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Analysis of Controllability of Fractional Functional Random Integroevolution Equations with Delay

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Abstract: Various scholars have lately employed a wide range of strategies to resolve two specific types of symmetrical fractional differential equations. The evolution of a number of real-world systems in the physical and biological sciences exhibits impulsive dynamical features that can be represented via impulsive differential equations. In this paper, we explore some existence and controllability theories for the Caputo order $q \in (1, 2)$ of delay- and random-effect-affected fractional functional integroevolution equations (FFIEEs). In order to prove that random solutions exist, we must prove a random fixed point theorem using a stochastic domain and the mild solution. Then we demonstrate that our solutions are controllable. At the end, applications and example is illustrated which indicates the applicability of this manuscript.

Keywords: random fixed point; state dependent delay; controllability; functional differential equation; mild solution; finite delay; cosine and sine family

MSC: 26A33; 34K37

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1. Introduction

Many different applications have been investigated through the theory of impulsive fractional differential equations (IFDEs) in the accurate mathematical representation of a wide variety of practical problems. It is acknowledged as a crucial area for research, as much as the modelling of impulsive issues in population dynamics, ecology, biotechnology, and other fields. In real-world situations, many processes and phenomena are characterised by rapid shifts in their states. The mentioned quick modifications are called impulsive effects within the system. Instantaneous and noninstantaneous impulses are the two main forms of impulses discussed in the literature to date. In contrast to the length of a whole evolution, such as that of shocks and natural disasters, the period of these fluctuations in instantaneous impulses is insignificant; in the case of noninstantaneous impulses, on the other hand, the duration of the changes exists throughout a finite time period.

Over the past three decades, the field of mathematical analysis has incorporated fractional calculus, FDEs, and integrodifferential equations, and the qualitative theory of these equations on both a theoretical and a practical level. Fundamentally, fractional calculus theory, the qualitative theory of FDEs and fractional integrodifferential equations, numerical simulations, and symmetry analysis are mathematical analytical tools used to study arbitrary-order integrals and derivatives that unify and generalise the conventional ideas of differentiation and integration. Compared to classical formulations, nonlinear operators with a fractional order are more useful. Throughout the development of emerging control theory, the controllability of DEs problems has played a major role. Typically, it means that a dynamical system may be moved from any initial state to the desired terminal state using a set of legal controls. Control theory places much emphasis on the qualitative characteristics of control systems. There has been particular focus on the controllability of linear and nonlinear systems in a finite-dimensional space that are described by ordinary DEs; see [1–4] for a list of researchers who have extended the idea to infinite-dimensional systems with bounded operators in Banach spaces (BS). The controllability problem was converted into a fixed-point problem by the authors of [5]. We advise reading [6,7] for additional information. The authors of [8,9] investigated a variety of functional DEs and inclusions, and proposed various controllability findings. A family of integrodifferential evolution equations' controllability was examined by Dilao et al. [10].

It is often advantageous to handle second-order abstract DEs explicitly rather than always reducing them to first-order systems. For the investigation of second-order issues, the theory of strongly continuous cosine families is an invaluable resource. We use some of the core ideas in cosine family theory [11]. Typically, this means that a dynamical system may be moved from any initial state to the desired terminal state using a set of legal controls. Control theory places much emphasis on the qualitative characteristics of control systems. There has been particular focus on the controllability of linear and nonlinear systems in finite-dimensional space that are described by ordinary DEs [12,13].

The reader is recommended to read [14–16] for more information on random differential equations, which are natural generalisations of deterministic DEs and appear in a variety of applications. The accuracy of our knowledge about the system's characteristics determines the nature of a dynamic system. When knowledge about a dynamic system is exact, a deterministic dynamical system emerges. Moreover, many of the available details for identifying and assessing dynamic system characteristics are incorrect, uncertain, or imprecise. To put it another way, determining the parameters of a dynamic system is highly risky. However, when we have probable knowledge and an understanding of statistical characteristics, we can use stochastic DEs in mathematically modelling such systems.

Ji-Huan He [17] studied fractal calculus. Wang et al. [18–20] worked on nondifferentiable exact solutions, the modification of the unsteady model, and diverse exact and explicit solutions. Mehmood et al. [21] worked on a partial DE. Niazi et al. [22], Shafqat et al. [23], Alnahdi [24], and Abuasbeh et al. [25] investigated the existence and uniqueness of FEEs. Inspired by the above studies [26], this paper deals with the controllability of the fractional functional integroevolution equation with delay and random effects:

$$\begin{aligned}
&{}_0^cD_\nu^q U(\chi,\xi) = B_1 U(\chi,\xi) + \varphi(\chi, U_\chi(.,\xi),\xi) + \int_0^\nu B_2 f(\chi,\xi) dC_v + Bx(\nu)Cx(\nu)d\nu,\ \xi \in \Theta := [0,\kappa],\ \nu \in [0,T] \\
&U(\chi,\xi) + m(U) = \varrho_1(\chi,\xi);\ \ \xi \in (-\infty, 0], \\
&U'(\chi,\xi) = \varrho_2(\xi)
\end{aligned} \tag{1}$$

Knowing that complete probability space $(\Phi, F, \wp)$ given functions $\varphi : \Theta \times D \times \Psi \to \Xi, \sigma_1 \in D \in D \times \Phi$, and infinitesimal generator $B_1 : D(B_1) \subset \Xi \to \Xi$ of a strongly continuous cosine family, the phase space is $(H_q(\chi))_{\chi \in \mathbf{R^m}}$ on Ξ, D, and a real BS is $(\Xi, |.|)$. Control function $\mathcal{P}(.,\xi)$ is specified in $L^2(\Theta,\Omega)$, a BS of possible control functions with Ω as a BS, and B_2 is a bounded linear operator (LO) from Ω into Ξ.

The component of $D \times \Phi$ determined with $D \times \Phi$, given by $U_\xi(\iota,\xi) = U(\xi+\iota,\xi), \iota \in (-\infty, 0]$ is denoted by $U_\chi(.,\xi)$. Here, the state's existence from the year $-\infty$ to the current day ξ is represented by the string $U_\chi(.,\xi)$. Eras $U_\chi(.,\xi)$ were presumptively part of some abstract phases D.

First, we suppose random issue

$$\begin{aligned}
&{}_0^cD_\nu^q U(\chi,\xi) = B_1 U(\chi,\xi) + \varphi(\chi, U_{\vartheta(\chi,U_\chi)}(.,\xi),\xi) + \int_0^\nu B_2 f(\chi,\xi) dC_v + Bx(\nu)Cx(\nu)d\nu,\ \xi \in \Theta := [0,\kappa],\ \nu \in [0,T] \\
&U(\chi,\xi) + m(U) = \varrho_1(\chi,\xi);\ \ \xi \in (-\infty, 0], \\
&U'(\chi,\xi) = \varrho_2(\xi)
\end{aligned} \tag{2}$$

where $\varphi : \Theta \times D \times \Psi \to \Xi, \sigma_1 \in D \in D \times \Phi$ are given random functions, $B_1 : D(B_1) \subset \Xi \to \Xi$ is as in problem (1), D is the phase space, $\psi; \Theta \times D \to (-\infty, \kappa]$, and $(\Xi, |.|)$ is a real

BS. For the key conclusions on Schauder's fixed theorem [27], and random fixed-point theorem paired with the family of cosine operators, we employ our' arguments.

The layout of this article is as follows. Section 2 contains some needed preliminaries and fundamental results. Sections 3 and 4 present our main results in two cases: infinite fixed delay and state-dependent delay, respectively. In Sections 5 and 6, we give applications and an example, respectively. In Section 7, we present the conclusion.

Motivation and Novelties

The incorporation of fractional-order derivatives in delay DEs provides a range of advantages, including hereditary properties, additional degrees of freedom, and other advantages of fractional modelling. As these equations are primarily used in control theory and robotics, the stability and asymptotics of these equations are of vital importance. However, stability and asymptotic analyses of fractional delay DEs are still in their early stages. Most of the current stability results on autonomous equations of this type are based on the root locus of their corresponding characteristic equations, and do not offer a universal and reliable way of assessing the stability of a given fractional delay DE.

FDEs with a time delay are widely used in natural phenomena, and the fields of science and engineering. To capture the dynamic behavior of travelling wave solutions on the basis of these equations, researchers have created algorithms with high performance for various spatial and time fractional delay DEs. However, there are still challenges to be addressed in the field of fractional delay DEs, such as the stability analysis of numerical time integration schemes and the numerical theory of the numerical scheme. Additionally, there is a need for stability and numerical simulations of travelling wave solutions, critical travelling wave solutions, and the design of compact fourth- and sixth-order schemes for fractional delay DEs with strong nonlinearity.

