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Harmonic Oscillator Coherent States from the Standpoint of Orbit Theory

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Abstract: We study the known coherent states of a quantum harmonic oscillator from the standpoint of the originally developed noncommutative integration method for linear partial differential equations. The application of the method is based on the symmetry properties of the Schrödinger equation and on the orbit geometry of the coadjoint representation of Lie groups. We have shown that analogs of coherent states constructed by the noncommutative integration can be expressed in terms of the solution to a system of differential equations on the Lie group of the oscillatory Lie algebra. The solutions constructed are directly related to irreducible representation of the Lie algebra on the Hilbert space functions on the Lagrangian submanifold to the orbit of the coadjoint representation.

Keywords: quantum harmonic oscillator; noncommutative integration method; coadjoint representation

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1. Introduction

The study of exact solutions to the Schrödinger equation for a general harmonic oscillator has attracted considerable interest in the literature thanks to the pivotal role of the oscillator in physics. Constructing exact solutions for a harmonic oscillator based on various ideas and methods and finding connections between them expands the knowledge about this fundamental system.

The well-known stationary states of a quantum harmonic oscillator in the coordinate representation are obtained by the separation of variables in the Schrödinger equation with the harmonic oscillator potential. Glauber proposed standard coherent states for a harmonic oscillator, which is the prototype for most coherent states [1,2]. The coherent states form a very convenient representation for problems in quantum mechanics. They can be created from the ground state by the displacement operator and can be expanded in terms of the harmonic oscillator Hamiltonian eigenstates. Coherent states are described in a wealth of superb reference books and papers, e.g., [3–5].

An alternative to the separation-of-variables method is the noncommutative integration method (NIM) proposed in [6] for linear partial differential equations and developed in [7–10] (see Ref. [11] for details). This method essentially uses the symmetry of a differential equation and its algebra of symmetry operators and allows one to construct a basis of solutions that, in general, differ from solutions constructed by the separation of variables and from coherent states. The NIM was effectively used to construct exact solutions to the Schrödinger, Klein–Gordon [6,8], and Dirac [9,10,12] equations, and also for classification of external fields in equations with symmetries in the Riemannian spaces of general relativity in [13–16].

This paper describes the development and application of the NIM for solving the Schrödinger equation for a quantum harmonic oscillator, using symmetry in this problem. We will be looking for the NIM solutions, which can be regarded as analogs of coherent states in the sense of [4,17] in view of the close relation of NIM with the group symmetry of the quantum harmonic oscillator.

The paper is organized as follows. Section 2 introduces the basic notation and a special λ-representation of Lie algebras necessary for applying the method of noncommutative integration. Section 3 considers the Schrödinger equation for a harmonic oscillator and shows that its symmetry algebra in the class of first-order linear differential operators forms the oscillatory Lie algebra $\mathfrak{g}_{osc}$. The next Section 4 considers the λ-representation of the oscillatory Lie algebra $\mathfrak{g}_{\mathfrak{osc}}$ and the generalized Fourier transform on the Lie group G_{osc} of the Lie algebra $\mathfrak{g}_{\mathfrak{osc}}$. Section 5 shows that the Schrödinger equation for an oscillator is equivalent to some right-invariant system of equations on the group G. Integrating this system by noncommutative integration, we obtain a basis of solutions and compare it with a system of coherent states. Section 6 contains some concluding remarks.

2. λ-Representation of Lie Groups

Using the orbit method [18,19], we define an irreducible λ representation of the Lie algebra $\mathfrak{g}$, which is a key element of the NIM.

First, we recall some necessary definitions related to the orbit method on Lie groups, which we will need in what follows. The degenerate Poisson–Lie bracket,

$$\{\phi,\psi\}(f)=\langle f,[d\phi(f),d\psi(f)]\rangle=C^c_{ab}f_c\frac{\partial\phi(f)}{\partial f_a}\frac{\partial\psi(f)}{\partial f_b},\quad \phi,\psi\in C^\infty(\mathfrak{g}^*), \tag{1}$$

defines a Poisson structure in the space $\mathfrak{g}^*$ [18]. Here, f_a are the coordinates of a linear functional $f=f_ae^a\in\mathfrak{g}^*$ relative to the dual basis $\{e^a\}$, $[\cdot,\cdot]$ is a commutator in the Lie algebra $\mathfrak{g}$, and the natural pairing between the spaces $\mathfrak{g}^*$ and $\mathfrak{g}$ is denoted by $\langle\cdot,\cdot\rangle$. The number ind$\mathfrak{g}$ of functionally independent Casimir functions $K_\mu(f)$, $\mu=1,\dots,\mathrm{ind}\mathfrak{g}$, with respect to the bracket (1) is called the *index of the Lie algebra* $\mathfrak{g}$.

