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On General Class of Z-Contractions with Applications to Spring Mass Problem

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Abstract: One of the latest techniques in metric fixed point theory is the interpolation approach. This notion has so far been examined using standard functional equations. A hybrid form of this concept is yet to be uncovered by observing the available literature. With this background information, and based on the symmetry and rectangular properties of generalized metric spaces, this paper introduces a novel and unified hybrid concept under the name interpolative Y-Hardy–Rogers–Suzuki-type Z-contraction and establishes sufficient conditions for the existence of fixed points for such contractions. As an application, one of the obtained results was employed to examine new criteria for the existence of a solution to a boundary valued problem arising in the oscillation of a spring. The ideas proposed herein advance some recently announced important results in the corresponding literature. A comparative example was constructed to justify the abstractions and pre-eminence of our obtained results.

Keywords: fixed point; admissible; interpolative; Y-metric; Z-contraction

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1. Introduction and Preliminaries

Banach [1] initiated one of the commonly utilized metric invariant point ideas, known as the Banach contraction principle. Meanwhile, the Banach contraction principle has been improved in several directions. Not long ago, Azmi [2] presented new contractive mappings and utilized the concept of triple-controlled metric-type space, which preserves the symmetry property to establish a new invariant point result. Bota and Micula [3] employed the recent Subrahmanyan contraction in the framework of a generalized metric to discuss the Ulam–Hyers stability property of an invariant point inclusion. For other refinements of the invariant point result due to Banach, we refer to [4–10] and the citations therein. Along these lines, one of the improvements of the contraction mapping principle was put forward by Khojasteh et al. [11] via a family of auxiliary functions under the name simulation functions. Shortly after, Argoubi et al. [12] observed that one of the axioms of simulation functions is redundant and hence came up with a variant of simulation functions. Mani et al. [13] modified the symmetry of orthogonal MS and applied the idea of simulation functions to set up new invariant point theorems in orthogonal rectangular metric space (MS). Throughout, $\mathbb{R}_+ = F_+$ so that $\mathbb{R} = F$. We record the definition of simulation functions as modified in [12] as follows:

Definition 1 ([12]). *A simulation function is a mapping $\eta : F_+ \times F_+ \longrightarrow F$ fulfilling the following criteria:*

(η_1) $\eta(\zeta, \hbar) < \hbar - \zeta$ *for all* $\zeta, \hbar \in F_+$;
(η_2) *if* $\{\zeta_n\}_{n\in\mathbb{N}}$ *and* $\{\hbar_n\}_{n\in\mathbb{N}}$ *are sequences in* $(0, \infty)$ *such that*

$$\lim_{n\longrightarrow\infty} \zeta_n = \lim_{n\longrightarrow\infty} \hbar_n, \text{ then}$$

$$\limsup_{n\longrightarrow\infty} \eta(\zeta_n, \hbar_n) < 0.$$

We depict the family of mappings obeying $(\eta_1) - (\eta_2)$ by Z_η.

Definition 2 ([11,12])**.** *Let $(\bigwedge, \rho)$ be an MS. A mapping $\Theta : \bigwedge \longrightarrow \bigwedge$ is called a Z-contraction with respect to $\eta \in Z_\eta$, if, for all $\varrho, \varsigma \in \bigwedge$,*

$$\eta(\rho(\Theta\varrho, \Theta\varsigma), \rho(\varrho, \varsigma)) \geq 0.$$

It is clear that, under the mapping $\eta(\xi, \hbar) = \lambda\hbar - \xi$ for all $\xi, \hbar \in \digamma_+$ and $\lambda \in (0, 1)$, every Banach contraction is a Z-contraction. For some examples of Z-contractions and related invariant point results, we refer to [11,14–16].

In 2014, Popescu [17] proposed the idea of τ-orbital admissible mapping as an extension of the concept of τ-admissible mapping due to Samet et al. [18]. In [17], it was demonstrated that every τ-admissible mapping is a τ-orbital admissible mapping, but the converse is not always valid.

Definition 3 ([17])**.** *Let $\Theta : \bigwedge \longrightarrow \bigwedge$ be a mapping and $\tau : \bigwedge \times \bigwedge \longrightarrow \digamma_+$ be a mapping. Then, Θ is called a τ-orbital admissible mapping, if, for all $\varrho \in \bigwedge$,*

$$\tau(\varrho, \Theta\varrho) \geq 1 \text{ implies } \tau(\Theta\varrho, \Theta^2\varrho) \geq 1.$$

It is pertinent to note that one of the applications of τ-admissible mappings is that it is τ-regular in the bodywork of MS. This property was employed to modify continuity criteria on point-to-point mappings coupled with some suitable conditions; for details, see [18].

Definition 4 ([18])**.** *An MS $(\bigwedge, \rho)$ is said to be τ-regular if every sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ in $\bigwedge$ has limit u in $\bigwedge$ and obeys $\tau(\varrho_n, \varrho_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, $\tau(\varrho_n, u) \geq 1$.*

In 2008, Suzuki [19] established an improvement in Edelstein's invariant point result in a compact MS. Meanwhile, the following mapping is known in the literature as a Suzuki-type contraction.

Definition 5 ([19])**.** *Let $(\bigwedge, \rho)$ be an MS. A mapping $\Theta : \bigwedge \longrightarrow \bigwedge$ is said to be a Suzuki-type contractio, if, for all $\varrho, \varsigma \in \bigwedge$ with $\varrho \neq \varsigma$,*

$$\frac{1}{2}\rho(\varrho, \Theta\varrho) \leq \rho(\varrho, \varsigma) \text{ implies } \rho(\Theta\varrho, \Theta\varsigma) \leq \rho(\varrho, \varsigma).$$

In 2018, Karapinar [20] complemented the classical invariant point result due to Kannan using interpolation theory in the following manner.

Definition 6 ([20])**.** *Let $(\bigwedge, \rho)$ be an MS. A mapping $\Theta : \bigwedge \longrightarrow \bigwedge$ is called an interpolative Kannan contraction if, for all $\varrho, \varsigma \in \bigwedge \setminus \mathcal{F}_{ix}(\Theta)$, we can find $\lambda, \zeta \in (0, 1)$ such that*

$$\rho(\Theta\varrho, \Theta\varsigma) \leq \lambda[\rho(\varrho, \Theta\varrho)]^\zeta [\rho(\varsigma, \Theta\varsigma)]^{1-\zeta},$$

where $\mathcal{F}_{ix}(\Theta)$ is the set of all invariant points of Θ.

