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Classification of the Lie and Noether Symmetries for the Klein–Gordon Equation in Anisotropic Cosmology

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Abstract: We carried out a detailed group classification of the potential in Klein–Gordon equation in anisotropic Riemannian manifolds. Specifically, we consider the Klein–Gordon equations for the four-dimensional anisotropic and homogeneous spacetimes of Bianchi I, Bianchi III and Bianchi V. We derive all the closed-form expressions for the potential function where the equation admits Lie and Noether symmetries. We apply previous results which connect the Lie symmetries of the differential equation with the collineations of the Riemannian space which defines the Laplace operator, and we solve the classification problem in a systematic way.

Keywords: Lie symmetries; Klein–Gordon; anisotropic spacetimes; Noether symmetries; conformal Killing vectors

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1. Introduction

A systematic approach for the study of nonlinear differential equations is the Lie symmetry analysis [1–4]. The novelty of the Lie symmetry approach is that through a systematic approach the existence of invariant transformations can be determined. The latter can be used to simplify the given differential equation with the use of similarity transformations. Under the application of similarity transformations in a given differential equation, we derive a new differential equation with less independent variables. Furthermore, conservation laws can construct which are essential for the study of the properties for the given differential equation [1]. Lie symmetries have been applied for the study of nonlinear differential equations in all areas of applied mathematics [5–13].

A systematic approach for the construction and the determination of conservation laws for differential equations was established by E. Noether. In Noether's famous work of 1918 [14], Noether showed that some of the Lie symmetries were related to symmetries of the variational principle. For each symmetry of the variation integral, Noether derived an exact formula for the derivation of the conservation law. That very simple method for the construction of conservation laws is very important in physical science and in other theories of applied mathematics.

In General Relativity the natural space is of four-dimensions described by a Riemannian manifold. In this work we investigate the Lie and Noether symmetries for the Klein–Gordon equation in anisotropic homogeneous geometries. Anisotropic homogeneous spacetimes are of special interest because they can describe the very early period of the universe, that is, before the inflationary era where anisotropies played an important role in the evolution of the physical variables. There is a plethora of studies in literature where symmetry analysis has been applied for the classification of the geodesic equations [15–18], the wave equation [19,20] in curved spaces and the gravitational field equations [21–24].

There has been investigated a relation between the symmetries of some differential equations of special interest and the collineations of the background geometry which provides the related differential operators. Indeed, the Lie symmetries of the geodesic equations in Riemannian manifolds are constructed by the elements of the projective group of the background spacetime [25,26]. The latter relation is true if we consider the existence of a force term in the field equations [27]. For the Noether symmetries of the geodesic Lagrangian, these are derived by the elements of the homothetic group of the background geometry [26]. That geometric results are extended and for higher-order symmetries, see for instance the discussion in [28] and references therein. As far as the case of partial differential equations is concerned, the symmetries of the Poisson equation are constructed by the elements of the conformal algebra of the Riemann metric which defines the Laplace operator [29]. Hence, it is clear that in order to solve the classification problem of our study we should present in detail the classification of the conformal algebra for the homogeneous and anisotropic spacetimes of our consideration. The structure of the paper is as follows.

In Section 2, we present in detail the theory of infinitesimal transformation and the definitions of basic motions in Riemannian manifolds. Moreover, we present the classification of the Killing symmetries, the Homothetic vector and the proper Conformal Killing vector for the Bianchi I, Bianchi III and Bianchi V spacetimes. In Section 3, we present the basic elements of the theory of differential equations. For the Poisson and the Klein–Gordon equation we recover previous results which show how the Lie symmetries are constructed directly from the Conformal Killing vectors of the background geometry. Moreover, a similar result is also presented and for the Noether symmetries of the Klein–Gordon equation. The classification problem of our study is solved in Section 4. We present all the functional forms of the potential function for the Klein–Gordon equation where nontrivial symmetry vectors exist for the Klein–Gordon equation. Finally, in Section 5, we summarise our results.

2. Infinitesimal Transformations and Motions of Riemannian Spaces

Assume the Riemannian manifold V^n, $\dim V^n = n$, with metric tensor $g_{\mu\nu}$. Consider now the one-parameter point transformation defined by the parametric equation $\bar{x}^\mu = \bar{x}^\mu(x^\nu, \varepsilon)$which defines a group orbit through the point $P(x^\mu, 0)$. Thus, the tangent vector at the point P is given by the following expression

$$X = \frac{\partial \bar{x}^\mu}{\partial \varepsilon}|_{\varepsilon\to 0}\partial_{x^\mu}|_P. \tag{1}$$

X is the generator vector of the infinitesimal transformation near the point P

$$\bar{x}^\mu = x^\mu + \varepsilon\xi^\mu(x^\nu), \tag{2}$$

in which $\xi^\mu = \frac{\partial \bar{x}^\mu}{\partial \varepsilon}|_{\varepsilon\to 0}$.

Let $F(x^\mu)$ be a function in the Riemannian manifold defined at the point P. Hence, under the action of the one-parameter point transformation (1) the function reads $\bar{F}(\bar{x}^\mu)$.

By definition, function F is invariant under the action of the one parameter point transformation (1) if and only if it has the same value/expression before and after the transformation. That is, $\bar{F}(\bar{x}^\mu) = F(x^\mu)$ or equivalently $\bar{F}(\bar{x}^\mu) = 0$ when$F(x^\mu) = 0$. The latter definition is described by the mathematical expression with the use of the infinitessimal generator

$$X(F) = 0. \tag{3}$$

equivalently $\xi^\mu \frac{\partial F}{\partial x^\mu} = 0$.

Expression (3) is the Lie symmetry condition for a function $F(x^\mu)$ to be invariant under the action of an one-parameter point transformation in the base manifold. If condition (3) is true for a specific vector field X, then X is a Lie symmetry vector for the function $F(x^\mu)$.

Consider now $\Omega^\mu(x^\nu)$ to be a geometric object with the generic transformation rule [30] $\bar{\Omega}^{\mu\prime} = \Phi^\mu(\Omega^\nu, x^\nu, \bar{x}^\nu)$. When $\Omega^\mu(x^\nu)$ is a linear homogeneous geometric object the transformation rule reads [30] $\Phi^\mu(\Omega^\nu, x^\nu, \bar{x}^\nu) = J^\mu_\lambda(x^\nu, \bar{x}^\nu)\Omega^\lambda$. Where $J^\mu_\lambda(x^\nu, \bar{x}^\nu)$ is the Jacobian matrix for the one-parameter point transformation with generator (1), that is $J^\mu_\lambda = \frac{\partial \bar{x}^\mu}{\partial x^\nu}$.

Similar, with the definition of functions, the a geometric object $\Omega^{\mu}(x^{\nu})$is invariant under a one parameter point transformation (3) if and only if $\bar{\Omega}^{\mu}(\bar{x}^{\nu}) = \Omega^{\mu}(x^{\nu})$ or

$$\mathcal{L}_X \Omega^{\mu}(x^{\nu}) = 0, \tag{4}$$

where $\mathcal{L}_X$ is the Lie derivative with respect to the vector field X. In the case where $\Omega^{\mu}(x^{\nu})$ is a function for the Lie derivative it holds $\mathcal{L}_X \Omega \equiv X(\Omega)$.

A more generalized concept of the Lie symmetries for geometric objects are summarized in the context of collineations. Consider now that for the geometric object Ω, the following expression holds

$$\mathcal{L}_X \Omega = \Psi \tag{5}$$

where Ψ is a tensor field and it has the same components and symmetries of the indices with Ω. If condition (5) is true, X is called a collineation for the geometric object Ω, then, the type of collineations is being defined by tensor field Ψ.

The metric tensor $g_{\mu\nu}$ of the Riemannian manifold is a linear homogeneous geometric object with definition for the Lie derivative

$$\mathcal{L}_X g_{\mu\nu} = X_{(\mu;\nu)} \tag{6}$$

where ; denotes covariant derivative with respect to the Levi–Civita connection.

For the metric tensor, the concept of collineations is expressed as

$$\mathcal{L}_X g_{\mu\nu} = 2\psi g_{\mu\nu} + 2H_{\mu\nu} \tag{7}$$

where ψ is the conformal function and $H_{\mu\nu}$ is a symmetric traceless tensor, i.e., $H^{\mu}_{\mu} = 0$. The most important collineations for the metric tensor are the motions with $H_{\mu\nu} = 0$. These are the Killing vectors, the Homothetic vectors and the Conformal killing vectors.

