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Enumerating Subtrees of Flower and Sunflower Networks

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Abstract: Symmetry widely exists in many complex and real-world networks, with flower networks and sunflower networks being two richly symmetric networks and having many practical applications due to their special structures. The number of subtrees (the subtree number index) is closely related to the reliable network design. Using a generating function, structural analysis techniques, and auxiliary structure introduction, this paper presents the subtree generating functions of flower networks $Fl_{n,m}(n \geq 3, m \geq 2)$ and sunflower networks $Sf_{n,m}(n \geq 3, m \geq 2)$ and, thus, solves the computation of subtree number indices of $Fl_{n,m}(n \geq 3, m \geq 2)$ and $Sf_{n,m}(n \geq 3, m \geq 2)$. The results provide a fundamental and efficient method for exploring novel features of symmetric complex cyclic networks from the structural subtree number index perspective. For instance, we conclude that under some parameter constraints, the flower networks are more reliable than sunflower networks.

Keywords: flower networks; sunflower networks; generating function; subtree number index; structure analysis; auxiliary cyclic chain

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1. Introduction

Local connections between the interacting system components and the dynamics of the underlying physical process can be illustrated effectively in a graph. It can also be used to illustrate the microscale channels in an aporous medium, with the vertices presenting as pores and the edges presenting as channels which connect the pores. The analysis of certain graph parameters is one of the central problems in theoretical computer science because of their extensive implicit applications; see [1–4]. Over the past few decades, a family of more than 400 topological indices (TIs) has substantially gained significance among the graph parameters. Due to their applications, TIs such as Wiener index [5–7], Szeged index [8,9], subtree index [10–13], and Zagreb index [14–16] received particular attention among these parameters. Among these structural TIs, the subtree number index of a graph (the number of all nonempty subtrees of a graph) plays a vital role in measuring the stability of a network [17], especially in the case of both vertex and edge failure. Moreover, it also presents applications in the multiple sequence parsimony alignment of pedigree trees (evolutionary trees) [18], characterizing the physio-chemical and structural characteristics of molecular graphs. According to earlier research, the well-known Wiener index, Randić index, and Harary index [19–21] are strongly related to the subtree number index.

It is well known that the problem of enumerating the number of subtrees of a general graph is NP complete. In fact, the problem of counting subtrees of most complex graphs has not been solved yet, except trees [11–13], unicyclic and bicyclic graphs [22], and regular chemical molecule [23]. In 2005, Székely et al. [10] comprehensively studied the subtree problems for trees. In 2006, Yan et al. [11] proposed an efficient linear time algorithm to enumerate subtrees of a tree using a generating function. Later, many researchers investigated the extremal problems for subtrees [24,25]. Moreover, through a weight contract and a generating function, Yang et al. [26–28] solved the subtree number of spiro-and polyphenyl hexagonal chains, hexagonal chain graphs, phenylene chain graphs, and tricyclic graphs. In 2018, Chin et al. [29], via generating functions, obtained the number of subtrees of complete (complete bipartite) graphs and theta graphs. More recently, in 2020, Dong et al. [17] proposed algorithms for computing the subtrees of the three-cactus network by assigning the two tuple weight to the vertices and using the technique of contracting the circle weight. In 2021, by constructing a new generating function, Yang et al. [12] proposed a new recursive (fast) algorithm to enumerate certain subtrees and BC-subtrees of the tree graph. By introducing a five tuple weight, in 2022, Abolfazl et al. [30] proposed a more efficient algorithm for computing subtrees and BC-subtrees of trees.

Symmetry is one of the core structural features of many complex and real-world networks, and it is closely related to the reliable network design; the study of symmetric complex network structures is attracting more and more attention worldwide [31–33]. On the other hand, the wheel graphs, flower networks, and sunflower networks are typical highly symmetric complex networks and have many practical applications due to their special structures. In 2017, Daoud [34] obtained the number of spanning trees and studied the asymptotic spanning tree properties of flower networks and sunflower networks. In 2018, Kaliraj et al. [35] obtained the star edge chromatic number of the corona product of a path with a wheel graph. In 2019, Zahid et al. [36] explored the combinatorial aspects of the spanning simplicial complex of a wheel graph. In 2020, Ali et al. [37] studied three-total edge product cordial labeling of sunflower networks. In 2022, Sathiya et al. [38] solved the equitable edge-coloring problem of wheel graphs. In the same year, Kaabar et al. [39] investigated the upper bounds of the radio number and radial radio number of sunflower-extended networks. However, there are few studies on the TIs of the sunflower networks and flower networks; moreover, it is well known that the subtree number index is closely related to the reliable network design. The key objective of this paper is to solve the subtree enumeration problem for flower networks and sunflower networks.

This paper is structured as follows: Section 2 presents notations, definitions, and lemmas. In Section 3, we started by solving the subtree generating function (SGF) of fan flower networks $FFl_{n,m}(n \geq 1, m \geq 2)$ and went on to solve the SGF problem of the flower networks $Fl_{n,m}(n \geq 3, m \geq 2)$. In Section 4, through an auxiliary cyclic chain and by computing its corresponding SGF, we solved the SGF problem of sunflower networks $Sf_{n,m}(n \geq 3, m \geq 2)$. We also briefly discussed the behavior of the subtree numbers in the flower graphs and sunflower graphs in Section 5. Lastly, we conclude our research in Section 6 and identify certain deep problems for upcoming work.

2. Terminologies and Notations

We start the section by briefly recalling the necessary notions and lemmas which will be utilized later. By $G = (V(G), E(G); f, g)$, we denote a simple graph with a vertex set V (with $|V| = n$) and an edge set E (with $|E| = m$), where f (g) is the vertex (edge) weight function. For a tree $T \in S(G)$, the weight of T is defined as $\omega(T) = \prod_{v \in V(T), e \in E(T)} f(v)g(e)$. Furthermore, the generating function of G (contain vertex v) is given by $F(G; f, g) = \prod_{T \in S(G)} \omega(T)$ $(F(G; f, g; v) = \prod_{T_s \in S(G;v)} \omega(T_s))$. With the above notations, we denote $\eta(G) = F(G; 1, 1)(\eta(G; v) = F(G; 1, 1; v))$ as all subtrees of G (containing v) in G. Some necessary notions and notations are listed below:

- $G \backslash X$ denotes the graph produced after excluding X from G, where X may be a vertex set or an edge set, or mixed set of vertices and edges).
- $S(G)$ denotes the subtree set of G.
- $S(G, X)$ denotes the subtree set containing X in G.

Definition 1. *Let $F_{n+1}(n \geq 1)$ be the fan graph that is constructed by connecting each vertex of path $P_n = c_1c_2\ldots c_n$ with c_0. Moreover, we call the graph constructed from $F_{n+1}(n \geq 1)$ the fan flower graph $FFl_{n,m}(n \geq 1, m \geq 2)$ (see Figure 1), by connecting each vertex pair c_i and c_0 with a path of length $m-1$.*

Definition 2. *Let W_{n+1} be a wheel graph that is constructed by connecting each vertex $c_i(i = 1, 2, \ldots, n)$ of the n vertex unicyclic graph with a vertex c_0. Let $Fl_{n,m}(n \geq 3, m \geq 2)$ (see Figure 2) be a flower network constructed by connecting each vertex pair c_i and center vertex c_0 of W_{n+1} with a path $P_{c_ic_0} = c_0u_{i,1}u_{i,2}u_{i,3}\ldots u_{i,m-2}c_i$ of length $m-1$. It is easy to see that $Fl_{n,m}(n \geq 3, m \geq 2)$ has $n(m+1)$ edges and $n(m-1)+1$ vertices.*

Definition 3. *Let $Sf_{n,m}(n \geq 3, m \geq 2)$ (see Figure 3) be a sunflower network that is constructed by connecting each adjacent vertex pair c_i and $c_{i+1}(1 \leq i \leq n)(c_{n+1} = c_1)$ of wheel graph W_{n+1} with a path $P_{c_ic_{i+1}} = c_iw_{i,1}w_{i,2}w_{i,3}\ldots w_{i,m-1}c_{i+1}$ of length $m-1$ sequentially. Obviously, $Sf_{n,m}(n \geq 3, m \geq 2)$ has $n(m+1)$ edges and $n(m-1)+1$ vertices.*

For the sake of brevity, we introduce some notations in the following Table 1.

Table 1. Some abbreviated symbols.