Following [20], more than a handful of invariant point ideas utilizing the interpolation approach have been advanced in the literature (see, e.g., [21–23]). Along these lines, by coupling the interpolation technique with Hardy-Rogers-type mapping, several authors have come up with new forms of useful contractions (for instance, see [24–26]). Recently, Maha [27] combined a Hardy–Rogers contraction of Suzuki-type with the notion of Z-contraction and launched the following novel concept:

Definition 7. *Let $(\bigwedge, \rho)$ be an MS and $\Theta : \bigwedge \longrightarrow \bigwedge$ be a mapping. Then, Θ is called an interpolative Hardy–Rogers–Suzuki-type Z-contraction with respect to $\eta \in Z_\eta$ if we can find $\theta, \sigma, \zeta \in (0,1)$ with $\theta + \sigma + \zeta < 1$, and a mapping $\tau : \bigwedge \times \bigwedge \longrightarrow F_+$ such that, for all $\varrho, \varsigma \in \bigwedge \setminus \mathcal{F}_{ix}(\Theta)$,*

$$\frac{1}{2}\rho(\varrho, \Theta\varrho) \leq \rho(\varrho, \varsigma)$$

implies

$$\eta(\tau(\varrho, \varsigma)\rho(\Theta\varrho, \Theta\varsigma), C(\varrho, \varsigma)) \geq 0,$$

where

$$C(\varrho, \varsigma) = [\rho(\varrho, \varsigma)]^\theta [\rho(\varrho, \Theta\varrho)]^\sigma [\rho(\varsigma, \Theta\varsigma)]^\zeta \left[\frac{1}{2}(\rho(\varrho, \Theta\varsigma) + \rho(\varsigma, \Theta\varrho))\right]^{1-\theta-\sigma-\zeta}.$$

On the other hand, due to enormous applications of MS, several versions have emerged in the literature. In particular, Mustafa and Sims [28] introduced the notion of generalized MS in the following sense:

Definition 8 ([28])**.** *Let $\bigwedge$ be a nonempty set and $\Upsilon : \bigwedge \times \bigwedge \times \bigwedge \longrightarrow F_+$ be a mapping obeying the following criteria:*

(Y1) $\Upsilon(\varrho, \varsigma, z) = 0$ if and only if $\varrho = \varsigma = z$;
(Y2) $0 < \Upsilon(\varrho, \varrho, \varsigma)$, for all $\varrho, \varsigma \in \bigwedge$, $\varrho \neq \varsigma$;
(Y3) $\Upsilon(\varrho, \varrho, \varsigma) \leq \Upsilon(\varrho, \varsigma, z)$, for all $\varrho, \varsigma, z \in \bigwedge$, $z \neq \varsigma$;
(Y4) $\Upsilon(\varrho, \varsigma, z) = \Upsilon(\varrho, z, \varsigma) = \Upsilon(\varsigma, z, \varrho) = \cdots$, for all $\varrho, \varsigma, z \in \bigwedge$; (symmetry in all three variables)
(Y5) $\Upsilon(\varrho, \varsigma, z) \leq \Upsilon(\varrho, a, a) + \Upsilon(a, \varsigma, z)$, for all $\varrho, \varsigma, z, a \in \bigwedge$; (rectangle inequality).

Then, Υ is called a generalized metric or an Υ-metric on $\bigwedge$, and $(\bigwedge, \Upsilon)$ is said to be an Υ-MS.

For some examples of Y-MS and related useful results, see [29–34]. We record a few more specific preliminaries of Y-MS as shown in [28,35,36] as follows.

Definition 9 ([28])**.** *Let $(\bigwedge, \Upsilon)$ be an Υ-MS and $\{\varrho_n\}_{n\in\mathbb{N}}$ be a sequence in $\bigwedge$. Then, $\{\varrho_n\}_{n\in\mathbb{N}}$ is said to be Υ-convergent to u if $\lim\limits_{n,m\longrightarrow\infty} \Upsilon(u, \varrho_n, \varrho_m) = 0$.*

Lemma 1 ([35], Propostion 1.4)**.** *Let $(\bigwedge, \Upsilon)$ be an Υ-MS and $\{\varrho_n\}_{n\in\mathbb{N}}$ be a sequence in $\bigwedge$. Then, the following statements are equivalent:*

(i) $\{\varrho_n\}_{n\in\mathbb{N}}$ is Υ-convergent to u.
(ii) $\Upsilon(\varrho_n, \varrho_n, u) \longrightarrow 0$ as $n \longrightarrow \infty$.
(iii) $\Upsilon(\varrho_n, u, u) \longrightarrow 0$ as $n \longrightarrow \infty$.

Definition 10 ([28])**.** *Let $(\bigwedge, \Upsilon)$ be an Υ-MS. A sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ in $\bigwedge$ is called Υ-Cauchy if, for any given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\Upsilon(\varrho_n, \varrho_m, \varrho_l) < \epsilon$, for all $n, m, l \geq n_0$; that is, $\Upsilon(\varrho_n, \varrho_m, \varrho_l) \longrightarrow 0$ as $n, m, l \longrightarrow \infty$. An Υ-MS is said to be Υ-complete if every Υ-Cauchy sequence in $\bigwedge$ is Υ-convergent in $\bigwedge$.*

Following the existing results and as far as our investigation reaches, we notice that a hybrid of interpolation theory and Hardy–Rogers and Suzuki, as well as Z-contractions in the framework of Y-MS has never been investigated, leaving some useful gaps in the literature. Hence, this paper introduces a new concept under the name interpolative Y-Hardy–Rogers–Suzuki-type Z-contraction based on the characterizations of generalized MS. Sufficient conditions for the existence of invariant points for such contractions were examined. A comparative example is provided to support the hypotheses of our proposed results and to show that the ideas developed herein improve and advance a few recently announced significant invariant points results.

2. Results

This section begins by introducing some auxiliary concepts as follows.

Definition 11. *Let $\Theta : \bigwedge \longrightarrow \bigwedge$ be a self-mapping on a nonempty set $\bigwedge$ and $\tau : \bigwedge \times \bigwedge \times \bigwedge \longrightarrow F_+$ be a mapping. Then, Θ is said to be Υ-τ-orbital admissible, if*

$$\tau(\varrho, \Theta\varrho, \Theta\varrho) \geq 1 \text{ implies } \tau(\Theta\varrho, \Theta^2\varrho, \Theta^2\varrho) \geq 1.$$

Definition 12. *An Υ-MS $(\bigwedge, \Upsilon)$ is said to be Υ-τ-regular if, for every sequence $\{\varrho_n\}_{n\in\mathrm{N}}$ in $\bigwedge$ that converges to $u \in \bigwedge$ and satisfies $\tau(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, we have $\tau(\varrho_n, u, u) \geq 1$ for all $n \in \mathbb{N}$.*