The generator (1) of the infinitesimal transformation (2) is called a Killing vector field (KV) for the Riemann space V^n, if and only if the metric tensor is invariant under the action of the transformation, that is,

$$\mathcal{L}_X g_{\mu\nu} = 0. \tag{8}$$

Moreover, the infinitesimal generator X is a Conformal Killing vector (CKV) for the Riemann space V^n if there exists a function $\psi(x^{\mu})$ such that

$$\mathcal{L}_X g_{\mu\nu} = 2\psi g_{\mu\nu} \tag{9}$$

where $\psi = \frac{1}{n} X^{\mu}_{;\mu}$.

An important class of collineations is when ψ is a constant, then the CKV becomes a Homothetic Killing Vector (HV). Moreover, when $\psi_{;\mu\nu} = 0$, the vector field X is a special CKV (sp. CKV) for the Riemann manifold. Indeed, when $\psi = 0$, the CKV is also a KV. With the term proper CKV we shall refer to CKVs which are not HVs or KVs.

The KVs, the HV and the CKVs form Lie algebras which are known as Killing algebra (G_{KV}), Homothetic algebra $(G_{HV})$and Conformal Killing algebra (G_{CV}). When for the dimensional of the Riemannian manifold V^n holds $n \geq 2$, then G_{KV} is a subalgebra of G_{HV} and the latter is a subalgebra of the Conformal Killing algebra, that is $G_{KV} \subseteq G_{HV} \subseteq G_{CV}$. For any Riemannian manifold, there exists at most one proper Homothetic vector. Moreover, the maximum dimensional Killing algebra is of $\frac{1}{2}n(n+1)$ and the maximum Conformal Killing algebra is of $\frac{1}{2}(n+1)(n+2)$ dimension.

Point transformations with a KV generator have the property to keep invariant the length and the angles of autoparallels, unlike of the homothetic vector where the angles remain invariant and the length is scaled with a constant parameter. However, in the case where the point transformation is generated by a CKV only the angles of autoparallels remain invariant.

The existence of collineations for the metric tensor is essential for the nature of the physical space which is described by Riemannian geometry. Indeed, our universe in large scales is described by the Friedmann–Lemaître–Robertson–Walker line element which has a maximal symmetric three-dimensional hypersurface. An important class of exact solutions in General Relativity are the self-similar spacetimes. This family of solutions has the main property to map to itself after an appropriate scale of the dependent or independent variables, thus a proper HV exists. Self-similar solutions of exact spacetimes describe the asymptotic behaviour of the most general solution of the gravitational theory [31,32]. Spacetimes with proper CKV are also of special interest; more details can be found in [33].

CKVs of Anisotropic Spacetimes

The Bianchi spacetimes describe anisotropic homogeneous cosmologies and they are of special interest, because they can describe the very early stage of the evolution of universe. In this family of spacetimes the line element of the metric tensor is foliated along the time axis, with three dimensional homogeneous hypersurfaces. The classification problem of all three dimensional real Lie algebras was solved by Bianchi and it has shown that there are nine Lie algebras. Thus, there are nine Bianchi models according to the admitted Killing algebra of the three-dimensional homogeneous hypersurface.

The generic line element for the Bianchi model is

$$ds^2 = -N^2(t)dt^2 + A^2(t)(\omega_1)^2 + B^2(t)(\omega_2)^2 + C^2(t)(\omega_3)^2 \,. \tag{10}$$

where ω_i, $i = 1,2,3$, are basic one-forms and $N(t), A(t)$, $B(t)$, $C(t)$ are functions which depend only on the time parameter, see [34]. In this study we are interested in the Bianchi I, Bianchi III and Bianchi V spacetimes. These spacetimes in terms of the coordinate expressions are diagonal.

Indeed, for these spacetimes the $1-$forms are

$$\begin{aligned} \text{Bianchi I} &: \omega_1 = dx, \omega_2 = dy, \omega_3 = dz, \\ \text{Bianchi III} &: \omega_1 = dx, \omega_2 = dy, \omega_3 = e^{-x}dz, \\ \text{Bianchi V} &: \omega_1 = dx, \omega_2 = e^{x}dy, \omega_3 = e^{x}dz. \end{aligned}$$

The Killing algebras of the Bianchi spacetimes are presented in [34]. However, the proper CKVs for the Bianchi I, Bianchi III and Bianchi V spacetimes have been derived before in [35,36].

The Bianchi I spacetime admits proper CKV when the metric tensor provides the line element

$$ds^2 = C^2(t)\left[-dt^2 + e^{-\frac{2}{c}t}dx^2 + e^{-\frac{2\alpha_1}{c}t}dy^2 + dz^2\right] \tag{11}$$

with proper CKV the vector field

$$X_1 = c\partial_t + x\partial_x + \alpha_1 y\partial_y \tag{12}$$

and conformal factor $\psi(X_1) = c(\ln|C|)_{,t}$. When $C(t) = e^{\psi_0 t}$, X_1 reduces to a proper HKV, while for $C(t) = C_0$, X_1 is a KV.

Moreover, when the line element is of the form

$$ds^2 = C^2(t)\left[-dt^2 + t^{2\frac{\alpha_2-1}{\alpha_2}}dx^2 + t^{2\frac{\alpha_2-\alpha_1}{\alpha_2}}dy^2 + dz^2\right] \tag{13}$$

the resulting CKV is

$$X_2 = \alpha_2 t\partial_t + x\partial_x + \alpha_1 y\partial_y + \alpha_2 z\partial_z \tag{14}$$

with corresponding conformal factor $\psi(X_2) = \alpha_2[1 + t(\ln|C|)_{,t}]$. Indeed, when $C(t) = t^{\psi_0 - 1}$, X_2 is a proper HKV, while when $C(t) = t^{-1}$, X_2 is reduced to a KV.

The Bianchi III spacetime admits a proper CKV when

$$ds^2 = A^2(t)\left[e^{m\lambda t}\left(-dt^2 + dx^2\right) + e^{m(\lambda-1)t}dy^2 + e^{-2x}dz^2\right]. \tag{15}$$

where now the corresponding vector field is

$$X_3 = \frac{2}{m}\partial_t + y\partial_y + \lambda z\partial_z \tag{16}$$

and $\psi_{(III)}(X_3) = \frac{2}{m}\frac{A_{,t}}{A} + \lambda$. Indeed, when $A(t) = e^{A_0 t}$, the vector field is reduced to a HV, and for $A(t) = e^{-\frac{\lambda}{2}mx}$, is reduced to a KV.

Finally, for the family of Bianchi V spacetimes it follows that the line element

$$ds^2 = A^2(t)\left[e^{m\lambda t}\left(-dt^2 + dx^2\right) + e^{2x}\left(e^{m(\lambda-1)t}dy^2 + dz^2\right)\right] \tag{17}$$

admits as proper CKV the vector field X_3 with the same conformal factor as before.

Recall that in the following we shall not investigate the case where the spacetimes reduce to locally rotational spaces or the scale factors are constant functions.

3. Symmetries of Differential Equations

In terms of geometry a differential equation can be considered as a function $H = H(x^\nu, u^A, u^A_{,\mu}, u^A_{,\mu\nu})$ in the space $B = B\left(x^\nu, u^A, u^A_{,\mu}, u^A_{,\mu\nu}\right)$, $u^A = u^A(x^\mu)$ denote the dependent variables, x^μ are the independent variables and $u^A_{,\mu} = \frac{\partial u^A}{\partial u^\mu}$.