The Lie group G acts on the coalgebra $\mathfrak{g}^*$ by the coadjoint representation Ad*: $G\times\mathfrak{g}^*\to\mathfrak{g}^*$ and splits $\mathfrak{g}^*$ into the coadjoint orbits (K-orbits). Orbits of maximum dimension $\dim\mathcal{O}^{(0)}=\dim\mathfrak{g}-\mathrm{ind}\mathfrak{g}$ are called *non-degenerate* [18].

Let $\mathcal{O}_\lambda$ be a non-degenerate coajoint orbit passing through a general covector $\lambda\in\mathfrak{g}^*$. Locally, one can always introduce the Darboux coordinates $(p,q)\in P\times Q$ on the orbit $\mathcal{O}_\lambda$ in which the Kirillov form ω_λ defining a symplectic structure on the coajoint orbits has the canonical form $\omega_\lambda=dp^a\wedge dq_a$, $a=1,\dots,\dim\mathcal{O}_\lambda/2$, and (p,q) are called the canonical coordinates. We assume that the transition from the local coordinates f on the orbit $\mathcal{O}_\lambda$ to the canonical coordinates (p,q) is given if the set of functions $f_X=f_X(p,q,\lambda)$, $X\in\mathfrak{g}^*$ is defined in such a way that

$$\frac{\partial f_X}{\partial p_\mu}\frac{\partial f_Y}{\partial q^\mu}-\frac{\partial f_X}{\partial p_\mu}\frac{\partial f_Y}{\partial q^\mu}=f_{[X,Y]},\quad f_X(0,0,\lambda)=\langle\lambda,X\rangle,$$
$$\mathrm{rank}\left\|\frac{\partial f_a}{\partial q^\mu},\frac{\partial f_a}{\partial p_\mu}\right\|=\dim\mathcal{O}_\lambda,\quad X,Y\in\mathfrak{g}.$$

Consider the functions $f_X=f_X(p,q,\lambda)$, which are linear in the variables p_a:

$$f_X(p,q,\lambda)=\alpha^a_X(q)p_a+\chi_X(q,\lambda),\quad q\in Q,\quad p\in P. \tag{2}$$

Denote by $\mathfrak{g}_{\mathbb{C}}$ a complex extension of the Lie algebra $\mathfrak{g}$. It was shown in Ref. [7] that canonical functions (2) can be constructed if for the functional λ there exists a subalgebra $\mathfrak{h}\subset\mathfrak{g}^{\mathbb{C}}$ in the complex extension $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ satisfying the conditions

$$\langle\lambda,[\mathfrak{h},\mathfrak{h}]\rangle=0,\quad \dim\mathfrak{h}=\dim\mathfrak{g}-\frac{1}{2}\dim\mathcal{O}_\lambda. \tag{3}$$

The subalgebra $\mathfrak{h}$ is called the *polarization* of the functional λ. In this case, the vector fields $\alpha_X(q)=\alpha^a_X(q)\partial_{q^a}$ are infinitesimal generators of a local transformation group $G_{\mathbb{C}}=\exp(\mathfrak{g}_{\mathbb{C}})$ of a partially holomorphic manifold Q. Equation (3) assumes that the functionals from $\mathfrak{g}^*$ can

be prolonged to $\mathfrak{g}^{\mathbb{C}}$ by linearity. Note that for non-degenerate coajoint orbits, there always exist the canonical functions with the form (2).

Let $L_2(Q, d\mu(q))$ be a space of complex functions on the manifold Q with a measure $d\mu(q)$ and inner product given by

$$(\psi_1, \psi_2) = \int_Q \overline{\psi_1(q)}\psi_2(q)d\mu(q), \quad d\mu(q) = \rho(q)dq, \tag{4}$$

where $\overline{\psi_1(q)}$ denotes the complex conjugate of $\psi_1(q)$. Functions of the space $L_2(Q, d\mu(q))$ are square-integrable on the manifold Q.