Symbol	**Explanation**
$F_n^{c_0}$	SGF of fan flower $FFl_{n,m}(n \geq 1, m \geq 2)$ containing c_0, for the case of $n = 0$, we let $F_0^{c_0}(FFl_{n,m}) = f(c_0)$
$F_{n,i}^{c_0}$	the SGF that contain both c_0 and path P_i, namely, $F(FFl_{n,m}; f, g; \{c_0, P_i\})(n \geq 2, m \geq 2, 2 \leq i \leq n)$.

Lemma 1. *Let $T = (V(T), E(T); f, g)$ denote $n(n > 1)$ a vertex-weighted tree, $v_i \in V(T)$; meanwhile, let $u \neq v_i$ be a leaf and $e = (u, v)$ be the corresponding edge of T. For the weighted tree $T' = (V(T'), E(T'); f', g')$ with vertex set $V(T') = V(T)\backslash\{u\}$, edge set $E(T') = E(T)\backslash\{e\}$ and*

$$f'(v_s) = \begin{cases} f(v)(1 + f(u)g(e)) & \text{if } v_s = v, \\ f(v_s) & \text{otherwise}. \end{cases} \tag{1}$$

where $v_s \in V(T'), g'(e) = g(e)(e \in E(T'))$. Then, we have $F(T; f, g; v_i) = F(T'; f'g'; v_i)$ [11].

Lemma 2. *Let $P_n = (V(P_n), E(P_n); f, g) = v_1, v_2, \ldots, v_n$ be the weighted path tree, with vertex set $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, edge set $E(P_n) = \{e_1, e_2, \ldots e_{n-1}\}$, and $f(v) = y(v \in V(P_n))$, $g(e) = z(e \in E(P_n))$ [11], with above notations; then, we have*

$$\begin{aligned} F(P_n; f, g) &= \sum_{i=1}^{n}(n-i+1)y^iz^{i-1} \\ F(P_n; f, g; v_1) &= \sum_{i=1}^{n} y^iz^{i-1} \end{aligned} \tag{2}$$

Let $U_n = (V(U_n), E(U_n); f, g)$ be the weighted unicyclic graph with n vertices and n edges; its vertex and edge weights are $f(v) = y(v \in V(U_n)), g(e) = z(e \in E(U_n))$, respectively, and the vertices on the cycle are labeled as $v_i(i = 1, 2, \ldots, n)$ sequentially. With the definition of SGF and structure analysis, it is easy to obtain the following Lemma.

Lemma 3. *Let $U_n = (V(U_n), E(U_n); f, g)$ be the above weighted unicyclic graph, for any vertex v_i and any continuous k edges $\bigcup_{j=1}^{k}(v_{i+j-1}, v_{i+j})(k = 1, 2, \ldots)(v_i = v_{n+i})$ [23]; then, we have*

$$
\begin{aligned}
F(U_n; f, g) &= n\sum_{i=1}^{n} y^i z^{i-1} \\
F(U_n; f, g; v_i) &= \sum_{i=1}^{n} i y^i z^{i-1} \\
F(U_n; f, g; \bigcup_{j=1}^{k}(v_{i+j-1}, v_{i+j})) &= \sum_{i=1}^{n-k} i y^{i+k} z^{i+k-1}
\end{aligned} \tag{3}
$$

Lemma 4. *Let $G = (V(G), E(G); f, g)$ denote a weighted graph, $(u, v) \in E(G)$; let $G/e = (V(G/e), E(G/e); f_e, g_e)$ be the graph that contract v along e to u, with $V(G/e) = V(G)\backslash v$, $E(G/e) = E(G)\backslash e$ [28]. Then, the vertex weight is*

$$
f_e(w) = \begin{cases} f(u)f(v)g(e) & \text{if } w = u, \\ f(w) & \text{otherwise.} \end{cases} \tag{4}
$$

and we have

$$
F(G; f, g) = F(G\backslash e; f, g) + F(G/e; f_e, g_e; u) \tag{5}
$$

Lemma 5. *For a weighted graph $G = (V(G), E(G); f, g)$ having vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$, where $G_1 \cap G_2 = (u, v)$, the vertex and edge generating functions are $f(v) = y(v \in V(G)), g(e) = z(e \in E(G))$, respectively [27]; then, we have*

$$
F(G; f, g) = \frac{F(G_1; f, g; (u, v))F(G_2; f, g; (u, v))}{y^2 z} + F(G\backslash(u, v); f, g) \tag{6}
$$

With Lemmas 3 and 5 and a structure analysis, we have the following Theorem immediately.

Theorem 1. *Let $FFl_{n,m} = (V(FFl_{n,m}), E(FFl_{n,m}); f, g)$ be the weighted fan flower graph as in Figure 1; P_n be the path connecting c_1 and c_n of $FFl_{n,m}$; and $FFl_{n,m}/P_n$ be the graph contracting c_n along P_n to c_1, with the notations given in Table 1; then, for $n \geq 2$, we have*

$$
\begin{aligned}
F_{n,n}^{c_0} &= F(FFl_{n,m}; f, g; \{c_0, P_n\}) = F(FFl_{n,m}/P_n; f, g; \{c_0, c_1\}) \\
&= n y^{n-1} z^{n-1} \left\{ \frac{(y^m z^{m-1} \sum_{i=1}^{m-1} i y^{i-1} z^{i-1})^{n-1}}{(y^m z^{m-1})^{n-2}} + \frac{(y^2 z \sum_{i=1}^{m-1} i y^{i-1} z^{i-1})^{n}}{(y^2 z)^{n-1}} \right\}
\end{aligned} \tag{7}
$$

For $n \geq 1$, denote by $FFl_{n,m}^{i+1,n}$ the weighted component that contains c_{i+1} of $FFl_{n,m}$ $\backslash\{\bigcup_{j=1}^{i}(c_0, c_j) \cup (c_0, u_{j,m-2}) \cup (c_i, c_{i+1})\}(i = 1, 2, \ldots, n-1)$. It is easy to know that $F(FFl_{n,m}^{i+1,n}$ and $f, g; c_0) = F(FFl_{n-i,m}; f, g; c_0)$; then, we have the following:

Theorem 2. *Let $FFl_{n,m} = (V(FFl_{n,m}), E(FFl_{n,m}); f, g)$ be the weighted fan flower graph, with the notations given in Table 1; for $n \geq 1$, we have*

$$
F_n^{c_0} = \begin{cases} \sum_{i=1}^{m} i y^i z^{i-1} & \text{if } n = 1, \\ \frac{F_1^{c_0} F_{n-1}^{c_0} + \sum_{i=2}^{n-1} F_{n-i}^{c_0} F_{i,i}^{c_0}}{y} + F_{n,n}^{c_0} & \text{otherwise.} \end{cases} \tag{8}
$$

where $F_{j,j}^{c_0}(j = 2, \ldots, n)$, as given in Theorem 2.

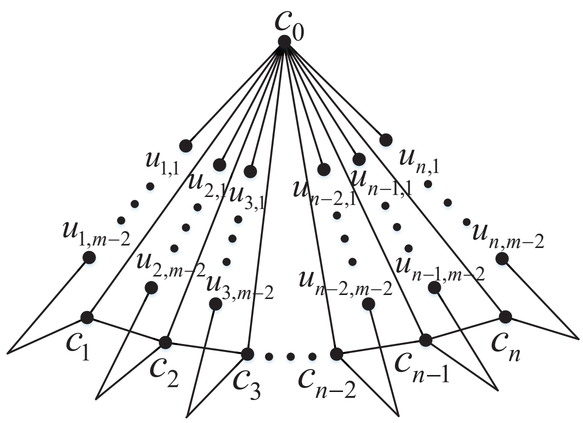

Figure 1. Fan flower graph $FFl_{n,m}(n \geq 1, m \geq 2)$.

Proof. When $n = 1$, $FFl_{1,m}$ is a unicyclic graph with m vertices, by Lemma 3, we have

$$F_n^{c_0} = \sum_{i=1}^{m} iy^i z^{i-1} \tag{9}$$

When $n > 1$, we divide the subtree set $S(FFl_{n,m}, c_0)$ into three cases

$$S(FFl_{n,m}; c_0) = \bigcup_{i=1}^{3} \tau_i$$

- τ_1: not containing (c_1, c_2);
- τ_2: contains $\bigcup\limits_{i=1}^{j}(c_i, c_{i+1})$ but not $(c_{j+1}, c_{j+2})(j = 1, \ldots, n-2)$;
- τ_3: contains $\bigcup\limits_{i=1}^{n-1}(c_i, c_{i+1})$, namely, the path P_n connecting c_1 and c_n of $FFl_{n,m}$.