Definition 13. *Let $(\bigwedge, \Upsilon)$ be an Υ-MS. We say that the mapping $\Theta : \bigwedge \longrightarrow \bigwedge$ is an interpolative Υ-Hardy–Rogers–Suzuki-type Z-contraction with respect to some $\eta \in Z_\eta$ if we can find $\theta, \sigma, \zeta \in (0,1)$ with $\theta + \sigma + \zeta < 1$ and a mapping $\tau : \bigwedge \times \bigwedge \times \bigwedge \longrightarrow F_+$ such that*

$$\frac{1}{2}\Upsilon(\varrho, \Theta\varrho, \Theta\varrho) \leq \Upsilon(\varrho, \varsigma, z) \tag{1}$$

implies

$$\eta\big(\tau(\varrho, \varsigma, z)\Upsilon(\Theta\varrho, \Theta\varsigma, \Theta z), C_\eta(\varrho, \varsigma, z)\big) \geq 0,$$

where

$$\begin{aligned} &C_\eta(\varrho, \varsigma, z) \\ &= [\Upsilon(\varrho, \varsigma, z)]^\theta [\Upsilon(\varrho, \Theta\varrho, \Theta\varrho)]^\sigma [\Upsilon(\varsigma, \Theta\varsigma, \Theta\varsigma)]^\zeta \\ &\cdot \left[\frac{1}{2}(\Upsilon(\varrho, \Theta\varsigma, \Theta\varsigma) + \Upsilon(\varsigma, \Theta\varrho, \Theta\varrho))\right]^{1-\theta-\sigma-\zeta}. \end{aligned}$$

We now present our main invariant point theorem.

Theorem 1. *Let $(\bigwedge, \Upsilon)$ be a complete Υ-MS and Θ be a self-mapping on $\bigwedge$. Suppose that the following conditions are satisfied:*

(i) *Θ is an interpolative Υ-Hardy–Rogers–Suzuki-type Z-contraction with respect to some $\eta \in Z_\eta$;*
(ii) *Θ is Υ-τ-orbital admissible;*
(iii) *we can find $u_0 \in \bigwedge$ such that $\tau(u_0, \Theta u_0, \Theta u_0) \geq 1$;*
(iv) *$\bigwedge$ is Υ-τ-regular.*

Then, Θ has an invariant point in $\bigwedge$.

Proof. Let $\varrho_0 \in \bigwedge$ be an arbitrary point. Define the sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ in $\bigwedge$ by $\varrho_n = \Theta^n(\varrho_0)$. Note that, if we can find $l \in \mathbb{N}$ such that $\varrho_l = \varrho_{l+1} = \Theta\varrho_l$, then the theorem is proved as ϱ_l is an invariant point of Θ. Suppose that $\varrho_n \neq \varrho_{n+1}$ for all $n \in \mathbb{N}$. Given that $\tau(\varrho_0, \Theta\varrho_0, \Theta\varrho_0) \geq 1$ and Θ is Υ-τ-orbital admissible, then $\tau(\varrho_1, \varrho_2, \varrho_2) = \tau(\Theta\varrho_0, \Theta\varrho_1, \Theta\varrho_1) \geq 1$. By following these steps, we obtain $\tau(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Since Θ obeys (1) with respect to some $\eta \in Z_\eta$ and

$$\frac{1}{2}\Upsilon(\varrho_n, \Theta\varrho_n, \Theta\varrho_n) = \frac{1}{2}\Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) \leq \Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1}),$$

then

$$\eta\big(\tau(\varrho_n, \varrho_{n+1}, \varrho_{n+1})\Upsilon(\varrho_{n+1}, \varrho_{n+2}, \varrho_{n+2}), C_\eta(\varrho_n, \varrho_{n+1}, \varrho_{n+1})\big) \geq 0;$$

which, by (η_1), gives

$$\begin{aligned}0 \leq \eta\big(\tau(\varrho_n,\varrho_{n+1},\varrho_{n+1})\Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2}), C_\eta(\varrho_n,\varrho_{n+1},\varrho_{n+1})\big)\\ < C_\eta(\varrho_n,\varrho_{n+1},\varrho_{n+1}) - \tau(\varrho_n,\varrho_{n+1},\varrho_{n+1})\Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2}).\end{aligned}$$

The above expression implies that

$$\tau(\varrho_n,\varrho_{n+1},\varrho_{n+1})\Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2}) < C_\eta(\varrho_n,\varrho_{n+1},\varrho_{n+1}).$$

Since $\tau(\varrho_n,\varrho_{n+1},\varrho_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2}) \leq\ & \tau(\varrho_n,\varrho_{n+1},\varrho_{n+1})\Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2})\\ <\ & C_\eta(\varrho_n,\varrho_{n+1},\varrho_{n+1}).\end{aligned} \tag{2}$$

It follows from (2) that

$$\begin{aligned}\Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2}) <\ & \Upsilon(\varrho_n,\varrho_{n+1},\varrho_{n+1})^\theta\Upsilon(\varrho_n,\varrho_{n+1},\varrho_{n+1})^\sigma\\ & \cdot \Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2})^\zeta\\ & \cdot \left[\frac{1}{2}(\Upsilon(\varrho_n,\varrho_{n+2},\varrho_{n+2}) + \Upsilon(\varrho_{n+1},\varrho_{n+1},\varrho_{n+1}))\right]^{1-\theta-\sigma-\zeta}.\end{aligned} \tag{3}$$

Applying (Y1) to (3) implies

$$\begin{aligned}\Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2}) <\ & \Upsilon(\varrho_n,\varrho_{n+1},\varrho_{n+1})^{\theta+\sigma}\Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2})^\zeta\\ & \cdot \left[\frac{1}{2}\Upsilon(\varrho_n,\varrho_{n+2},\varrho_{n+2})\right]^{1-\theta-\sigma-\zeta}.\end{aligned} \tag{4}$$

Taking help from (Y5) and the fact that $\varpi(\varrho) = \varrho^{1-\theta-\sigma-\zeta}$ is a nondecreasing function for $\varrho > 0$, we have

$$\left[\frac{1}{2}\Upsilon(\varrho_n,\varrho_{n+2},\varrho_{n+2})\right]^{1-\theta-\sigma-\zeta} \leq \left[\frac{1}{2}(\Upsilon(\varrho_n,\varrho_{n+1},\varrho_{n+1}) + \Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2}))\right]^{1-\theta-\sigma-\zeta}. \tag{5}$$

Hence, (4) and (5) give

$$\begin{aligned}\Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2}) <\ & \Upsilon(\varrho_n,\varrho_{n+1},\varrho_{n+1})^{\theta+\sigma}\Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2})^\zeta\\ & \cdot \left[\frac{1}{2}(\Upsilon(\varrho_n,\varrho_{n+1},\varrho_{n+1}) + \Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2}))\right]^{1-\theta-\sigma-\zeta}.\end{aligned} \tag{6}$$

Assuming that

$$\Upsilon(\varrho_n,\varrho_{n+1},\varrho_{n+1}) < \Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2}),\ \text{for all } n \in \mathbb{N},$$

then, it turns (6) into

$$\begin{aligned}\Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2}) &< \Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2})^{\theta+\sigma+\zeta+1-\theta-\sigma-\zeta}\\ &= \Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2}),\end{aligned}$$

a contradiction. Therefore,

$$\Upsilon(\varrho_{n+1},\varrho_{n+2},\varrho_{n+2}) \leq \Upsilon(\varrho_n,\varrho_{n+1},\varrho_{n+1}),\ \text{for all } n \in \mathbb{N}.$$

