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On Some Generalizations of Cauchy–Schwarz Inequalities and Their Applications

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Abstract: The aim of this paper is to provide new upper bounds of $\omega(T)$, which denotes the numerical radius of a bounded operator T on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. We show the Aczél inequality in terms of the operator $|T|$. Next, we give certain inequalities about the A-numerical radius $\omega_A(T)$ and the A-operator seminorm $\|T\|_A$ of an operator T. We also present several results related to the $\mathbb{A}$-numerical radius of 2×2 block matrices of semi-Hilbert space operators, by using symmetric 2×2 block matrices.

Keywords: numerical radius; positive operator; 2×2-operator matrix; semi-inner product

MSC: 46C05; 47A12; 47A05; 47B65

Citation: Altwaijry, N.; Feki, K.; Minculete, N. On Some Generalizations of Cauchy–Schwarz Inequalities and Their Applications. *Symmetry* **2023**, *15*, 304. https://doi.org/10.3390/sym15020304

Academic Editor: Ioan Rașa

Received: 11 December 2022
Revised: 29 December 2022
Accepted: 6 January 2023
Published: 21 January 2023

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space, endowed with the inner product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$. We denote by $\mathbb{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$ with identity I. For $T \in \mathbb{B}(\mathcal{H})$, the nullspace and the range of T are, respectively, denoted by $\mathcal{N}(T)$ and $\mathcal{R}(T)$. If $\mathcal{S}$ is any closed linear subspace of $\mathcal{H}$, then $P_{\mathcal{S}}$ stands for the orthogonal projection onto $\mathcal{S}$. If we have $T = T^*$, then a bounded linear operator T on $\mathcal{H}$ is called selfadjoint . We denote by $\mathbb{B}_h(\mathcal{H})$ the semi-space of all selfadjoint operators in $\mathbb{B}(\mathcal{H})$. We remark that $T \in \mathbb{B}_h(\mathcal{H})$ if and only if $\langle Tx, x\rangle \in \mathbb{R}$, for any vector $x \in \mathcal{H}$. We define by $\mathbb{B}(\mathcal{H})^+$ the cone of positive (semi-definite) operators of $\mathbb{B}(\mathcal{H})$, namely,

$$\mathbb{B}(\mathcal{H})^+ = \{T \in \mathbb{B}(\mathcal{H})\,;\, \langle Tx, x\rangle \geq 0,\ \forall\, x \in \mathcal{H}\ \} \subseteq \mathbb{B}_h(\mathcal{H}).$$

In [1], for any unit vector $x \in \mathcal{H}$ and $T \in \mathbb{B}(\mathcal{H})^+$ we have the McCarthy inequality

$$\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle, \quad r \geq 1. \tag{1}$$

If $T \in \mathbb{B}(\mathcal{H})^+$, then we can say that there exists a unique positive bounded linear operator $T^{1/2}$ such that $T = (T^{1/2})^2$.

For $T \in \mathbb{B}(\mathcal{H})$, the absolute value $|T|$ is defined by $|T| = (T^*T)^{1/2}$. Notice that $|T|$ is a positive operator.

For $T \in \mathbb{B}(\mathcal{H})$, we recall the following values: $\|T\| := \sup\left\{\|Tx\|\,;\ x \in \mathcal{H}, \|x\| = 1\right\}$ (the operator norm of T) and $\omega(T) := \sup\left\{|\langle Tx, x\rangle|\,;\ x \in \mathcal{H}, \|x\| = 1\right\}$ (the numerical radius of the operator T). It is easy to see that $\omega(T) \leq \|T\|$. If T is a normal operator ($T^*T = TT^*$), then we deduce $\omega(T) = \|T\|$. In [2], Kittaneh showed that

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\|$$

and in [3] the same author proved that:

$$\frac{1}{4}\||T|^2 + |T^*|^2\| \leq \omega^2(T) \leq \frac{1}{2}\||T|^2 + |T^*|^2\|. \tag{2}$$

For $T, S \in \mathbb{B}(\mathcal{H})$,

$$\omega^r(S^*T) \leq \frac{1}{2}\||T|^{2r} + |S|^{2r}\|, \quad (r \geq 2). \tag{3}$$

This represents an inequality given by Dragomir in [4].

Next, we present an improvement of the above inequality for $r = 2$, given by Kittaneh and Moradi in [5]:

$$\omega^2(S^*T) \leq \frac{1}{6}\||T|^4 + |S|^4\| + \frac{1}{3}\omega(S^*T)\||T|^2 + |S|^2\| \leq \frac{1}{2}\||T|^4 + |S|^4\|. \tag{4}$$

Some results related to the numerical radius are given in [6,7].

The Moore-Penrose inverse of T denoted by $T^\dagger$ has the properties studied in several papers (see [8]).

In [8], it is given that $T^\dagger \in \mathbb{B}(\mathcal{H})$ if and only if T has closed range in $\mathcal{H}$, that is, $\overline{\mathcal{R}(T)} = \mathcal{R}(T)$, where $\overline{\mathcal{R}(T)}$ means the closure of $\mathcal{R}(T)$ in the norm topology of $\mathcal{H}$.

From now on, we assume that $A \in \mathbb{B}(\mathcal{H})^+$ is a nonzero operator which defines the following positive semidefinite sesquilinear form

$$\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}, \ (x, y) \longmapsto \langle x, y \rangle_A := \langle Ax, y \rangle.$$

Notice that the seminorm induced by $\langle \cdot, \cdot \rangle_A$ is given by $\|x\|_A = \sqrt{\langle x, x \rangle_A} = \|A^{1/2}x\|$ for every $x \in \mathcal{H}$. We remark that $\| \cdot \|_A$ is a norm on $\mathcal{H}$ if and only if A is an injective operator, and that the semi-Hilbert space $(\mathcal{H}, \| \cdot \|_A)$ is complete if and only if $\mathcal{R}(A)$ is a closed subspace in $\mathcal{H}$. It is easy to see that if $A = I$, then $\langle x, y \rangle_A = \langle x, y \rangle$ and $\|x\|_A = \|x\|$.

The numerical radius plays an important role in various fields of operator theory and matrix analysis (cf. [9,10]). We remark that $|\langle x, y \rangle_A| \leq \|A\|$ and $\|x\|_A \leq \sqrt{\|A\|}$, for $x \in \mathcal{H}$ with $\|x\| = \|y\| = 1$.

Certain generalizations for the notion of the numerical radius have recently been introduced (cf. [11–13]). Among these generalizations is the A-numerical radius of an operator $T \in \mathbb{B}(\mathcal{H})$, which was firstly defined by Saddi in [13] as

$$\omega_A(T) = \sup\left\{ \left|\langle Tx, x \rangle_A\right| ; \ x \in \mathcal{H}, \ \|x\|_A = 1 \right\}.$$

There are many other results, in numerous recent papers, related to the A-numerical radius (cf. [14–19]) and the references therein.

An operator $S \in \mathbb{B}(\mathcal{H})$ is called an A-adjoint operator of T, where $T \in \mathbb{B}(\mathcal{H})$, if the identity $\langle Tx, y \rangle_A = \langle x, Sy \rangle_A$ holds for every $x, y \in \mathcal{H}$, therefore, S is the solution of the following operator equation $AX = T^*A$. This equation can be investigated by using a theorem due to Douglas [20]. We denote by $\mathbb{B}_A(\mathcal{H})$ and $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ the sets of all operators that admit A–adjoints and $A^{1/2}$–adjoints, respectively. From Douglas's theorem we deduce that

$$\mathbb{B}_A(\mathcal{H}) = \left\{ T \in \mathbb{B}(\mathcal{H}) ; \ \mathcal{R}(T^*A) \subseteq \mathcal{R}(A) \right\},$$

and

$$\mathbb{B}_{A^{1/2}}(\mathcal{H}) = \left\{T \in \mathbb{B}(\mathcal{H})\,;\ \exists\, c > 0;\ \|Tx\|_A \leq c\|x\|_A,\ \forall x \in \mathcal{H}\right\}.$$

We observed that $\mathbb{B}_A(\mathcal{H})$ and $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ are two subalgebras of $\mathbb{B}(\mathcal{H})$ which are neither closed nor dense in $\mathbb{B}(\mathcal{H})$. Moreover, the following proper inclusion $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H})$ holds (see [18]).

An operator $T \in \mathbb{B}(\mathcal{H})$ is called A–bounded if $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. On the set $\mathbb{B}_{A^{1/2}}(\mathcal{H})$, the following semi-norm is defined

$$\|T\|_A := \sup_{\substack{x \in \overline{\mathcal{R}(A)},\\ x \neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup\left\{\|Tx\|_A\,;\ x \in \mathcal{H}, \|x\|_A = 1\right\} < +\infty,$$

(see [18] and the references therein). It is easy to see that for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, $\|T\|_A = 0$ if and only if $AT = 0$. We also observe that for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, $\|Tx\|_A \leq \|T\|_A\|x\|_A$, for all $x \in \mathcal{H}$. This immediately yields $\|TS\|_A \leq \|T\|_A\|S\|_A$, for all $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$.