Assume now the infinitesimal transformation in the base manifold of the differential equation H,

$$\bar{x}^\mu = x^\mu + \varepsilon\xi^\mu(x^\nu, u^B), \tag{18}$$

$$\bar{u}^A = \bar{u}^A + \varepsilon\eta^A(x^\nu, u^B), \tag{19}$$

with vector generator

$$\mathbf{X} = \xi^\mu(x^\nu, u^B)\partial_{x^\mu} + \eta^A(x^\nu, u^B)\partial_{u^A}. \tag{20}$$

Similarly to the case of functions, the geometric vector field X is a Lie symmetry of $H = H(x^\nu, u^A, u^A_{,\mu}, u^A_{,\mu\nu})$ if and only if the following is true [2,3]

$$X^{[2]}(H) = 0, \tag{21}$$

in which $X^{[2]}$ is the second extension of the vector field X in the space $B = B\left(x^\nu, u^A, u^A_{,\mu}, u^A_{,\mu\nu}\right)$ defined as

$$X^{[2]} = X + \eta^A_\mu\partial_{u^A_\mu} + \eta^A_{\mu\nu}\partial_{u^A_{\mu\nu}} \tag{22}$$

in which

$$\eta^A_\mu = D_\mu\eta^A - D_\mu\xi^\nu u^A_{,\nu}, \tag{23}$$

and

$$\eta^A_{\mu\nu} = D_\nu\eta^A_\mu - D_\nu D_\mu\xi^\kappa u^A_{,\kappa},$$

where D_μ is the total derivative.

A straightforward application of the Lie symmetries for a given differential equation is the construction of invariant functions by deriving the characteristic functions. The characteristic functions can be used to define similarity transformations which can be used to reduce the number of the indepedent variables in the case of partial differential equations.

The invariants are determined by the solution of the following Lagrangian system.

$$\frac{dx^\mu}{\xi^\mu} = \frac{du^A}{\eta^A} = \frac{du^A_\mu}{\eta^A_\mu} = \frac{du^A_{\mu\nu}}{\eta^A_{\mu\nu}}. \quad (24)$$

In the case where the differential equation H follows from a variational principle with Lagrangian function $L = L(x^\mu, u^A, u^A_{,\mu})$ such as $H \equiv \mathbf{E}(L) = 0$, where $\mathbf{E}$ is the Euler operator. The Lie point symmetry $\mathbf{X}$ of the DE H is a Noether point symmetry of H, if and only if the following condition is satisfied

$$\mathbf{X}^{[1]}L + LD_i\xi^i = D_iA^i\left(x^k, u^C\right), \quad (25)$$

where $\mathbf{X}^{[1]}$ is the first prolongation of $\mathbf{X}$, and A^μ is a vector field which should be determined. Condition (25) is Noether's second theorem. The second theorem of Noether states that for any vector field X where condition (25) is true the following function is a conservation law

$$I^\mu = \xi^\nu\left(u^A_\nu \frac{\partial L}{\partial u^A_\mu} - \delta^\mu_\nu L\right) - \eta^A \frac{\partial L}{\partial u^A_\mu} + A^\mu, \quad (26)$$

that is, $D_\mu I^\mu = 0$, [2,3].

Poisson Equation

Let Δ be the Laplace operator in the Riemannian manifold V^n,

$$\Delta = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right),$$

then the Poisson equation reads

$$\Delta u = f(x^\mu, u), \quad (27)$$

or equivalently

$$g^{\mu\nu}u_{\mu\nu} - \Gamma^\mu u_\mu = f(x^\nu, u), \quad (28)$$

where $\Gamma^\mu = \Gamma^\mu_{\nu\kappa}g^{\nu\kappa}$ and $\Gamma^\mu_{\nu\kappa}$ are the Christoffel symbols for the Levi–Civita connection of the metric tensor $g_{\mu\nu}$.

The Lie symmetry analysis for the Poisson equation when $f = f(u)$ have been given in [37], and for $f = f\left(x^i, u\right)$ are presented in [29]. Indeed, the Lie (point) symmetries for the Poisson equation are related to the elements of the conformal algebra for the Riemannian manifold as described in the following.

Theorem 1. *The Lie symmetries for the Poisson equation are constructed by the generic CKV of the background metric tensor $g_{\mu\nu}$ of the Riemannian manifold V^n:*

(a) For $n > 2$, the Lie symmetry vector has the generic form

$$X = \xi^\mu(x^\nu)\partial_\mu + \left(\frac{2-n}{2}\psi(x^\nu)u + a_0u + b(x^\nu)\right)\partial_u, \quad (29)$$

where $\xi^i(x^\nu)$ is a CKV of the Riemannian manifold with conformal factor $\psi(x^\nu)$ and the following condition holds

$$\frac{2-n}{2}\Delta\psi u + g^{\mu\nu}b_{\mu;\nu} - \xi^\nu f_{,\nu} - \frac{2-n}{2}\psi u f_{,u} - \frac{n+2}{2}\psi f - bf_{,u} = 0, \quad (30)$$

(b) For $n = 2$, the generic Lie symmetry vector is

$$X = \xi^\mu(x^\nu)\partial_\mu + (a_0u + b(x^\nu))\partial_u, \quad (31)$$

where ξ^{μ} is a CKV and the following conditions are satisfied

$$g^{\mu\nu}b_{;\mu\nu} - \xi^{\nu}f_{,\nu} - a_0 u f_{,u} + (a_0 - 2\psi)f - bf_{,u} = 0, \tag{32}$$

that is, the function b is solution of the Laplace equation.

A special case of the Poisson equation is the Klein–Gordon equation with $f(x^{\nu}, u) = V(x^{\nu})u$, that is,

$$\Delta u - V(x^{\nu})u = 0, \tag{33}$$

where $V(x^{\mu})$ is the potential function. For the Lie symmetries of the Klein–Gordon equation it follows

Theorem 2. *For the Klein–Gordon Equation (33) the Lie symmetries are constructed by the elements of the conformal algebra of the Riemannian manifold:*
(a) for $n > 2$, the generic symmetry vector is expressed as

$$X = \xi^{\mu}(x^{\nu})\partial_{\mu} + \left(\frac{2-n}{2}\psi(x^{\nu})u + a_0 u + b(x^{\nu})\right)\partial_u, \tag{34}$$

where now ξ^{μ} is a CKV with conformal factor $\psi(x^{\nu})$, $b(x^{\nu})$ solves Equation (33) with constraint condition

$$\xi^{\nu}V_{,\nu} + 2\psi V - \frac{2-n}{2}\Delta\psi = 0, \tag{35}$$

(b) for $n = 2$, the generic symmetry vector is written

$$X = \xi^{\mu}(x^{\nu})\partial_{\mu} + (a_0 u + b(x^{\nu}))\partial_u, \tag{36}$$

where ξ^{μ} is a CKV with conformal factor $\psi(x^{\nu})$, $b(x^{\nu})$ solves Equation (33) with constrain

$$\xi^{\nu}V_{,\nu} + 2\psi V = 0. \tag{37}$$

The Klein–Gordon Equation (33) can be reproduced by the variation of the Lagrangian function

$$L(x^{\nu}, u, u_{,\nu}) = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}u_{,\mu}u_{,\nu} + \frac{1}{2}\sqrt{|g|}V(x^{\nu})u^2. \tag{38}$$

Therefore, for the Noether symmetries of the Klein–Gordon Lagrangian (38) the following Theorem holds.

Theorem 3. *The Lie point symmetries of the Klein–Gordon Equation (33) are generated from the elements of the conformal algebra of the Riemannian manifold, where the generic Noether symmetry is of the form*

$$X_N = \xi^{\mu}(x^{\nu})\partial_{\mu} + \left(\frac{2-n}{2}\psi(x^{\nu})u\right)\partial_u,$$

where the corresponding vector $A_{\mu} = \frac{2-n}{4}\sqrt{|g|}\psi_{,\mu}(x^{\nu})u^2$, in which $\xi^{\mu}(x^{\nu})$ is a CKV with conformal factor $\psi(x^{\nu})$. The constraint equation is

$$\xi^{\nu}V_{,\nu} + 2\psi V - \frac{2-n}{2}\Delta\psi = 0. \tag{39}$$

We remark that for the Klein–Gordon equation all the non-trivial Lie symmetries are also Noether symmetries. The resulting conservation law is of the form

$$I^{\mu} = \sqrt{|g|}\left(\left(\frac{1}{2}g^{\kappa\nu}u_{,\kappa}u_{,\nu} - \frac{1}{2}V(x^{\nu})u^2\right)\xi^{\mu} - \eta\frac{1}{2}g^{\mu\nu}u_{,\nu} + \frac{2-n}{4}\psi_{,\mu}(x^{\nu})u^2\right). \tag{40}$$

4. Klein–Gordon Equation in Anisotropic Geometries

We proceed with the solution of the classification problem for the potential function $V(x^\mu)$ for the Klein–Gordon Equation (33) in the case of anisotropic cosmologies where the Klein–Gordon equation admits non-trivial Lie symmetries. The trivial symmetries are the vector fields $X_u = u\partial_u$, $X_b = b\partial_u$ which exist for any potential function $V(x^\nu)$.