For τ_1, by Lemma 5, we can obtain its SGF as

$$\frac{F_1^{c_0} F_{n-1}^{c_0}}{y} \tag{10}$$

Similarly, the SGF of τ_2 is

$$\sum_{i=2}^{n-1} \frac{F_{n-i}^{c_0} F_{i,i}^{c_0}}{y} \tag{11}$$

and SGF of τ_3 is

$$F_{n,n}^{c_0} \tag{12}$$

Combining Equations (9)–(12), we can obtain Equation (8); the theorem is thus proved. □

Lemma 6. *Let $FFl_{n,m} = (V(FFl_{n,m}), E(FFl_{n,m}); f, g)$ be the weighted fan flower graph and let $P_i(2 \leq i \leq n)$ be the path connecting c_1 with c_i of $FFl_{n,m}$, with notations given in Table 1; then, we have*

$$F_{n,i}^{c_0} = \frac{\sum\limits_{j=i}^{n} F_{j,j}^{c_0} F_{n-j}^{c_0}}{y} \tag{13}$$

Proof. Firstly, we divide the subtree set of $FFl_{n,m}$ containing $P_i(2 \leq i \leq n)$ into three cases:
(i) Not containing (c_i, c_{i+1});
(ii) Contains $\bigcup\limits_{j=0}^{k}(c_{i+j}, c_{i+j+1})(k = 0, 1, 2, \ldots, n-i-2)$ but not (c_{i+j+1}, c_{i+j+2});

(iii) Contains P_n.

For case (i), using a structure analysis, and Lemmas 1 and 2, we obtain the SGF of case (i) as

$$\frac{F^{c_0}_{i,i}F^{c_0}_{n-i}}{y} \tag{14}$$

and the SGF of case (ii) as

$$\sum_{k=0}^{n-i-2} \frac{F^{c_0}_{i+k+1,i+k+1}F^{c_0}_{n-(i+k+1)}}{y} \tag{15}$$

For case (iii), by using Theorem 1, we obtain its SGF as

$$\frac{F^{c_0}_{n,n}F^{c_0}_{0}}{y} \tag{16}$$

Combining Equations (14)–(16), we can obtain Equation (13); the theorem thus holds. □

3. The Subtree Number of Flower Network

Theorem 3. *Let $Fl_{n,m}(n \geq 3, m \geq 2)$ be the weighted flower network (see Figure 2); with the notations given in Table 1, we have the SGF of $Fl_{n,m}(n \geq 3)$ as*

$$\begin{aligned} F(Fl_{n,m}; f, g) =& n(\sum_{i=1}^{m-2}(m-1-i)y^i z^{i-1} + \sum_{i=1}^{n}(\sum_{i=1}^{m-1} y^i z^{i-1})^i z^{i-1}) \\ &+ \sum_{j=1}^{n-1}\frac{\sum\limits_{s=j+1}^{n} F^{c_0}_{s,s}F^{c_0}_{n-s}}{y} + \frac{F^{c_0}_{1}F^{c_0}_{n-1} + \sum\limits_{i=2}^{n-1} F^{c_0}_{n-i}F^{c_0}_{i,i}}{y} + F^{c_0}_{n,n} \end{aligned} \tag{17}$$

where $F^{c_0}_{j,j}(j = 2, \ldots, n)$ and $F^{c_0}_{i}(i = 1, 2, \ldots, n-1)$ are as given in Theorems 1 and 2, respectively.

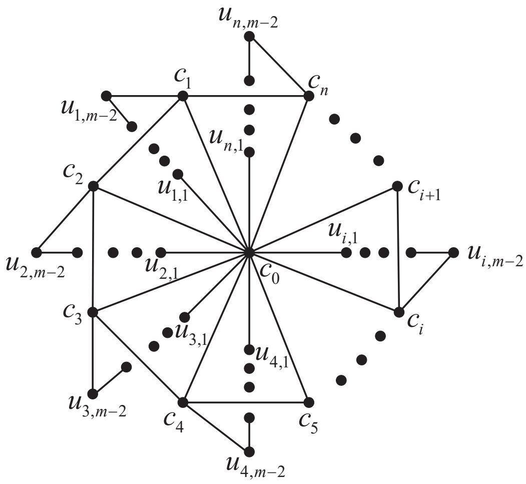

Figure 2. Flower network $Fl_{n,m}(n \geq 3, m \geq 2)$.

Proof. We characterize the subtrees of $Fl_{n,m}(n \geq 3, m \geq 2)$ into two cases:

(1) Not containing c_0;

(2) Contains c_0.

For case (1), it is easy to know that $Fl_{n,m}\backslash\{c_0\}$ is a unicyclic graph; with Lemma 1 and Lemma 3, we can obtain its SGF as

$$n(\sum_{i=1}^{m-2}(m-1-i)y^i z^{i-1} + \sum_{i=1}^{n}(\sum_{i=1}^{m-1} y^i z^{i-1})^i z^{i-1}) \tag{18}$$

We further divide case (2) into two cases:

- $\tau_{2,1}$: not containing (c_1, c_2);
- $\tau_{2,2}$: contains $\bigcup\limits_{k=1}^{j}(c_k, c_{k+1})$ but not $(c_{j+1}, c_{j+2})(c_{n+1} = c_1)(j = 1, 2, \ldots, n-1)$.

For $\tau_{2,1}$, with the structure analysis and Theorem 2, we have its SGF as

$$F_n^{c_0} \tag{19}$$

For $\tau_{2,2}$, using the structure analysis and Lemma 6, we can obtain its SGF as

$$\begin{aligned} &F(Fl_{n,m} \backslash (c_{j+1}, c_{j+2}); f, g; \{c_0, \bigcup_{k=1}^{j}(c_k, c_{k+1})\}) \\ &= \sum_{j=1}^{n-1} F(FFl_{n,m}; f, g; \{c_0, P_{j+1}\}) \\ &= \sum_{j=1}^{n-1} F_{n,j+1}^{c_0} = \sum_{j=1}^{n-1} \frac{\sum\limits_{s=j+1}^{n} F_{s,s}^{c_0} F_{n-s}^{c_0}}{y} \end{aligned} \tag{20}$$

Combining Equations (18)–(20), we can obtain Equation (17); the theorem thus holds. □

4. The Subtree Number of Sunflower Network

Let $Sf_{n,m} = (V(Sf_{n,m}), E(Sf_{n,m}); f, g)(n \geq 3, m \geq 2)$ denote the weighted sunflower network; before solving the problem of enumerating subtrees of sunflower network, we introduce some Lemmas.

Let $U_n = (V(U_n), E(U_n); f, g)$ be a weighted graph, and $P_{v_i v_j} = v_i \ldots v_j$ be the path connecting v_i and v_j of U_n; we define $U_n^c = (V(U_n^c), E(U_n^c). Let f^c, g^c)$ be the weighted graph that contracts $P_{v_i v_j} = v_i \ldots v_j$ to v_i, $V(U_n^c) = \{v_i\} \cup \{V(U_n) \backslash V(P_{v_i v_j}), E(U_n^c) = E(U_n) \backslash E(P_{v_i v_j})$, where

$$f^c(v_i) = \prod_{v \in V(P_{v_i v_j})} f(v) \prod_{e \in E(P_{v_i v_j})} g(e)$$

$f^c(v) = f(v)$ for $v \in V(U_n^c) \backslash v_i$ and $g^c(e) = g(e)$ for $e \in E(U_n^c)$.

Lemma 7. *Assume that U_n and U_n^c, the weighted unicyclic graphs, are defined as above and $P_{v_i v_j} = v_i \ldots v_j$ is the path connecting v_i and v_j of U_n [23]; then,*

$$F(U_n; f, g; P_{v_i v_j}) = F(U_n^c; f^c, g^c; v_i) \tag{21}$$

Let

$$P_1 = v_i v_{(i+1)(mod\ n)} \cdots v_{(j-1)(mod\ n)} v_j$$

and

$$P_2 = v_i v_{(i-1)(mod\ n)} \cdots v_{(j+1)(mod\ n)} v_j$$

be the two paths connecting v_i and v_j of U_n; moreover, define the weighted unicyclic graph $U_n^{c_1} = (V(U_n^{c_1}), E(U_n^{c_1}); f_1^c, g_1^c)$ as the graph contracting the path P_1 of U_n to v_i and $U_n^{c_2} = (V(U_n^{c_2}), E(U_n^{c_2}); f_2^c, g_2^c)$ be the graph contracting the path P_2 of U_n to v_i.