This paper aims to investigate the existence and controllability of solutions to FDEs with delay and random effects. While the majority of results in the literature have focused on first-order equations, some researchers produced FDE results. In our study, we obtained findings for Caputo derivatives of order (1,2) using a mild solution. Stability is a major area of research in DE theory, and over the past 20 years, stability for FDE has been a major focus of research. In order to illustrate this, we consider the prerequisites for solution stability and FDE asymptotic stability. We also examine delay fractional functional random integroevolution equations.

2. Preliminaries

We discuss a few of the abbreviations, definitions, and theorems that are used throughout the work in this part. Considering the BS $D(\Xi)$ of bounded LOs from Ξ into Ξ, where $\Theta := [0,\kappa], \kappa > 0$,

$$||\aleph||_{D(\Xi)} = \sup_{||\chi||=1} ||\aleph(U)||.$$

Let $\mathcal{C} := \mathcal{C}(I,\Xi)$ be the Banach space of continuous functions $U : \Theta \to \Xi$ with the norm

$$||U||_{\mathcal{C}} = \sup_{\chi\in\Theta} |U(\chi)|.$$

We follow to the methodology described in [28] and apply the axiomatic description of the phase space D given in [29]. Once $(D, ||.||_D)$ is defined as a seminormed linear space of functions translating $(-\infty, 0]$ into Ξ, we have

(J_1) Let $U : (-\infty,\kappa) \to \Xi, \kappa > 0$, is a continuous function on Θ and $U_0 \in D$, then, for every $\chi \in \Theta$, the following hold.

- (a) $U_\chi \in D$;
- (b) There $\exists$ a positive constant ρ, $|U(\chi)| \leq \varpi||U_\chi||_D$.

(c) There $\exists$ two functions $\beta(.), \omega(.) : \mathbf{R}^{\mathbf{m}}_{+} \to \mathbf{R}^{\mathbf{m}}_{+}$ independent of U with β continuous and bounded and ω locally bounded where:

$$||U_\chi||_D \leq \beta(\chi)\sup\{|U(\rho)| : 0 \leq \rho \leq \rho\} + \omega(\chi)||U_0||_D.$$

(J_2) For function U in (A_1), U_χ is a D-valued continuous function on Θ.
(J_3) The space D is complete.

Set

$$\varsigma = \sup\{\beta(\chi) : \chi \in \Theta\},\ and\ \omega = \sup\{\omega(\chi) : \chi \in \Theta\}.$$

Remark 1. *1. (2) is equivalent to $|\varrho_1||_D \leq \varpi||\varrho_1||_D \forall \varrho_1 \in D$.*

2. *$||.||_D$ is a seminorm, this implies that the two elements $\varrho_1, \eta \in D$ satisfy $||\varrho_1 - \eta||_D = 0$ not necessarily that $\varrho_1(\iota) = \eta(\iota)\forall \iota \leq 0$.*
3. *For all $\varrho_1, \eta \in D$ where $||\varrho_1 - \chi||_D = 0. \Rightarrow \varrho_1(0) = \eta(0)$.*
Let us present the space

$$\Xi := \{U : (-\infty, \kappa] : U|_{(\infty,0]} \in D\ and\ U|_\Theta \in C\},$$

and let $||U||_\Xi$ be the seminorm in Ξ given by

$$||U||_\Xi = ||\varrho_1||_D + ||U||_C.$$

Definition 1. *Let $\{H_q(\chi) : \chi \in \mathbf{R}^{\mathbf{m}}\}$ be a family of bounded LOs in the Banach space Ψ, which is a strongly continuous cosine family if*

- *$H_q(0) = I$.*
- *$H_q(\chi)_\eta$ is strongly continuous in χ on $\mathbf{R}^{\mathbf{m}}$ for each fixed $\eta \in \Psi$.*
- *$H_q(\chi - \rho) = 2H_q(\chi)H_q(\rho)\forall \chi, \rho \in \mathbf{R}^{\mathbf{m}}$.*

Let $\{H_q(\chi) : \chi \in \mathbf{R}^{\mathbf{m}}\}$ be a strongly continuous cosine family in Ψ. Define the sine family $\{K_q(\chi) : \chi \in \mathbf{R}^{\mathbf{m}}\}$ with

$$K_q(\chi)\eta = \int_0^\chi H_q(\rho)\eta d\rho,\ \eta \in \Xi,\ \chi \in \mathbf{R}^{\mathbf{m}}.$$

The infinitesimal generator $B_1 : \Xi \to \Xi$ of the cosine family $\{S_{(\chi)} : \chi \in \mathbf{R}^{\mathbf{m}}\}$ is defined by

$$B_1\eta = \frac{d^2}{d\chi^2} H_q(\chi)\eta|_{\chi=0},\ \eta \in D(B_1),$$

where

$$D(B_1) = \{\eta \in \Xi : H_q(.)\eta \in C^2(\mathbf{R}^{\mathbf{m}}, \Xi)\}.$$

Definition 2. *Consider the map $\phi : \Theta \times D \times \psi \to \Xi$ is random Caratheodory if*

(i) *$\chi \to \phi(\chi, U, \Delta)$, this map measurable $\forall\ U \in D$ and for all $\Delta \in \psi$.*
(ii) *$U \to \phi(\chi, U, \Delta)$ is measurable $\forall\ U \in D$ and for all $\Delta \in \psi$.*
(iii) *$\Delta \to \phi(\chi, U, \Delta)$ is measurable $\forall\ U \in D$, and almost $\chi \in \Theta$.*

Let D_Ξ be the Borel σ-algebra in separable BS Ξ. If, for each $\Pi \in D_\Xi$, $p^{-1}(\Pi) \in F$, then the map $p : \psi \to \Xi$ is a random variable. If $G(., p)$, written as $G(\Delta, p) = G(\Delta)p$, is measurable for each $p \in \Xi$, then $G : \psi \times \Xi \to \Xi$ is a random operator.

Definition 3 ([30])**.** *Let $\acute{G}$ be a mapping from ψ into 2^Ξ. A mapping $G : \{(\Delta, p) : \Delta \in \psi \wedge p \in \acute{G}(\Delta)\} \to \Xi$ is a random operator with stochastic domain $\acute{G}$ if and only if, for all closed $\Pi_1 \subseteq \Xi$, $\{\Delta \in \psi : \acute{G}(\Delta) \cap \acute{G}_1 \neq \emptyset\} \in F$, and for all open $\Pi_2 \subseteq \Xi$ and all $p \in \Xi$, $\{\Delta \in \psi : p \in \acute{G}(\Delta) \wedge G(\Delta, p) \in \Pi_2\} \in F$. G is continuous if every $G(\Delta)$ is continuous. A mapping $p : \psi \to \Xi$ is a random fixed point of G if and only if for all $\Delta \in \psi$, $p(\Delta) \in \acute{G}(\Delta)$ and $G(\Delta)p(\Delta) = p(\Delta)$ and p is measurable if for all open $\Pi_2 \subseteq \Xi$, $\{\Delta \in \psi : p(\Delta) \in \Pi_2\} \in F$.*

Lemma 1 ([30])**.** *Let $\acute{G} : \psi \to 2^{\Xi}$ be measurable for every $\Delta \in \psi$ with $\acute{G}(\Delta)$ closed, convex, and solid (i.e., $\int G(\Delta) \neq \emptyset$). We assumed the existence of a measurable $p_0 : \psi \to \Xi$ with $p_0 \in \int \acute{G}(\Delta)$ for all $\Delta \in \psi$. Assume that G is a continuous random operator with the stochastic domain $\acute{G}$; as such, $G(\Delta)p = p \neq \emptyset$ for any $\Delta \in \psi, \{p \in \acute{G}(\Delta)$. Once this happens, G has a stochastic fixed point. If the function $p(\chi, .)$ is measurable for each $\chi \in \Theta$, then the mapping of p of $\Theta \times \psi$ into Ξ is stochastic.*

Definition 4 ([31])**.** *Assume that U is a BS, and ϕ_U is the bounded subsets of Ξ. The Kuratowski measure of noncompactness is map $\mu : \psi_U \to [0, \infty)$ given by $\mu(\Pi) = \inf\{\epsilon > 0 : \Pi \subseteq \cup_{i=1}^{n}$ and $diam(\Pi_i) \leq \epsilon\}$; here $\Pi \in \psi_U$ and verifies the following properties:*

(a) $\mu(\Pi) = 0 \Leftrightarrow \bar{\Pi}$ *is compact.*
(b) $\mu(\Pi) = \mu(\bar{\Pi})$.
(c) $\tilde{\Pi} \subset \Pi \Rightarrow \mu(\tilde{\Pi}) \leq (\Pi)$.
(d) $\mu(\tilde{\Pi} + \Pi) \leq \mu(\tilde{\Pi} + \mu(\Pi))$.
(e) $\mu(\epsilon\Pi) = |\epsilon|\mu(\Pi); \epsilon \in \mathbf{R^m}$.
(f) $\mu(conv\Pi) = \mu(B)$.