4.1. Bianchi I

In a Bianchi I spacetime, the Klein–Gordon equation is written

$$\left(-\frac{u_{,tt}}{N^2}+\frac{u_{,xx}}{A^2}+\frac{u_{,yy}}{B^2}+\frac{u_{,zz}}{C^2}\right)+\frac{1}{N^2}\left(\frac{N_{,t}}{N}u_{,t}-\frac{A_{,t}}{A}u_{,x}-\frac{C_{,t}}{C}u_{,y}-\frac{C_{,t}}{C}u_{,z}\right)-\frac{V(t,x,y,z)}{A^2B^2C}u^2=0. \quad (41)$$

For arbitrary function forms of the scale factors the background space admits the three KVs, $\xi_I^1=\partial_x, \xi_I^2=\partial_y$ and $\xi_I^3=\partial_z$.

Hence, from Theorem 2 it follows that: (i) ξ_I^1, is a Lie symmetry when $V_i^I(t,x,y,z)=V(t,y,z)$; (ii) ξ_I^2 is a Lie symmetry when $V(t,x,y,z)=V_{ii}^I(t,x,z)$; (iii) ξ_I^3 is a Lie symmetry when $V(t,x,y,z)=V_{iii}^I(t,x,y)$; and (iv) $\alpha\xi_I^1+\beta\xi_I^2+\gamma\xi_I^3$ is a Lie symmetry when $V_{iv}^I(t,x,y,z)=V\left(t,y-\frac{\beta}{\alpha}x,z-\frac{\gamma}{\alpha}x\right)$.

In the special case where the line element is that of (11) the CKV X_1 produces the Lie symmetry vector $X=X_1+\left(-2c(\ln|C|)_{,t}u\right)\partial_u$ for the Klein–Gordon equation, if and only if

$$V_v^I(t,x,y,z)=\frac{cC_{,tt}-(\alpha_1+1)C_{,t}}{cC^3}+\frac{1}{C^2}U\left(xe^{-\frac{t}{c}},ye^{-\frac{a_1}{c}t},z\right). \quad (42)$$

Similarly, the vector field $X=X_1+\alpha\xi_I^1+\beta\xi_I^2+\gamma\xi_I^3+\left(-2c(\ln|C|)_{,t}u\right)\partial_u$ is a Lie symmetry for the Klein–Gordon equation in a Bianchi I spacetime with line element (11) if and only if

$$V_{vi}^I(t,x,y,z)=\frac{cC_{,tt}-(\alpha_1+1)C_{,t}}{cC^3}+\frac{1}{C^2}U\left((x+\alpha)e^{-\frac{t}{c}},\left(y+\frac{\beta}{a_1}\right)e^{-\frac{a_1}{c}t},z-\frac{\gamma}{c}\tau\right). \quad (43)$$

On the other hand, for the Bianchi I line element (13) the CKV X_2 is the generator of the Lie symmetry vector $X=X_2-\alpha_2[1+t(\ln|C|)_{,t}]u\partial_u$ for the Klein–Gordon Equation (41) for the potential function

$$V_{vii}^I(t,x,y,z)=\frac{1}{t^2C^2}\left(U\left(xt^{-\frac{1}{a_2}},yt^{-\frac{a_1}{a_2}},\frac{z}{t}\right)+F(t)\right) \quad (44)$$

where

$$\begin{aligned} F(t) &= -\frac{1}{a_2C^2}\left(a_2t^2CC_{,ttt}+tC_{,tt}(a_2C_{,t}+C(a_1-4a_2+1))\right)+ \\ &\quad -\frac{1}{a_2C^2}(C_{,t}(1+a_1-2a_2)(C-tC_{,t})). \end{aligned} \quad (45)$$

Moreover, the vector field $X=X_2+\alpha\xi_I^1+\beta\xi_I^2+\gamma\xi_I^3-\alpha_2[1+t(\ln|C|)_{,t}]\partial_u$ is a Lie symmetry for the Klein–Gordon equation when

$$V_{viii}^I(t,x,y,z)=\frac{1}{t^2C^2}\left(U\left((x+\alpha)t^{-\frac{1}{a_2}},\left(y+\frac{\beta}{a_1}\right)t^{-\frac{a_1}{a_2}},\left(z+\frac{\gamma}{a_2}\right)t^{-1}\right)+F(t)\right). \quad (46)$$

4.1.1. Invariant Functions

Let us now determine the invariant functions which correspond to each admitted Lie point symmetry. The invariant functions can be used to determine similarity transformations whenever they are applied the number of dependent variables of the Klein–Gordon equation is reduced.

For the vector field ξ_I^1, the invariant functions are $\{t, y, z, u\}$. Similarly, for the vector field ξ_I^2 we determine the Lie invariants $\{t, x, z, u\}$. Moreover, for ξ_I^3 the Lie invariants are $\{t, x, y, u\}$, while for the vector field $\alpha\xi_I^1 + \beta\xi_I^2 + \gamma\xi_I^3$, the Lie invariants are $\left\{t, y - \frac{\beta}{\alpha}x, z - \frac{\gamma}{\alpha}x, u\right\}$.

Furthermore, for the potential function V_v^I where $X = X_1 + \left(-2c(\ln|C|)_{,t}u\right)\partial_u$ is a Lie symmetry, the resulting Lie invariants are calculated $\left\{xe^{-\frac{t}{c}}, ye^{-\frac{a_1}{c}t}, z, uC(t)^2\right\}$. For the potential V_{vi}^I, the Lie invariants are $\left\{(x+\alpha)e^{-\frac{t}{c}}, \left(y + \frac{\beta}{a_1}\right)e^{-\frac{a_1}{c}t}, z - \frac{\gamma}{c}\tau, uC(t)^2\right\}$. In a similar way, for the potential functions $V_{vii}^I(t,x,y,z)$ and $V_{viii}^I(t,x,y,z)$ the admitted Lie invariants are $\left\{xt^{-\frac{1}{a_2}}, yt^{-\frac{a_1}{a_2}}, \frac{z}{t}, ut^2C(t)^2\right\}$ and $\left\{(x+\alpha)t^{-\frac{1}{a_2}}, \left(y+\frac{\beta}{a_1}\right)t^{-\frac{a_1}{a_2}}, \left(z+\frac{\gamma}{a_2}\right)t^{-1}, ut^2C(t)^2\right\}$ provided by the Lie symmetries $X_2 - \alpha_2[1 + t(\ln|C|)_{,t}]u\partial_u$ and $X_2 + \alpha\xi_I^1 + \beta\xi_I^2 + \gamma\xi_I^3 - \alpha_2[1 + t(\ln|C|)_{,t}]\partial_u$ respectively.