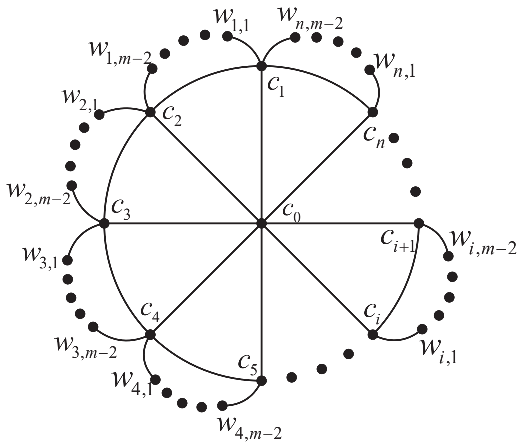

Figure 3. Sunflower network $Sf_{n,m}(n \geq 3, m \geq 2)$.

Lemma 8. *Assume that U_n, $U_n^{c_1}, U_n^{c_2}$ are the unicyclic graphs defined as above and v_i and v_j are two distinct vertices of U_n [23]; then,*

$$F(U_n; f, g; v_i, v_j) = F(U_n^{c_1}; f_1^c, g_1^c; v_i) + F(U_n^{c_2}; f_2^c, g_2^c; v_i) \tag{22}$$

Before proving the SGF of sunflower network, we first introduce and solve the subtree number enumeration problems of an auxiliary cyclic chain network that is defined as follows.

Definition 4. *Let a, b be positive integers and $a + b = n$, the cyclic chain network $G_t(a, b) = (V(G_t(a, b)), E(G_t(a, b)); f, g)(t \geq 1)$, be constructed as follows:*

- *For $t = 1$, G_1 is an unicyclic graph with length n;*
- *For $t \geq 2$, let U_t be an unicyclic graph with length t, the vertex set be $V(U_t) = \{v_1, v_2 \ldots, v_t\}$, and G_t be derived from U_t by replacing each existing edge (v_i, v_{i+1}) in U_t by two parallel paths of length a and b (see Figure 4).*

Lemma 9. *Let $G_t(t \geq 1) = (V(G_t), E(G_t); f, g)$ (see Figure 4) be the weighted auxiliary cyclic chain network defined above; then,*

$$F(G_1; f, g) = n \sum_{i=1}^{n} y^i z^{i-1} \tag{23}$$

and

$$\begin{aligned} F(G_t; f, g) =& t \Bigg(\sum_{i=1}^{a-1} (a-i) y^i z^{i-1} + \sum_{j=1}^{b-1} (b-j) y^j z^{j-1} + \alpha\beta^2\lambda + (\alpha\beta)^2 \sum_{s=2}^{t-1} y^{-s}\gamma^{s-1} + \\ & y^{1-t}\gamma^{t-1} \Big(\alpha\beta + \beta \sum_{p=1}^{a-1}\sum_{i=1}^{a-p} y^{p+i} z^{p+i-1} + \lambda \sum_{q=1}^{b-1}\sum_{j=1}^{b-q} y^{q+j} z^{q+j-1} + \sum_{p=1}^{a-1}\sum_{q=1}^{b-1} \\ & \sum_{i=1}^{a-p} (1 + z \sum_{j=1}^{b-q-1} y^j z^{j-1}) y^{p+q+i} z^{p+q+i-1} \Big) \Bigg) \end{aligned} \tag{24}$$

for $t > 1$, where $\alpha = \sum_{i=1}^{a} y^i z^{i-1}$, $\beta = 1 + \sum_{j=1}^{b-1} y^j z^j$, $\gamma = y^{a+1} z^a \sum_{j=1}^{b} j y^{j-1} z^{j-1} + y^{b+1} z^b \sum_{i=1}^{a} i y^i z^{i-1}$, $\lambda = 1 + z \sum_{i=1}^{a-1} y^i z^{i-1}$.

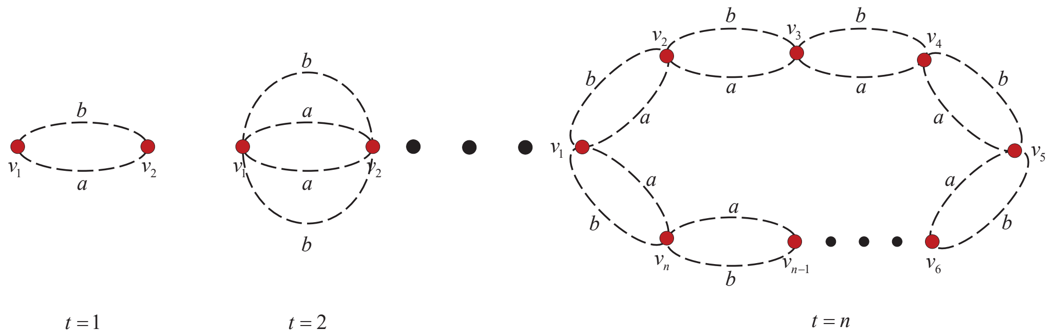

Figure 4. Cyclic chain network G_t.

Proof. For the case of $t = 1$, with Lemma 3, we have

$$F(G_1; f, g) = n \sum_{i=1}^{n} y^i z^{i-1} \tag{25}$$

When $t > 1$, we divide the subtrees of G_t into four cases:
(1) Not containing any vertex of $\{v_1, v_2 \ldots, v_t\}$;
(2) Contains only one vertex of $\{v_1, v_2 \ldots, v_t\}$;
(3) Contains only consecutive $s(s = 2, 3, \ldots, t-1)$ vertices in $\{v_1, v_2 \ldots, v_t\}$;
(4) Contains all vertices of $\{v_1, v_2 \ldots, v_t\}$.
For the sake of brevity, we claim that $\alpha, \beta, \lambda, \gamma$ are notations given in Lemma 9.
For case (1), by observing the structure pattern with Lemma 2, we can the SGF of case (1) as

$$t\Big(\sum_{i=1}^{a-1}(a-i)y^i z^{i-1} + \sum_{j=1}^{b-1}(b-j)y^j z^{j-1}\Big) \tag{26}$$

Moreover, by observing the structure patterns with Lemma 1, we can obtain the SGF of case (2) as

$$t(\alpha\beta^2\lambda) \tag{27}$$

With Lemmas 1, 5, 7, and 8, we can obtain the SGF of case (3) as

$$t\Big((\alpha\beta)^2 \sum_{s=2}^{t-1} y^{-s}\gamma^{s-1}\Big) \tag{28}$$

For case (4), let $P_1 = (v_1)u_1u_2\ldots u_a u_{a+1}(v_n), P_2 = (v_1)w_1w_2\ldots w_b w_{b+1}(v_n)$ be the two paths connecting v_1 and v_n of graph $G_t \backslash v_2$, where $v_1 = u_1 = w_1, v_n = u_{a+1} = w_{b+1}$.

- τ_1: not containing (u_1, u_2) and (w_1, w_2);
- τ_2: contains $\bigcup\limits_{i=1}^{p}(u_i, u_{i+1})$ but not (w_1, w_2) and $(u_p, u_{p+1})(p = 1, 2, \ldots, a-1)$;
- τ_3: contains $\bigcup\limits_{i=1}^{j}(w_i, w_{i+1})$ but not (u_1, u_2) and $(w_j, w_{j+1})(j = 1, 2, \ldots, b-1)$;
- τ_4: contains $\bigcup\limits_{i=1}^{p}\bigcup\limits_{i=1}^{j}(u_i, u_{i+1}) \cup (w_i, w_{i+1})$ but not $(u_p, u_{p+1}) \cup (w_j, w_{j+1})(p = 1, 2, \ldots, a-1)(j = 1, 2, \ldots, b-1)$.

