} \geq 0 \tag{13}$$

is in the class $\mathcal{TS}_{c(\alpha,k,q)}$, then

$$\mathcal{F}(w) = \sum_{j=1}^{m} u_j \xi_j(w), \tag{14}$$

also belongs to $\mathcal{TS}_{c(\alpha,k,q)}$, where

$$\sum_{j=1}^{m} u_j = 1. \tag{15}$$

Proof. Based on (13)–(15), we found

$$\mathcal{F}(w) = w - \frac{c(1-\alpha)}{\left\{\widetilde{[2]}_q(1+k)-(\alpha+k)\right\}}w^2 - \sum_{l=3}^{\infty}\left(\sum_{j=1}^{m} u_j a_{l,j}\right)w^l.$$

For every $j = 1, 2, 3, \ldots, m$, $\xi_j(w) \in \mathcal{TS}_{c(\alpha,k,q)}$. Theorem 3 yields

$$\sum_{l=3}^{\infty}\left\{\widetilde{[l]}_q(1+k)-(\alpha+k)\right\}a_{l,j} \leq (1-c)(1-\alpha). \tag{16}$$

To prove $\xi(w) \in \mathcal{TS}_{c(\alpha,k,q)}$, it was sufficient to show that $\mathcal{F}(w)$ satisfied the condition of (8). Therefore,

$$\begin{aligned} & \sum_{l=3}^{\infty}\left\{\widetilde{[l]}_q(1+k)-(\alpha+k)\right\}\left(\sum_{j=1}^{m} u_j a_{l,j}\right) \\ = \quad & \sum_{j=1}^{m} u_j\left(\sum_{l=3}^{\infty}\left\{\widetilde{[l]}_q(1+k)-(\alpha+k)\right\}a_{l,j}\right). \end{aligned} \tag{17}$$

After combining (15) and (16) in (17), we obtained

$$\sum_{l=3}^{\infty}\left\{\widetilde{[l]}_q(1+k)-(\alpha+k)\right\}\left(\sum_{j=1}^{m} u_j a_{l,j}\right) \leq (1-c)(1-\alpha).$$

Therefore, $\mathcal{F}(w) \in \mathcal{TS}_{c(\alpha,k,q)}$. □

Theorem 6. *Let*

$$\xi_2(w) = w - \frac{c(1-\alpha)}{\left\{\widetilde{[2]}_q(1+k)-(\alpha+k)\right\}}w^2 \tag{18}$$

and

$$\xi_l(w) = w - \frac{c(1-\alpha)}{\left\{\widetilde{[2]}_q(1+k) - (\alpha+k)\right\}} w^2 - \frac{(1-c)(1-\alpha)}{\left\{\widetilde{[l]}_q(1+k) - (\alpha+k)\right\}} w^l, \tag{19}$$

for $l = 3, 4, \ldots$. *Then* $\xi(w) \in \mathcal{TS}_{c(\alpha,k,q)}$, *if, and only if,*

$$\xi(w) = \sum_{l=2}^{\infty} \lambda_l \xi_l(w), \tag{20}$$

where

$$\lambda_l \geq 0 \text{ and } \sum_{l=2}^{\infty} \lambda_l = 1.$$

Proof. Using (18) and (19) in (20), we obtained

$$\xi(w) = w - \sum_{l=2}^{\infty} \mathcal{B}_l w^l, \tag{21}$$

where

$$\mathcal{B}_2 = \frac{c(1-\alpha)}{\left\{\widetilde{[2]}_q(1+k) - (\alpha+k)\right\}} \tag{22}$$

and

$$\mathcal{B}_l = \frac{(1-c)(1-\alpha)}{\left\{\widetilde{[l]}_q(1+k) - (\alpha+k)\right\}}, \quad l \geq 3. \tag{23}$$