The linear operators

$$\ell_X(q,\lambda) = \frac{i}{\hbar} f_X(-i\hbar\partial_q, q, \lambda) = \alpha^a_X(q)\partial_{q^a} + \frac{i}{\hbar}[\chi_X(q,\lambda) + i\hbar\beta_X], \quad \partial_{q^a} = \frac{\partial}{\partial q^a}$$
$$K_\mu(-i\hbar\ell(q,\lambda)) = K_\mu(\lambda), \tag{5}$$
$$[\ell_X(q,\partial_q,\lambda), \ell_Y(q,\partial_q,\lambda)] = \ell_{[X,Y]}(q,\partial_q,\lambda), \quad \beta_{\overline{\alpha}} = -\frac{1}{2}\mathrm{Tr}(\mathrm{ad}_{\overline{\alpha}}|_{\mathfrak{h}}), \quad X, Y \in \mathfrak{g}, \tag{6}$$

implement the irrep λ-representation of a Lie algebra $\mathfrak{g}$ in $L_2(Q, d\mu(q))$ and can be seen as the result of qp-quantization on the coajoint orbit $\mathcal{O}_\lambda$ [6,7].

We assume that with respect to the inner product (4) the operators $-i\hbar\ell_X(q, \partial_q, \lambda)$ are Hermitian.

Let $\xi_X(g) = (L_g)_* X$ and $\eta_X(g) = -(R_g)_* X$ be left- and right-invariant vector fields on a Lie group G, respectively. We will be interested in generalized solutions of the system of equations

$$\big(\eta_X(g) + \ell_X(q, \partial_q, \lambda)\big)\mathscr{D}^\lambda_{qq'}(g) = 0, \tag{7}$$
$$\Big(\xi_X(g) + \overline{\ell_X(q', \partial_{q'}, \lambda)}\Big)\mathscr{D}^\lambda_{qq'}(g) = 0, \quad \mathscr{D}^\lambda_{qq'}(e) = \delta(q, q'), \tag{8}$$

The functions $\mathscr{D}^\lambda_{qq'}(g)$ provide the lift of the λ-representation of the Lie algebra $\mathfrak{g}$ to the local unitary representation T^λ of its Lie group G,

$$(T^\lambda_g\psi)(q) = \int_Q \psi(q')\mathscr{D}^\lambda_{qq'}(g)d\mu(q), \quad \frac{d}{dt}\Big(T^\lambda_{\exp(tX)}\varphi\Big)\Big|_{t=0}(q) = \ell_X(q, \partial_q, \lambda)\varphi(q), \tag{9}$$

and satisfy the relations

$$\mathscr{D}^\lambda_{qq'}(g_1g_2) = \int_Q \mathscr{D}^\lambda_{qq''}(g_1)\mathscr{D}^\lambda_{q''q'}(g_2)d\mu(q''), \quad \mathscr{D}^\lambda_{qq'}(g) = \overline{\mathscr{D}^\lambda_{q'q}(g^{-1})},$$

where $g_1, g_2 \in G$. Note that by properly defining the measure $d\mu(\lambda)$ in the parameter space J, we can check the completeness and orthogonality properties for the functions $\mathscr{D}^\lambda_{qq'}(g)$,

$$\int_G \overline{\mathscr{D}^{\tilde\lambda}_{\tilde q\tilde q'}(g)}\mathscr{D}^\lambda_{qq'}(g)d\mu(g) = \delta(q,\tilde q)\delta(\tilde q', q')\delta(\tilde\lambda, \lambda),$$
$$\int_{Q\times Q\times J} \overline{\mathscr{D}^\lambda_{qq'}(\tilde g)}\mathscr{D}^\lambda_{qq'}(g)d\mu(q)d\mu(\lambda) = \delta(\tilde g g^{-1}), \tag{10}$$

where $\delta(g)$ is the generalized Dirac delta function with respect to the right Haar measure $d\mu(g)$ on the Lie group G.