Lemma 2 ([32])**.** *$\mu(g(\chi))$ is continuous on theta if and only if $g \subset C(\Theta, \Xi)$ is bounded and equicontinuous:*

$$\mu\left(\left\{ \int_{\Theta} \eta(\rho)d\rho : \eta \in g\right\}\right) \leq \int_{\Theta} \mu(g(\rho))d\rho,$$

where $g(\chi) = \{\eta(\chi) : \eta \in g\}, \chi \in \Theta$.

Lemma 3 (Gronwall lemma [28])**.** *Assume $\mu, y \in \mathcal{H}([0,1], \mathbb{R}_+)$ and let μ be increasing. If $\mathrm{u} \in \mathcal{H}([0,1], \mathbb{R}_+)$ satisfies*

$$\mathrm{u}(\omega) \leqslant \mu(\omega) + \int_0^{\omega} y(s)\mathrm{u}(s)ds,\ \omega \in [0,1],$$

then

$$\mathrm{u}(\omega) \leqslant \mu(\omega) \exp \int_0^{\omega} y(s)\mathrm{u}(s)ds,\ \omega \in [0,1].$$

Definition 5 ([30])**.** *The fractional Riemann–Liouville (RL) derivative is defined as follows.*

$$\begin{aligned} {}_aD_{\omega}^{p}\chi(\omega) = \frac{1}{\Gamma(n-p+1)}\left(\frac{d}{d\omega}\right)^{n+1} \\ \int_a^{\omega} (\omega-\tau)^{n-p}\chi(\tau)d\tau,\ n \leqslant p \leqslant n+1. \end{aligned}$$

Definition 6 ([30])**.** *Caputo fractional derivatives ${}_a^CD_{\omega}^{\alpha}\chi(\omega)$ of order $\alpha \in \mathbb{R}^+$ are defined by*

$${}_a^CD_{\omega}^{\alpha}\chi(\omega) = {}_aD_{\omega}^{\alpha}(\chi(\omega) - \sum_{\jmath=0}^{k-1} \frac{\chi^{(\jmath)}(a)}{\jmath!}(\omega-a)^{\jmath}),$$

in which $k = [\alpha] + 1$.

Definition 7 ([31])**.** *Wright function ψ_{α} is defined by*

$$\begin{aligned} \psi_{\alpha}(\kappa) &= \sum_{\jmath=0}^{\infty} \frac{(-\kappa)^{\jmath}}{\jmath!\Gamma(-\alpha\jmath+1-\alpha)} \\ &= \frac{1}{\pi}\sum_{\jmath=1}^{\infty} \frac{(-\kappa)^{\jmath}}{(\jmath-1)!}\Gamma(\jmath\alpha)\sin(\jmath\pi\alpha), \end{aligned}$$

$\alpha \in (0,1), \kappa \in \mathbb{C}$.

3. Results of Controllability for the Steady Delay Case

Definition 8. *Equation (1) is controllable on the interval $(-\infty, \kappa]$ if, for all final state $U^1(\xi)$, there $\exists$ a control $\mathcal{P}(.,\xi)$ in $L^2(\Theta,\Omega)$, such that the solution $U(\chi,\xi)$ of (1) satisfies $U(\kappa,\xi) = U^1(\xi)$.*

Definition 9. *A stochastic process $U : (-\infty,\kappa] \times \Phi \to \Xi$ is a random mild solution of Problem (1) if $U(\chi,\xi) = \varrho_1(\chi,\xi); \chi \in (-\infty,\chi], U^\infty(0,\xi) = \varrho_2(\xi)$, and the restriction of $U(.,\xi)$ to the interval Θ is continuous and verifies:*

$$\begin{aligned} U(\chi,\xi) \quad = \quad & H_q(\chi)(\varrho_1(\chi,\xi) - m(U)) + K_q(\chi)\varrho_2(\chi) + \int_0^\nu (\chi-\rho)P_q(\chi-\rho)B_1U(\chi,\xi)d\rho + \int_0^\nu (\chi-\rho)P_q(\chi-\rho) \\ & [\varphi(\chi,U_\chi(.,\xi),\xi)]d\rho + \int_0^\chi \left((\chi-\rho)P_q(\chi-\rho)\int_0^\nu B_2 f(\chi,\xi)dC_v + Bx(\rho)Cx(\rho) \right) d\rho \end{aligned}$$

Let

$$\omega = \sup\{||H_q(\chi)||_{D(\Xi)} : \chi \geq 0\}$$

and

$$\omega = \sup\{||K_q(\chi)||_{D(\Xi)} : \chi \geq 0\}.$$

The following hypotheses must be introduced:

(H_1) $H_q(\chi)$ is compact for $\chi > 0$,
(H_2) The function $\phi : \Theta \times D \times \psi \to \Psi$ is random Caratheodory.
(H_3) There $\exists$ functions $\eta : \Theta \times \phi \to \mathbf{R}_+^\mathbf{m}$ and $p : \Theta \times \psi \to \mathbf{R}_+^\mathbf{m}$ for each $\Delta \in \psi, \eta(.,\Delta)$ is continuous nondecreasing and $p(.,\Delta)$ integrable with:

$$|\phi(\chi,\mathcal{P},\Delta)| \leq p(\chi,\Delta)\eta(||\mathcal{P}||_D,\Delta) \; fora.e. \; \chi \in \Theta \; and \; each \; \mathcal{P} \in D,$$

(H_4) There $\exists$ a random function $Q : \psi \to \mathbf{R}_+^\mathbf{m} \; \{0\}$ where:

$$\omega(1+\kappa\omega\zeta(||\varrho_1||_D + \eta(D,\Delta||p||_{L^1}) + \kappa\omega\zeta||\eta^1|| + \omega'(1+\kappa\omega\zeta)|\varrho_2| \leq Q(\Delta)$$

where

$$D := \zeta Q(\Delta) + \sigma||\varrho_1||_D,$$

(H_5) The linear $\daleth : L^2(\Theta,\Omega) \to \Psi$ given by

$$\daleth\mathcal{P} = \int_0^\kappa H_q(\kappa-\rho)B_2\mathcal{P}(\rho,\Delta)d\rho$$

has an inverse operator $\daleth^{-1}$ in $L^2(\Theta,\Omega)/\ker\daleth$, and there $\exists$ a positive constant ζ, such that $||B_2\daleth^{-1}|| \leq \zeta$,
(H_6) for each $\Delta \in \psi, \varrho(.,\Delta)$ is continuous and $\chi, \varrho_1(\chi,.)$ and $\Delta \in \psi, \varrho_2(\Delta)$ are measurable.

Theorem 1. *Assume that (H_1)–(H_2) are met; then Problem (1) is controllable on Θ.*

Proof. Define the control:

$$\begin{aligned} \mathcal{P}(\chi,\Delta) \quad = \quad & \daleth^{-1}\Big(p^1(\Theta) - H_q(\chi)(\varrho_1(\chi,\xi) - m(U)) - K_q(\chi)\varrho_2(\chi) - \int_0^\nu (\chi-\rho)P_q(\chi-\rho)B_1U(\chi,\xi)d\rho \\ & - \int_0^\nu (\chi-\rho)P_q(\chi-\rho)[\varphi(\chi,U_\chi(.,\xi),\xi)]d\rho \Big). \end{aligned}$$

The operator $I : \psi \times \Xi \to \Xi$ defined by $(I(\xi)p)(\chi) = \varrho_1(\chi,\xi)$, if $\chi \in (-\infty,0]$, and for $\chi \in \Theta$:

$$
\begin{aligned}
U(\chi,\xi) \;=\;& H_q(\chi)(\varrho_1(\chi,\xi)-m(U)) + K_q(\chi)\varrho_2(\chi) + \int_0^{\nu}(\chi-\rho)P_q(\chi-\rho)B_1U(\chi,\xi)d\rho + \int_0^{\nu}(\chi-\rho)P_q(\chi-\rho) \\
& [\varphi(\chi,U_\chi(.,\xi),\xi)]d\rho + \int_0^{\chi}\bigg((\chi-\rho)P_q(\chi-\rho)B_{\daleth}^{-1}\bigg(U^1(\Theta) - H_q(\chi)(\varrho_1(\chi,\xi)-m(U)) \\
& -K_q(\chi)\varrho_2(\chi) - \int_0^{\nu}(\chi-\rho)P_q(\chi-s)B_1U(\chi,\xi)d\rho - \int_0^{\nu}(\chi-\rho)P_q(\chi-\rho) \\
& [\varphi(\chi,U_\chi(.,\xi),\xi)]dC_\rho\bigg) + Bx(\rho)Cx(\rho)\bigg)d\rho. \qquad (3)
\end{aligned}
$$