4.1.2. Conservation Laws

We apply Noether's theorem and expression (40) hence the resulting conservation laws related to the admitted Lie symmetries for the Klein–Gordon Equation (41) are

$$I^x\left(\xi_I^1\right) = \frac{NABC}{2}\left(\left(\left(-\frac{1}{N^2}u_{,t}^2 + \frac{1}{A^2}u_{,x}^2 + \frac{1}{B^2}u_{,y}^2 + \frac{1}{C^2}u_{,z}^2\right) - V_i^I(t,x,y,z)u^2\right)\right). \tag{47}$$

$$I^t\left(\xi_I^1\right) = 0, I^y\left(\xi_I^1\right) = 0 \text{ and } I^z\left(\xi_I^1\right) = 0. \tag{48}$$

For the vector field ξ_I^2 there exists the conservation law

$$I^y\left(\xi_I^2\right) = \frac{NABC}{2}\left(\left(\left(-\frac{1}{N^2}u_{,t}^2 + \frac{1}{A^2}u_{,x}^2 + \frac{1}{B^2}u_{,y}^2 + \frac{1}{C^2}u_{,z}^2\right) - V_{ii}^I(t,x,y,z)u^2\right)\right). \tag{49}$$

$$I^t\left(\xi_I^2\right) = 0, I^x\left(\xi_I^2\right) = 0 \text{ and } I^z\left(\xi_I^2\right) = 0. \tag{50}$$

While for the vector field ξ_I^3 the resulting Noetherian conservation law is

$$I^z\left(\xi_I^3\right) = \frac{NABC}{2}\left(\left(\left(-\frac{1}{N^2}u_{,t}^2 + \frac{1}{A^2}u_{,x}^2 + \frac{1}{B^2}u_{,y}^2 + \frac{1}{C^2}u_{,z}^2\right) - V_{iii}^I(t,x,y,z)u^2\right)\right). \tag{51}$$

$$I^t\left(\xi_I^3\right) = 0, I^x\left(\xi_I^3\right) = 0 \text{ and } I^y\left(\xi_I^3\right) = 0. \tag{52}$$

For the generic vector field $\alpha\xi_I^1 + \beta\xi_I^2 + \gamma\xi_I^3$ we calculate the conservation law

$$I^x\left(\alpha\xi_I^1 + \beta\xi_I^2 + \gamma\xi_I^3\right) = \alpha\frac{NABC}{2}\left(\left(\left(-\frac{1}{N^2}u_{,t}^2 + \frac{1}{A^2}u_{,x}^2 + \frac{1}{B^2}u_{,y}^2 + \frac{1}{C^2}u_{,z}^2\right) - V_{iv}^I(t,x,y,z)u^2\right)\right), \tag{53}$$

$$I^y\left(\alpha\xi_I^1 + \beta\xi_I^2 + \gamma\xi_I^3\right) = \beta\frac{NABC}{2}\left(\left(\left(-\frac{1}{N^2}u_{,t}^2 + \frac{1}{A^2}u_{,x}^2 + \frac{1}{B^2}u_{,y}^2 + \frac{1}{C^2}u_{,z}^2\right) - V_{iv}^I(t,x,y,z)u^2\right)\right), \tag{54}$$

$$I^z\left(\alpha\xi_I^1 + \beta\xi_I^2 + \gamma\xi_I^3\right) = \gamma\frac{NABC}{2}\left(\left(\left(-\frac{1}{N^2}u_{,t}^2 + \frac{1}{A^2}u_{,x}^2 + \frac{1}{B^2}u_{,y}^2 + \frac{1}{C^2}u_{,z}^2\right) - V_{iv}^I(t,x,y,z)u^2\right)\right) \tag{55}$$

$$I^t\left(\alpha\xi_I^1 + \beta\xi_I^2 + \gamma\xi_I^3\right) = 0. \tag{56}$$

For the potential $V_v^I(t,x,y,z)$ there exists the conservation law

$$I^t(X_1) = c\frac{e^{-\frac{t(1+\alpha_1)}{c}}C^4}{2}\left(H_v^I - \left((\ln|C|)_{,t}u\right)\frac{1}{C}u^{,t} - \frac{1}{2C}(\ln|C|)_{,tt}u^2\right). \tag{57}$$

$$I^x(X_1) = \frac{e^{-\frac{t(1+\alpha_1)}{c}}C^4}{2}\left(xH_v^I - \left(c(\ln|C|)_{,t}u\right)\frac{1}{C}u^{,x}\right). \tag{58}$$

$$I^y(X_1) = \frac{e^{-\frac{t(1+\alpha_1)}{c}}C^4}{2}\left(\alpha_1 yH_v^I - \left(c(\ln|C|)_{,t}u\right)\frac{1}{C}u^{,y}\right). \tag{59}$$

$$I^z(X_1) = \frac{e^{-\frac{t(1+\alpha_1)}{c}}C^4}{2}\left(-\left(c(\ln|C|)_{,t}u\right)\frac{1}{N}u^{,z}\right). \tag{60}$$

where

$$H_v^I = \left(\left(-\frac{1}{C^2}u_{,t}^2 + \frac{e^{\frac{2t}{c}}}{C^2}u_{,x}^2 + \frac{e^{\frac{2\alpha_1}{c}t}}{C^2}u_{,y}^2 + \frac{1}{C^2}u_{,z}^2\right) - V_v^I(t,x,y,z)u^2\right) \tag{61}$$

For the potential function $V_{vi}^I(t,x,y,z)$ the resulting conservation law is derived

$$I^t = I^t(X_1), \tag{62}$$

$$I^x = I^x(X_1) + \alpha I^x\left(\xi_I^1\right), \tag{63}$$

$$I^y = I^y(X_1) + \beta I^y\left(\xi_I^1\right), \tag{64}$$

$$I^z = I^z(X_1) + \gamma I^z\left(\xi_I^1\right), \tag{65}$$

for $N(t) = C(t), A(t) = e^{-\frac{t}{c}}C(t)$and $B(t) = e^{-\frac{\alpha_1}{c}t}C(t)$ with potential function $V_{vi}^I(t,x,y,z)$

For the potential functionV_{vii}^I where X_2 is the generator of the Lie symmetry vector the resulting Noetherian conservation law is

$$I^t(X_2) = \frac{C^4}{2}t^{2-\frac{1+\alpha_1}{\alpha_2}}\left(\alpha_2 tH_{vii}^I - \alpha_2\left((\ln|C|)_{,t}u\right)\frac{1}{C}u^{,t} - \frac{1}{2C}\alpha_2[1+t(\ln|C|)_{,t}]_{,t}u^2\right). \tag{66}$$

$$I^x(X_2) = \frac{C^4}{2}t^{2-\frac{1+\alpha_1}{\alpha_2}}\left(xH_{vii}^I - (\alpha_2[1+t(\ln|C|)_{,t}]u)\frac{1}{C}u^{,x}\right). \tag{67}$$

$$I^y(X_2) = \frac{C^4}{2}t^{2-\frac{1+\alpha_1}{\alpha_2}}\left(\alpha_1 yH_{vii}^I - (\alpha_2[1+t(\ln|C|)_{,t}]u)\frac{1}{C}u^{,y}\right). \tag{68}$$

$$I^z(X_2) = \frac{C^4}{2}t^{2-\frac{1+\alpha_1}{\alpha_2}}\left(\alpha_2 zH_{vii}^I - (\alpha_2[1+t(\ln|C|)_{,t}]u)\frac{1}{N}u^{,z}\right). \tag{69}$$

where

$$H_{vii}^I = \left(\left(-\frac{1}{C^2}u_{,t}^2 + \frac{1}{C^2t^{2-\frac{2}{\alpha_2}}}u_{,x}^2 + \frac{1}{C^2t^{2-\frac{2\alpha_1}{\alpha_2}}}u_{,y}^2 + \frac{1}{C^2}u_{,z}^2\right) - V_{vii}^I(t,x,y,z)u^2\right). \tag{70}$$

Finally, for the potential function $V_{viii}^I(t,x,y,z)$ the conservation law is

$$I^t = I^t(X_2), \tag{71}$$

$$I^x = I^x(X_2) + \alpha I^x\left(\xi_I^1\right), \tag{72}$$

$$I^y = I^y(X_2) + \beta I^y\left(\xi_I^1\right), \tag{73}$$

$$I^z = I^z(X_2) + \gamma I^z\left(\xi_I^1\right), \tag{74}$$

for $N(t) = C(t), A(t) = t^{1-\frac{1}{\alpha_2}}C(t)$,$B(t) = t^{1-\frac{\alpha_1}{\alpha_2}}C(t)$ and potential function $V_{viii}^I(t,x,y,z)$

4.2. Bianchi III

In the Bianchi III geometry, the Klein–Gordon equation reads

$$\left(-\frac{u_{,tt}}{N^2} + \frac{u_{,xx}}{A^2} + \frac{u_{,yy}}{B^2} + e^{2x}\frac{u_{,zz}}{C^2}\right) + \frac{1}{N^2}\left(\frac{N_{,t}}{N}u_{,t} - \frac{A_{,t}}{A}u_{,x} - \frac{C_{,t}}{C}u_{,y} - \frac{C_{,t}}{C}u_{,z}\right) - \frac{1}{A^4}u_{,x} - \frac{V(t,x,y,z)}{A^2B^2C}u^2 = 0. \tag{75}$$

The three KVs of the Bianchi III spacetime are $\xi_{III}^1 = \partial_x + z\partial_z, \xi_{III}^2 = \partial_y$ and $\xi_{III}^3 = \partial_z$. Hence, (i) ξ_{III}^1 is a Lie symmetry for Equation (75) when $V_i^{III}(t,x,y,z) = V(t,y,ze^{-x})$; (ii) ξ_{III}^2 is a Lie symmetry for $V_{ii}^{III}(t,x,y,z) = V(t,x,z)$; (iii) ξ_{III}^3 is a Lie symmetry when $V_{iii}^{III}(t,x,y,z) = V(t,x,y)$; (iv) $\alpha\xi_{III}^1 + \beta\xi_{III}^2 + \gamma\xi_{III}^3$ is a Lie symmetry when $V_{iv}^{III}(t,x,y,z) = V\left(t, y - \frac{\beta}{\alpha}x, (z+\frac{\gamma}{\alpha})e^{-x}\right)$.