If $T \in \mathbb{B}_A(\mathcal{H})$, then the Douglas solution of the equation $AX = T^*A$ (see [20]) will be denoted by $T^{\sharp_A}$. Note that $T^{\sharp_A} = A^{\dagger}T^*A$. Furthermore, if $T \in \mathbb{B}_A(\mathcal{H})$, then $T^{\sharp_A} \in \mathbb{B}_A(\mathcal{H})$, $(T^{\sharp_A})^{\sharp_A} = P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}$ and $\left((T^{\sharp_A})^{\sharp_A}\right)^{\sharp_A} = T^{\sharp_A}$. Let $T \in \mathbb{B}(\mathcal{H})$. The operator T is called A–selfadjoint if $AT \in \mathbb{B}_h(\mathcal{H})$, that is, $AT = T^*A$. Further, T is called A–positive if $AT \geq 0$ and we write $T \geq_A 0$. Clearly, A–positive operators are always A–selfadjoint. It is obvious that if T is A–selfadjoint, then $T \in \mathbb{B}_A(\mathcal{H})$. However, in general, the equality $T = T^{\sharp_A}$ may not hold. We also note that if $T \in \mathbb{B}_A(\mathcal{H})$, then $T = T^{\sharp_A}$ if and only if T is A–selfadjoint and $\mathcal{R}(T) \subseteq \overline{\mathcal{R}(A)}$. Furthermore, it was shown in [21] that if T is an A–selfadjoint operator, then $T^{\sharp_A}$ is A–selfadjoint and

$$(T^{\sharp_A})^{\sharp_A} = T^{\sharp_A}. \tag{5}$$

Moreover, it was proven in [18] that if T is A–selfadjoint, then

$$\|T\|_A = \omega_A(T). \tag{6}$$

For proofs and other related results, the reader is referred to [8,21–23] and the references therein.

Before we move on, it must be emphasized that $\omega_A(T)$ may be equal to $+\infty$ for some $T \in \mathbb{B}(\mathcal{H})$ (see [21]). Furthermore, it can be checked that for all $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, $\left|\langle Tx, x\rangle_A\right| \leq \omega_A(T)\|x\|_A^2$, for every $x \in \mathcal{H}$, holds. It is known that $\omega_A(\cdot)$ defines a semi-norm on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ such that for all $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ the following inequality holds,

$$\frac{1}{2}\|T\|_A \leq \omega_A(T) \leq \|T\|_A. \tag{7}$$

Some improvements of the inequalities (7) have been recently established by many authors (e.g., see [15,21], and their references). In particular, it has been shown in [24,25] that

$$\frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A \leq \omega_A^2(T) \leq \frac{1}{2}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A \tag{8}$$

for all $T \in \mathbb{B}_A(\mathcal{H})$.

When $A = I$, we deduce the well-known inequalities proved by Kittaneh in [3, Theorem 1], given in (2). Notice that the second author showed that $\omega_A(\cdot)$ satisfies the power property, that is, for every $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ and all positive integers n,

$$\omega_A(T^n) \leq \omega_A^n(T). \tag{9}$$

In [5], a new improvement of the Cauchy–Schwarz inequality (in short (C-S)) is given by:

$$|\langle x,y\rangle| \le \sqrt{\frac{1}{2}(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2) + |\langle x,y\rangle|\|x\|\|y\|} \le \|x\|\|y\|, \tag{10}$$

for any $x,y \in \mathcal{H}$. This inequality provides refinements of some numerical radius inequalities for Hilbert space operators. Another inequality of the type above is given by Alomari [26]

$$\big|\langle x,y\rangle\big|^2 \le \lambda\|x\|^2\|y\|^2 + (1-\lambda)\big|\langle x,y\rangle\big|\|x\|\|y\| \le \|x\|^2\|y\|^2, \tag{11}$$

for any $x,y \in \mathcal{H}$ and $\lambda \in [0,1]$.

The main objective of the present paper is to study some new improvements of the upper bounds of $\omega(T)$, $\|T\|$ and $\omega(S^*T)$, of the type given in (2)–(4). Next, we give certain inequalities about the A-numerical radius $\omega_A(T)$ and the A-operator seminorm $\|T\|_A$ of an operator T defined on the semi-Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_A)$, respectively, where $\langle x,y\rangle_A := \langle Ax,y\rangle$ for all $x,y \in \mathcal{H}$. One of the main purposes of this paper is to prove some refinements of the inequalities (8). We also present several results related to the $\mathbb{A}$-numerical radius for 2×2 block matrices of semi-Hilbert space operators, where $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ denotes the 2×2 diagonal operator matrix whose each diagonal entry is the operator A.

2. Inequalities about $\omega(S^*T)$

In this section, our first results are given. To begin with, a result which generalizes inequality (10) is presented:

Lemma 1. *Let $\lambda \in [0,1]$. Then*

$$|\langle x,y\rangle|^2 \le \lambda(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2) + |\langle x,y\rangle|^{2\lambda}\|x\|^{2-2\lambda}\|y\|^{2-2\lambda} \le \|x\|^2\|y\|^2 \tag{12}$$

for any $x,y \in \mathcal{H}$.

Proof. For $\lambda = 0$ and $\lambda = 1$ in the inequality (12) we obtain the inequality (C-S), $|\langle x,y\rangle| \le \|x\|\|y\|$. Therefore the inequality of the statement is true. However, we have

$$\lambda(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2) \ge 0,$$

for all $x,y \in \mathcal{H}$ and $\lambda \in (0,1)$, which means that

$$\lambda(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2) + |\langle x,y\rangle|^{2\lambda}\|x\|^{2-2\lambda}\|y\|^{2-2\lambda} \ge |\langle x,y\rangle|^{2\lambda}|\langle x,y\rangle|^{2-2\lambda}.$$

This yields that

$$|\langle x,y\rangle|^2 \le \lambda(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2) + |\langle x,y\rangle|^{2\lambda}\|x\|^{2-2\lambda}\|y\|^{2-2\lambda}, \tag{13}$$

for every $x,y \in \mathcal{H}$ and $\lambda \in (0,1)$. Using Young's inequality,

$$a^\lambda b^{1-\lambda} \le \lambda a + (1-\lambda)b,$$

for every $a,b > 0$ and $\lambda \in (0,1)$, we obtain

$$|\langle x,y\rangle|^{2\lambda}\|x\|^{2-2\lambda}\|y\|^{2-2\lambda} \le \lambda|\langle x,y\rangle|^2 + (1-\lambda)\|x\|^2\|y\|^2,$$

which is equivalent to

$$\begin{aligned}&\lambda(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2) + |\langle x,y\rangle|^{2\lambda}\|x\|^{2-2\lambda}\|y\|^{2-2\lambda}\\ &\le \lambda(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2) + \lambda|\langle x,y\rangle|^2 + (1-\lambda)\|x\|^2\|y\|^2,\end{aligned}$$

whence

$$\lambda(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2) + |\langle x,y\rangle|^{2\lambda}\|x\|^{2-2\lambda}\|y\|^{2-2\lambda} \le \|x\|^2\|y\|^2, \tag{14}$$

for all $x, y \in \mathcal{H}$ and $\lambda \in (0,1)$. Consequently, we obtain the desired inequality by taking (13) and (14) into consideration. □

Remark 1. *Another form of inequality* (12) *can be given as:*

$$|\langle x,y\rangle| \le \sqrt{\lambda(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2) + |\langle x,y\rangle|^{2\lambda}\|x\|^{2-2\lambda}\|y\|^{2-2\lambda}} \le \|x\|\|y\| \tag{15}$$

for any $x, y \in \mathcal{H}$ and $\lambda \in [0,1]$, being an improvement on the inequality (C-S). For $\lambda = \frac{1}{2}$ in inequality (15)*, we obtain inequality* (10)*.*

Theorem 1. *Let $T, S \in \mathbb{B}(\mathcal{H})$, $r \ge 1$ and $\lambda \in [0,1]$. Then the inequality*

$$\omega^{2r}(S^*T) \le \frac{1}{2}\||T|^{4r} + |S|^{4r}\| - \frac{\lambda(1-\lambda)}{1+\lambda-\lambda^2}\left(\||T|^{4r} + |S|^{4r}\| - \omega(S^*T)\||T|^{2r} + |S|^{2r}\|\right) \tag{16}$$

holds.

Proof. Taking the first inequality from Lemma 1, we have

$$|\langle x,y\rangle|^2 \le \frac{\lambda}{1+\lambda}\|x\|^2\|y\|^2 + \frac{1}{1+\lambda}|\langle x,y\rangle|^{2\lambda}\|x\|^{2-2\lambda}\|y\|^{2-2\lambda}$$

for every $x, y \in \mathcal{H}$ and $\lambda \in [0,1]$. From the power-mean inequality [27] given by

$$\lambda a + (1-\lambda)b \le (\lambda a^r + (1-\lambda)b^r)^{\frac{1}{r}}$$

for all $\lambda \in [0,1]$, $a, b \ge 0$ and $r \ge 1$, we show that

$$|\langle x,y\rangle|^{2r} \le \frac{\lambda}{1+\lambda}\|x\|^{2r}\|y\|^{2r} + \frac{1}{1+\lambda}|\langle x,y\rangle|^{2\lambda r}\|x\|^{(2-2\lambda)r}\|y\|^{(2-2\lambda)r} \tag{17}$$

for every $x, y \in \mathcal{H}$, $r \ge 1$ and $\lambda \in [0,1]$. If we replace x and y by Tx and Sx, respectively, in (17), and we assume that $\|x\| = 1$, we then have

$$|\langle S^*Tx,x\rangle|^{2r} \le \frac{\lambda}{1+\lambda}\|Tx\|^{2r}\|Sx\|^{2r} + \frac{1}{1+\lambda}|\langle S^*Tx,x\rangle|^{2\lambda r}\|Tx\|^{(2-2\lambda)r}\|Sx\|^{(2-2\lambda)r}.$$

Using the same idea as in [5] or [26], for the above inequality, we deduce

$$\begin{aligned}|\langle S^*Tx,x\rangle|^{2r} &\le \frac{1}{1+\lambda}|\langle S^*Tx,x\rangle|^{2\lambda r}\left(\langle|T|^2x,x\rangle\langle|S|^2x,x\rangle\right)^{(1-\lambda)r}\\ &\quad + \frac{\lambda}{1+\lambda}\langle|T|^2x,x\rangle^r\langle|S|^2x,x\rangle^r\\ &= \frac{1}{1+\lambda}|\langle S^*Tx,x\rangle|^{\lambda r}\langle|T|^2x,x\rangle^{(1-\lambda)r}|\langle S^*Tx,x\rangle|^{\lambda r}\langle|S|^2x,x\rangle^{(1-\lambda)r}\\ &\quad + \frac{\lambda}{1+\lambda}\langle|T|^2x,x\rangle^r\langle|S|^2x,x\rangle^r.\end{aligned}$$