We use (H_5) to show that I has a fixed point $U(\chi,\xi)$ that is a mild solution of (1). This suggests that Issue (1) is manageable on Θ. Additionally, we establish that I is a random operator. To prove this, we show that $I(.)(U):\psi\to\Xi$ is a random variable for any $U\in\Xi$. The measurement of $I(.)(U):\psi\to\Xi$ is then shown. Because of the assumptions (H_2) and (H_6), the mapping $\varphi(\chi,U,.),\chi\in\Theta,U\in\Xi$ is measurable. Assume that $D:\psi\to 2^{\Xi}$ is provided by:

$$D(\xi)=\{U\in\Xi:\|U\|_{\Xi}\le Q(\xi)\}.$$

$D(\chi)$ is bounded, convex, closed, and solid for all $\xi\in\psi$. So, D is measurable. Suppose $\xi\in\psi$ is fixed; then, $U\in D(\xi)$ and by (A_1), we obtain:

$$
\begin{aligned}
\|U_\rho\|_D &\le \beta(\rho)|U(\rho)|W + \omega(\rho)\|U_0\|_D \\
&\le \zeta_\kappa|U(\rho)| + \omega_\kappa\|\varrho_1\|_D,
\end{aligned}
$$

and via (H_3) and (H_4), we have

$$
\begin{aligned}
|(I(\xi)U)(\chi)| \;\le\;& \omega\|\varrho_1\|_D + \omega'|\varrho_2| + \omega\int_0^{\chi}|\varphi(\rho,U_\rho,\xi)|d\rho + \omega\zeta\int_0^{\chi}|U^1(\xi)| + \omega\|\varrho_1\|_D \\
& +\omega'|\varrho_2|d\rho\omega\zeta\int_0^{\chi}\int_0^{\kappa}\|H_q(\epsilon-\rho)\||\varphi(\epsilon,U_\epsilon(.,\xi),\xi)|d\epsilon d\rho \\
\le\;& \omega\|\varrho_1\|_D + \omega'|\varrho_2| + \omega\int_0^{\kappa}p(\varrho,\xi)\chi(\|U_\chi\|_D,\xi)d\rho + \kappa\omega\zeta|U^1(\xi)| + \kappa\omega^2\zeta\|\varrho_1\|_D + \kappa\omega\omega'\zeta|\varrho_2| \\
& +\kappa\omega^2\zeta\int_0^{\kappa}p(\epsilon,\xi)U(\|U_\epsilon\|_D,\omega)d\epsilon \\
\le\;& \omega(1+\kappa\omega\zeta)\|\varrho_1|_D + \kappa\omega\zeta|U^1(\xi)| + \omega'(1+\kappa\omega\zeta)|\varrho_2| + \omega(1+\kappa\omega\zeta)\int_0^{\kappa}p(\rho,\xi)U(\|p_\rho\|_D,\xi)d\rho \\
\le\;& \omega(1+\kappa\omega\zeta)\bigg(\|\varrho_1|_D + U(D_\kappa,\xi)\int_0^{\kappa}p(\rho,\xi)d\rho\bigg)\kappa\omega\zeta\|U^1(\xi)\| + \omega'(1+\kappa\omega\zeta)|\varrho_2|.
\end{aligned}
$$

Set

$$D_\kappa := \zeta_\kappa Q(\xi) + \rho_\kappa\|\varrho_1\|_D.$$

Then, we have

$$|(I(\xi)U(\chi)| \;\le\; \omega(1+\kappa\omega\zeta)\bigg(\|\varrho_1|_D + U(D_\kappa,\xi)\int_0^{\kappa}p(\rho,\xi)d\rho\bigg)\kappa\omega\zeta\|p^1(\xi)\| + \omega'|\varrho_2|(1+\kappa\omega\zeta).$$

Thus

$$
\begin{aligned}
\|I(\xi)U\|_{\Xi} &\le \omega(1+\kappa\omega\zeta)(\|\varrho_1\|_D + U(D_\kappa,\omega)\|\varrho\|_L^1)\kappa\omega\zeta|U^1(\xi)| + \omega'(1+\kappa\omega\zeta)|\varrho_2| \\
&\le Q(\omega).
\end{aligned}
$$

Thus, we deduce that, with stochastic domain D, I is a random operator and $I(\xi):D(\xi)\to D(\xi)$ for each $\xi\in\psi$.

Claim 1: I is continuous.

Assume that U^n is a sequence where $U^n\to U$ in Y. Then,

$$|(I(\xi)U^n)(\chi)-(I(\xi)U(\chi)| \leq \omega\int_0^\chi |\varphi(\rho,U^n_\rho,\xi)-\varphi(\rho,U_\rho,\xi)|d\epsilon d\rho+\zeta\omega\int_0^\chi\int_0^\kappa \|H_q(\kappa-\epsilon)\| |\varphi(\epsilon,U^n_\epsilon(.,\xi)-\varphi(\epsilon,U_\epsilon,\xi)|d\epsilon d\rho$$
$$\leq \omega\int_0^\chi |\varphi(\rho,U^n_\rho,\xi)-\varphi(\rho,U_\rho,\xi)|d\epsilon d\rho+\kappa\omega^2\zeta\int_0^\kappa |\varphi(\epsilon,U^n_\epsilon(.,\xi)-\varphi(\epsilon,U_\epsilon,\xi)|d\epsilon$$
$$\leq \omega(1+\kappa\omega\zeta)\int_0^\kappa |\varphi(\epsilon,U^n_\epsilon(.,\xi)-\varphi(\epsilon,U_\epsilon,\xi)|d\epsilon$$

As $\varphi(\chi,.,\xi)$ is continuous, we obtain

$$\|\varphi(.,U^n,\xi)-\varphi(.,U,\xi)\|_{L^1}\to 0 \; as \; n\to+\infty.$$

I is continuous.

Claim 2: we show that $\xi\in\psi,\{U\in D(\xi):I(\xi)U=U\}\neq\varnothing$ by applying Schauder's theorem.

(a) I maps bounded sets into equicontinuous sets in $D(\xi)$.
Assume that $\epsilon_1,\epsilon_2\in[0,\kappa]$ with $\epsilon_2>\epsilon_1, D(\xi)$ are a bounded set, as in Claim 2, and $U\in D(\xi)$. Now,