For τ_1, with Lemmas 1, 5, 7, and 8, we have the SGF of τ_1 as

$$y^{1-t}\gamma^{t-1}(\alpha\beta) \tag{29}$$

Similarly, the SGF of τ_2 is

$$y^{1-t}\gamma^{t-1}(\beta\sum_{p=1}^{a-1}\sum_{i=1}^{a-p}y^{p+i}z^{p+i-1}) \tag{30}$$

Similarly, the SGF of τ_3 is

$$y^{1-t}\gamma^{t-1}(\lambda\sum_{q=1}^{b-1}\sum_{j=1}^{b-q}y^{q+j}z^{q+j-1}) \tag{31}$$

The SGF of τ_4 is

$$y^{1-t}\gamma^{t-1}\Big(\sum_{p=1}^{a-1}\sum_{q=1}^{b-1}\sum_{i=1}^{a-p}(1+z\sum_{j=1}^{b-q-1}y^jz^{j-1})y^{p+q+i}z^{p+q+i-1}\Big) \tag{32}$$

With structure analysis and Equations (29)–(32), we can obtain the SGF of case (4) as

$$\begin{aligned}&t\gamma^{t-1}y^{1-t}\Big(\alpha\beta+\beta\sum_{p=1}^{a-1}\sum_{i=1}^{a-p}y^{p+i}z^{p+i-1}+\lambda\sum_{q=1}^{b-1}\sum_{j=1}^{b-q}y^{q+j}z^{q+j-1}+\sum_{p=1}^{a-1}\sum_{q=1}^{b-1}\sum_{i=1}^{a-p}\\&(1+z\sum_{j=1}^{b-q-1}y^jz^{j-1})y^{p+q+i}z^{p+q+i-1}\Big)\end{aligned} \tag{33}$$

With Equations (26)–(28) and (33), we can obtain the SGF of $G_t(t>1)$ as

$$\begin{aligned}F(G_t;f,g)=&t\Big(\sum_{i=1}^{a-1}(a-i)y^iz^{i-1}+\sum_{j=1}^{b-1}(b-j)y^jz^{j-1}+\alpha\beta^2\lambda+(\alpha\beta)^2\sum_{s=2}^{t-1}y^{-s}\gamma^{s-1}+\\&y^{1-t}\gamma^{t-1}\Big(\alpha\beta+\beta\sum_{p=1}^{a-1}\sum_{i=1}^{a-p}y^{p+i}z^{p+i-1}+\lambda\sum_{q=1}^{b-1}\sum_{j=1}^{b-q}y^{q+j}z^{q+j-1}+\sum_{p=1}^{a-1}\sum_{q=1}^{b-1}\\&\sum_{i=1}^{a-p}(1+z\sum_{j=1}^{b-q-1}y^jz^{j-1})y^{p+q+i}z^{p+q+i-1}\Big)\Big)\end{aligned} \tag{34}$$

□

Lemma 10. *Let $G_t(t\geq 1)=(V(G_t),E(G_t);f,g)$ (see Figure 4) be the weighted auxiliary cyclic chain network defined above; then,*

- *for $t=1$,*

$$F(G_1;f,g;v_k)=\sum_{i=1}^{n}iy^iz^{i-1} \tag{35}$$

  where $v_k\in\{v_1,v_2\}$;
- *for $t\geq 2$,*

$$\begin{aligned}F(G_t;f,g;v_k)=&\alpha\beta^2\lambda+(\alpha\beta)^2\sum_{s=2}^{t-1}sy^{-s}\gamma^{s-1}+ty^{1-t}\gamma^{t-1}\Big(\alpha\beta+\beta\sum_{p=1}^{a-1}\sum_{i=1}^{a-p}y^{p+i}\\&z^{p+i-1}+\lambda\sum_{q=1}^{b-1}\sum_{j=1}^{b-q}y^{q+j}z^{q+j-1}+\sum_{p=1}^{a-1}\sum_{q=1}^{b-1}\sum_{i=1}^{a-p}(1+z\sum_{j=1}^{b-q-1}y^jz^{j-1})\\&y^{p+q+i}z^{p+q+i-1}\Big)\end{aligned} \tag{36}$$

where $v_k\in\{v_1,v_2,\ldots,v_t\}$, $\alpha=\sum_{i=1}^{a}y^iz^{i-1}$, $\beta=1+\sum_{j=1}^{b-1}y^jz^j$, $\gamma=y^{a+1}z^a\sum_{j=1}^{b}jy^{j-1}z^{j-1}+y^{b+1}z^b\sum_{i=1}^{a}iy^iz^{i-1}$, $\lambda=1+\sum_{i=1}^{a-1}y^iz^i$.

Proof. For $t=1$, $v_k\in\{v_1,v_2\}$, with Lemma 3, we can obtain the SGF of G_1 containing v_k as

$$F(G_1; f, g; v_k) = \sum_{i=1}^{n} i y^i z^{i-1} \tag{37}$$

For $t > 1$, we divide the subtrees of G_t containing $v_k \in \{v_1, v_2, \ldots, v_t\}$ into three cases:
(1) Containing only v_k;
(2) Containing only consecutive $i(= 2, 3, \ldots, t-1)$ vertices in $\{v_1, v_2, \ldots, v_t\}$ including v_k;
(3) Containing all vertices in $\{v_1, v_2, \ldots, v_t\}$.
For the sake of brevity, we claim that $\alpha, \beta, \lambda, \gamma$ are the notations given in Lemma 10.
For case (1), with structure analysis and Lemma 1 , we can obtain its SGF as

$$\alpha\beta^2\lambda \tag{38}$$

With a structure analysis and Lemmas 1, 5, 7, and 8, we obtain the SGF of case (2) as

$$(\alpha\beta)^2 \sum_{s=2}^{t-1} s y^{-s} \gamma^{s-1} \tag{39}$$

For case (3), with a similar analysis of case (3) in Lemma 9, we have its SGF as

$$\begin{aligned} &t\gamma^{t-1} y^{1-t} \Big(\alpha\beta + \beta \sum_{p=1}^{a-1} \sum_{i=1}^{a-p} y^{p+i} z^{p+i-1} + \lambda \sum_{q=1}^{b-1} \sum_{j=1}^{b-q} y^{q+j} z^{q+j-1} + \sum_{p=1}^{a-1} \sum_{q=1}^{b-1} \sum_{i=1}^{a-p} \\ &(1 + z \sum_{j=1}^{b-q-1} y^j z^{j-1}) y^{p+q+i} z^{p+q+i-1} \Big) \end{aligned} \tag{40}$$

Combining Equations (38)–(40), we can obtain the SGF of G_t containing $v_k \in \{v_1, v_2, \ldots, v_t\}$ as

$$\begin{aligned} F(G_t; f, g; v_k) =& \alpha\beta^2\lambda + (\alpha\beta)^2 \sum_{s=2}^{t-1} s y^{-s} \gamma^{s-1} + t y^{1-t} \gamma^{t-1} \Big(\alpha\beta + \beta \sum_{p=1}^{a-1} \sum_{i=1}^{a-p} y^{p+i} \\ & z^{p+i-1} + \lambda \sum_{q=1}^{b-1} \sum_{j=1}^{b-q} y^{q+j} z^{q+j-1} + \sum_{p=1}^{a-1} \sum_{q=1}^{b-1} \sum_{i=1}^{a-p} (1 + \sum_{j=1}^{b-q-1} y^j z^j) \\ & y^{p+q+i} z^{p+q+i-1} \Big) \end{aligned} \tag{41}$$

The theorem thus follows. $\square$

Next, we solve the subtree enumeration problem of sunflower network.

Theorem 4. *Let $Sf_{n,m} = (V(Sf_{n,m}), E(Sf_{n,m}); f, g)(n \geq 3, m \geq 2)$ be the weighted sunflower network; then, we have its SGF as*

$$\begin{aligned} F(Sf_{n,m}; f, g) =& n \Big(\sum_{j=1}^{m-2} (m-j-1) y^j z^{j-1} + y\zeta_1^2 + \sum_{s=2}^{n-1} (y^{1-s} \zeta_1^2 \zeta_2^{s-1}) + y^{1-n} \zeta_2^{n-1} \\ & (y\zeta_1 + \sum_{q=1}^{m-2} \sum_{j=1}^{m-q-1} y^{q+j} z^{q+j-1}) \Big) + y + nyz \Big(y\zeta_1^2 + \sum_{s=1}^{n-2} (s+1)(y^{-s} \zeta_1^2 \zeta_2^s) \\ & + n y^{1-n} \zeta_2^{n-1} (y\zeta_1 + \sum_{q=1}^{m-2} \sum_{j=1}^{m-q-1} y^{q+j} z^{q+j-1}) \Big) + \frac{(\sum_{i=1}^{m-1} i y^i z^{i-1})^n}{(y^3 z^2)^{n-1}} + \\ & \sum_{j=1}^{n-2} \frac{\sum_{d_1=1}^{n-1} \sum_{d_2=1}^{n-d_1-1} \cdots \sum_{d_j=1}^{n-\sum_{k=1}^{j-1} d_k - 1} (n - \sum_{s=1}^{j} d_s) \prod_{s=1}^{j} \phi(d_s) \phi(n - \sum_{p=1}^{j} d_p)}{(y^3 z^2)^j} \end{aligned} \tag{42}$$

where $\zeta_1 = 1 + \sum\limits_{j=1}^{m-2} y^j z^j$, $\zeta_2 = \sum\limits_{j=1}^{m-1} j y^{j+1} z^j + y^m z^{m-1}$,

$$\phi(k) = \begin{cases} \sum\limits_{i=1}^{m-1} i y^i z^{i-1} & \textit{if } k = 1, \\ y\zeta_1^2 + \sum\limits_{s=1}^{k-2} (s+1)(y^{-s}\zeta_1^2\zeta_2^s) & \textit{if } 2 \le k \le n-1. \\ +k y^{1-k}\zeta_2^{k-1}\Big(y\zeta_1 + \sum\limits_{q=1}^{m-2}\sum\limits_{j=1}^{m-q-1} y^{q+j} z^{q+j-1}\Big) & \end{cases} \tag{43}$$

Proof. We divide the subtrees of $Sf_{n,m}(n \ge 3, m \ge 2)$ into two cases:

(a) Not containing c_0;

(b) Containing c_0.