This shows that $\{\Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1})\}_{n\in\mathbb{N}}$ is a decreasing sequence in $(\bigwedge, \Upsilon)$. Since $0 \leq \Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1})$, for all $n \in \mathbb{N}$, then $\{\Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1})\}_{n\in\mathbb{N}}$ is a bounded monotonic sequence of real numbers and hence converges to some $p \in F_+$; that is,

$$\lim_{n\longrightarrow\infty} \Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) = p. \tag{7}$$

We claim that p must be equal to zero and the sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ is Y-Cauchy. First, notice that

$$\lim_{n\longrightarrow\infty} C_\eta(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) = p. \tag{8}$$

Hence, by Sandwich theorem for functions of several variables, it follows from (2) that

$$\lim_{n\longrightarrow\infty} \tau(\varrho_n, \varrho_{n+1}, \varrho_{n+1})\Upsilon(\varrho_{n+1}, \varrho_{n+2}, \varrho_{n+2}) = p.$$

If we assume the contrary that $p > 0$, then utilizing (η_2), gives

$$\begin{aligned}0 &\leq \eta\big(\tau(\varrho_n, \varrho_{n+1}, \varrho_{n+1})\Upsilon(\varrho_{n+1}, \varrho_{n+2}, \varrho_{n+2}), C_\eta(\varrho_n, \varrho_{n+1}, \varrho_{n+1})\big)\\ &< 0,\end{aligned}$$

a contradiction. Thus,

$$\lim_{n\longrightarrow\infty} \Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) = \lim_{n\longrightarrow\infty} C_\eta(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) = 0. \tag{9}$$

Now, to prove that $\{\varrho_n\}_{n\in\mathbb{N}}$ is an Y-Cauchy sequence of points in $(\bigwedge, \Upsilon)$, assume on the contrary that $\{\varrho_n\}_{n\in\mathbb{N}}$ is not an Y-Cauchy sequence. Then, we can find $\epsilon > 0$ and sequences $\{m(k)\}_{k\in\mathbb{N}}$ and $\{n(k)\}_{k\in\mathbb{N}}$ such that, for all $k \in \mathbb{N}$, $m(k) > n(k) > k$, $\Upsilon(\varrho_{n(k)}, \varrho_{m(k)}, \varrho_{m(k)}) \geq \epsilon$, and $\Upsilon(\varrho_{n(k)}, \varrho_{m(k)-1}, \varrho_{m(k)-1}) < \epsilon$. Now, for all $k \in \mathbb{N}$, we obtain

$$\begin{aligned}\epsilon \leq \Upsilon(\varrho_{n(k)}, \varrho_{m(k)}, \varrho_{m(k)}) \leq\ & \Upsilon(\varrho_{n(k)}, \varrho_{m(k)-1}, \varrho_{m(k)-1})\\ &+ \Upsilon(\varrho_{m(k)-1}, \varrho_{m(k)}, \varrho_{m(k)})\\ <\ & \epsilon + \Upsilon(\varrho_{m(k)-1}, \varrho_{m(k)}, \varrho_{m(k)}).\end{aligned} \tag{10}$$

Letting $k \longrightarrow \infty$ in (10) and using (9), we have

$$\lim_{k\longrightarrow\infty} \Upsilon(\varrho_{n(k)}, \varrho_{m(k)}, \varrho_{m(k)}) = \epsilon. \tag{11}$$

Again, we have

$$\begin{aligned}\Upsilon(\varrho_{n(k)}, \varrho_{m(k)}, \varrho_{m(k)}) \quad \leq \quad & \Upsilon(\varrho_{n(k)}, \varrho_{n(k)+1}, \varrho_{n(k)+1})\\ &+\Upsilon(\varrho_{n(k)+1}, \varrho_{m(k)-1}, \varrho_{m(k)-1})\\ &+\Upsilon(\varrho_{m(k)-1}, \varrho_{m(k)}, \varrho_{m(k)})\end{aligned}$$

and

$$\begin{aligned}\Upsilon(\varrho_{n(k)+1}, \varrho_{m(k)+1}, \varrho_{m(k)+1}) \quad \leq \quad & \Upsilon(\varrho_{n(k)+1}, \varrho_{m(k)-1}, \varrho_{m(k)-1})\\ &+\Upsilon(\varrho_{m(k)-1}, \varrho_{m(k)}, \varrho_{m(k)})\\ &+\Upsilon(\varrho_{m(k)}, \varrho_{m(k)+1}, \varrho_{m(k)+1}).\end{aligned} \tag{12}$$

As $k \longrightarrow \infty$ in (12), by (9) and (11), we infer that

$$\lim_{k\longrightarrow\infty} \Upsilon(\varrho_{n(k)+1}, \varrho_{m(k)+1}, \varrho_{m(k)+1}) = \epsilon.$$

From $\lim_{k\longrightarrow\infty} \Upsilon(\varrho_{m(k)}, \varrho_{m(k)+1}, \varrho_{m(k)+1}) = 0$ and $\lim_{k\longrightarrow\infty} \Upsilon(\varrho_{m(k)}, \varrho_{n(k)}, \varrho_{n(k)}) \geq \epsilon > 0$, we conclude that we can find $n_0 \in \mathbb{N}$ such that, for all $k \geq n_0$,

$$\frac{1}{2}\Upsilon(\varrho_{m(k)}, \varrho_{m(k)+1}, \varrho_{m(k)+1}) \leq \Upsilon(\varrho_{m(k)}, \varrho_{m(k)+1}, \varrho_{m(k)+1}).$$

Since Θ is Υ-τ-orbital admissible, then, using the same steps for obtaining (2), we obtain

$$\begin{aligned} 0 <\ & \Upsilon(\varrho_{m(k)+1}, \varrho_{m(k)+2}, \varrho_{m(k)+2}) \\ \leq\ & \tau(\varrho_{m(k)}, \varrho_{m(k)+1}, \varrho_{m(k)+1})\Upsilon(\varrho_{m(k)+1}, \varrho_{m(k)+2}, \varrho_{m(k)+2}) \\ <\ & C_\eta(\varrho_{m(k)}, \varrho_{m(k)+1}, \varrho_{m(k)+1}). \end{aligned} \tag{13}$$

Taking the limit in (13) as $k \longrightarrow \infty$ and using (9) implies $0 < 0$, which is a contradiction. Therefore, $\{\varrho_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\bigwedge, \Upsilon)$. By the completeness of this space, we can find $u \in \bigwedge$ such that $\{\varrho_n\}_{n\in\mathbb{N}}$ is Υ-convergent to u. We now show that u is an invariant point of Θ. Observe that, since $(\bigwedge, \Upsilon)$ is Υ-τ-regular and $\tau(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\tau(\varrho_n, u, u) \geq 1$. Now, either

$$\frac{1}{2}\Upsilon(\varrho_n, \Theta\varrho_n, \Theta\varrho_n) \leq \Upsilon(\varrho_n, u, u), \tag{14}$$

or

$$\frac{1}{2}\Upsilon(\Theta\varrho_n, \Theta^2\varrho_n, \Theta^2\varrho_n) \leq \Upsilon(u, \Theta\varrho_n, \Theta\varrho_n). \tag{15}$$