To prove $\xi(w) \in \mathcal{TS}_{c(\alpha,k,q)}$, it was sufficient to show that it satisfied the condition of Theorem 3. Therefore,

$$\begin{aligned} & \sum_{l=2}^{\infty} \left\{\widetilde{[l]}_q(1+k) - (\alpha+k)\right\} \mathcal{B}_l \\ = \quad & c(1-\alpha) + \sum_{l=3}^{\infty} \lambda_l (1-c)(1-\alpha). \end{aligned}$$

Since

$$\sum_{l=2}^{\infty} \lambda_l = 1,$$

the above equation can be written as:

$$\begin{aligned} & \sum_{l=2}^{\infty} \left\{\widetilde{[l]}_q(1+k) - (\alpha+k)\right\} \mathcal{B}_l \\ = \quad & (1-\alpha)[c + (1-\lambda_2)(1-c)] \leq (1-\alpha). \end{aligned}$$

Hence, $\xi(w) \in \mathcal{TS}_{c(\alpha,k,q)}$.

Conversely, if $\xi(w)$ is defined by (6), belonging to the class $\mathcal{TS}_{c(\alpha,k,q)}$, then by using (8), the results are

$$a_l \leq \frac{(1-c)(1-\alpha)}{\left\{\widetilde{[l]}_q(1+k) - (\alpha+k)\right\}}, \quad l \geq 3. \tag{24}$$

By taking

$$\lambda_l = \frac{\left\{\widetilde{[l]}_q(1+\beta) - (\alpha+\beta)\right\} a_l}{(1-c)(1-\alpha)} \tag{25}$$

and

$$\lambda_2 = 1 - \sum_{l=3}^{\infty} \lambda_l \tag{26}$$

we obtained (20). Therefore, the proof is complete. □

Corollary 4. *The extreme points for the class $\mathcal{TS}_{c(\alpha,k,q)}$ are given by Theorem 6.*

The Class $\mathcal{TS}_{c_l,i(\alpha,k,q)}$

By fixing several coefficients finitely, we introduced a new subclass $TS_{c_l,i(\alpha,k,q)}$ as follows:

Definition 8. *Let $\mathcal{TS}_{c_l,i(\alpha,k,q)}$ denote the class of functions in $\mathcal{TS}_{c(\alpha,k,q)}$ of the form*

$$\xi(w) = w - \sum_{l=2}^{i} \frac{c_l(1-\alpha)}{\left\{[\widetilde{l}]_{p,q}(1+k) - (\alpha+k)\right\}} w^l - \sum_{l=i+1}^{\infty} a_l w^l,$$

where

$$0 \leq \sum_{l=2}^{i} c_l = c \leq 1.$$

Note that

$$\mathcal{TS}_{c_2,2(\alpha,k,q)} = \mathcal{TS}_{c(\alpha,k,q)}.$$

Theorem 7. *The extreme points of the class $\mathcal{TS}_{c_l,i(\alpha,k,q)}$ are*

$$\xi_i(w) = w - \sum_{l=2}^{i} \frac{c_l(1-\alpha)}{\left\{[\widetilde{l}]_q(1+k) - (\alpha+k)\right\}} w^l$$

and

$$\xi_l(w) = w - \sum_{l=2}^{i} \frac{c_l(1-\alpha)}{\left\{[\widetilde{l}]_q(1+k) - (\alpha+k)\right\}} w^l - \sum_{l=i+1}^{\infty} \frac{(1-c)(1-\alpha)}{\left\{[\widetilde{2}]_q(1+k) - (\alpha+k)\right\}} w^l.$$

Proof. The details of the proof have been omitted. □

3. Conclusions

By using a q-differential operator several new subclasses of convex and starlike functions have been defined to obtain many interesting results. In this article, we used symmetric quantum calculus to define a new subclass of starlike functions in the open symmetric unit disc ∇, and we employed coefficient estimates, closure theorems, and extreme points to obtain our results. By fixing several coefficients finitely, we were able to generalize our approach for any subclass of analytic functions.

The mathematical strategy outlined in this study can also be modified to define new subfamilies of starlike functions related to q-symmetric calculus. The application of q-symmetric calculus operator theory should assist researchers in defining additional new subclasses of analytic functions.

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