For the sake of brevity, we declare that $\zeta_1, \zeta_2, \phi(k)(k = 1, 2, \ldots, n-1)$ are the notations given in Theorem 4.

For case (a), with the structure analysis and Lemma 9 , we can obtain its SGF as

$$\begin{aligned} & n\Big(\sum_{j=1}^{m-2}(m-j-1)y^j z^{j-1} + y\zeta_1^2 + \sum_{s=2}^{n-1}(y^{1-s}\zeta_1^2\zeta_2^{s-1}) + y^{1-n}\zeta_2^{n-1} \\ & (y\zeta_1 + \sum_{q=1}^{m-2}\sum_{j=1}^{m-q-1} y^{q+j}z^{q+j-1})\Big) \end{aligned} \tag{44}$$

For case (b), we further divide it into two cases:

(i) Not containing any edge of $\bigcup\limits_{i=1}^{n}(c_0, c_i)$;

(ii) Contains at least one edge of $\bigcup\limits_{i=1}^{n}(c_0, c_i)$.

It is easy to obtain that the SGf of case (i) is

$$y \tag{45}$$

For case (ii), we first consider the case of containing only one edge in $\bigcup\limits_{i=1}^{n}(c_0, c_i)$; with the structure analysis and Lemmas 5, 7, 8, and 10, we can obtain the SGF of this case as

$$nyz\Big(y\zeta_1^2 + \sum_{s=1}^{n-2}(s+1)(y^{-s}\zeta_1^2\zeta_2^s) + ny^{1-n}\zeta_2^{n-1}(y\zeta_1 + \sum_{q=1}^{m-2}\sum_{j=1}^{m-q-1} y^{q+j}z^{q+j-1})\Big). \tag{46}$$

With Lemmas 5 and 10, we can obtain the SGF of all n edges in $\bigcup\limits_{i=1}^{n}(c_0, c_i)$ as

$$\frac{(\sum\limits_{i=1}^{m-1} i y^i z^{i-1})^n}{(y^3 z^2)^{n-1}} \tag{47}$$

For the case containing only $s(2 \le s \le n-1)$ edges in $\bigcup\limits_{i=1}^{n}(c_0, c_i)$, whether (c_0, c_1) is in the s edges or not, we define (c_0, c_1) as the boundary; then, we label the s edges with $e_1, e_2, \ldots, e_s(2 \le s \le n-1)$ in a counterclockwise manner. Relabeling the vertex on the cycle adjacent to e_i as $\tilde{c}_i$, we define d_i the counterclockwise distance between the vertices $\tilde{c}_i$ and $c_{\tilde{i+1}}$ on the circle.

Let $\phi(k) = F(G_k(1, m-1); f, g; v_j)(2 \le k \le n-1)$, where $v_j \in \{v_1, v_2, \ldots, v_k\}$.

Combining Lemmas 3, 5, 7, 8, 10, and the structure analysis, we can obtain the SGF of $Sf_{n,m}(n \geq 3, m \geq 2)$ containing $s(2 \leq s \leq n-1)$ edges in $\bigcup_{i=0}^{n}(c_0, c_i)$

$$\sum_{j=1}^{n-2} \frac{\sum_{d_1=1}^{n-1} \sum_{d_2=1}^{n-d_1-1} \cdots \sum_{d_j=1}^{n-\sum_{k=1}^{j-1} d_k - 1} (n - \sum_{s=1}^{j} d_s) \prod_{s=1}^{j} \phi(d_s)\phi(n - \sum_{p=1}^{j} d_p)}{(y^3 z^2)^j} \tag{48}$$

where

$$\phi(k) = \begin{cases} \sum_{i=1}^{m-1} i y^i z^{i-1} & \text{if } k = 1, \\ y\zeta_1^2 + \sum_{s=1}^{k-2} (s+1)(y^{-s}\zeta_1^2 \zeta_2^s) & \text{if } 2 \leq k \leq n-1. \\ +ky^{1-k}\zeta_2^{k-1}\Big(y\zeta_1 + \sum_{q=1}^{m-2} \sum_{j=1}^{m-q-1} y^{q+j} z^{q+j-1}\Big). & \end{cases} \tag{49}$$

With Equations (44)–(48), we can obtain the Equation (42); the theorem thus holds. □

With the subtree generating functions of flower networks $Fl_{n,m}(n \geq 3, m \geq 2)$ and sunflower networks $Sf_{n,m}(n \geq 3, m \geq 2)$, we can easily obtain their exact subtree number indices by taking $y = 1$ and $z = 1$ into their subtree generating functions.

5. Results and Discussions

It is well known that the subtree number index is closely related to the reliable network design [17,40], and it is an important parameter to measure the reliability of a network for both vertex and edge failures. Namely, networks with more subtrees are more reliable. As an application, in this section, we briefly study the behavior of the subtree number in the flower networks and sunflower networks and discuss the difference of the subtree numbers between flower networks and sunflower networks with some lower-order n and m. From Equation (7) in Theorem 1, Equation (8) in Theorem 2, and Equation (17) in Theorem 3, we can obtain the subtree numbers of flower networks, as shown in Tables 2–4; similarly, with Equations (42) and (43) in Theorem 4, we can obtain the subtree numbers of sunflower networks, as shown in Tables 2–4.

Table 2. The subtree numbers of flower and sunflower networks when $n = 12, m = 3, 4, \ldots, 18$.

n	m	$\eta(Sf_{n,m})$	$log_{10}^{\eta(Sf_{n,m})}$	$\eta(Fl_{n,m})$	$log_{10}^{\eta(Fl_{n,m})}$
12	3	808,994,334,366	11.90794548	112,238,363,816,4	12.05014133
12	4	998,610,060,032,920	14.99939594	151,308,674,603,059,0	15.17986383
12	5	265,280,251,112,307,000	17.42370492	404,849,273,581,826,000	17.60729336
12	6	256,550,685,339,735,000,00	19.40917318	384,650,010,292,476,000,00	19.58506575
12	7	122,063,734,917,778,000,000,0	21.08658665	178,548,004,931,270,000,000,0	21.251755
12	8	344,273,389,923,669,000,000,00	22.53690346	490,934,797,202,023,000,000,00	22.69102382
12	9	650,645,762,570,061,000,000,000	23.81334461	905,809,267,379,347,000,000,000	23.95703676
12	10	896,628,634,179,120,000,000,000,0	24.9526126	122,119,065,856,034,000,000,000,00	25.08678347
12	11	957,232,024,063,896,000,000,000,00	25.98101722	127,822,905,177,000,000,000,000,000	26.10660868
12	12	827,993,950,521,421,000,000,000,000	26.91802716	108,623,009,982,456,000,000,000,000,0	27.03592183
12	13	600,392,405,365,584,000,000,000,000,0	27.77843519	775,222,331,646,122,000,000,000,000,0	27.88942627
12	14	374,744,209,248,582,000,000,000,000,00	28.57373493	477,004,278,516,100,000,000,000,000,00	28.67852227
12	15	205,602,572,796,967,000,000,000,000,000	29.31302854	258,359,754,119,165,000,000,000,000,000,0	30.41222486
12	16	100,842,557,399,484,000,000,000,000,000,0	30.00364385	125,251,904,753,382,000,000,000,000,000,0	30.09778434
12	17	448,296,476,198,180,000,000,000,000,000,0	30.65156533	550,957,046,116,700,000,000,000,000,000,0	30.74111774
12	18	182,701,553,239,954,000,000,000,000,000,00	31.26174224	222,390,527,574,997,000,000,000,000,000,00	31.34711629

Table 3. The subtree numbers of flower and sunflower networks when $m = 12, n = 3, 4, \ldots, 12$.