Applying Young's inequality in the above relation, we find the following inequality:

$$\begin{aligned}
&\left|\langle S^*Tx,x\rangle\right|^{2r}\\
&\le \frac{1}{1+\lambda}\left(\lambda\left|\langle S^*Tx,x\rangle\right| + (1-\lambda)\langle |T|^2x,x\rangle\right)^r\left(\lambda\left|\langle S^*Tx,x\rangle\right| + (1-\lambda)\langle |S|^2x,x\rangle\right)^r\\
&\qquad + \frac{\lambda}{1+\lambda}\langle |T|^2x,x\rangle^r\langle |S|^2x,x\rangle^r\\
&\le \frac{1}{1+\lambda}\left(\lambda\left|\langle S^*Tx,x\rangle\right|^r + (1-\lambda)\langle |T|^2x,x\rangle^r\right)\left(\lambda\left|\langle S^*Tx,x\rangle\right|^r + (1-\lambda)\langle |S|^2x,x\rangle^r\right)\\
&\qquad + \frac{\lambda}{1+\lambda}\langle |T|^2x,x\rangle^r\langle |S|^2x,x\rangle^r\\
&\le \frac{1}{1+\lambda}\left(\lambda\left|\langle S^*Tx,x\rangle\right|^r + (1-\lambda)\langle |T|^{2r}x,x\rangle\right)\left(\lambda\left|\langle S^*Tx,x\rangle\right|^r + (1-\lambda)\langle |S|^{2r}x,x\rangle\right)\\
&\qquad + \frac{\lambda}{1+\lambda}\langle |T|^{2r}x,x\rangle\langle |S|^{2r}x,x\rangle\\
&= \frac{\lambda(1-\lambda)}{1+\lambda}\left|\langle S^*Tx,x\rangle\right|^r\langle (|T|^{2r}+|S|^{2r})x,x\rangle + \frac{(1-\lambda)^2}{1+\lambda}\langle |T|^{2r}x,x\rangle\langle |S|^{2r}x,x\rangle\\
&\qquad + \frac{\lambda}{1+\lambda}\langle |T|^{2r}x,x\rangle\langle |S|^{2r}x,x\rangle + \frac{\lambda^2}{1+\lambda}\left|\langle S^*Tx,x\rangle\right|^{2r},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&(1+\lambda-\lambda^2)\left|\langle S^*Tx,x\rangle\right|^{2r}\\
&\le (1-\lambda+\lambda^2)\langle |T|^{2r}x,x\rangle\langle |S|^{2r}x,x\rangle + \lambda(1-\lambda)|\langle S^*Tx,x\rangle|^r\langle (|T|^{2r}+|S|^{2r})x,x\rangle\\
&\le \frac{1-\lambda+\lambda^2}{2}\langle (|T|^{4r}+|S|^{4r})x,x\rangle + \lambda(1-\lambda)|\langle S^*Tx,x\rangle|^r\langle (|T|^{2r}+|S|^{2r})x,x\rangle.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&2(1+\lambda-\lambda^2)\left|\langle S^*Tx,x\rangle\right|^{2r}\\
&\le (1-\lambda+\lambda^2)\langle (|T|^{4r}+|S|^{4r})x,x\rangle + 2\lambda(1-\lambda)\left|\langle S^*Tx,x\rangle\right|^r\langle (|T|^{2r}+|S|^{2r})x,x\rangle.
\end{aligned}$$

If we take the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ in the above inequality, then we obtain the inequality

$$\omega^{2r}(S^*T) \le \frac{1-\lambda+\lambda^2}{2(1+\lambda-\lambda^2)}\left\||T|^{4r}+|S|^{4r}\right\| + \frac{\lambda(1-\lambda)}{1+\lambda-\lambda^2}\omega^r(S^*T)\left\||T|^{2r}+|S|^{2r}\right\|. \tag{18}$$

Inequality (18), can be rewritten, rearranging the terms, as the inequality of the statement. □

Remark 2. *For $\lambda = 0$ or $\lambda = 1$ in (16), we deduce the following inequality:*

$$\omega^{2r}(S^*T) \le \frac{1}{2}\left\||T|^{4r}+|S|^{4r}\right\|, \tag{19}$$

when $r \ge 1$. This represents inequality (3) given by Dragomir in [4].

Because $\lambda \in [0,1]$ and the following inequality:

$$\left\||T|^{4r}+|S|^{4r}\right\| \ge \frac{1}{2}\left\||T|^{2r}+|S|^{2r}\right\|^2 \ge \omega(S^*T)\left\||T|^{2r}+|S|^{2r}\right\|$$

holds when $r \ge 1$, then (16) is an improvement of inequality (19).

In [28], Buzano proved an interesting inequality:

$$|\langle x,u\rangle\langle u,y\rangle| \le \frac{1}{2}(\|x\|\|y\| + |\langle x,y\rangle|), \tag{20}$$

where $x, y, u \in \mathcal{H}$ and $\|u\| = 1$. We apply this result in order to give another inequality related to the numerical radius.

Theorem 2. *Let $T \in \mathbb{B}(\mathcal{H})$ and $\lambda \in [0,1]$. Then the inequalities*

$$\begin{aligned} \omega^4(T) &\le \frac{\lambda}{8}\||T|^4 + |T^*|^4\| + \left[\frac{\lambda}{4}\omega(T^2) + \frac{1-\lambda}{4}\omega^2(T)\right]\||T|^2 + |T^*|^2\| \\ &\quad + \frac{\lambda}{4}\omega^2(T^2) + \frac{1-\lambda}{2}\omega(T^2)\omega^2(T) \\ &\le \frac{1}{2}\||T|^4 + |T^*|^4\| \end{aligned}$$

hold.

Proof. Let $x, y, u \in \mathcal{H}$ with $\|u\| = 1$ and $\lambda \in [0,1]$. Using inequality (20), we have

$$\begin{aligned} |\langle x,u\rangle\langle u,y\rangle|^2 &\le |\langle x,u\rangle\langle u,y\rangle|\frac{1}{2}(\|x\|\|y\| + |\langle x,y\rangle|) \\ &\le \frac{\lambda}{4}(\|x\|\|y\| + |\langle x,y\rangle|)^2 + \frac{1-\lambda}{2}|\langle x,u\rangle\langle u,y\rangle|(\|x\|\|y\| + |\langle x,y\rangle|). \end{aligned}$$

Thus, we deduce

$$\begin{aligned} |\langle x,u\rangle\langle u,y\rangle|^2 &\le \frac{\lambda}{4}|\langle x,y\rangle|^2 + \frac{1-\lambda}{2}|\langle x,u\rangle\langle u,y\rangle|(\|x\|\|y\| + |\langle x,y\rangle|) \\ &\quad + \frac{\lambda}{4}\|x\|^2\|y\|^2 + \frac{\lambda}{2}\|x\|\|y\||\langle x,y\rangle|. \end{aligned} \tag{21}$$

If we replace u by x where $\|x\| = 1$, x by Tx and y by T^*x in the above inequality (21), we obtain

$$\begin{aligned} |\langle Tx,x\rangle|^4 &= |\langle Tx,x\rangle\langle x,T^*x\rangle|^2 \\ &\le \frac{\lambda}{4}|\langle Tx,T^*x\rangle|^2 + \frac{1-\lambda}{2}|\langle Tx,x\rangle\langle x,T^*x\rangle|(\|Tx\|\|T^*x\| + |\langle Tx,T^*x\rangle|) \\ &\quad + \frac{\lambda}{4}\|Tx\|^2\|T^*x\|^2 + \frac{\lambda}{2}\|Tx\|\|T^*x\||\langle Tx,T^*x\rangle| \\ &\le \frac{\lambda}{8}\||T|^4 + |T^*|^4\| + \left[\frac{\lambda}{4}\omega(T^2) + \frac{1-\lambda}{4}\omega^2(T)\right]\||T|^2 + |T^*|^2\| \\ &\quad + \frac{\lambda}{4}\omega^2(T^2) + \frac{1-\lambda}{2}\omega(T^2)\omega^2(T). \end{aligned} \tag{22}$$

We take into account the following sequence of inequalities:

$$\begin{aligned} \|Tx\|^2\|T^*x\|^2 &= \langle Tx,Tx\rangle\langle T^*x,T^*x\rangle \\ &= \langle |T|^2x,x\rangle\langle |T^*|^2x,x\rangle \\ &\le \frac{1}{4}\left(\langle |T|^2x,x\rangle + \langle |T^*|^2x,x\rangle\right)^2 \\ &\le \frac{1}{2}\left(\langle |T|^2x,x\rangle^2 + \langle |T^*|^2x,x\rangle^2\right) \\ &\le \frac{1}{2}\langle (|T|^4 + |T^*|^4)x,x\rangle \le \frac{1}{2}\||T|^4 + |T^*|^4\|. \end{aligned}$$

Consequently, taking the supremum for $\|x\| = 1$ in inequality (22), we find the first inequality of the statement.