However, if we assume that

$$\frac{1}{2}\Upsilon(\varrho_n, \Theta\varrho_n, \Theta\varrho_n) > \Upsilon(\varrho_n, u, u)$$

and

$$\frac{1}{2}\Upsilon(\Theta\varrho_n, \Theta^2\varrho_n, \Theta^2\varrho_n) > \Upsilon(u, \Theta\varrho_n, \Theta\varrho_n),$$

then utilizing (Y5) implies

$$\begin{aligned} \Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) =\ & \Upsilon(\varrho_n, \Theta\varrho_n, \Theta\varrho_n) \\ \leq\ & \Upsilon(\varrho_n, u, u) + \Upsilon(u, \Theta\varrho_n, \Theta\varrho_n) \\ <\ & \frac{1}{2}\Upsilon(\varrho_n, \Theta\varrho_n, \Theta\varrho_n) + \frac{1}{2}\Upsilon(\Theta\varrho_n, \Theta^2\varrho_n, \Theta^2\varrho_n) \\ =\ & \frac{1}{2}\Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) + \frac{1}{2}\Upsilon(\varrho_{n+1}, \varrho_{n+2}, \varrho_{n+2}). \end{aligned} \tag{16}$$

Since the sequence $\{\Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1})\}_{n\in\mathbb{N}}$ is nonincreasing, (16) gives

$$\begin{aligned} \Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) <\ & \frac{1}{2}\Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) + \frac{1}{2}\Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) \\ =\ & \Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1}), \end{aligned}$$

a contradiction. Hence, either (14) or (15) is valid. Now, suppose that (14) is true and $\Theta u \neq u$. Then, applying the Υ-τ-regularity of $(\bigwedge, \Upsilon)$ implies

$$0 \leq \eta\big(\tau(\varrho_n, u, u)\Upsilon(\Theta\varrho_n, \Theta u, \Theta u), C_\eta(\varrho_n, u, u)\big). \tag{17}$$

Employing (η_2) in (17) leads to

$$0 \leq C_\eta(\varrho_n, u, u) - \tau(\varrho_n, u, u)\Upsilon(\Theta\varrho_n, \Theta u, \Theta u),$$

from which, we have

$$\begin{aligned} \Upsilon(\varrho_{n+1}, \Theta u, \Theta u) \leq & \ \tau(\varrho_n, u, u)\Upsilon(\varrho_{n+1}, \Theta u, \Theta u) \\ \leq & \ C_\eta(\varrho_n, u, u) \\ = & \ [\Upsilon(\varrho_n, u, u)]^\theta [\Upsilon(\varrho_n, \varrho_{n+1}, \varrho_{n+1})]^\sigma [\Upsilon(u, \Theta u, \Theta u)]^\zeta \\ & \cdot \left[\frac{1}{2}(\Upsilon(\varrho_n, \Theta u, \Theta u) + \Upsilon(u, \varrho_{n+1}, \varrho_{n+1}))\right]^{1-\theta-\sigma-\zeta}. \end{aligned} \tag{18}$$

Letting $n \longrightarrow \infty$ in (18), and using the sandwich theorem for functions of several variables, we reach $0 = \lim_{n \longrightarrow \infty} \Upsilon(\varrho_{n+1}, \Theta u, \Theta u) = \Upsilon(u, \Theta u, \Theta u)$. Hence, (Y1) can be employed to infer that $u = \Theta u$. Using similar steps, if (15) is satisfied, we can conclude as well that u is an invariant point of Θ. $\square$

Using the same steps as employed in proving Theorem 1, we can also establish the following results by reducing the terms in Theorem 1.

Theorem 2. *Let $(\bigwedge, \Upsilon)$ be a complete Υ-MS and Θ be a self-mapping on $\bigwedge$. Suppose that we can find $\theta, \sigma \in (0,1)$ with $\theta + \sigma < 1$, $\eta \in Z_\eta$ and a mapping $\tau : \bigwedge \times \bigwedge \times \bigwedge \longrightarrow \digamma_+$ such that*

(i) $\frac{1}{2}\Upsilon(\varrho, \Theta\varrho, \Theta\varrho) \leq \Upsilon(\varrho, \varsigma, z)$ *implies*

$$\eta\big(\tau(\varrho, \varsigma, z)\Upsilon(\Theta\varrho, \Theta\varsigma, \Theta z), \Gamma_\eta(\varrho, \varsigma, z)\big) \geq 0,$$

for all $\varrho, \varsigma, z \in \bigwedge \setminus \mathcal{F}_{ix}(\Theta)$, where

$$\Gamma_\eta(\varrho, \varsigma, z) = [\Upsilon(\varrho, \varsigma, z)]^\theta [\Upsilon(\varrho, \Theta\varrho, \Theta\varsigma)]^\sigma [\Upsilon(\varsigma, \Theta\varsigma, \Theta\varsigma)]^{1-\theta-\sigma}; \tag{19}$$

(ii) *Θ is Υ-τ-orbital admissible;*
(iii) *we can find $u_0 \in \bigwedge$ such that $\tau(u_0, \Theta u_0, \Theta u_0) \geq 1$;*
(iv) *$\bigwedge$ is Υ-τ-regular.*

Then, Θ has an invariant point in $\bigwedge$.

Theorem 3. *Let $(\bigwedge, \Upsilon)$ be a complete Υ-MS and Θ be a self-mapping on $\bigwedge$. Suppose that we can find $\kappa \in \left(0, \frac{1}{2}\right]$, $\eta \in Z_\eta$ and a mapping $\tau : \bigwedge \times \bigwedge \times \bigwedge \longrightarrow \digamma_+$ such that*

(i) $\frac{1}{2}\Upsilon(\varrho, \Theta\varrho, \Theta\varrho) \leq \Upsilon(\varrho, \varsigma, z)$ *implies*

$$\eta(\tau(\varrho, \varsigma, z)\Upsilon(\Theta\varrho, \Theta\varsigma, \Theta z), \kappa(\Upsilon(\varrho, \Theta\varsigma, \Theta\varsigma) + \Upsilon(\varsigma, \Theta\varrho, \Theta\varrho))) \geq 0, \tag{20}$$

for all $\varrho, \varsigma, z \in \bigwedge \setminus \mathcal{F}_{ix}(\Theta)$;
(ii) *Θ is Υ-τ-orbital admissible;*
(iii) *we can find $u_0 \in \bigwedge$ such that $\tau(u_0, \Theta u_0, \Theta u_0) \geq 1$;*
(iv) *$\bigwedge$ is Υ-τ-regular.*

Then, Θ has an invariant point in $\bigwedge$.