$$|(I(\xi)U)(\epsilon_2)-(I(\xi)U)(\epsilon_1)| \leq \|H_q(\epsilon_2)-H_q(\epsilon_1)\|_{D(\Psi)}\|\varrho_1\|_D+\|K_q(\epsilon_2)-K_q(\epsilon_1)\|_{D(\Psi)}|\varrho+\int_0^{\epsilon_1}\|H_q(\epsilon_2 -\rho)-H_q(\epsilon_1-\rho)\|_{D(\Psi)}|\varphi(\rho,U_\rho,\xi)|d\rho+\int_{\epsilon_1}^{\epsilon_2}\|C(\epsilon_2-\rho)\|_{D(\Psi)}|\varphi(\rho,U_\rho,\xi)|d\rho$$
$$+\zeta\int_0^{\epsilon_1}\|H_q(\epsilon_2-\rho)-H_q(\epsilon_1-\rho)\|_{D(\Psi)}\times[|U^1(\xi)|+\|H_q(\kappa)\|_{D(\Psi)}\|\varrho_1\|_D+ \|K_q(\kappa)\|_{D(\Psi)}|\varrho_2|]d\rho+\zeta\int_0^{\epsilon_1}\|H_q(\epsilon_2-\rho)-H_q(\epsilon_1-\rho)\|_{D(\Psi)}\int_0^\kappa\|H_q(\kappa-\epsilon)\|_{D(\Psi)}|$$
$$\varphi(\epsilon,U_\epsilon(.,\xi),\xi)|d\epsilon d\rho+\zeta\int_{\epsilon_1}^{\epsilon_2}\|C(\epsilon_2-\rho)\|_{D(\Psi)}[|U^1(\xi)|+\|H_q(\kappa)\|_{D(\Psi)}\|\varrho_1\|_D+\|H_q(\kappa)\|_{D(\Psi)}|\varrho_2|]d\rho$$
$$+\zeta\int_{\epsilon_1}^{\epsilon_2}\|C(\epsilon_2-\rho)\|_{D(\Psi)}\int_0^\kappa\|H_q(\kappa-\epsilon)\|_{D(\Psi)}|\varphi(\epsilon,U_\epsilon(.,\xi)\xi)|d\epsilon d\rho$$
$$\leq \|H_q(\epsilon-\rho)-H_q(\epsilon_1-\rho)\|_{D(\Psi)}\|\varrho_1\|_D+\|K_q(\epsilon_2)-K_q(\epsilon_1)\|_{D(\Psi)}|\varrho_2|U(D_\kappa,\xi)\int_0^{\epsilon_1}\|H_q(\epsilon_2-\rho) -H_q(\epsilon_1-\rho)\|_{D(\Psi)}U(\rho,\xi)d\rho+\omega x(D_\kappa,\xi)\int_{\epsilon_1}^{\epsilon_2}p(\rho,\xi)d\rho+\zeta\int_0^{\epsilon_1}\|H_q(\epsilon_2-\rho)-H_q(\epsilon_1-\rho)\|_{D(\Psi)}$$
$$\times[|U^1(\xi)|+\|H_q(\kappa)\|_{D(\Psi)}\|\varrho_1\|_D+\|K_q(\kappa)\|_{D(\Psi)}|\varrho_2|]d\rho+\zeta\omega U(D_\kappa,\xi)\int_0^{\epsilon_1}\|H_q(\epsilon_2-\rho) -H_q(\epsilon_1-\rho\|_{D(\Psi)}\int_0^\kappa U(\epsilon,\xi)d\epsilon d\rho\zeta\omega\int_{\epsilon_1}^{\epsilon_2}(|U^1(\xi)|+\|H_q(\kappa)\|_{D(\Psi)}\|\varrho_1\|_D+\|K_q(\kappa)\|_{D(\Psi)}|\varrho_2|$$
$$+\omega U(D_\kappa,\xi)\int_0^\kappa U(\epsilon,\xi)d\epsilon d\rho.$$

In the above inequality, right-hand side tends to zero as $\epsilon_2-\epsilon_1\to 0$, since $H_q(\chi),K_q(\chi)$ are compact for $\chi>0$ and strongly continuous; then, we obtain the continuity in the uniform operator topology [12,33].

(b) Assume that $\chi\in[0,\kappa]$ is, fixed and $U\in D(\xi)$: by assumption $(H_3),(H_5)$; since $H_q(\chi)$ is compact, the set

$$\left\{\int_0^\chi H_q(\chi-\rho)\varphi(\rho,U_\rho(.,\xi),\xi)d\rho\int_0^\chi H_q(\chi-\rho)B_2\mathfrak{p}(\chi,\xi)d\rho\right\}$$

is precompact in Ψ; then, the set

$$\left\{H_q(\chi)(\varrho_1(\chi,\xi)-m(U))+K_q(\chi)\varrho_2(\chi)+\int_0^\chi(\chi-\rho)P_q(\chi-s)B_1U(\chi,\xi)d\rho+\int_0^\chi(\chi-\rho)P_q(\chi-s)\right.$$
$$\left.[\varphi(\chi,U_2(.,\xi),\xi)]d\rho+\int_0^\chi\left((\chi-\rho)P_q(\chi-\rho)\int_0^\nu B_2f(\chi,\xi)dC_v+Bx(\rho)Cx(\rho)\right)d\rho\right\}$$

is precompact in Ψ. Thus, $I(\xi):D(\xi)\to D(\xi)$ is continuous. Through compact Schauder's theorem, we obtain that $I(\xi)$ has a fixed point $U(\xi)$ in $D(\xi)$. Since $\cap_{\xi\in\psi}D(\xi)\neq\varnothing$, and a measurable selector of $\int D$ exists, then via Lemma 4, I has a stochastic fixed point $U^*(\xi)$, which is a random mild solution of (1).

□

4. Results for State-Dependent Delay Case Controllability

Definition 10. *A stochastic process $U:(-\infty,\kappa]\times\psi\to\Psi$ is a random mild solution of Problem (2) if $U(\chi,\xi)=\varrho(\chi,\xi);\chi\in(-\infty,0],U'(0,\xi)=\varpi_2(\xi)$, and the restriction of $U(.,\xi)$ to the interval Θ is continuous and verifies the following equation:*

$$\begin{aligned}U(\chi,\xi) \quad = \quad & H_q(\chi)(\varrho_1(\chi,\xi)-m(U))+K_q(\chi)\varrho_2(\chi)+\int_0^\chi(\chi-\rho)P_q(\chi-s)B_1U(\chi,\xi)d\rho+\int_0^\chi(\chi-\rho)\\ & P_q(\chi-\rho)[\varphi(\chi,U_2(.,\xi),\xi)]d\rho+\int_0^\chi\left((\chi-\rho)P_q(\chi-\rho)\int_0^\nu B_2f(\chi,\xi)dC_v+BU(\rho)CU(\rho)\right)d\rho\end{aligned}$$

Set

$$Q(\theta^{-1})=\{\theta(\rho,\varrho_2):(\rho,\varrho_2)\in\Theta\times D,\theta(\rho,\varrho_2)\leq 0\}.$$

Suppose that $\theta:\Theta\to(-\infty,\kappa]$ is continuous. (H_{ϱ_1}) the function $\chi\to\varrho_{1\chi}$ is continuous from $Q(\theta^{-1})$ into D, and there exists a continuous and bounded function $\beta^{\varrho_1}:Q(\theta^-)\to(0,\infty)$ where $\beta^{\varrho_1}(\chi)||\varrho_1||_D$ for every $\chi\in Q(\theta^-)$.

Remark 2 ([28]). *Hypothesis H_{ϱ_1} is satisfied through continuous and bounded functions.*

Lemma 4 ([34]). *If $U:(-\infty,\kappa]\to\Psi$ is a function, such that $U_0=\varrho_1$, then*

$$\|U_\varrho\|_D\leq(\omega_\kappa+\beta^{\varrho_1})\|\varrho_1\|_D+\zeta_\kappa\sup\{|U(i)|;I\in[0,\max\{0,\rho\}]\},\varrho\in Q(\theta^-)\bigcup\Theta.$$

where $\beta^{\varrho_1}=\sup_{\chi\in Q(\theta^{-1})}\beta^{\varrho_1}(\chi)$.

The hypotheses

(H_1') $H_q(\chi)$ is compact for $\chi>0$ in Ψ.

(H_2') The function $\varphi:\Theta\times D\times\psi\to\Psi$ is random Caratheodory.

(H_3') There $\exists$ a function $\eta:\Theta\times\psi\to\mathbf{R}_+^{\mathbf{m}}$ and $p:\Theta\times\to\mathbf{R}_+^{\mathbf{m}}$, such that $\xi\in\psi,U(.,\xi)$ is a continuous nondecreasing function and $p(.,\xi)$ integrable with:

$$|\phi(\chi,\mathcal{P},\Delta)|\leq p(\chi,\Delta)\eta(||\mathcal{P}||_D,\Delta)\ for\ a.e.\ \chi\in\Theta\ and\ each\ \mathcal{P}\in D,$$

(H_4') There $\exists$ a random function $\alpha:\Theta\times\psi\to\mathbf{R}_+^{\mathbf{m}}$ with $\alpha(.,\chi)\in L^1(\Theta,\mathbf{R}_+^{\mathbf{m}})$ for each $\xi\in\psi$ such that for any bounded $B\subseteq\Psi$.

$$\mu(\varphi(\chi,B,\chi))\leq\alpha(\chi,\xi)\mu(B).$$

(H_5') There $\exists$ a random function $Q:\psi\to\mathbf{R}_+^{\mathbf{m}}\ \{0\}$ where:

$$\omega(1+\kappa\omega\lambda)\left(\|\varrho_1\|_D+\eta(\omega_\kappa+\beta^{\varrho_1})\|\varrho_1\|_D+\zeta_\kappa Q(\chi),\chi)\int_0^\kappa p(\rho,\chi)d\rho\right)+\kappa\omega\lambda\|U^1(\chi)\|+\omega'(1+\kappa\omega\lambda)|\varrho_2|\leq Q(\xi).$$

(H_6') The linear LO $\daleth : L^2(\Theta, \Omega) \to \Psi$ defined by:

$$\daleth U = \int_0^{\kappa} H_q(\kappa - \rho) B_2 U(\rho, \xi) d\rho$$

has an inverse operator $\daleth^{-1}$ that takes values in $L^2(\Theta, \Omega)/ker\daleth$, and there $\exists$ a positive constant λ, such that $\|B_2\daleth^{-1}\| \leq \lambda$.