Now, we have

$$\begin{aligned}
&\frac{\lambda}{8}\||T|^4 + |T^*|^4\| + \left[\frac{\lambda}{4}\omega(T^2) + \frac{1-\lambda}{4}\omega^2(T)\right]\||T|^2 + |T^*|^2\| \\
&+ \frac{\lambda}{4}\omega^2(T^2) + \frac{1-\lambda}{2}\omega(T^2)\omega^2(T) \\
&\le \frac{\lambda}{8}\||T|^4 + |T^*|^4\| + \left[\frac{\lambda}{4}\omega^2(T) + \frac{1-\lambda}{4}\omega^2(T)\right]\||T|^2 + |T^*|^2\| \\
&+ \frac{\lambda}{4}\omega^4(T) + \frac{1-\lambda}{2}\omega^4(T) \\
&\le \frac{\lambda}{8}\||T|^4 + |T^*|^4\| + \frac{1}{4}\omega^2(T)\||T|^2 + |T^*|^2\| + \frac{2-\lambda}{4}\omega^4(T) \le \frac{1}{2}\||T|^4 + |T^*|^4\|.
\end{aligned}$$

In the above sequence of inequalities, we used inequality (2) and inequality (9) for $A = I$ and $n = 2$, hence

$$\omega(T^2) \le \omega^2(T) \le \frac{1}{2}\||T|^2 + |T^*|^2\|,$$

and the results hold. □

Remark 3. *For $\lambda = \frac{1}{3}$ in Theorem 2, we find an inequality given in [5], namely:*

$$\begin{aligned}
\omega^4(T) \le \frac{1}{24}\||T|^4 + |T^*|^4\| &+ \left[\frac{1}{12}\omega(T^2) + \frac{1}{6}\omega^2(T)\right]\||T|^2 + |T^*|^2\| \\
&+ \frac{1}{12}\omega^2(T^2) + \frac{1}{3}\omega(T^2)\omega^2(T).
\end{aligned}$$

If we take $\lambda = \frac{1}{2}$ in Theorem 2, we also obtain:

$$\begin{aligned}
\omega^4(T) \le \frac{1}{16}\||T|^4 + |T^*|^4\| &+ \frac{1}{8}\left[\omega(T^2) + \omega^2(T)\right]\||T|^2 + |T^*|^2\| \\
&+ \frac{1}{8}\omega^2(T^2) + \frac{1}{4}\omega(T^2)\omega^2(T).
\end{aligned}$$

3. Some Inequalities About to the A–Numerical Radius

Next, we give several results related to the seminorm $\|\cdot\|_A$ induced by A.

Theorem 3. *If $a, b \in \mathbb{R}$, $x, y \in \mathcal{H}$ and $|a| \ge \|x\|_A > 0$ and $|b| \ge \|y\|_A > 0$, then*

$$\|bx - ay\|_A^2 + |\langle x, y\rangle_A|^2 \ge \|x\|_A^2\|y\|_A^2. \tag{23}$$

Proof. Let $\Re z$ denote the real part of any complex number z. Clearly, the inequality (23) can be written as

$$a^2\|y\|_A^2 - 2ab\Re\langle x, y\rangle_A + b^2\|x\|_A^2 + |\langle x, y\rangle_A|^2 - \|x\|_A^2\|y\|_A^2 \ge 0.$$

Assume that $b \ge \|y\|_A > 0$. We consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(a) = a^2\|y\|_A^2 - 2ab\Re\langle x, y\rangle_A + b^2\|x\|_A^2 + |\langle x, y\rangle_A|^2 - \|x\|_A^2\|y\|_A^2.$$

This means that

$$\begin{aligned}
\Delta_a &= 4b^2(\Re\langle x, y\rangle_A)^2 - 4b^2\|x\|_A^2\|y\|_A^2 - 4\|y\|_A^2(|\langle x, y\rangle_A|^2 - \|x\|_A^2\|y\|_A^2) \\
&\le 4(b^2 - \|y\|_A^2)(|\langle x, y\rangle_A|^2 - \|x\|_A^2\|y\|_A^2),
\end{aligned}$$

where, we used the inequality $(\Re\langle x,y\rangle_A)^2 = (\Re\langle Ax,y\rangle)^2 \le |\langle Ax,y\rangle|^2 = |\langle x,y\rangle_A|^2$.

Therefore, because we have $b^2 \ge \|y\|_A^2 > 0$ and using the well-known inequality (C-S), $|\langle x,y\rangle_A|^2 \le \|x\|_A^2\|y\|_A^2$, we find that the discriminant Δ_a is negative, and hence $f(a) \ge 0$, for all $a \in \mathbb{R}$. Consequently, the inequality from the statement is valid. □

Corollary 1. *If $a,b \in \mathbb{R}$, $x,y \in \mathcal{H}$ and $|a| \ge \|x\|_A > 0$ and $|b| \ge \|y\|_A > 0$, then the inequality*

$$\|bx - ay\|_A \ge \|x\|_A\|y\|_A - |\langle x,y\rangle_A| \tag{24}$$

holds.

Proof. By using inequality (23) and the following algebraic inequality $\sqrt{\alpha^2-\beta^2} \ge \alpha - \beta$, where $\alpha \ge \beta \ge 0$, for $\alpha = \|x\|_A\|y\|_A$ and $\beta = |\langle x,y\rangle_A|$, we find the inequality of the statement. □

Theorem 4. *If $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ and $a,b \in \mathbb{R}$, $|b| \ge 1$, then we have*

$$\|bT - aI\|_A + \omega_A(T) \ge \|T\|_A. \tag{25}$$

Proof. In inequality (24), replace x by Tx and y by x. Thus,

$$\|bTx - ax\|_A + |\langle Tx,x\rangle_A| \ge \|Tx\|_A\|x\|_A,$$

for $|b| \ge \|x\|_A$. If we take the supremum over $\|x\|_A = 1$, then we get the inequality of the statement, when $|b| \ge 1$. □

Theorem 5. *Let $x,y \in \mathcal{H}$ and $a,b \in \mathbb{R}$. Then, the equality*

$$(a^2 - \|x\|_A^2)(b^2 - \|y\|_A^2) = |ab - \langle x,y\rangle_A|^2 + \|x\|_A^2\|y\|_A^2 - |\langle x,y\rangle_A|^2 - \|bx - ay\|_A^2 \tag{26}$$

holds.

Proof. We remark that $\overline{\langle Ax,y\rangle} = \langle y,Ax\rangle = \langle Ay,x\rangle$. Next, we have the following calculations:

$$\begin{aligned} \left|ab - \langle x,y\rangle_A\right|^2 &= \big(ab - \langle Ax,y\rangle\big)\big(ab - \overline{\langle Ax,y\rangle}\big) \\ &= (ab)^2 - ab\big(\langle Ax,y\rangle + \langle Ay,x\rangle\big) + \left|\langle Ax,y\rangle\right|^2, \end{aligned}$$

and

$$\begin{aligned} \|bx - ay\|_A^2 &= \langle bAx - aAy, bx - ay\rangle \\ &= b^2\|y\|_A^2 - ab\big(\langle Ax,y\rangle + \langle Ay,x\rangle\big) + a^2\|x\|_A^2, \end{aligned}$$

which means that

$$\left|ab - \langle x,y\rangle_A\right|^2 - \|bx - ay\|_A^2 = \big(a^2 - \|x\|_A^2\big)\big(b^2 - \|y\|_A^2\big) + \left|\langle x,y\rangle_A\right|^2 - \|x\|_A^2\|y\|_A^2.$$

Therefore, the equality of the statement is true. □

Corollary 2. *If $a,b \in \mathbb{R}$, $x,y \in \mathcal{H}$ and $|a| \ge \|x\|_A > 0$ and $|b| \ge \|y\|_A > 0$, then the inequality*

$$(a^2 - \|x\|_A^2)(b^2 - \|y\|_A^2) \le |ab - \langle x,y\rangle_A|^2 \tag{27}$$

holds.

Proof. Using inequality (23) and equality (26) we deduce the inequality of the statement. □

Remark 4. *This inequality is the Aczél inequality in vectorial form (see, e.g., [29]).*

Theorem 6. *Let $x, y \in \mathcal{H}$ with $\|x\| = 1$. Then we have*

$$\left(\|A\| - \|x\|_A^2\right)\left(\|A\| - \|y\|_A^2\right) \leq \left|\|A\| - \langle x, y\rangle_A\right|^2.$$

Proof. Using inequality (27) for $a = b = \sqrt{\|A\|}$ and taking into account the fact that $\|x\|_A \leq \sqrt{\|A\|}$, for $x \in \mathcal{H}$ with $\|x\| = 1$, we deduce the inequality of the statement. □

To establish our next result which covers and extends a well-known theorem by Kittaneh et al. in [5], we need the following two lemmas.

Lemma 2. *Let $T, S \in \mathbb{B}(\mathcal{H})$ be A-positive operators. Then*

$$\left\|\frac{T+S}{2}\right\|_A^n \leq \left\|\frac{T^n + S^n}{2}\right\|_A, \quad \forall\, n \in \mathbb{N}^*,$$

where $\mathbb{N}^*$ denotes the set of all positive integers.

To prove Lemma 2, we require the following lemma which was recently proven in [16]:

Lemma 3. *Let $T \in \mathbb{B}(\mathcal{H})$ be such that $T \geq_A 0$. Then, we have*

$$\langle Tx, x\rangle_A^n \leq \langle T^n x, x\rangle_A, \quad \forall\, n \in \mathbb{N}^*. \tag{28}$$

for every $x \in \mathcal{H}$ with $\|x\|_A = 1$.