In the following, we construct an example to support the hypotheses of Theorem 1.

Example 1. *Let $\bigwedge = \digamma$ and $\Upsilon : \bigwedge \times \bigwedge \times \bigwedge \longrightarrow \digamma$ be given by*

$$\Upsilon(\varrho, \varsigma, z) = |\varrho - \varsigma| + |\varsigma - z| + |\varrho - z|, \text{ for all } \varrho, \varsigma, z \in \bigwedge.$$

Then, $(\bigwedge, \Upsilon)$ is a complete Υ-MS (see [35], Example 2.2). Define the mappings $\Theta : \bigwedge \longrightarrow \bigwedge$, $\tau : \bigwedge \times \bigwedge \times \bigwedge \longrightarrow \digamma_+$ and $\eta : \digamma_+ \times \digamma_+ \longrightarrow \digamma$ as follows:

$$\Theta\varrho = \begin{cases} \frac{1}{2} - \varrho, & \text{if } \varrho \in \{\frac{1}{4}, 2, 3, 4\} \\ \frac{1}{5}, & \text{if } \varrho \in \{\frac{1}{5}, 5, 10, 20\} \\ 0, & \text{elsewhere}, \end{cases}$$

$$\tau(\varrho, \varsigma, z) = \begin{cases} 1, & \text{if } \varrho = \frac{1}{5}, \varsigma = 10, z = 20 \text{ or } \varrho, \varsigma, z \in \left\{\frac{1}{5}\right\} \\ 0, & \text{elsewhere}, \end{cases}$$

$\eta(\xi, \hbar) = \lambda\hbar - \xi$ for all $\xi, \hbar \in \digamma_+$ and $\lambda \in (0,1)$. Clearly, $\eta \in Z_\eta$. First, to see that Θ is not a Z-contraction in the sense of Khojasteh et al. [11], take $\varrho = 2, \varsigma = z = 3$; then,

$$\begin{aligned} \eta(\Upsilon(\Theta 2, \Theta 3, \Theta 3), \Upsilon(2,3,3)) &= \lambda\Upsilon(2,3,3) - \Upsilon(\Theta 2, \Theta 3, \Theta 3) \\ &= 2\lambda - 2\left|\left(\frac{1}{2} - 2\right) - \left(\frac{1}{2} - 3\right)\right| \\ &= 2(\lambda - 1) < 0. \end{aligned}$$

Moreover, since the mapping Θ is not Υ-continuous, none of the results in [28,35] are applicable to this example. However, in order to understand that Θ is an interpolative Υ-Hardy–Rogers–Suzuki-type Z-contraction on $(\bigwedge, \Upsilon)$, notice that, for all $\varrho, \varsigma, z \in \bigwedge \setminus \left\{\frac{1}{5}, 2, 3, 4, 5, 10, 20\right\}$ such that

$$\frac{1}{2}\Upsilon(\varrho, \Theta\varrho, \Theta\varrho) \leq \Upsilon(\varrho, \varsigma, z), \text{ we have}$$

$$\eta(\tau(\varrho, \varsigma, z)\Upsilon(\Theta\varrho, \Theta\varsigma, \Theta z), C(\varrho, \varsigma, z)) = \lambda C(\varrho, \varsigma, z) \geq 0.$$

Now, if $\varrho, \varsigma, z \in \bigwedge$ with $\varrho = 5, \varsigma = 10, z = 20$ such that

$$\frac{1}{2}\Upsilon(\varrho, \Theta\varrho, \Theta\varrho) \leq \Upsilon(\varrho, \varsigma, z),$$

direct calculation verifies that

$$\eta(\tau(\varrho, \varsigma, z)\Upsilon(\Theta\varrho, \Theta\varsigma, \Theta z), C(\varrho, \varsigma, z)) = \lambda C(\varrho, \varsigma, z) \geq 0.$$

Let $\varrho, \varsigma, z \in \bigwedge$ such that $\tau(\varrho, \varsigma, z) \geq 1$. This implies that $\varrho, \varsigma, z \in \left\{\frac{1}{5}, 5, 10, 20\right\}$, and, by the definition of τ, we have $\Theta\varrho = \Theta\varsigma = \Theta z = \frac{1}{5}$ and $\tau(\Theta\varrho, \Theta\varsigma, \Theta z) = 1$. It follows that Θ is Υ-τ-orbital admissible. Moreover, we can find $u_0 = 5 \in \bigwedge$ such that $\tau(u_0, \Theta u_0, \Theta u_1) \geq 1$. Further, consider a sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ in $(\bigwedge, \Upsilon)$, such that $\tau(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{\varrho_n\}_{n\in\mathbb{N}}$ is Υ-convergent to $u \in (\bigwedge, \Upsilon)$. Since $\tau(\varrho_n, \varrho_{n+1}, \varrho_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then, by the definition of τ, $\varrho_n \in \left\{\frac{1}{5}, 5, 10, 20\right\}$, which implies that $\tau(\varrho_n, u, u) \geq 1$ for all $n \in \mathbb{N}$. Hence, $(\bigwedge, \Upsilon)$ is Υ-τ-regular. Consequently, all of the assumptions of Theorem 1 are satisfied, and Θ has an invariant point. In this case, we see that $\mathcal{F}_{ix}(\Theta) = \left\{0, \frac{1}{5}, \frac{1}{4}\right\}$.

Consequences

By using variants of simulation functions, some particular cases of our main results can be highlighted as follows.

Corollary 1. *Let $(\bigwedge, \Upsilon)$ be a complete Υ-MS and Θ be a self-mapping on $\bigwedge$. Suppose that*

(i) we can find $\theta, \sigma, \zeta \in (0,1)$ with $\theta + \sigma + \zeta < 1$, $\lambda \in [0,1)$ and a mapping $\tau : \bigwedge \times \bigwedge \times \bigwedge \longrightarrow \digamma_+$ such that

$$\frac{1}{2}\Upsilon(\varrho, \Theta\varrho, \Theta\varrho) \leq \Upsilon(\varrho, \varsigma, z) \text{ implies}$$

$$\tau(\varrho,\varsigma,z)\Upsilon(\Theta\varrho,\Theta\varsigma,\Theta z)\leq\lambda C_{\eta}(\varrho,\varsigma,z);$$

(ii) Θ *is* Υ-τ*-orbital admissible;*
(iii) *we can find* $u_0\in\bigwedge$ *such that* $\tau(u_0,\Theta u_0,\Theta u_0)\geq 1$;
(iv) $\bigwedge$ *is* Υ-τ*-regular.*

Then, Θ *has an invariant point in* $\bigwedge$.