For the line (15) where the Bianchi III spacetime admits the additional CKV X_3, it follows that the vector field $X = X_3 - \left(\frac{2}{m}\frac{A_{,t}}{A} + \lambda\right)u\partial_u$ is a Lie symmetry vector for the Klein–Gordon Equation (75) when

$$V_v^{III}(t,x,y,z) = \frac{1}{A^2}e^{-m\lambda t}U\left(x, ye^{-\frac{m}{2}t}, ze^{-\frac{m\lambda}{2}t}\right) + \frac{m(\lambda-1)A_{,t} + 2A_{,tt}}{2A^3}. \tag{76}$$

Hence, the vector field $X = X_3 + \alpha\xi_{III}^1 + \beta\xi_{III}^2 + \gamma\xi_{III}^3 - \left(\frac{2}{m}\frac{A_{,t}}{A} + \lambda\right)u\partial_u$ is a Lie symmetry of Equation (75) for the potential function

$$V_{vi}^{III}(t,x,y,z) = \frac{1}{A^2}e^{-m\lambda t}U\left(x - \frac{\alpha m}{2}t, (y+\beta)e^{-\frac{m}{2}t}, \left(z + \frac{\gamma}{\lambda+\alpha}\right)e^{-\frac{m(\lambda+\alpha)}{2}t}\right) + \frac{m(\lambda-1)A_{,t} + 2A_{,tt}}{2A^3}. \tag{77}$$

4.2.1. Invariant Functions

We proceed with the derivation of the invariant functions provided by each case for the above potential functions. For $V_i^{III}(t,x,y,z)$ the Lie invariants are $\{t,y,ze^{-x},u\}$, for $V_{ii}^{III}(t,x,y,z)$ and the Lie symmetry vector ξ_{III}^2 we determine the Lie invariants $\{t,x,z,u\}$ while from ξ_{III}^3 for the potential $V_{iii}^{III}(t,x,y,z)$ the Lie invariants are $\{t,x,y,u\}$. Similarly, for the generic vector field $\alpha\xi_{III}^1 + \beta\xi_{III}^2 + \gamma\xi_{III}^3$ and potential $V_{iv}^{III}(t,x,y,z)$ the corresponding Lie invariants are $\left(t, y - \frac{\beta}{\alpha}x, (z+\frac{\gamma}{\alpha})e^{-x}, u\right)$.

In the case where the proper CKV produces a Lie symmetry, then for the Klein–Gordon Equation (75) with potential function $V_v^{III}(t,x,y,z)$ the Lie invariants are $\left\{x, ye^{-\frac{m}{2}t}, ze^{-\frac{m\lambda}{2}t}, e^{m\lambda t}u\right\}$, while for the potential function $V_v^{III}(t,x,y,z)$ the resulting Lie invariants are $\left\{x - \frac{\alpha m}{2}t, (y+\beta)e^{-\frac{m}{2}t}, \left(z + \frac{\gamma}{\lambda+\alpha}\right)e^{-\frac{m(\lambda+\alpha)}{2}t}, e^{m\lambda t}u\right\}$.

4.2.2. Conservation Laws

For the Noetherian conservation laws for the Klein–Gordon Equation (75) it follows that for $V_i^{III}(t,x,y,z)$ it follows

$$I^x\left(\xi_{III}^1\right) = \frac{NABC}{2}e^{-x}\left(\left(\left(-\frac{1}{N^2}u_{,t}^2 + \frac{1}{A^2}u_{,x}^2 + \frac{1}{B^2}u_{,y}^2 + \frac{e^{2x}}{C^2}u_{,z}^2\right) - V_i^{III}(t,x,y,z)u^2\right)\right) \tag{78}$$

$$I^z\left(\xi_{III}^1\right) = \frac{NABC}{2}e^{-x}\left(z\left(\left(-\frac{1}{N^2}u_{,t}^2 + \frac{1}{A^2}u_{,x}^2 + \frac{1}{B^2}u_{,y}^2 + \frac{e^{2x}}{C^2}u_{,z}^2\right) - V_i^{III}(t,x,y,z)u^2\right)\right) \tag{79}$$

$$I^t\left(\xi_{III}^1\right) = 0 \text{ and } I^y\left(\xi_{III}^1\right) = 0. \tag{80}$$

For the potential function $V_{ii}^{III}(t,x,y,z)$ the Noetherian conservation law has the following components

$$I^{y}\left(\xi_{III}^{2}\right)=\frac{NABC}{2}e^{-x}\left(\left(\left(-\frac{1}{N^{2}}u_{,t}^{2}+\frac{1}{A^{2}}u_{,x}^{2}+\frac{1}{B^{2}}u_{,y}^{2}+\frac{e^{2x}}{C^{2}}u_{,z}^{2}\right)-V_{ii}^{III}(t,x,y,z)u^{2}\right)\right) \tag{81}$$

$$I^{t}\left(\xi_{III}^{2}\right)=0\,,\,I^{x}\left(\xi_{III}^{2}\right)=0\text{and }I^{z}\left(\xi_{III}^{2}\right)=0. \tag{82}$$

Similarly, for $V_{iii}^{III}(t,x,y,z)$ we determine the conservation law

$$I^{z}\left(\xi_{III}^{3}\right)=\frac{NABC}{2}e^{-x}\left(\left(\left(-\frac{1}{N^{2}}u_{,t}^{2}+\frac{1}{A^{2}}u_{,x}^{2}+\frac{1}{B^{2}}u_{,y}^{2}+\frac{e^{2x}}{C^{2}}u_{,z}^{2}\right)-V_{ii}^{III}(t,x,y,z)u^{2}\right)\right) \tag{83}$$

$$I^{t}\left(\xi_{III}^{3}\right)=0\,,\,I^{x}\left(\xi_{III}^{3}\right)=0\text{ and }I^{y}\left(\xi_{III}^{3}\right)=0. \tag{84}$$

For the Klein–Gordon equation with potential $V_{iv}^{III}(t,x,y,z)$ the conservation law has the components

$$I^{x}\left(\alpha\xi_{III}^{1}+\beta\xi_{III}^{2}+\gamma\xi_{III}^{3}\right)=\alpha I^{x}\left(\xi_{III}^{1}\right), \tag{85}$$

$$I^{y}\left(\alpha\xi_{III}^{1}+\beta\xi_{III}^{2}+\gamma\xi_{III}^{3}\right)=\beta I^{y}\left(\xi_{III}^{2}\right), \tag{86}$$

$$I^{z}\left(\alpha\xi_{III}^{1}+\beta\xi_{III}^{2}+\gamma\xi_{III}^{3}\right)=\alpha I^{z}\left(\xi_{III}^{1}\right)+\gamma I^{z}\left(\xi_{III}^{3}\right), \tag{87}$$

and

$$I^{t}\left(\alpha\xi_{III}^{1}+\beta\xi_{III}^{2}+\gamma\xi_{III}^{3}\right)=0, \tag{88}$$

with potential function $V_{iv}^{III}(t,x,y,z)$.