(H_7') For each $\Delta \in \psi, \varrho(., \Delta)$ is continuous and, for each $\chi, \varrho_1(\chi, .)$, is measurable, and, for each $\Delta \in \psi, \varrho_2(\Delta)$, is measurable.

Theorem 2. *Suppose that (H_1')–(H_7') and (H_{ϱ_1}) hold. If*

$$\omega(1 + \omega\lambda\kappa) \int_0^{\kappa} \alpha(\rho)\xi(\rho) d\rho < 1. \tag{4}$$

Therefore, Theta can be used to control Random Problem (2).

Proof. Using (H_6), the control is

$$U(\chi, \xi) = \daleth^{-1}\Big(U^1(\xi) - H_q(\kappa)\varrho_1(0, \xi) - K_q(\kappa)\varrho_2(\xi) - \int_0^{\kappa} H_q(\kappa - \rho) B_2 U(\chi, \xi) d\rho - \int_0^{\kappa} H_q(\kappa - \rho)\varphi(\rho, U_{\theta(\rho, U_\rho)}(., \xi), \xi) d\rho\Big).$$

The operator $I : \psi \times \Xi \to \Xi$ given by: $(I(\xi)U)(\chi) = \varrho_1(\chi, \xi)$, if $\chi \in (-\infty, 0]$, and for $\chi \in \Theta$:

$$\begin{aligned} U(\chi, \xi) \quad = \quad & H_q(\chi)(\varrho_1(\chi, \xi) - m(U)) + K_q(\chi)\varrho_2(\chi) + \int_0^{\chi} (\chi - \rho) P_q(\chi - s) B_1 U(\chi, \xi) d\rho + \int_0^{\chi} (\chi - \rho) P_q(\chi - \rho) \\ & [\varphi(\chi, U_2(., \xi), \xi)] d\rho + \int_0^{\chi} \Big((\chi - \rho) P_q(\chi - \rho) B_{\daleth}^{-1} \Big(p^1(\Theta) - H_q(\chi)(\varrho_1(\chi, \xi) - m(U)) \\ & - K_q(\chi)\varrho_2(\chi) - \int_0^{\chi} (\chi - \rho) P_q(\chi - s)[\varphi(\chi, U_2(., \xi), \xi)] dC_\rho \Big) + BU(\rho) CU(\rho) \Big) d\rho \end{aligned} \tag{5}$$

This proves that I has a fixed point $U(\chi, \xi)$, and that (2) is controllable. Moreover, we demonstrate that I is a random operator by showing that, for any $U \in \Xi, I(.)(U) : \psi \to \Xi$ is a random variable. We also show that $I(.)(U) : \psi \to \Xi$ is measurable, as a mapping $\varphi(\chi, U, .), \chi \in \Theta, U \in \Xi$ is measurable through assumptions (H_2') and (H_6'). Assume that $D : \psi \to 2^{\Xi}$ is given by:

$$D(\xi) = \{U \in \Xi : \|U\|_{\Xi} \leq Q(\xi)\}.$$

$D(\chi)$ is bounded, convex, closed and solid for all $\xi \in \psi$. Then, D is measurable. Let $\xi \in \psi$ be fixed; if $p \in D(\xi)$, then

$$\|U_{\varrho(\chi, U_\chi)}\|_D \quad = \quad (\omega_\kappa + L^{\varrho_1})\|\varrho_1\|_D + \zeta_\kappa Q(\xi),$$

For each $U \in D(\xi)$, (H_3'), and (H_4'), for each $\chi \in \Theta$, we have

$$
\begin{aligned}
|(I(\xi)U)(\chi)| &\le \omega\|\varrho_1\|_D + \omega'|\varrho_2| + \omega\int_0^{\chi}|\varphi(\rho, U_{\varrho(\chi,U_\chi)},\xi)|d\rho + \omega\zeta\int_0^{\chi}|U^1(\xi)| + \omega\|\varrho_1\|_D \\
&\quad +\omega'|\varrho_2|d\rho\omega\zeta\int_0^{\chi}\int_0^{\kappa}\|H_q(\epsilon-\rho)\|\,|\varphi(\epsilon, U_{\varrho(\chi,U_\chi)}(.,\xi),\xi)|d\epsilon d\rho \\
&\le \omega\|\varrho_1\|_D + \omega'|\varrho_2| + \omega\int_0^{\kappa}p(\varrho,\xi)\eta(\|U_\chi\|_D,\xi)d\rho + \kappa\omega\zeta|U^1(\xi)| + \kappa\omega^2\zeta\|\varrho_1\|_D + \kappa\omega\omega'\zeta|\varrho_2| \\
&\quad +\kappa\omega^2\zeta\int_0^{\kappa}p(\epsilon,\xi)\eta(\|p_\epsilon\|_D,\omega)d\epsilon \\
&\le \omega(1+\kappa\omega\lambda)\|\varrho_1|_D + \kappa\omega\lambda|U^1(\xi)| + \omega'(1+\kappa\omega\lambda)|\varrho_2| + \omega(1+\kappa\omega\lambda)\int_0^{\kappa}p(\rho,\xi)\eta(\|U_{\varrho(\chi,U_\chi)}\|_D,\xi)d\rho \\
&\le \omega(1+\kappa\omega\lambda)\times\left(\|\varrho_1\|_D + \eta(\omega_\kappa+\beta^{\varrho_1})\|\varrho_1\|_D + \zeta_\kappa Q(\xi),\xi)\int_0^{\kappa}p(\rho,\xi)d\rho\right)\kappa\omega\lambda\|U^1(\xi)\| \\
&\quad +\omega'(1+\kappa\omega\lambda)|\varrho_2|.
\end{aligned}
$$

Thus, with stochastic domain D, I is a random operator and $I(\xi): D(\xi) \to D(\xi)$ for each $\xi \in \psi$.

Claim 1: I is continuous.

Suppose that U^n is a sequence where $U^n \to U$ in Ξ. Then,

$$
\begin{aligned}
|(I(\xi)U^n)(\chi) - (I(\xi)U(\chi)| &\le \omega\int_0^{\chi}|\varphi(\rho, U_\vartheta(\chi, U_\chi^n)^n,\xi) - \varphi(\rho, U_{\vartheta(\chi,U_\chi)},\xi)|d\epsilon d\rho \\
&\quad +\zeta\omega\int_0^{\chi}\int_0^{\kappa}\|H_q(\kappa-\epsilon)\|\,|\varphi(\epsilon, p_\epsilon^n(.,\xi) - \varphi(\epsilon, p_\epsilon,\xi))|d\epsilon d\rho \\
&\le \omega\int_0^{\chi}|\varphi(\rho, U_\vartheta(\chi, U_\chi^n),\xi)^n) - \varphi(\rho, U_\vartheta(\chi, U_\chi),\xi))|d\epsilon d\rho \\
&\quad \kappa\omega^2\zeta\int_0^{\kappa}|\varphi(\epsilon, U_\vartheta(\chi, U_\chi^n)^n(.,\xi)) - \varphi(\epsilon U_\vartheta(\chi, U_\chi),\xi)|d\epsilon \\
&\le \omega(1+\kappa\omega\zeta)\int_0^{\kappa}|\varphi(\epsilon, U^n_{\vartheta(\chi,U_\chi^n)}(.,\xi) - \varphi(\epsilon U_\vartheta(\chi, U_\chi),\xi)|d\epsilon
\end{aligned}
$$

As $\varphi(\chi, ., \xi)$ is continuous, we have

$$\|\varphi(., U^n, \xi) - \varphi(., U, \xi)\|_\Xi \to 0 \; as \; n \to +\infty.$$

I is continuous.

Claim 2: We show that $\xi \in \psi, \{U \in D(\xi) : I(\xi)U = U\} \ne \varnothing$. We apply Mönch fixed point theorem [35,36].