Proof. It is enough to take $\eta(t,s)=\lambda s-t$ for all $t,s\in F_+$ in Theorem 1. □

Corollary 2. *Let* $(\bigwedge,\Upsilon)$ *be a complete* Υ*-MS and* Θ *be a self-mapping on* $\bigwedge$*. Suppose that*

(i) *we can find* $\theta,\sigma,\zeta\in(0,1)$ *with* $\theta+\sigma+\zeta<1$*, a mapping* $\tau:\bigwedge\times\bigwedge\times\bigwedge\longrightarrow F_+$ *and an upper semi-continuous mapping* $\varsigma:F_+\longrightarrow F_+$ *with* $\varsigma(t)<t$ *for all* $t>0$ *and* $\varsigma(0)=0$ *if and only if* $t=0$ *such that*

$$\frac{1}{2}\Upsilon(\varrho,\Theta\varrho,\Theta\varrho)\leq\Upsilon(\varrho,\varsigma,z)\text{ implies}$$

$$\tau(\varrho,\varsigma,z)\Upsilon(\Theta\varrho,\Theta\varsigma,\Theta z)\leq\varsigma(C_{\eta}(\varrho,\varsigma,z));$$

(ii) Θ *is* Υ-τ*-orbital admissible;*
(iii) *we can find* $u_0\in\bigwedge$ *such that* $\tau(u_0,\Theta u_0,\Theta u_0)\geq 1$;
(iv) $\bigwedge$ *is* Υ-τ*-regular.*

Then, Θ *has an invariant point in* $\bigwedge$.

Proof. Take $\eta(t,s)=\varsigma(s)-t$ for all $t,s\in F_+$ in Theorem 1. □

Corollary 3. *Let* $(\bigwedge,\Upsilon)$ *be a complete* Υ*-MS and* Θ *be a self-mapping on* $\bigwedge$*. Suppose that*

(i) *we can find* $\theta,\sigma\in(0,1)$ *with* $\theta+\sigma<1$*,* $\lambda\in[0,1)$ *and a mapping* $\tau:\bigwedge\times\bigwedge\times\bigwedge\longrightarrow F_+$ *such that*

$$\frac{1}{2}\Upsilon(\varrho,\Theta\varrho,\Theta\varrho)\leq\Upsilon(\varrho,\varsigma,z)\text{ implies}$$

$$\tau(\varrho,\varsigma,z)\Upsilon(\Theta\varrho,\Theta\varsigma,\Theta z)\leq\lambda\Gamma_{\eta}(\varrho,\varsigma,z),$$

where $\Gamma_{\eta}(\varrho,\varsigma,z)$ *is as defined in* (19)*;*

(ii) Θ *is* Υ-τ*-orbital admissible;*
(iii) *we can find* $u_0\in\bigwedge$ *such that* $\tau(u_0,\Theta u_0,\Theta u_0)\geq 1$;
(iv) $\bigwedge$ *is* Υ-τ*-regular.*

Then, Θ *has an invariant point in* $\bigwedge$.

Proof. It is enough to take $\eta(t,s)=\lambda s-t$ for all $t,s\in F_+$ in Theorem 2. □

Corollary 4. *Let* $(\bigwedge,\Upsilon)$ *be a complete* Υ*-MS and* Θ *be a self-mapping on* $\bigwedge$*. Suppose that*

(i) *we can find* $\theta,\sigma\in(0,1)$ *with* $\theta+\sigma<1$*, a mapping* $\tau:\bigwedge\times\bigwedge\times\bigwedge\longrightarrow F_+$ *and an upper semi-continuous mapping* $\beth:F_+\longrightarrow F_+$ *with* $\beth(t)<t$ *for all* $t>0$ *and* $\beth(0)=0$ *if and only if* $t=0$ *such that*

$$\frac{1}{2}\Upsilon(\varrho,\Theta\varrho,\Theta\varrho)\leq\Upsilon(\varrho,\varsigma,z)\text{ implies}$$

$$\tau(\varrho,\varsigma,z)\Upsilon(\Theta\varrho,\Theta\varsigma,\Theta z)\leq\beth(\Gamma_{\eta}(\varrho,\varsigma,z));$$

(ii) Θ *is* Υ-τ*-orbital admissible;*
(iii) *we can find* $u_0\in\bigwedge$ *such that* $\tau(u_0,\Theta u_0,\Theta u_0)\geq 1$;
(iv) $\bigwedge$ *is* Υ-τ*-regular.*

Then, Θ *has an invariant point in* $\bigwedge$.

Proof. Put $\eta(t,s)=\beth(s)-t$ for all $t,s\in F_+$ in Theorem 2. □

3. Application to Spring Mass Problem

Considering the movement of a spring that is under the influence of a frictional force (with respect to a horizontal spring) or a damping force (with respect to a vertical movement through a fluid; an example is the damping force provided by a shock absorber in a car). Besides this, the motion of the spring is acted upon by an external force. This type of damped motion is described by the boundary value problem (BVP):

$$\begin{cases} \frac{d^2u}{dt^2} + \frac{a}{b}\frac{du}{dt} & = M(t, u(t)) \\ u(0) = 0, & u'(0) = c, \end{cases} \tag{21}$$

where $M : [0, \Lambda] \times \digamma_+ \longrightarrow \digamma$ is a continuous function and $\Lambda > 0$. The integral reformulation of (21) is given by

$$u(t) = \int_0^t \Xi(t, s)M(s, u(s))ds,\ t \in [0, \Lambda] = \Delta, \tag{22}$$

and the Green's function $\Xi(t, s)$ is given by

$$\Xi(t, s) = \begin{cases} (t - s)e^{\delta(t-s)}, & \text{if } 0 \le s \le t \le \Lambda \\ 0, & \text{if } 0 \le t \le s \le \Lambda, \end{cases} \tag{23}$$

where $\delta > 0$ is a constant, evaluated in terms of a and b in (21). The solvability criteria for Problem (21) were examined by Deepak et al. [37] using invariant point results of F-contractions. In this section, we continue this study under new assumptions that complement ([37], Theorem 4.1).

Let $C(\Delta, \digamma_+)$ be the set of all non-negative continuous functions defined on Δ. For an arbitrary $u \in C(\Delta, \digamma_+)$, define

$$\|u\| = \sup\{u(t) : t \in \Delta\}. \tag{24}$$

Then, define the function $\Upsilon : C(\Delta, \digamma_+)^3 \longrightarrow \digamma_+$ by

$$\Upsilon(u, v, w) = \max\{\|u - v\|, \|v - w\|, \|w - u\|\}, \tag{25}$$

where $\|u\|$ is given by (24). Then, obviously, $C((\Delta, \digamma_+), \Upsilon)$ is a complete Υ-MS.

Define a self-mapping $\Theta : C(\Delta, \digamma_+) \longrightarrow C(\Delta, \digamma_+)$ by

$$\Theta u(t) = \int_0^t \Xi(t, s)M(s, u(s))\,ds\,, t \in \Delta. \tag{26}$$

Clearly, the invariant point of Θ in (26) corresponds to the solution of the BVP (21). Below, we examine conditions for the existence of an invariant point of Θ.