Proof of Lemma 2. We consider $x \in \mathcal{H}$ with $\|x\|_A = 1$. From the convexity of $h(t) = t^n$ with $t \geq 0$, we get

$$\begin{aligned}
\left(\left\langle \frac{T+S}{2}x, x\right\rangle_A\right)^n &= \left(\frac{\langle Tx, x\rangle_A + \langle Sx, x\rangle_A}{2}\right)^n \\
&\leq \frac{\langle Tx, x\rangle_A^n + \langle Sx, x\rangle_A^n}{2} \\
&\leq \frac{\langle T^n x, x\rangle_A + \langle S^n x, x\rangle_A}{2} \quad \text{(by Lemma 3)} \\
&= \left\langle \frac{T^n + S^n}{2}x, x\right\rangle_A \\
&\leq \left\|\frac{T^n + S^n}{2}\right\|_A.
\end{aligned}$$

Therefore, we obtain

$$\left(\left\langle \frac{T+S}{2}x, x\right\rangle_A\right)^n \leq \left\|\frac{T^n + S^n}{2}\right\|_A.$$

Hence, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the above inequality we get

$$\omega_A^n\left(\frac{T+S}{2}\right) \leq \left\|\frac{T^n + S^n}{2}\right\|_A.$$

Therefore, the proof is complete by using (6) since $\dfrac{T+S}{2} \geq_A 0$. □

Lemma 4. *Let $x, y, u \in \mathcal{H}$ be such that $\|u\|_A = 1$. Then*

$$12\left|\langle x,u\rangle_A\langle u,y\rangle_A\right|^2 \leq \|x\|_A^2\|y\|_A^2 + \left|\langle x,y\rangle_A\right|^2 + 2\|x\|_A\|y\|_A\left|\langle x,y\rangle_A\right| + 4\left|\langle x,u\rangle_A\langle u,y\rangle_A\right|\left(\|x\|_A\|y\|_A + |\langle x,y\rangle_A|\right).$$

Proof. Let $x, y, u \in \mathcal{H}$ be such that $\|u\|_A = 1$. It follows from [13] that

$$2|\langle x,u\rangle_A\langle u,y\rangle_A| \leq |\langle x,y\rangle_A| + \|x\|_A\|y\|_A. \tag{29}$$

By using (29), we see that

$$\begin{aligned}
&\left|\langle x,u\rangle_A\langle u,y\rangle_A\right|^2\\
&= \frac{1}{3}\left|\langle x,u\rangle_A\langle u,y\rangle_A\right|^2 + \frac{2}{3}\left|\langle x,u\rangle_A\langle u,y\rangle_A\right|^2\\
&\leq \frac{1}{12}\left(\|x\|_A\|y\|_A + |\langle x,y\rangle_A|\right)^2 + \frac{2}{6}\left|\langle x,u\rangle_A\langle u,y\rangle_A\right|\left(\|x\|_A\|y\|_A + |\langle x,y\rangle_A|\right).
\end{aligned}$$

This immediately proves the desired result. □

Now, we will give an inequality concerning $\omega_A^4(T)$.

Theorem 7. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then the following inequality*

$$\begin{aligned}
\omega_A^4(T) &\leq \frac{1}{24}\left\|(T^{\sharp_A}T)^2 + (TT^{\sharp_A})^2\right\|_A + \frac{1}{12}\omega_A^2(T^2) + \frac{1}{3}\omega_A^2(T)\omega_A(T^2)\\
&\quad + \frac{1}{12}\left\|T^{\sharp_A}T + TT^{\sharp_A}\right\|_A\left(\omega_A(T^2) + 2\omega_A^2(T)\right)\\
&\leq \frac{1}{2}\left\|(T^{\sharp_A}T)^2 + (TT^{\sharp_A})^2\right\|_A
\end{aligned}$$

holds.

Proof. Let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. By putting $u = x$ and then replacing x and y by Tx and $T^{\sharp_A}x$, respectively, in Lemma 4 we see that

$$\begin{aligned}
&12\left|\langle Tx,x\rangle_A\right|^4\\
&\leq \|Tx\|_A^2\|T^{\sharp_A}x\|_A^2 + \left|\langle Tx,T^{\sharp_A}x\rangle_A\right|^2 + 2\|Tx\|_A\|T^{\sharp_A}x\|_A\left|\langle Tx,T^{\sharp_A}x\rangle_A\right|\\
&+ 4\left|\langle Tx,x\rangle_A\right|^2\left(\|Tx\|_A\|T^{\sharp_A}x\|_A + |\langle Tx,T^{\sharp_A}x\rangle_A|\right)\\
&= \langle T^{\sharp_A}Tx,x\rangle_A\langle TT^{\sharp_A}x,x\rangle_A + 2\sqrt{\langle T^{\sharp_A}Tx,x\rangle_A\langle TT^{\sharp_A}x,x\rangle_A}\left|\langle T^2x,x\rangle_A\right|\\
&+ 4\left|\langle Tx,x\rangle_A\right|^2\left(\sqrt{\langle T^{\sharp_A}Tx,x\rangle_A\langle TT^{\sharp_A}x,x\rangle_A} + |\langle T^2x,x\rangle_A|\right) + \left|\langle T^2x,x\rangle_A\right|^2.
\end{aligned}$$

Further, by applying the arithmetic–geometric mean inequality, we have

$$\begin{aligned}
&12\left|\langle Tx,x\rangle_A\right|^4\\
&\leq \frac{1}{2}\left(\langle T^{\sharp_A}Tx,x\rangle_A^2 + \langle TT^{\sharp_A}x,x\rangle_A^2\right) + \left|\langle T^2x,x\rangle_A\right|\left(\langle T^{\sharp_A}Tx,x\rangle_A + \langle TT^{\sharp_A}x,x\rangle_A\right)\\
&+ 2\left|\langle Tx,x\rangle_A\right|^2\left(\langle T^{\sharp_A}Tx,x\rangle_A^2 + \langle TT^{\sharp_A}x,x\rangle_A^2 + 2|\langle T^2x,x\rangle_A|\right) + \left|\langle T^2x,x\rangle_A\right|^2\\
&\leq \frac{1}{2}\left(\langle [(T^{\sharp_A}T)^2 + (TT^{\sharp_A})^2]x,x\rangle_A\right) + \omega_A(T^2)\left(\langle [T^{\sharp_A}T + TT^{\sharp_A}]x,x\rangle_A\right)\\
&+ 2\omega_A^2(T)\left(\langle (T^{\sharp_A}T + TT^{\sharp_A}),x\rangle_A + 2\omega_A(T^2)\right) + \omega_A^2(T^2),
\end{aligned}$$

where the last inequality follows by applying Lemma 3 since the operators $T^{\sharp_A}T$ and $TT^{\sharp_A}$ are A-positive. In addition, by using the inequality (C-S), we see that

$$\begin{aligned}12\left|\langle Tx,x\rangle_A\right|^4 &\leq \frac{1}{2}\left\|(T^{\sharp_A}T)^2+(TT^{\sharp_A})^2\right\|_A+\omega_A(T^2)\left\|T^{\sharp_A}T+TT^{\sharp_A}\right\|_A\\&\quad+2\omega_A^2(T)\Big(\left\|T^{\sharp_A}T+TT^{\sharp_A}\right\|_A+2\omega_A(T^2)\Big)+\omega_A^2(T^2)\\&=\frac{1}{2}\left\|(T^{\sharp_A}T)^2+(TT^{\sharp_A})^2\right\|_A+\omega_A^2(T^2)+4\omega_A^2(T)\omega_A(T^2)\\&\quad+\left\|T^{\sharp_A}T+TT^{\sharp_A}\right\|_A\Big(\omega_A(T^2)+2\omega_A^2(T)\Big).\end{aligned}$$

This gives the following:

$$\begin{aligned}\left|\langle Tx,x\rangle_A\right|^4 &\leq \frac{1}{24}\left\|(T^{\sharp_A}T)^2+(TT^{\sharp_A})^2\right\|_A+\frac{1}{12}\omega_A^2(T^2)+\frac{1}{3}\omega_A^2(T)\omega_A(T^2)\\&\quad+\frac{1}{12}\left\|T^{\sharp_A}T+TT^{\sharp_A}\right\|_A\Big(\omega_A(T^2)+2\omega_A^2(T)\Big).\end{aligned}$$

This proves the first inequality in Theorem 7 by taking the supremum over $x\in\mathcal{H}$ with $\|x\|_A=1$ in the last inequality. On the other hand, by applying (9) together with (8), we see that

$$\begin{aligned}&\frac{1}{24}\left\|(T^{\sharp_A}T)^2+(TT^{\sharp_A})^2\right\|_A+\frac{1}{12}\omega_A^2(T^2)+\frac{1}{3}\omega_A^2(T)\omega_A(T^2)\\&+\frac{1}{12}\left\|T^{\sharp_A}T+TT^{\sharp_A}\right\|_A\Big(\omega_A(T^2)+2\omega_A^2(T)\Big)\\&\leq\frac{1}{24}\left\|(T^{\sharp_A}T)^2+(TT^{\sharp_A})^2\right\|_A+\frac{5}{12}\omega_A^4(T)\\&+\frac{1}{4}\left\|T^{\sharp_A}T+TT^{\sharp_A}\right\|_A\omega_A^2(T)\\&\leq\frac{1}{24}\left\|(T^{\sharp_A}T)^2+(TT^{\sharp_A})^2\right\|_A+\frac{11}{48}\left\|T^{\sharp_A}T+TT^{\sharp_A}\right\|_A^2\\&=\frac{1}{24}\left\|(T^{\sharp_A}T)^2+(TT^{\sharp_A})^2\right\|_A+\frac{11}{48}\left\|\frac{2T^{\sharp_A}T+2TT^{\sharp_A}}{2}\right\|_A^2.\end{aligned}$$

Thus, by applying Lemma 2 for $n=2$, we get

$$\begin{aligned}&\frac{1}{24}\left\|(T^{\sharp_A}T)^2+(TT^{\sharp_A})^2\right\|_A+\frac{1}{12}\omega_A^2(T^2)+\frac{1}{3}\omega_A^2(T)\omega_A(T^2)\\&+\frac{1}{12}\left\|T^{\sharp_A}T+TT^{\sharp_A}\right\|_A\Big(\omega_A(T^2)+2\omega_A^2(T)\Big)\\&\leq\frac{1}{24}\left\|(T^{\sharp_A}T)^2+(TT^{\sharp_A})^2\right\|_A+\frac{11}{24}\left\|(T^{\sharp_A}T)^2+(TT^{\sharp_A})^2\right\|_A\\&=\frac{1}{2}\left\|(T^{\sharp_A}T)^2+(TT^{\sharp_A})^2\right\|_A.\end{aligned}$$

□

Remark 5. *(i)* *Note that the inequalities in Theorem 7 are sharp. Indeed, it suffices to consider any A-normal operator T, i.e.,* $T^{\sharp_A}T=TT^{\sharp_A}$*, then by using the following properties:* $\omega_A(T^2)=\omega_A^2(T)=\|T\|_A^2$ *and*

$$\left\|(T^{\sharp_A}T)^2\right\|_A=\left\|(TT^{\sharp_A})^2\right\|_A=\left\|T^{\sharp_A}T\right\|_A^2=\|T\|_A^4,$$

from [18], it is clear that no superior values exist.