(a) In $D(\xi)$, I transforms bounded sets into equicontinuous sets.
Let $\epsilon_1, \epsilon_2 \in [0, \kappa]$ with $\epsilon_2 > \epsilon_1, D(\xi)$ be a bounded set as in Claim 2, and $U \in D(\xi)$. Then,

$$
\begin{aligned}
|(I(\xi)U)(\epsilon_2) - (I(\xi)U)(\epsilon_1)| &\le \|H_q(\epsilon_2) - H_q(\epsilon_1)\|_{D(\Psi)}\|\varrho_1\|_D + \|K_q(\epsilon_2) - K_q(\epsilon_1)\|_{D(\Psi)}|\varrho \\
&\quad + \int_0^{\epsilon_1}\|H_q(\epsilon_2-\rho) - H_q(\epsilon_1-\rho)\|_{D(\Psi)}|\varphi(\rho, U_{\vartheta(\chi,U_\chi)},\xi)|d\rho \\
&\quad + \int_{\epsilon_1}^{\epsilon_2}\|C(\epsilon_2-\rho)\|_{D(\Psi)}|\varphi(\chi, U_{\vartheta(\chi,U_\chi)},\xi)|d\rho + \zeta\int_0^{\epsilon_1}\|H_q(\epsilon_2-\rho) - H_q(\epsilon_1-\rho)\|_{D(\Psi)} \\
&\quad \times[|p^1(\xi)| + \|H_q(\kappa)\|_{D(\Psi)}\|\varrho_1\|_D + \|K_q(\kappa)\|_{D(\Psi)}|\varrho_2|]d\rho \\
&\quad +\zeta\int_0^{\epsilon_1}\|H_q(\epsilon_2-\rho) - S_1(\epsilon_1-\rho)\|_{D(\Psi)}\int_0^{\kappa}\|H_q(\kappa-\epsilon)\|_{D(\Psi)}|\varphi(\epsilon, U_{\vartheta(\chi,U_\chi)},\xi)|d\epsilon d\rho \\
&\quad +\zeta\int_{\epsilon_1}^{\epsilon_2}\|C(\epsilon_2-\rho)\|_{D(\Psi)}[|U^1(\xi)| + \|H_q(\kappa)\|_{D(\Psi)}\|\varrho_1\|_D + \|H_q(\kappa)\|_{D(\Psi)}|\varrho_2|]d\rho \\
&\quad +\zeta\int_{\epsilon_1}^{\epsilon_2}\|C(\epsilon_2-\rho)\|_{D(\Psi)}\int_0^{\kappa}\|H_q(\kappa-\epsilon)\|_{D(\Psi)}|\varphi(\epsilon, U_{\vartheta(\chi,U_\chi)},\xi)|d\epsilon d\rho
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|(I(\xi)U)(\epsilon_2) - (I(\xi)U)(\epsilon_1)| \leq\ & |H_q(\epsilon_2) - H_q(\epsilon_1)| \|\varrho_1\|_D + \|K_q(\epsilon_2) - K_q(\epsilon_1)\|_{D(\psi)} |\varrho_2| \\
& + \int_0^{\epsilon_1} \|H_q(\epsilon_2 - \rho) - H_q(\epsilon_1 - \rho)\|_{D(\psi)} \varphi(\rho, U_{\vartheta(\chi, U_\chi)}, \xi) d\rho + \int_{\epsilon_1}^{\epsilon_2} \|H_q(\epsilon_2 - \rho)\|_{D(\psi)} \varphi(\rho, U_{\vartheta(\chi, U_\chi)}, \xi) d\rho \\
& + \lambda \int_0^{\epsilon_1} \|H_q(\epsilon_2 - \rho) - H_q(\epsilon_1 - \rho)\|_{D(\psi)} [\|p^1(\xi)\| + \|H_q(\kappa)\|_{D(\psi)} |\varrho_1(0, \xi)|] d\rho \\
& + \lambda \int_0^{\epsilon_1} \|H_q(\epsilon_2 - \rho) - H_q(\epsilon_1 - \rho)\|_{D(\psi)} \eta((\omega_\kappa + \beta^{\varrho_1}) \|\varrho_1\|_D + \zeta_\kappa Q(\xi)) \times \int_0^\kappa p(\epsilon, \xi) d\epsilon d\rho + \\
& \lambda \omega \int_{\epsilon_1}^{\epsilon_2} \|U^1\| + \|H_q(\kappa)\|_{D(\psi)} |\varrho_1(0, \xi)| + \omega \eta((\omega_\kappa + \beta^{\varrho_1}) \|\varrho_1\|_D + \zeta_\kappa Q(\xi)) \times \int_0^\kappa p(\epsilon, \xi) d\epsilon d\rho
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|(I(\xi)U)(\epsilon_2) - (I(\xi)U)(\epsilon_1)| \leq\ & |H_q(\epsilon_2) - H_q(\epsilon_1)|_{D(\psi)} \|\varrho_1\|_D + \|K_q(\epsilon_2) - K_q(\epsilon_1)\|_{D(\psi)} |\varrho_2| \\
& + \eta(\omega_\kappa + \beta^{\varrho_1} \|\varrho_1\|_D + \zeta_\kappa Q(\varpi)) \int_0^{\epsilon_1} \|H_q(\epsilon_2 - \rho) - H_q(\epsilon_1 - \rho)\|_{D(\psi)} p(\chi, \xi) d\rho \\
& + \eta((\omega_\kappa + \beta^{\varrho_1} \|\varrho_1\|_D + \zeta_\kappa Q(\varpi), \varpi) \int_{\epsilon_1}^{\epsilon_2} p(\chi, \xi) d\rho \\
& + \lambda \int_0^{\epsilon_1} \|H_q(\epsilon_2 - \rho) - H_q(\epsilon_1 - \rho)\|_{D(\psi)} [\|U^1(\xi)\| + \|H_q(\kappa)\|_{D(\psi)} |\varrho_1(0, \xi)|] d\rho \\
& + \lambda \int_0^{\epsilon_1} \|H_q(\epsilon_2 - \rho) - H_q(\epsilon_1 - \rho)\|_{D(\psi)} \eta((\omega_\kappa + \beta^{\varrho_1}) \|\varrho_1\|_D + \zeta_\kappa Q(\xi)) \\
& \times \int_0^\kappa p(\epsilon, \xi) d\epsilon d\rho + \lambda \omega \int_{\epsilon_1}^{\epsilon_2} \|U^1(\varpi)\| + \|H_q(\kappa)\|_{D(\psi)} |\varrho_1(0, \xi)| + \\
& \omega \eta((\omega_\kappa + \beta^{\varrho_1}) \|\varrho_1\|_D + \zeta_\kappa Q(\xi)) \times \int_0^\kappa p(\epsilon, \xi) d\epsilon d\rho
\end{aligned}
$$

In the previous inequality, the right-hand side went to zero as $\epsilon_2 - \epsilon_1 \to 0$, $H_q(\chi), K_q(\chi)$ are a strongly continuous operator, and $H_q(\chi)$ and $K_q(\chi)$ for $\chi > 0$ are compact, which implies that uniform operator topology is continuous. Suppose that $\xi \in \psi$ is fixed.

(b) Suppose that Λ is a subset of $D(\xi)$ where $\Lambda \subset \overline{conv}(I(\Lambda) \bigcup \{0\})$. Λ is bounded and equicontinuous, and function $\chi \to v(\chi) = \varsigma(\Lambda(\chi))$ is continuous on $(-\infty, \kappa]$. Via (H_2), and by considering the characteristics of the measure Λ, we have $\chi \in (-\infty, \kappa]$:

$$
\begin{aligned}
v &\leq \varsigma(I(\Lambda))(\chi)\bigcup\{0\}) \\
&\leq \varsigma(I(\Lambda)(\chi)) \\
&\leq \varsigma(H_q(\chi)\varrho_1(0,\xi)) + \varsigma(K_q(\chi)\varrho_2(\xi)) + \varsigma\bigg(\int_0^\chi H_q(\chi-\rho)\varphi(\epsilon, U_{\vartheta(\chi,U_\chi)}(.,\xi)d\rho\bigg) + \omega\lambda\int_0^\chi \varsigma(U^1(\xi) \\
&\quad -H_q(\kappa)\varrho_1(0,\xi) - K_q(\kappa)\varrho_2(\xi)) + \varsigma\bigg(\int_0^\kappa H_q(\kappa-\epsilon)\varphi(\epsilon, U_{\vartheta(\chi,U_\chi)}(.,\xi),\xi\bigg)d\rho \\
&\leq \omega\int_0^\chi \varsigma(\varphi(\rho, U_{\vartheta(\chi,U_\chi)}(.,\xi),\xi))d\rho\omega\lambda\int_0^\chi\int_0^\kappa \varsigma(H_q(\kappa-\epsilon)\varphi(\epsilon, U_{\vartheta(\chi,U_\chi)}(.,\xi),\xi)d\epsilon d\rho \\
&\leq \omega\int_0^\chi \alpha(\rho)\varsigma(\{U_{\vartheta(\chi,p_\chi)} : p\in\Lambda\})d\rho\omega\lambda\int_0^\chi\int_0^\kappa \varsigma(H_q(\kappa-\epsilon)\varphi(\epsilon, U_{\vartheta(\chi,U_\chi)}(.,\xi),\xi)d\epsilon d\rho \\
&\leq \omega\int_0^\chi \gamma(\rho)\zeta(\rho)\sup_{0\leq\epsilon\leq\rho}\varsigma(\Lambda(\epsilon))\rho + \omega^2\lambda\int_0^\chi\int_0^\kappa \varsigma(\varphi(\epsilon, U_{\vartheta(\chi,U_\chi)},\xi)d\epsilon d\rho \\
&\leq \omega\int_0^\chi \gamma(\rho)\zeta(\rho)\varsigma(\Lambda(\rho))d\rho + \omega^2\lambda\kappa\int_0^\kappa \alpha(\epsilon)\varsigma(\varphi(\{U_{\vartheta(\chi,U_\chi)} : U\in\Lambda)d\epsilon \\
&\leq \omega\int_0^\chi v(\rho)\alpha(\rho)\zeta(\rho)d\rho + \omega^2\lambda\kappa\int_0^\kappa \alpha(\epsilon)\zeta(\epsilon)\varsigma(\Lambda(\epsilon))d\epsilon \\
&= \omega\int_0^\chi \alpha(\rho)\zeta(\rho)v(\rho)d\rho + \omega^2\lambda\kappa\int_0^\kappa \alpha(\epsilon)\zeta(\epsilon)v(\epsilon))d\epsilon \\
&\leq \omega\int_0^\chi \alpha(\rho)\zeta(\rho)v(\rho)d\rho + \omega^2\lambda\kappa\int_0^\kappa \alpha(\epsilon)\zeta(\epsilon)v(\epsilon))d\epsilon \\
&\leq \omega(1+\omega\lambda\kappa)\int_0^\kappa \alpha(\rho)\zeta(\rho)v(\rho))d\rho \\
&\leq \omega(1+\omega\lambda\kappa)\int_0^\kappa \alpha(\rho)\zeta(\rho)\sup_{0\leq\epsilon\leq\rho} v(\epsilon))d\rho \\
&\leq \omega(1+\omega\lambda\kappa)\|v\|_\infty\int_0^\kappa \alpha(\rho)\zeta(\rho)d\rho.
\end{aligned}
$$

Thus,

$$\|v\|_\infty \leq \omega(1+\omega\lambda\kappa)\|v\|_\infty\int_0^\kappa \alpha(\rho)\zeta(\rho)d\rho$$

Then,

$$\|v\|_\infty\bigg(1-\omega(1+\omega\lambda\kappa)\int_0^\kappa \alpha(\rho)\zeta(\rho)d\rho\bigg) \leq 0.$$

Hereby, $\|v\|_\infty = 0$; thus, $v(\chi) = 0$ for each $\chi \in \Theta$, this implies $\Lambda(\chi)$ is relatively compact in Ψ. Through the result of Ascoli-Arzel *à* theorem, Λ is relatively compact in $D(\xi)$. Via *Mönch* fixed-point theorem, we show that I has a fixed point $U(\xi) \in D(\xi)$. As $\bigcap_{\xi\in\varphi} D(\xi) \neq \emptyset$; moreover, a measurable selector of $\int D$ exists. Lemma implies that I has a stochastic fixed point $U^*(\xi)$, which is a mild solution of (2).

□

5. Applications

The qualitative theory of FDEs, fractional integrodifferential equations, and fractional-order operators can be applied to a wide range of scientific fields, including fluid mechanics, viscoelasticity, physics, biology, chemistry, dynamical systems, signal processing, and entropy theory. Due to this, academics from all over the world have become interested in the applications of the theory of fractional calculus and the qualitative theory of the aforementioned equations, and many researchers have included them into their most recent research.

For a very long time, DEs driven by a Brownian motion (or Wiener process) have been the focus of study on the qualitative characteristics of stochastic DEs and their applications. Furthermore, applications from a variety of domains, including storage, queueing, economic, and neurophysiological systems, can be found frequently in stochastic DEs driven by a Poisson process. Additionally, stochastic DEs with Poisson jumps have gained much traction in modelling phenomena from a variety of disciplines, especially economics, where jump processes are frequently used to describe asset and commodity price dynamics. These factors are sufficient for the existence and uniqueness of non-Lipschitz stochastic neutral delay DEEs driven by Poisson jumps.

Levy procedures are becoming increasingly significant in the world of banking. While Levy processes are often employed in newer models to accommodate jumps (which can be regarded as external shocks) and achieve a better fit to empirical data, Brownian motion is still frequently used in older models as a source of randomness. As a result, Levy process applications in finance are simple to locate. There have been numerous applications of the theory of impulsive DEs of an integer order in accurate mathematical modelling. It has recently become a crucial subject of research due to the large range of practical problems. This is because many evolutionary systems' states are frequently exposed to rapid disturbances and undergo abrupt shifts from time to time. These changes have a very brief and insignificant length when compared to the lifespan of the process under consideration, and can be viewed as impulses. Due to the lack of effective methods, the control analysis of problems, including the impulse effect, fractional calculus, and white noise, is challenging.

6. Example

Consider

$$\begin{array}{l} {}^{c}_{0}D^{q}_{\nu}U(\chi,\xi,\varsigma) = \varphi(\chi,U(\chi,\xi,\varsigma),\varsigma) + \int_0^{\nu} B_2 f(\chi,\varsigma)dC_v,\ \xi \in \Theta := [0,\kappa],\ \nu \in [0,T] \\ U(\chi,\pi,\varsigma) + m(U) = U_1(\chi,2\pi,\varsigma);\ \ \xi \in [0,\kappa], \\ U'(\chi,\xi,\varsigma) = U_2(\xi), \end{array} \tag{6}$$

where $\Phi : \Theta \times R \times \zeta \to \mathbf{R^m}$ is a given function. If $\Xi = L^2[\pi, 2\pi]$, and $B_1 : \Xi \to \Xi$ given by $B_1 U = U'$ with domain $D(B_1) = \{U \in \Phi; U, U'$ *are absolutely continuous*, $U' \in \Xi, U(\pi) = U(2\pi) = 0\}$. Let the strongly continuous cosine function $(H_q(\chi))_{\chi \in \mathbf{R^m}}$ on Φ be infinitesimally generated by the operator B_1. Furthermore, B_1 has a discrete spectrum, and the eigenvalues are $-n^2, n \in IN$ with corresponding normalized eigenvectors

$$U_n(\varepsilon) := \left(\frac{2}{2\pi}\right)^{\frac{1}{2}} cos(n\varepsilon),$$

and

(i) $\{U_n : n \in IN\}$ is an orthonormal basis of Φ,
(ii) If $x \in \Phi$, then $B_1 x = -\sum_{n=1}^{\infty} n^2 \langle x, U_n \rangle U_n$,
(iii) For $x \in \Phi, H_q(\vartheta)x = \sum_{n=1}^{\infty} \sin(nt)\langle x, U_n \rangle U_n$, and the associated cosine family is

$$K_q(\vartheta)x = \sum_{n=1}^{\infty} \frac{\cos(nt)}{n} \langle x, U_n \rangle U_n.$$

Consequently, $K_q(\chi)$ is compact for all $\chi > 0$ and

$$\|H_q(\vartheta)\| = \|K_q(\chi)\| \le 1, \forall \chi \ge 0.$$

(iv) Let the group of translation be denoted by Φ:

$$\overline{\psi}(\chi)x(U,\varsigma) = \widetilde{x}(U+\chi,\varsigma),$$

where $\widetilde{x}$ is the extension of x with period 4π. Then,

$$H_q(\chi) = \frac{1}{2}(\overline{\psi} + \psi(-\chi)); U_1 = D,$$

where D is the infinitesimal generator of the group on

$$X = \{x(.,\varsigma) \in H^1(\pi, 2\pi) : x(\pi, \varsigma) = x(2\pi, \varsigma) = 0\}.$$

Suppose that B_2 is a bounded LO from Ω into Ξ and the linear operator $K : L^2(\Theta, \Omega) \to \Xi$ given by:

$$Kf = \int_0^k H_q(k - \varrho) B_2 f(\varrho, \varsigma) d\rho,$$

has an inverse operator K^{-1} in $L^2(\Theta, \Omega)/\ker K$. We deduce that Equation (1) is an abstract formulation of Equation (6) if H_1 to H_6 are met. Via Theorem 1, we conclude that Equation (6) is controllable.

7. Conclusions

Existence and controllability results were presented for a couple of classes of second-order fractional functional differential equations. A stochastic random fixed-point theorem established the basis for our claims. Then, we demonstrated that our problems were controllable. Some of the findings in this area form the basis of our future research plans. New results can be obtained by either changing or generalising the conditions and the functional spaces, or even by involving some fractional differential problems.

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