Moreover, for $V_{v}^{III}(t,x,y,z)$ the conservation law has the following components

$$I^{t}(X_{3})=\frac{\bar{A}^{4}e^{\frac{(3m\lambda-1)}{2}t}e^{-x}}{2}\left(\frac{2}{m}H_{V}^{III}-\left(\left(\frac{2}{m}\frac{\bar{A}_{,t}}{\bar{A}}+\lambda\right)u\right)\frac{1}{A}u^{,t}-\frac{1}{2\bar{A}e^{\frac{m}{2}\lambda t}}\left(\frac{2}{m}\frac{\bar{A}_{,t}}{\bar{A}}+\lambda\right)_{t}u^{2}\right). \tag{89}$$

$$I^{x}(X_{3})=\frac{\bar{A}^{4}e^{\frac{(3m\lambda-1)}{2}t}e^{-x}}{2}\left(-\left(\left(\frac{2}{m}\frac{\bar{A}_{,t}}{\bar{A}}+\lambda\right)u\right)\frac{1}{\bar{A}e^{\frac{1}{2}\lambda t}}u^{,x}\right). \tag{90}$$

$$I^{y}(X_{3})=\frac{\bar{A}^{4}e^{\frac{(3m\lambda-1)}{2}t}e^{-x}}{2}\left(yH_{V}^{III}-\left(\left(\frac{2}{m}\frac{A_{,t}}{A}+\lambda\right)u\right)\frac{1}{\bar{A}e^{m(\lambda-1)t}}u^{,y}\right). \tag{91}$$

$$I^{z}(X_{3})=\frac{\bar{A}^{4}e^{\frac{(3m\lambda-1)}{2}t}e^{-x}}{2}\left(\lambda zH_{V}^{III}-\left(\left(\frac{2}{m}\frac{\bar{A}_{,t}}{\bar{A}}+\lambda\right)u\right)\frac{1}{\bar{A}e^{-x}}u^{,z}\right). \tag{92}$$

where now

$$H_{V}^{III}=\left(\left(-\frac{1}{\bar{A}^{2}e^{m\lambda t}}u_{,t}^{2}+\frac{1}{\bar{A}^{2}}u_{,x}^{2}+\frac{1}{\bar{A}^{2}e^{m(\lambda-1)t}}u_{,y}^{2}+\frac{e^{2x}}{C^{2}}u_{,z}^{2}\right)-V_{v}^{III}(t,x,y,z)u^{2}\right)$$

Finally, for the potential $V_{vi}^{III}(t,x,y,z)$ the conservation law for the Klein–Gordon Equation (75) related to the generic symmetry vector $X_{3}+\alpha\xi_{III}^{1}+\beta\xi_{III}^{2}+\gamma\xi_{III}^{3}-\left(\frac{2}{m}\frac{A_{,t}}{A}+\lambda\right)u\partial_{u}$ has the following components

$$I^t = I^t(X_3), \quad (93)$$
$$I^x = I^x(X_3) + \alpha I^x\left(\xi^1_{III}\right), \quad (94)$$
$$I^y = I^y(X_3) + \beta I^y\left(\xi^2_{III}\right), \quad (95)$$
$$I^z = I^z(X_3) + \alpha I^z\left(\xi^1_{III}\right) + \gamma I^z\left(\xi^3_{III}\right), \quad (96)$$

with $N = \bar{A}(t)e^{\frac{m}{2}\lambda t}$, $B = \bar{A}(t)e^{\frac{m}{2}(\lambda-1)t}$, $C(t) = \bar{A}(t)$ and $A(t) = e^{m\lambda t}\bar{A}(t)$ and potential function $V^{III}_{vii}(t,x,y,z)$.

4.3. Bianchi V

For the Bianchi V spacetime the Klein–Gordon equation is

$$\left(-\frac{u_{,tt}}{N^2} + \frac{u_{,xx}}{A^2} + e^{-2x}\left(\frac{u_{,yy}}{B^2} + \frac{u_{,zz}}{C^2}\right)\right) + \frac{1}{N^2}\left(\frac{N_{.t}}{N}u_{,t} - \frac{A_{,t}}{A}u_{,x} - \frac{C_{,t}}{C}u_{,y} - \frac{C_{,t}}{C}u_{,z}\right) + \frac{2}{A^4}u_{,x} - \frac{V(t,x,y,z)}{A^2B^2C}u^2 = 0. \quad (97)$$

The KVs of the Bianchi V spacetime are $\xi^1_V = \partial_x - y\partial_z - z\partial_z$, $\xi^2_V = \partial_y$ and $\xi^3_V = \partial_z$.

Therefore, from Theorem 2, we find that (i) ξ^1_{III} is a Lie symmetry for the Klein–Gordon Equation (97) when $V^V_i(t,x,y,z) = V(t,e^x y, e^x z)$; (ii) ξ^2_V is a Lie symmetry for $V^V_{ii}(t,x,y,z) = V(t,x,z)$; (iii) ξ^3_V is a Lie symmetry when $V^V_{iii}(t,x,y,z) = V(t,x,y)$; (iv) $\alpha\xi^1_V + \beta\xi^2_V + \gamma\xi^3_V$ is a Lie symmetry when $V^V_{iv}(t,x,y,z) = V\left(t, \left(y - \frac{\beta}{\alpha}\right)e^x, \left(z - \frac{\gamma}{\alpha}\right)e^x\right)$.

Finally, for the line element (17) the vector field $X = X_3 - \left(\frac{2}{m}\frac{A_{,t}}{A} + \lambda\right)u\partial_u$ is a Lie symmetry for the Klein–Gordon equation when the potential is of the form of function (76), while $X = X_3 + \alpha\xi^1_V + \beta\xi^2_V + \gamma\xi^3_V - \left(\frac{2}{m}\frac{A_{,t}}{A} + \lambda\right)u\partial_u$ is a Lie symmetry when

$$V^V_v(t,x,y,z) = \frac{1}{A^2}e^{-m\lambda t}U\left(x - \frac{\alpha m}{2}t, \left(y + \frac{\beta}{\alpha-1}\right)e^{\frac{m(\alpha-1)}{2}t}, \left(z + \frac{\gamma}{\alpha-\lambda}\right)e^{-\frac{m(\alpha-\lambda)}{2}t}\right) + \frac{m(\lambda-1)A_{,t} + 2A_{,tt}}{2A^3}. \quad (98)$$

4.3.1. Invariant Functions

As previously, we determine the Lie invariants related to the admitted symmetry vectors for each potential functional. Indeed, for the potential $V^V_i(t,x,y,z)$ the invariant functions related to the Lie symmetry ξ^1_{III} are $\{t, e^x y, e^x z, u\}$. For the potential function $V^V_{ii}(t,x,y,z)$ we determine the invariants $\{t,x,z,u\}$ while for $V^V_{iii}(t,x,y,z)$ the invariants are $\{t,x,y,u\}$. Moreover, for $V^V_{iv}(t,x,y,z)$ the corresponding invariant functions related to the generic vector field $\alpha\xi^1_V + \beta\xi^2_V + \gamma\xi^3_V$ are $\left\{t, \left(y - \frac{\beta}{\alpha}\right)e^x, \left(z - \frac{\gamma}{\alpha}\right)e^x, u\right\}$.

Finally, for the remaining cases where the proper CKV generates Lie symmetries, it follows that for potential $V^{III}_v(t,x,y,z)$ the Lie invariants are $\left\{x, ye^{-\frac{m}{2}t}, ze^{-\frac{m\lambda}{2}t}, e^{m\lambda t}u\right\}$ while for the potential $V^V_v(t,x,y,z)$ and the Lie symmetry $X_3 + \alpha\xi^1_V + \beta\xi^2_V + \gamma\xi^3_V - \left(\frac{2}{m}\frac{A_{,t}}{A} + \lambda\right)u\partial_u$ the corresponding Lie invariants are $\left\{x - \frac{\alpha m}{2}t, \left(y + \frac{\beta}{\alpha-1}\right)e^{\frac{m(\alpha-1)}{2}t}, \left(z + \frac{\gamma}{\alpha-\lambda}\right)e^{-\frac{m(\alpha-\lambda)}{2}t}, e^{m\lambda t}u\right\}$.

We proceed with the derivation of the conservation laws.