(ii) *Note that Theorem 3 in [5] follows from Theorem 7 by letting* $A=I$*.*

4. On Inequalities about the $\mathbb{A}$–Numerical Radius of 2×2 Block Matrices

We consider $\mathbb{A}$, the 2×2 diagonal operator matrix given as $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. It is obvious that $\mathbb{A} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H})^+$ and $\mathbb{A}$ induces the semi-inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{A}} = \langle \mathbb{A}\mathbf{x}, \mathbf{y} \rangle = \langle x_1, y_1 \rangle_A + \langle x_2, y_2 \rangle_A,$$

for every $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathcal{H} \oplus \mathcal{H}$. In recent literature, some bounds concerning the $\mathbb{A}$-numerical radius of 2×2 block matrices are given (see for example [30] and the reference therein). In the present section, we continue working in this direction and we prove new inequalities involving $\omega_{\mathbb{A}}(\mathbb{T})$, where $\mathbb{T}$ is a 2×2-operator matrix.

To prepare the framework in which we will work, we need the following lemmas, the first of which was proven in [19,31].

Lemma 5 ([19,31])**.** *Let $T, S, X, Y \in \mathbb{B}_A(\mathcal{H})$. Then*

(i) $\begin{pmatrix} T & X \\ Y & S \end{pmatrix}^{\sharp_{\mathbb{A}}} = \begin{pmatrix} T^{\sharp_A} & Y^{\sharp_A} \\ X^{\sharp_A} & S^{\sharp_A} \end{pmatrix}$.

(ii) $\left\| \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\|_{\mathbb{A}} = \left\| \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \right\|_{\mathbb{A}} = \max\left\{ \|X\|_A, \|Y\|_A \right\}$.

(iii) $\omega_{\mathbb{A}}\left[\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right] = \max\left\{ \omega_A(X), \omega_A(Y) \right\}$.

(iv) $\omega_{\mathbb{A}}\left[\begin{pmatrix} X & Y \\ Y & X \end{pmatrix} \right] = \max\left\{ \omega_A(X+Y), \omega_A(X-Y) \right\}$. *In particular, we have*

$$\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & Y \\ Y & 0 \end{pmatrix} \right] = \omega_A(Y). \tag{30}$$

The second lemma is a straightforward application of (29) and is stated as follows.

Lemma 6 ([32])**.** *Let $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{H} \oplus \mathcal{H}$ with $\|\mathbf{z}\|_{\mathbb{A}} = 1$. Then*

$$\left| \langle \mathbf{x}, \mathbf{z} \rangle_{\mathbb{A}} \langle \mathbf{z}, \mathbf{y} \rangle_{\mathbb{A}} \right| \le \frac{1}{2}\left(\|\mathbf{x}\|_{\mathbb{A}} \|\mathbf{y}\|_{\mathbb{A}} + |\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{A}}| \right).$$

Now, we state the following results related to the $\mathbb{A}$-numerical radius of 2×2 block matrices of semi-Hilbert space operators.

Theorem 8. *Let $X, Y \in \mathbb{B}_A(\mathcal{H})$ and $\lambda \in [0,1]$. Then the inequality*

$$\begin{aligned}
&\omega_{\mathbb{A}}^4\left[\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \right] \\
&\le \frac{\lambda}{8} \max\left\{ \left\| (XX^{\sharp_A})^2 + (Y^{\sharp_A}Y)^2 \right\|_A, \left\| (YY^{\sharp_A})^2 + (X^{\sharp_A}X)^2 \right\|_A \right\} \\
&+ \frac{\lambda}{4} \max\left\{ \left\| XX^{\sharp_A} + Y^{\sharp_A}Y \right\|_A, \left\| YY^{\sharp_A} + X^{\sharp_A}X \right\|_A \right\} \max\left\{ \omega_A(XY), \omega_A(YX) \right\} \\
&+ \frac{1-\lambda}{4} \max^2\left\{ \omega_A(X), \omega_A(Y) \right\} \max\left\{ \left\| XX^{\sharp_A} + Y^{\sharp_A}Y \right\|_A, \left\| YY^{\sharp_A} + X^{\sharp_A}X \right\| \right\} \\
&+ \frac{1-\lambda}{2} \max^2\left\{ \omega_A(X), \omega_A(Y) \right\} \max\left\{ \omega_A(XY), \omega_A(YX) \right\} + \frac{\lambda}{4} \max^2\left\{ \omega_A(XY), \omega_A(YX) \right\}
\end{aligned}$$

holds.

Proof. Using the inequality from Lemma 6, we have

$$\begin{aligned}
&\left|\langle \mathbf{x},\mathbf{z}\rangle_{\mathbb{A}}\langle \mathbf{z},\mathbf{y}\rangle_{\mathbb{A}}\right|^2 \\
&\le \left|\langle \mathbf{x},\mathbf{z}\rangle_{\mathbb{A}}\langle \mathbf{z},\mathbf{y}\rangle_{\mathbb{A}}\right|\frac{1}{2}\left(\|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}}+|\langle \mathbf{x},\mathbf{y}\rangle_{\mathbb{A}}|\right) \\
&\le \frac{\lambda}{4}\left(\|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}}+|\langle \mathbf{x},\mathbf{y}\rangle_{\mathbb{A}}|\right)^2+\frac{1-\lambda}{2}\left|\langle \mathbf{x},\mathbf{z}\rangle_{\mathbb{A}}\langle \mathbf{z},\mathbf{y}\rangle_{\mathbb{A}}\right|\left(\|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}}+|\langle \mathbf{x},\mathbf{y}\rangle_{\mathbb{A}}|\right) \\
&= \frac{\lambda}{4}\|\mathbf{x}\|^2{}_{\mathbb{A}}\|\mathbf{y}\|^2{}_{\mathbb{A}}+\frac{\lambda}{2}|\langle \mathbf{x},\mathbf{y}\rangle_{\mathbb{A}}|\|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}} \\
&+\frac{\lambda}{4}\left|\langle \mathbf{x},\mathbf{y}\rangle_{\mathbb{A}}\right|^2+\frac{1-\lambda}{2}\left|\langle \mathbf{x},\mathbf{z}\rangle_{\mathbb{A}}\langle \mathbf{z},\mathbf{y}\rangle_{\mathbb{A}}\right|\left(|\langle \mathbf{x},\mathbf{y}\rangle_{\mathbb{A}}|+\|\mathbf{x}\|_{\mathbb{A}}\|\mathbf{y}\|_{\mathbb{A}}\right),
\end{aligned}$$

where $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathcal{H}\oplus\mathcal{H}$ with $\left\|\mathbf{z}\right\|_{\mathbb{A}}=1$ and $\lambda\in[0,1]$.

Let us consider $\mathbb{M}=\begin{pmatrix}0 & X\\ Y & 0\end{pmatrix}$. By using Lemma 6, it follows that $\mathbb{M}^{\sharp_{\mathbb{A}}}=\begin{pmatrix}0 & Y^{\sharp_A}\\ X^{\sharp_A} & 0\end{pmatrix}$, $\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}}=\begin{pmatrix}XX^{\sharp_A} & 0\\ 0 & YY^{\sharp_A}\end{pmatrix}$, $\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M}=\begin{pmatrix}Y^{\sharp_A}Y & 0\\ 0 & X^{\sharp_A}X\end{pmatrix}$ and $\mathbb{M}^2=\begin{pmatrix}XY & 0\\ 0 & YX\end{pmatrix}$.

If we replace $\mathbf{z}$ by $\mathbf{x}$ with $\left\|\mathbf{x}\right\|_{\mathbb{A}}=1$, $\mathbf{x}$ by $\mathbb{M}\mathbf{x}$ and $\mathbf{y}$ by $\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x}$, then the above inequality becomes

$$\begin{aligned}
&|\langle \mathbb{M}\mathbf{x},\mathbf{x}\rangle_{\mathbb{A}}|^4=|\langle \mathbb{M}\mathbf{x},\mathbf{x}\rangle_{\mathbb{A}}\langle \mathbf{x},\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x}\rangle_{\mathbb{A}}|^2 \\
&\le \frac{\lambda}{4}\|\mathbb{M}\mathbf{x}\|_{\mathbb{A}}^2\|\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x}\|_{\mathbb{A}}^2+\frac{\lambda}{2}\|\mathbb{M}\mathbf{x}\|_{\mathbb{A}}\|\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x}\|_{\mathbb{A}}|\langle \mathbb{M}\mathbf{x},\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x}\rangle_{\mathbb{A}}| \\
&+\frac{\lambda}{4}|\langle \mathbb{M}\mathbf{x},\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x}\rangle_{\mathbb{A}}|^2+\frac{1-\lambda}{2}|\langle \mathbb{M}\mathbf{x},\mathbf{x}\rangle_{\mathbb{A}}\langle \mathbf{x},\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x}\rangle_{\mathbb{A}}|\Big(\|\mathbb{M}\mathbf{x}\|_{\mathbb{A}}\|\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x}\|_{\mathbb{A}}+|\langle \mathbb{M}\mathbf{x},\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x}\rangle_{\mathbb{A}}|\Big).
\end{aligned}$$

This implies that

$$\begin{aligned}
|\langle \mathbb{M}\mathbf{x},\mathbf{x}\rangle_{\mathbb{A}}|^4 \quad &\le \tfrac{\lambda}{8}\left\|\left(\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M}\right)^2+\left(\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}}\right)^2\right\|_{\mathbb{A}}+\tfrac{\lambda}{4}\omega_{\mathbb{A}}^2(\mathbb{M}^2)+\tfrac{1-\lambda}{2}\omega_{\mathbb{A}}(\mathbb{M}^2)\omega^2(\mathbb{M}) \\
&+\left(\tfrac{\lambda}{4}\omega_{\mathbb{A}}(\mathbb{M}^2)+\tfrac{1-\lambda}{4}\omega_{\mathbb{A}}^2(\mathbb{M})\right)\left\|\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M}+\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}}\right\|_{\mathbb{A}}. \qquad (31)
\end{aligned}$$