Theorem 4. *Consider the hypotheses:*

(C1) *we can find a mapping $\xi : \digamma^3 \longrightarrow \digamma$ and $u_0 \in C(\Delta, \digamma_+)$ such that, for all $t \in \Delta$,*

$$\xi\left(u_0(t), \int_0^t \Xi(t, s)M(s, u_0(s))\,ds, \int_0^t \Xi(t, s)M(s, u_0(s))\,ds\right) \ge 0;$$

(C2) *For all $t \in \Delta$ and $u, v, w \in C(\Delta, \digamma_+)$,*

$$\xi\left(u(t), \int_0^t \Xi(t, s)M(s, u(s))\,ds, \int_0^t \Xi(t, s)M(s, u(s))\,ds\right) \ge 0$$

implies

$$\xi\left(\int_0^t \Xi(t,s)M(s,u(s))\,ds, \int_0^t\int_0^s \Xi(t,s)M(s,u(s))\,ds\,dt,\right.$$
$$\left.\int_0^t\int_0^s \Xi(t,s)M(s,u(s))\,ds\,dt\right) \geq 0;$$

(C3) *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $C(\Delta, \digamma_+)$ such that $u_n \longrightarrow u \in C(\Delta, \digamma_+)$ as $n \longrightarrow \infty$. Suppose that, for all $t \in \Delta$,*

$$\xi(u_n(t), u_{n+1}(t), u_{n+1}(t)) \geq 0 \text{ for all } n \in \mathbb{N}$$

implies

$$\xi(u_n(t), u(t), u(t)) \geq 0 \text{ for all } n \in \mathbb{N};$$

(C4) *we can find a constant $\alpha \in \left(0, \frac{1}{4}\right]$ such that, for all $u, v \in C(\Delta, \digamma_+)$,*

$$\begin{aligned} \Big|M(s,u(s)) - M(s,v(s))\Big| \leq \alpha\Bigg(&\left|u(s) - \int_0^t \Xi(t,s)M(s,v(s))\,ds\right| \\ &+\left|v(s) - \int_0^t \Xi(t,s)M(s,u(s))\,ds\right|\Bigg); \end{aligned} \tag{27}$$

(C5) $\sup_{s\in\Delta}\left(\int_0^t \Xi(s,t)\,ds\right) \leq 1.$

Under (C1)–(C5), the BVP (21) has a solution in $C(\Delta, \digamma_+)$.

Proof. For all $u, v \in C(\Delta, \digamma_+)$ such that $\Theta u(t) \neq \Theta v(t)$, using (27), we obtain

$$\begin{aligned} &|\Theta u(t) - \Theta v(t)| \\ &\leq \left|\int_0^t \Xi(t,s)M(s,u(s))\,ds - \int_0^t \Xi(t,s)M(s,v(s))\,ds\right| \\ &\leq \int_0^t \Xi(t,s)|M(s,u(s)) - M(s,v(s))|\,ds \\ &\leq \alpha\int_0^t \Xi(t,s)\,ds\left(\left|u(s) - \int_0^t \Xi(t,s)M(s,v(s))\,ds\right|\right) \\ &+\left|v(s) - \int_0^t \Xi(t,s)M(s,u(s))\,ds\right| \\ &\leq \alpha\sup_{s\in\Delta}\left(\int_0^t \Xi(t,s)\,ds\right)\Big(\|u - \Theta v\| + \|v - \Theta u\|\Big) \\ &\leq \alpha(\|u - \Theta v\| + \|v - \Theta u\|). \end{aligned}$$

That is,

$$\|\Theta u - \Theta v\| \leq \alpha(\|u - \Theta v\| + \|v - \Theta u\|). \tag{28}$$

Using similar steps as above, we can have

$$\|\Theta v - \Theta w\| \leq \alpha(\|\Theta v - w\| + \|\Theta w - v\|). \tag{29}$$

Hence,

$$\begin{aligned}
&\max\{\|\Theta u-\Theta v\|,\|\Theta v-\Theta w\|,\|\Theta w-\Theta u\|\}\\
&\leq\ (2\alpha)\max\{\|u-\Theta v\|,\|\Theta v-u\|\}+(2\alpha)\max\{\|v-\Theta u\|,\|\Theta u-v\|\}\\
&=\ \kappa\bigg(\max\{\|u-\Theta v\|,\|\Theta v-u\|\}+\max\{\|v-\Theta u\|,\|\Theta u-v\|\}\bigg),
\end{aligned} \tag{30}$$

where $\kappa=2\alpha$. The inequality (30) implies that

$$\Xi(\Theta u,\Theta v,\Theta w)\leq\kappa(\Xi(u,\Theta v,\Theta v)+\Xi(v,\Theta u,\Theta u)). \tag{31}$$

Consider a mapping $\sigma: C(\Delta,\digamma_+)^3\longrightarrow\digamma_+$ defined by

$$\sigma(u(t),v(t),w(t))=\begin{cases}1, & \text{if } \xi(u(t),v(t),w(t))\geq 0, \text{ for all } t\in\Delta\\ 0, & \text{elsewhere.}\end{cases}$$

Then, for all $u,v,w\in C(\Delta,\digamma_+)$, we derive from (31) that

$$\begin{aligned}
\sigma(u,v,w)\Upsilon(\Theta u,\Theta v,\Theta w) &\leq \kappa(\Upsilon(u,\Theta v,\Theta v)+\Upsilon(v,\Theta u,\Theta u))\\
&\leq \frac{1}{2}(\Upsilon(u,\Theta v,\Theta v)+\Upsilon(v,\Theta u,\Theta u)).
\end{aligned} \tag{32}$$

Observe that (32) coincides with (20) with $\eta(t,s)=\lambda s-t$ for all $t,s\in\digamma_+$ and $\lambda\in[0,1)$. Clearly, all other hypotheses of Theorem 3 are easily verifiable. As a consequence of Theorem 3, we can find $u^*\in C(\Delta,\digamma_+)$ such that $\Theta u^*=u^*$. □

4. Conclusions

Based on the properties of generalized MS, interpolative, Suzuki, Hardy-Rogers, and Z-contractions, a new notion under the name interpolative Y-Hardy-Rogers–Suzuki-type Z-contraction is initiated in this paper. Sufficient criteria for the existence of invariant points for such contractions were established. By using variants of simulation functions, a few special cases of the main results obtained herein are deduced. As an application, one of our results is utilized to study new conditions for the existence of a solution to a boundary valued problem arising in the oscillation of a spring. In particular, the principal idea of this manuscript is an extension of some recently announced results in [11,27,28,35] and a few related references therein.

Knowing that invariant point theory in Y-MS is still at the outset, many potential results can be suggested as some future work. Accordingly, for robustness, the ideas proposed herein can be explored in other approaches, such as fixed-circle problems, hybrid, fuzzy, and crisp multi-valued contractions, as well as related problems.

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