4.3.2. Conservation Laws

The conservation law for the Klein–Gordon Equation (97) and potential function $V^V_i(t,x,y,z)$ has the following components

$$I^x\left(\xi_V^1\right) = \frac{NABC}{2}e^x\left(\left(\left(-\frac{1}{N^2}u_{,t}^2 + \frac{1}{A^2}u_{,x}^2 + \frac{e^{-x}}{B^2}u_{,y}^2 + \frac{e^{-x}}{C^2}u_{,z}^2\right) - V_i^V(t,x,y,z)u^2\right)\right), \tag{99}$$

$$I^y\left(\xi_V^1\right) = \frac{NABC}{2}e^x\left(-y\left(\left(-\frac{1}{N^2}u_{,t}^2 + \frac{1}{A^2}u_{,x}^2 + \frac{e^{-x}}{B^2}u_{,y}^2 + \frac{e^{-x}}{C^2}u_{,z}^2\right) - V_i^V(t,x,y,z)u^2\right)\right), \tag{100}$$

$$I^z\left(\xi_V^1\right) = \frac{NABC}{2}e^x\left(-z\left(\left(-\frac{1}{N^2}u_{,t}^2 + \frac{1}{A^2}u_{,x}^2 + \frac{e^{-x}}{B^2}u_{,y}^2 + \frac{e^{-x}}{C^2}u_{,z}^2\right) - V_i^V(t,x,y,z)u^2\right)\right), \tag{101}$$

and

$$I^t\left(\xi_V^1\right) = 0\,. \tag{102}$$

For $V_{ii}^V(t,x,y,z)$ it follows

$$I^y\left(\xi_V^2\right) = \frac{NABC}{2}e^x\left(\left(\left(-\frac{1}{N^2}u_{,t}^2 + \frac{1}{A^2}u_{,x}^2 + \frac{e^{-x}}{B^2}u_{,y}^2 + \frac{e^{-x}}{C^2}u_{,z}^2\right) - V_{ii}^V(t,x,y,z)u^2\right)\right), \tag{103}$$

$$I^t\left(\xi_V^2\right) = 0\,,\ I^x\left(\xi_V^2\right) = 0 \text{ and } I^z\left(\xi_V^2\right) = 0\,. \tag{104}$$

For $V_{iii}^V(t,x,y,z)$ we find

$$I^z\left(\xi_V^3\right) = \frac{NABC}{2}e^x\left(\left(\left(-\frac{1}{N^2}u_{,t}^2 + \frac{1}{A^2}u_{,x}^2 + \frac{e^{-x}}{B^2}u_{,y}^2 + \frac{e^{-x}}{C^2}u_{,z}^2\right) - V_{iii}^V(t,x,y,z)u^2\right)\right), \tag{105}$$

$$I^t\left(\xi_V^3\right) = 0\,,\ I^x\left(\xi_V^3\right) = 0 \text{ and } I^y\left(\xi_V^3\right) = 0\,. \tag{106}$$

For the generic vector field $\alpha\xi_V^1 + \beta\xi_V^2 + \gamma\xi_V^3$ and potential $V_{iv}^V(t,x,y,z)$ it follows

$$I^x\left(\alpha\xi_V^1 + \beta\xi_V^2 + \gamma\xi_V^3\right) = \alpha I^x\left(\xi_V^1\right)$$

$$I^y\left(\alpha\xi_V^1 + \beta\xi_V^2 + \gamma\xi_V^3\right) = \alpha I^y\left(\xi_V^1\right) + \beta I^y\left(\xi_V^2\right)$$

$$I^z\left(\alpha\xi_V^1 + \beta\xi_V^2 + \gamma\xi_V^3\right) = \alpha I^z\left(\xi_V^1\right) + \gamma I^z\left(\xi_V^3\right)$$

and

$$I^t\left(\alpha\xi_V^1 + \beta\xi_V^2 + \gamma\xi_V^3\right) = 0,$$

with potential function $V_{iv}^V(t,x,y,z)$.

From the Lie symmetry vector $X = X_3 - \left(\frac{2}{m}\frac{A_{,t}}{A} + \lambda\right)u\partial_u$ we determine the conservation law

$$I^t(X_3) = \frac{\bar{A}^4 e^{\frac{(3m\lambda-1)}{2}t}e^x}{2}\left(\frac{2}{m}H_V^V - \left(\left(\frac{2}{m}\frac{\bar{A}_{,t}}{\bar{A}} + \lambda\right)u\right)\frac{1}{A}u^{,t} - \frac{1}{2\bar{A}e^{\frac{m}{2}\lambda t}}\left(\frac{2}{m}\frac{\bar{A}_{,t}}{\bar{A}} + \lambda\right)_t u^2\right). \tag{107}$$

$$I^x(X_3) = \frac{\bar{A}^4 e^{\frac{(3m\lambda-1)}{2}t}e^x}{2}\left(-\left(\left(\frac{2}{m}\frac{\bar{A}_{,t}}{\bar{A}} + \lambda\right)u\right)\frac{1}{\bar{A}e^{\frac{1}{2}\lambda t}}u^{,x}\right). \tag{108}$$

$$I^y(X_3) = \frac{\bar{A}^4 e^{\frac{(3m\lambda-1)}{2}t}e^x}{2}\left(yH_V^V - \left(\left(\frac{2}{m}\frac{A_{,t}}{A} + \lambda\right)u\right)\frac{1}{\bar{A}e^{m(\lambda-1)t}e^x}u^{,y}\right). \tag{109}$$

$$I^z(X_3) = \frac{\bar{A}^4 e^{\frac{(3m\lambda-1)}{2}t}e^x}{2}\left(\lambda z H_V^V - \left(\left(\frac{2}{m}\frac{\bar{A}_{,t}}{\bar{A}} + \lambda\right)u\right)\frac{1}{\bar{A}e^x}u^{,z}\right). \tag{110}$$

in which

$$H_V^V = \left(\left(-\frac{1}{\bar{A}^2 e^{m\lambda t}} u_{,t}^2 + \frac{1}{\bar{A}^2} u_{,x}^2 + \frac{e^{-x}}{\bar{A}^2 e^{m(\lambda-1)t}} u_{,y}^2 + \frac{e^{-x}}{C^2} u_{,z}^2 \right) - V_v^{III}(t,x,y,z) u^2 \right).$$

Finally, the Klein–Gordon equation in the Bianchi V background space with potential function $V_v^V(t,x,y,z)$ admits the conservation law with components

$$I^t = I^t(X_3), \quad (111)$$

$$I^x = I^x(X_3) + \alpha I^x\left(\xi_V^1\right), \quad (112)$$

$$I^y = I^y(X_3) + \alpha I^y\left(\xi_V^1\right) + \beta I^y\left(\xi_V^2\right), \quad (113)$$

$$I^z = I^z(X_3) + \alpha I^z\left(\xi_V^1\right) + \gamma I^z\left(\xi_V^3\right). \quad (114)$$

5. Conclusions

We performed a detailed study for infinitesimal transformations which leave invariant the Klein–Gordon equation with a non-constant potential function in curved spacetimes. Specifically, we determined all the admitted Lie and Noether symmetries for the Klein–Gordon equation. We considered four-dimensional Riemannian manifolds which describe homogeneous and anisotropic cosmologies. We wrote the Klein–Gordon equation in the case of Bianchi I, Bianchi III and Bianchi V spacetimes and we determined all the unknown functional forms of the potential function where the Klein–Gordon equations admit non-trivial Lie and Noether symmetries.

We made use of some previous results which relate the infinitesimal transformations, i.e., the Lie and Noether symmetries, for the Klein–Gordon equation to the elements of the conformal algebra for the metric tensor of the Riemannian manifold where the Laplace operator is defined. Thus, we performed a detailed presentation of the CKVs for the three spacetimes of our consideration. These spacetimes for arbitrary scale factors have a three-dimensional conformal algebra which consists of these KVs. However, for special functions of the scale factors the spacetimes admit a proper CKV. There are two forms for the line-element of Bianchi I spacetime where a proper CKV exists, and there is one specific form for the line element of Bianchi III and one specific line element for the Bianchi V spacetime where one proper CKV exist.

Thus, for all the specific line elements we present in a systematic way all the functional forms for the potential for the Klein–Gordon equation where Lie and Noether symmetries exist. Such an analysis is important in order to understand the relation of symmetries of differential equations with the background geometry, as it also shows how symmetries can be derived in a simple and systematic approach by using tools from differential geometry. Last but not least, the Noetherian conservation laws can be easily constructed with the application of Noether's second theorem.

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