We take into account the following sequence of the inequalities, taking into account that $\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M}$ and $\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}}$ are $\mathbb{A}$–positive:

$$\begin{aligned}
\left\|\mathbb{M}\mathbf{x}\right\|_{\mathbb{A}}^2\left\|\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x}\right\|_{\mathbb{A}}^2 &= \langle \mathbb{M}\mathbf{x},\mathbb{M}\mathbf{x}\rangle_{\mathbb{A}}\langle \mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x},\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x}\rangle_{\mathbb{A}} \\
&= \langle \mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M}\mathbf{x},\mathbf{x}\rangle_{\mathbb{A}}\langle \mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x},\mathbf{x}\rangle_{\mathbb{A}} \\
&\le \frac{1}{4}\left(\langle \mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M}\mathbf{x},\mathbf{x}\rangle_{\mathbb{A}}+\langle \mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x},\mathbf{x}\rangle_{\mathbb{A}}\right)^2 \\
&\le \frac{1}{2}\left(\langle \mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M}\mathbf{x},\mathbf{x}\rangle_{\mathbb{A}}^2+\langle \mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbf{x},\mathbf{x}\rangle_{\mathbb{A}}^2\right) \\
&\le \frac{1}{2}\langle\left((\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M})^2+(\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}})^2\right)\mathbf{x},\mathbf{x}\rangle_{\mathbb{A}} \\
&\le \frac{1}{2}\left\|(\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M})^2+(\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}})^2\right\|_{\mathbb{A}}.
\end{aligned}$$

Consequently, taking the supremum over $\left\|\mathbf{x}\right\|_{\mathbb{A}}=1$ in inequality (31), we obtain the inequality of the statement. □

Remark 6. *For $\lambda=0$ in inequality (31), we deduce*

$$\omega_{\mathbb{A}}^2(\mathbb{M})\le\frac{1}{4}\left\|\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M}+\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}}\right\|_{\mathbb{A}}+\frac{1}{2}\omega_{\mathbb{A}}(\mathbb{M}^2).$$

So, by taking Lemma 6 into account, we get the inequality recently established by Xu et al. in [32]:

$$\omega_{\mathbb{A}}^2\left[\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}\right] \leq \frac{1}{4}\max\{\|XX^{\sharp_A}+Y^{\sharp_A}Y\|_A, \|YY^{\sharp_A}+X^{\sharp_A}X\|_A\} + \frac{1}{2}\max\{\omega_A(XY), \omega_A(YX)\},$$

for all $X, Y \in \mathbb{B}_A(\mathcal{H})$.

Corollary 3. *Let $X, Y \in \mathbb{B}_A(\mathcal{H})$ and $\lambda \in [0,1]$. Then inequalities*

$$\begin{aligned}
&\omega_{\mathbb{A}}^4\left[\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}\right] \\
&\leq \frac{\lambda}{8}\max\left\{\|(XX^{\sharp_A})^2+(Y^{\sharp_A}Y)^2\|_A, \|(YY^{\sharp_A})^2+(X^{\sharp_A}X)^2\|_A\right\} \\
&+\frac{\lambda}{4}\max\left\{\|XX^{\sharp_A}+Y^{\sharp_A}Y\|_A, \|YY^{\sharp_A}+X^{\sharp_A}X\|_A\right\}\max\{\omega_A(XY), \omega_A(YX)\} \\
&+\frac{\lambda}{4}\max{}^2\{\omega_A(XY), \omega_A(YX)\}+\frac{1-\lambda}{2}\max{}^2\{\omega_A(X), \omega_A(Y)\}\max\left\{\omega_A(XY), \omega_A(YX)\right\} \\
&+\frac{1-\lambda}{4}\max{}^2\{\omega_A(X), \omega_A(Y)\}\max\left\{\|XX^{\sharp_A}+Y^{\sharp_A}Y\|_A, \|YY^{\sharp_A}+X^{\sharp_A}X\|\right\} \\
&\leq \frac{1}{2}\max\left\{\|(XX^{\sharp_A})^2+(Y^{\sharp_A}Y)^2\|_A, \|(YY^{\sharp_A})^2+(X^{\sharp_A}X)^2\|_A\right\}
\end{aligned}$$

hold.

Proof. From inequality (31), we have

$$\begin{aligned}
&\omega_{\mathbb{A}}^4(\mathbb{M}) \\
&\leq \frac{\lambda}{8}\|(\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M})^2+(\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}})^2\|_{\mathbb{A}}+\left[\frac{\lambda}{4}\omega_{\mathbb{A}}(\mathbb{M}^2)+\frac{1-\lambda}{4}\omega_{\mathbb{A}}^2(\mathbb{M})\right]\|\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M}+\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}}\|_{\mathbb{A}} \\
&+\frac{\lambda}{4}\omega_{\mathbb{A}}^2(\mathbb{M}^2)+\frac{1-\lambda}{2}\omega_{\mathbb{A}}(\mathbb{M}^2)\omega_{\mathbb{A}}^2(\mathbb{M}) \\
&\leq \frac{\lambda}{8}\|(\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M})^2+(\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}})^2\|_{\mathbb{A}}+\left[\frac{\lambda}{4}\omega_{\mathbb{A}}^2(\mathbb{M})+\frac{1-\lambda}{4}\omega_{\mathbb{A}}^2(\mathbb{M})\right]\|\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M}+\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}}\|_{\mathbb{A}} \\
&+\frac{\lambda}{4}\omega_{\mathbb{A}}^4(\mathbb{M})+\frac{1-\lambda}{2}\omega_{\mathbb{A}}^4(\mathbb{M}) \\
&\leq \frac{\lambda}{8}\|(\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M})^2+(\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}})^2\|_{\mathbb{A}}+\frac{1}{4}\omega_{\mathbb{A}}^2(\mathbb{M})\|\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M}+\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}}\|_{\mathbb{A}}+\frac{2-\lambda}{4}\omega_{\mathbb{A}}^4(\mathbb{M}) \\
&\leq \frac{1}{2}\|(\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M})^2+(\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}})^2\|_{\mathbb{A}}.
\end{aligned}$$

In the above sequence of inequalities, we used the fact that $\omega_{\mathbb{A}}(\mathbb{M}^2) \leq \omega_{\mathbb{A}}^2(\mathbb{M})$ and the inequality:

$$\omega_{\mathbb{A}}^2(\mathbb{M}) \leq \frac{1}{2}\|\mathbb{M}^{\sharp_{\mathbb{A}}}\mathbb{M}+\mathbb{M}\mathbb{M}^{\sharp_{\mathbb{A}}}\|_{\mathbb{A}}.$$

Therefore, by applying Lemma 6, it is clear that the inequalities of the statement are true. □

Remark 7. *By taking $X = Y$ in Corollary 3 and then using (30), we obtain:*

$$\begin{aligned}\omega_A^4(X) &\le \frac{\lambda}{8}\|(X^{\sharp_A}X)^2 + (XX^{\sharp_A})^2\|_A + \frac{\lambda}{4}\|X^{\sharp_A}X + XX^{\sharp_A}\|_A\omega_A(X^2)\\ &+ \frac{\lambda}{4}\omega_A^2(X^2) + \frac{1-\lambda}{4}\omega_A^2(X)\|X^{\sharp_A}X + XX^{\sharp_A}\|_A + \frac{1-\lambda}{2}\omega_A^2(X)\omega_A(X^2)\\ &\le \frac{1}{2}\|(X^{\sharp_A}X)^2 + (XX^{\sharp_A})^2\|_A^2.\end{aligned}$$

If we take $A = I$ in the last inequalities, then since $X^{\sharp_I} = X^*$, the inequality in Theorem 2 is true.

Our next theorem provides an extension of a recent result by Bani-Domi et al. in [33].

Theorem 9. *Let $X, Y, T, S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\begin{aligned}\omega_{\mathbb{A}}^2\left[\begin{pmatrix} T & X \\ Y & S\end{pmatrix}\right] &\le \frac{1}{2}\max\left\{\|X^{\sharp_A}X + YY^{\sharp_A}\|_A, \|XX^{\sharp_A} + Y^{\sharp_A}Y\|_A\right\}\\ &+ 2\max\left\{\omega_A^2(T), \omega_A^2(S)\right\} + \max\left\{\omega_A(XY), \omega_A(YX)\right\}.\end{aligned}$$

Proof. Consider the matrices $\mathbb{M} = \begin{pmatrix} T & 0 \\ 0 & S\end{pmatrix}$ and $\mathbb{N} = \begin{pmatrix} 0 & X \\ Y & 0\end{pmatrix}$. Let $\mathbf{x} \in \mathcal{H} \oplus \mathcal{H}$ be such that $\|\mathbf{x}\|_{\mathbb{A}} = 1$. By using the convexity of the function $t \mapsto t^2$, we deduce that

$$\begin{aligned}\left|\left\langle\begin{pmatrix} T & X \\ Y & S\end{pmatrix}\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}}\right|^2 &= |\langle\mathbb{M}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}} + \langle\mathbb{N}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}|^2\\ &\le 2\left(|\langle\mathbb{M}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}|^2 + |\langle\mathbb{N}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}|^2\right)\\ &= 2\left(|\langle\mathbb{M}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}|^2 + \left|\langle\mathbb{N}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\langle\mathbf{x}, \mathbb{N}^{\sharp_{\mathbb{A}}}\mathbf{x}\rangle_{\mathbb{A}}\right|\right).\end{aligned}$$