n	m	$\eta(Sf_{n,m})$	$log_{10}^{\eta(Sf_{n,m})}$	$\eta(Fl_{n,m})$	$log_{10}^{\eta(Fl_{n,m})}$
12	3	569,170,9	6.755,242,688	517,527,0	6.713933012
12	4	980,681,131	8.991527819	990,109,132	8.995683066
12	5	168,515,441,557	11.2266397	181,560,385,458	11.2590211
12	6	291,422,645,379,48	13.4645233	327,929,007,285,78	13.51577983
12	7	506,766,497,010,234,0	15.7048079	589,052,387,641,233,0	15.77015392
12	8	884,180,126,963,189,000	17.94654075	105,597,948,140,887,000,0	18.02365548
12	9	154,536,766,330,510,000,000	20.18903182	189,163,078,432,656,000,000	20.27683637
12	10	270,327,973,135,294,000,000,00	22.43189099	338,765,521,831,683,000,000,00	22.5298992
12	11	473,064,933,203,821,000,000,000,0	24.67492076	606,622,526,889,045,000,000,000,0	24.78291853
12	12	828,000,782,196,875,000,000,000,000	26.91803075	108,623,009,982,456,000,000,000,000,0	27.03592183

Table 4. The subtree numbers of flower and sunflower networks when $m = n = 3, 4, \ldots, 12$.

n	m	$\eta(Sf_{n,m})$	$log_{10}^{\eta(Sf_{n,m})}$	$\eta(Fl_{n,m})$	$log_{10}^{\eta(Fl_{n,m})}$
3	3	102,7	3.011570444	101,7	3.007320953
4	4	105,451	5.023050703	112,636	5.051677219
5	5	187,184,91	7.272270835	215,012,25	7.332463204
6	6	513,612,006,6	9.710635168	617,934,704,1	9.790942587
7	7	201,051,445,463,5	12.3033072	249,018,559,016,5	12.39623172
8	8	106,896,683,359,669,0	15.02896423	134,000,723,234,807,0	15.12710714
9	9	725,328,036,700,938,000	17.86053446	928,294,891,665,960,000	17.96768596
10	10	622,370,323,833,015,000,000	20.79404888	804,6744,796,912,410,000,00	20.90562023
11	11	654,629,293,559,492,000,000,000	23.81599544	853,195,489,291,013,000,000,000	23.93104855
12	12	827,993,950,521,421,000,000,000,000	27.03592183	108,623,009,982,456,000,000,000,000,0	27.03592183

With Table 2, we can obtain Figure 5; with Table 3, we can obtain Figure 6; and with Table 4, we can obtain Figure 7. Observing Tables 2–4 and Figures 5–7, we find that the subtree numbers of flower networks are always greater than that of sunflower networks. Therefore, the network reliability of flower networks are higher than that of sunflower networks. When designing a more reliable network, the flower network is a better choice in terms of reliability.

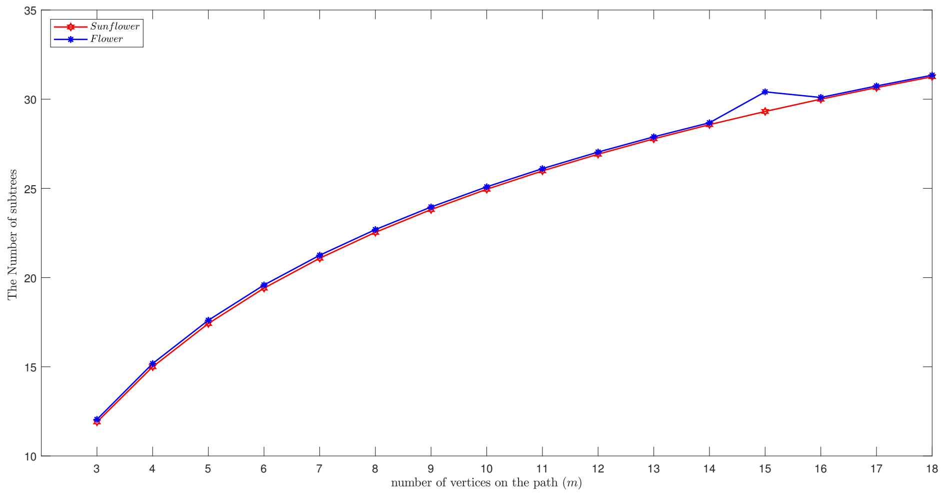

Figure 5. The subtree number of flower network $Fl_{n,m}(n = 12, 3 \leq m \leq 18)$ and sunflower network $Sf_{n,m}(n = 12, 3 \leq m \leq 18)$, in semi-log(Log-Y) coordinates.

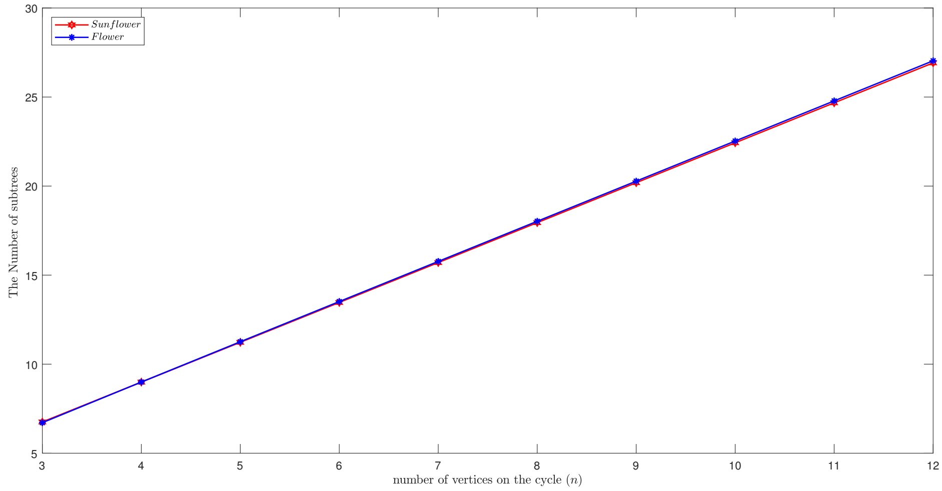

Figure 6. The subtree number of flower network $Fl_{n,m}(m = 12, 3 \leq n \leq 12)$ and sunflower network $Sf_{n,m}(m = 12, 3 \leq n \leq 12)$, in semi-log(Log-Y) coordinates.

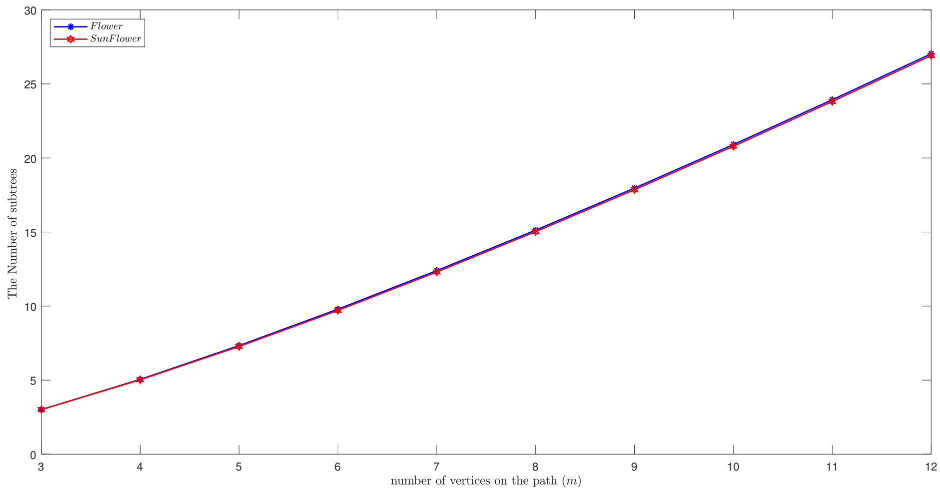

Figure 7. The subtree number of flower network $Fl_{n,m}(m = n = 3, 4 \cdots, 12)$ and sunflower network $Sf_{n,m}(m = n = 3, 4 \cdots, 12)$, in semi-log(Log-Y) coordinates.

6. Concluding Remarks

Using a generating function, structure analysis, and auxiliary structure introduction, we obtained the subtree generating function of flower networks $Fl_{n,m}(n \geq 3, m \geq 2)$. Moreover, by introducing and solving the subtrees of an auxiliary cyclic chain network, we presented the subtree generating function of sunflower networks $Sf_{n,m}(n \geq 3, m \geq 2)$. We also briefly discussed the behavior of the subtree numbers in the flower graphs and sunflower graphs. Additionally, we obtain the conclusion that under some parameter constraints, the flower networks are more reliable than sunflower networks. These findings are likely useful in designing reliable networks. Our study provides a theoretical basis for exploring new structural properties of complex networks and chemical molecules. Many important networks are derived from the flower and sunflower networks; for future work, we intend to investigate other topological indices of flower networks, sunflower networks, and their derived networks and analyze the relationships between them.