Further, by applying Lemma 6 we get

$$\begin{aligned}&\left|\left\langle\begin{pmatrix} T & X \\ Y & S\end{pmatrix}\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}}\right|^2\\ &\le 2|\langle\mathbb{M}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}|^2 + \left|\langle\mathbb{N}\mathbf{x}, \mathbb{N}^{\sharp_{\mathbb{A}}}\mathbf{x}\rangle_{\mathbb{A}}\right| + \|\mathbb{N}\mathbf{x}\|_{\mathbb{A}}\|\mathbb{N}^{\sharp_{\mathbb{A}}}\mathbf{x}\|_{\mathbb{A}}\\ &= 2|\langle\mathbb{M}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}|^2 + \left|\langle\mathbb{N}^2\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right| + \sqrt{\langle\mathbb{N}^{\sharp_{\mathbb{A}}}\mathbb{N}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}}\sqrt{\langle\mathbb{N}\mathbb{N}^{\sharp_{\mathbb{A}}}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}}\\ &\le 2|\langle\mathbb{M}\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}|^2 + \left|\langle\mathbb{N}^2\mathbf{x}, \mathbf{x}\rangle_{\mathbb{A}}\right| + \frac{1}{2}\left\langle\left(\mathbb{N}^{\sharp_{\mathbb{A}}}\mathbb{N} + \mathbb{N}\mathbb{N}^{\sharp_{\mathbb{A}}}\right)\mathbf{x}, \mathbf{x}\right\rangle_{\mathbb{A}}\\ &\le 2\omega_{\mathbb{A}}^2(\mathbb{M}) + \omega_{\mathbb{A}}(\mathbb{N}^2) + \frac{1}{2}\left\|\mathbb{N}^{\sharp_{\mathbb{A}}}\mathbb{N} + \mathbb{N}\mathbb{N}^{\sharp_{\mathbb{A}}}\right\|_{\mathbb{A}},\end{aligned}$$

where the last inequality is deduced from (6) since $\mathbb{N}^{\sharp_{\mathbb{A}}}\mathbb{N} + \mathbb{N}\mathbb{N}^{\sharp_{\mathbb{A}}}$ is an $\mathbb{A}$–selfadjoint operator. We take the supremum over all $x \in \mathcal{H} \oplus \mathcal{H}$ with $\|\mathbf{x}\|_{\mathbb{A}} = 1$ in the above inequality, implies that

$$\omega_{\mathbb{A}}^2\left[\begin{pmatrix} T & X \\ Y & S\end{pmatrix}\right] \le 2\omega_{\mathbb{A}}^2(\mathbb{M}) + \omega_{\mathbb{A}}(\mathbb{N}^2) + \frac{1}{2}\left\|\mathbb{N}^{\sharp_{\mathbb{A}}}\mathbb{N} + \mathbb{N}\mathbb{N}^{\sharp_{\mathbb{A}}}\right\|_{\mathbb{A}}. \tag{32}$$

On the other hand, it can be seen that

$$\mathbb{N}^2 = \begin{pmatrix} XY & 0 \\ 0 & YX\end{pmatrix} \text{ and } \mathbb{N}^{\sharp_{\mathbb{A}}}\mathbb{N} + \mathbb{N}\mathbb{N}^{\sharp_{\mathbb{A}}} = \begin{pmatrix} XX^{\sharp_A} + Y^{\sharp_A}Y & 0 \\ 0 & X^{\sharp_A}X + YY^{\sharp_A}\end{pmatrix}.$$

Therefore, the desired result is obtained by taking (32) into consideration and then applying Lemma 6. $\square$

We remark that the following corollary considerably improves the second inequality in (8) and was already proven by the second author in [25]. This corollary was also approached by Bhunia et al. in [34], when the operator A is assumed to be injective.

Corollary 4. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then, the inequality*

$$\omega_A(T) \leq \frac{1}{2}\sqrt{\left\|TT^{\sharp_A} + T^{\sharp_A}T\right\|_A + 2\omega_A(T^2)} \tag{33}$$

holds.

Proof. By letting $T = S = X = Y$ in Theorem 9 and then using Lemma 5 (iv), we obtain the desired result. $\square$

As an application of (33), we derive the following result which extends a recent theorem stated by Kittaneh et al. in [35].

Theorem 10. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then, the following inequality*

$$\begin{aligned}
&\|T + S^{\sharp_A}\|_A \\
&\leq \sqrt{\max\left\{\left\|T^{\sharp_A}T + SS^{\sharp_A}\right\|_A, \left\|T^{\sharp_A}T + SS^{\sharp_A}\right\|_A\right\} + 2\max\left\{\omega_A(TS), \omega_A(ST)\right\}} \\
&\leq \|T\|_A + \|S\|_A
\end{aligned}$$

holds.

To prove Theorem 10, we need the following Lemma.

Lemma 7. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix}\right] = \frac{1}{2}\sup_{\theta \in \mathbb{R}}\left\|e^{i\theta}T + e^{-i\theta}S^{\sharp_A}\right\|_A.$$

We are now able to prove Theorem 10.

Proof of Theorem 10. Let $\mathbb{T} = \begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix}$ and $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Clearly, $\mathbb{T}^2 = \begin{pmatrix} TS & 0 \\ 0 & ST \end{pmatrix}$. Further, by using Lemma 5 (i), we see that

$$\mathbb{T}\mathbb{T}^{\sharp_{\mathbb{A}}} + \mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T} = \begin{pmatrix} TT^{\sharp_A} + S^{\sharp_A}S & 0 \\ 0 & SS^{\sharp_A} + T^{\sharp_A}T \end{pmatrix}.$$

Hence, an application of (33) together with Lemma 7 gives

$$\begin{aligned}
&\|T + S^{\sharp_A}\|_A \\
&\leq 2\omega_{\mathbb{A}}\left[\begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix}\right] \\
&\leq \sqrt{\left\|\mathbb{T}\mathbb{T}^{\sharp_{\mathbb{A}}} + \mathbb{T}^{\sharp_{\mathbb{A}}}\mathbb{T}\right\|_{\mathbb{A}} + 2\omega_{\mathbb{A}}(\mathbb{T}^2)} \\
&= \sqrt{\max\left\{\left\|T^{\sharp_A}T + SS^{\sharp_A}\right\|_A, \left\|T^{\sharp_A}T + SS^{\sharp_A}\right\|_A\right\} + 2\max\left\{\omega_A(TS), \omega_A(ST)\right\}},
\end{aligned}$$

where the last equality follows by applying Lemma 5 (ii) and (iii). Furthermore, we can see that

$$\begin{aligned}&\sqrt{\max\left\{\left\|T^{\sharp_A}T+SS^{\sharp_A}\right\|_A,\left\|T^{\sharp_A}T+SS^{\sharp_A}\right\|_A\right\}+2\max\left\{\omega_A(TS),\omega_A(ST)\right\}}\\&\le\sqrt{\|T\|_A^2+\|S\|_A^2+2\|T\|_A\|S\|_A}=\|T\|_A+\|S\|_A.\end{aligned}$$

This completes the proof. □

Corollary 5. *If $T, S \in \mathbb{B}(\mathcal{H})$ are A–selfadjoint operators, then we have*

$$\|T+S\|_A \le \sqrt{\|T^2+S^2\|_A+2\omega_A(TS)} \le \|T\|_A+\|S\|_A. \tag{34}$$

Proof. Notice that since T and S are A–selfadjoint operators, then so are $T^{\sharp_A}$ and $S^{\sharp_A}$. Thus, by (5) we get

$$(T^{\sharp_A})^{\sharp_A}=T^{\sharp_A} \quad \text{and} \quad (S^{\sharp_A})^{\sharp_A}=S^{\sharp_A}.$$

Therefore, by replacing T and S by $T^{\sharp_A}$ and $S^{\sharp_A}$ in Theorem 10, respectively, and then using the fact that $\|X^{\sharp_A}\|_A=\|X\|_A$ for all $X \in \mathbb{B}_A(\mathcal{H})$, we obtain (34) as required. □

5. Conclusions

The main objective of the present paper is to present new upper bounds of $\omega(T)$, which denotes the numerical radius of a bounded operator T on a Hilbert space $(\mathcal{H},\langle\cdot,\cdot\rangle)$. The study's motivation is given by the multitude of recent papers that refer to the numerical radius, see [11,17,24,26,31,34]. The large number of papers published in this area demonstrates the relevance of this field of research. The main objective is focused on the study of some new improvements of the upper bounds of $\omega(T)$, $\|T\|$ and $\omega(S^*T)$, of the type given in (2)–(4). We show the Aczél inequality in terms of the operator $|T|$.

Next, we give certain inequalities about the A-numerical radius $\omega_A(T)$ and the A-operator seminorm $\|T\|_A$ of an operator T from the semi-Hilbert space $(\mathcal{H},\langle\cdot,\cdot\rangle_A)$, where $\langle x,y\rangle_A := \langle Ax,y\rangle$ for all $x,y \in \mathcal{H}$.

Furthermore, we present several results related to the $\mathbb{A}$-numerical radius of 2×2 block matrices of semi-Hilbert space operators, by using symmetric 2×2 block matrices. The symmetric 2×2 block matrices are very important in our study because they are easy to use.

As a future approach, we will study better estimates of the $\mathbb{A}$-numerical radius for the symmetric 2×2 operator matrix and we will investigate new inequalities involving a d-tuple of operators $\mathbf{T}=(T_1,\ldots,T_d)\in\mathbb{B}_A(\mathcal{H})^d$.

Author Contributions: The work presented here was carried out in collaboration between all authors. All authors contributed equally and significantly in writing this article. All authors have contributed to the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: The first author extends her appreciation to the Distinguished Scientist Fellowship Program at King Saud University, Riyadh, Saudi Arabia, for funding this work through Researchers Supporting Project number (RSP2023R187).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to express their most sincere thanks to the anonymous reviewers for their constructive comments regarding the improvement of the original draft.

Conflicts of Interest: The authors declare no conflict of interest.

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