Author Contributions: Y.Y. conceived of the presented idea. L.L. developed the theory and performed the computations. F.D. and Z.J. verified the analytical methods. H.L., W.Z., and Y.Y. investigated and supervised the findings of this work. All authors discussed the results and contributed to the final manuscript. All authors have read and agreed to the published version of the manuscript.

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References

1. Allen, B.; Lippner, G.; Chen, Y.T.; Fotouhi, B.; Momeni, N.; Yau, S.T.; Nowak, M.A. Evolutionary dynamics on any population structure. *Nature* **2017**, *544*, 227–230. [CrossRef] [PubMed]
2. Park, H.J.; Friston, K. Structural and functional brain networks: From connections to cognition. *Science* **2013**, *342*, 1238411. [CrossRef]
3. Débarre, F.; Hauert, C.; Doebeli, M. Social evolution in structured populations. *Nat. Commun.* **2014**, *5*, 1–7. [CrossRef]
4. Wang, Y.; Hafeez, S.; Akhter, S.; Iqbal, Z.; Aslam, A. The Generalized Inverse Sum Indeg Index of Some Graph Operations. *Symmetry* **2022**, *14*, 2349. [CrossRef]
5. Harry, W. Structural determination of paraffin boiling points. *J. Am. Chem. Soc.* **1947**, *69*, 17–20.
6. Dobrynin, A.A.; Entringer, R.; Gutman, I. Wiener index of trees: Theory and applications. *Acta Appl. Math.* **2001**, *66*, 211–249. [CrossRef]
7. Ali, F.; Rather, B.A.; Fatima, N.; Sarfraz, M.; Ullah, A.; Alharbi, K.A.M.; Dad, R. On the Topological Indices of Commuting Graphs for Finite Non-Abelian Groups. *Symmetry* **2022**, *14*, 1266. [CrossRef]
8. Khadikar, P.V.; Deshpande, N.V.; Kale, P.P.; Dobrynin, A.; Gutman, I.; Domotor, G. The Szeged index and an analogy with the Wiener index. *J. Chem. Inf. Comput. Sci.* **1995**, *35*, 547–550. [CrossRef]
9. Al-Fozan, T.; Manuel, P.; Rajasingh, I.; Rajan, R.S. Computing Szeged index of certain nanosheets using partition technique. *Match Commun. Math. Comput. Chem.* **2014**, *72*, 339–353.
10. Székely, L.A.; Wang, H. On subtrees of trees. *Adv. Appl. Math.* **2005**, *34*, 138–155. [CrossRef]
11. Yan, W.; Yeh, Y.N. Enumeration of subtrees of trees. *Theor. Comput. Sci.* **2006**, *369*, 256–268. [CrossRef]
12. Yang, Y.; Li, X.; Jin, M.; Li, L.; Wang, H.; Zhang, X. Enumeration of subtrees and BC-subtrees with maximum degree no more than k in trees. *Theor. Comput. Sci.* **2021**, *892*, 258–278. [CrossRef]
13. Yang, Y.; Liu, H.; Wang, H.; Zhang, X.D.; Chen, C.P. On enumerating algorithms of novel multiple leaf-distance granular regular α-subtrees of trees. *Inf. Comput.* **2022**, *289*, 104942. [CrossRef]
14. Furtula, B.; Graovac, A.; Vukičević, D. Augmented zagreb index. *J. Math. Chem.* **2010**, *48*, 370–380. [CrossRef]
15. Gutman, I. An exceptional property of first Zagreb index. *Match Commun. Math. Comput. Chem.* **2014**, *72*, 733–740.
16. Kazemi, R. Note on the multiplicative Zagreb indices. *Discret. Appl. Math.* **2016**, *198*, 147–154. [CrossRef]
17. Dong, L.; Zhao, H.; Lai, H.J. Entropy and Enumeration of Subtrees in a Cactus Network. *Front. Phys.* **2020**, *8*, 575648. [CrossRef]
18. Knudsen, B. Optimal multiple parsimony alignment with affine gap cost using a phylogenetic tree. In Proceedings of the International Workshop on Algorithms in Bioinformatics, Budapest, Hungary, 15–20 September 2003; pp. 433–446.
19. Wagner, S.G. Correlation of graph-theoretical indices. *SIAM J. Discret. Math.* **2007**, *21*, 33–46. [CrossRef]
20. Zhang, X.; Zhang, X.; Gray, D.; Wang, H. The number of subtrees of trees with given degree sequence. *J. Graph Theory* **2013**, *73*, 280–295. [CrossRef]
21. Andriantiana, E.O.D.; Wagner, S.; Wang, H. Greedy trees, subtrees and antichains. *Electron. J. Comb.* **2013**, *20*, P28. [CrossRef]
22. Yang, Y.; Liu, H.; Wang, H.; Feng, S. On algorithms for enumerating BC-subtrees of unicyclic and edge-disjoint bicyclic graphs. *Discret. Appl. Math.* **2016**, *203*, 184–203. [CrossRef]
23. Yang, Y.; Sun, X.; Cao, J.; Wang, H.; Zhang, X. The expected subtree number index in random polyphenylene and spiro chains. *Discret. Appl. Math.* **2020**, *285*, 483–492. [CrossRef]
24. Székely, L.A.; Wang, H. Binary trees with the largest number of subtrees. *Discret. Appl. Math.* **2007**, *155*, 374–385. [CrossRef]
25. Zhang, X.; Zhang, X.; Gray, D.; Wang, H. Trees with the most subtrees—An algorithmic approach. *J. Comb.* **2012**, *3*, 207–223.
26. Yang, Y.; Liu, H.; Wang, H.; Fu, H. Subtrees of spiro and polyphenyl hexagonal chains. *Appl. Math. Comput.* **2015**, *268*, 547–560. [CrossRef]
27. Yang, Y.; Liu, H.; Wang, H.; Deng, A.; Magnant, C. On Algorithms for Enumerating Subtrees of Hexagonal and Phenylene Chains. *Comput. J.* **2017**, *60*, 690–710. [CrossRef]
28. Yang, Y.; Chen, B.; Zhang, G.; Li, Y.; Sun, D.; Liu, H. Algorithms based on path contraction carrying weights for enumerating subtrees of tricyclic graphs. *Comput. J.* **2022**, *65*, 554–572. [CrossRef]

29. Chin, A.J.; Gordon, G.; MacPhee, K.J.; Vincent, C. Subtrees of graphs. *J. Graph Theory* **2018**, *89*, 413–438. [CrossRef]
30. Poureidi, A. On computing the number of (BC-) subtrees, eccentric subtree number, and global and local means of trees. *Inf. Process. Lett.* **2022**, *178*, 106302. [CrossRef]
31. Chen, Y.; Zhao, Y.; Han, X. Characterization of Symmetry of Complex Networks. *Symmetry* **2019**, *11*, 692. [CrossRef]
32. MacArthur, B.D.; Sánchez-García, R.J.; Anderson, J.W. Symmetry in complex networks. *Discret. Appl. Math.* **2008**, *156*, 3525–3531. [CrossRef]
33. Xiao, Y.H.; Wu, W.T.; Wang, H.; Xiong, M.; Wang, W. Symmetry-based structure entropy of complex networks. *Phys. A Stat. Mech. Its Appl.* **2008**, *387*, 2611–2619. [CrossRef]
34. Daoud, S. Complexity of graphs generated by wheel graph and their asymptotic limits. *J. Egypt. Math. Soc.* **2017**, *25*, 424–433. [CrossRef]
35. Kaliraj, K.; Sivakami, R.; Vernold, V.J. Star edge coloring of corona product of path and wheel graph families. *Proyecciones* **2018**, *37*, 593–601. [CrossRef]
36. Zahid, A.; Saleem, M.; Kashif, A.; Khan, M.; Meraj, M.; Irfan, R. Spanning Simplicial Complex of Wheel Graph W_n. *Algebra Colloq.* **2019**, *26*, 309–320. [CrossRef]
37. Ali, U.; Ahmad, Y.; Sardar, M.S. On 3-total edge product cordial labeling of tadpole, book and flower graphs. *Open J. Math. Sci.* **2020**, *4*, 48–55. [CrossRef]
38. Sathiya, D.; Senthamilselvi, S. On Equitable Edge Coloring of Wheel Graph Families. *J. Algebr. Stat.* **2022**, *13*, 1791–1797.
39. Kaabar, M.K.; Yenoke, K. Radio and Radial Radio Numbers of Certain Sunflower Extended Graphs. *Int. J. Math. Math. Sci.* **2022**, *2022*, 9229409. [CrossRef]
40. Xiao, Y.; Zhao, H.; Liu, Z.; Mao, Y. Trees with large numbers of subtrees. *Int. J. Comput. Math.* **2017**, *94*, 372–385. [CrossRef]