Note that the functions $\mathscr{D}^\lambda_{qq'}(g)$ are defined globally on the Lie group G iff the Kirillov condition of integerness of the orbit $\mathcal{O}_\lambda$ holds [18]:

$$\frac{1}{2\pi}\int_{\gamma\in H_1(\mathcal{O}_\lambda)} \omega_\lambda = n_\gamma \in \mathbb{Z}.. \tag{11}$$

Here, $H_1(\mathcal{O}_\lambda)$ is a one-dimensional homology group of the stationarity subgroup $G^\lambda = \{g \in G \mid Ad^*_g\lambda = \lambda\}$.

Let $L_2(G, d\mu(g))$ be the space of functions having the form

$$\psi(g) = \int_Q \psi(q, q', \lambda)\mathscr{D}^\lambda_{qq'}\left(g^{-1}\right)d\mu(q')d\mu(q)d\mu(\lambda). \tag{12}$$

The function $\psi(q, q', \lambda)$ belongs to the space $L_2(Q, \mathfrak{h}, \lambda)$ with respect to the variables q and q'. Within the framework of the NIM, the relation (12) is a generalization of the Fourier transform with kernel $\mathscr{D}^\lambda_{qq'}(g^{-1})$ to the case of the Lie group G. From the relations, (10) follows the expression for the inverse Fourier transform in the form

$$\psi(q, q', \lambda) = \int_G \psi^\lambda(g)\overline{\mathscr{D}^\lambda_{qq'}(g^{-1})}d\mu(g). \tag{13}$$

The transformations (12) and (13) induce the corresponding transformation of the left- and right-invariant fields on the Lie group G:

$$\begin{aligned}
&\xi_X(g)\psi^\lambda(g) \Longleftrightarrow \overline{\ell^\dagger_X(q, \partial_q, \lambda)}\psi(q, q', \lambda),\\
&\eta_X(g)\psi^\lambda(g) \Longleftrightarrow \ell_X(q', \partial_{q'}, \lambda)\psi(q, q', \lambda).
\end{aligned} \tag{14}$$

It follows from (14) that functions from the space $L_2(G, d\mu(g))$ that have the form (12) are eigenfunctions of the Casimir operators $K^{(s)}_{mu}(i\hbar\xi) = K^{(s)}_\mu(-i\hbar\eta)$:

$$\begin{aligned}
&K^{(s)}_\mu(i\hbar\xi)\psi^\lambda(g) \Longleftrightarrow \kappa^{(s)}_\mu(\lambda)\psi(q, q', \lambda),\\
&K^{(s)}_\mu(-i\hbar\ell(q', \partial_{q'}, \lambda)) = \kappa^{(s)}_\mu(\lambda), \quad \overline{\kappa^{(s)}_\mu(\lambda)} = \kappa^{(s)}_\mu(\lambda), \quad \lim_{\hbar\to 0}\kappa^{(s)}_\mu(\lambda) = \omega^{(s)}_\mu(\lambda).
\end{aligned}$$

Since the left- and right-invariant vector fields on the Lie group G are transformed into λ-representation operators under the transformation (12), the Casimir operators in the λ-representation are constants.

3. Symmetry Algebra of a Quantum Harmonic Oscillator

The states of a one-dimensional quantum harmonic oscillator in the coordinate representation $\hat{x} = x, \quad \hat{p} = -i\hbar\partial_x$ are described by the wave function $\psi = \psi(t, x)$, which satisfies the nonstationary Schrödinger equation

$$i\hbar\partial_t = \hat{H}\psi, \quad \hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2, \tag{15}$$

where $m > 0$ is the mass of the quantum particle, $\omega > 0$ is the frequency of the harmonic oscillator, and $\hbar$ is Planck's constant.

The well-known wave functions of the harmonic oscillator in terms of the Hermite polynomials $H_n(z)$ are [4]

$$\begin{aligned}
\psi_n(t, x) &= \exp\left(-i\frac{E_n}{\hbar}t\right)\psi_n(x),\\
\psi_n(x) &= \langle x \mid n\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right),\\
E_n &= \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots.
\end{aligned} \tag{16}$$

The eigenstates $|n\rangle$ for the Hamiltonian $\hat{H}$ are called Fock's or number states, $\hat{H}|n\rangle = E_n|n\rangle$. The Fock states are orthonormal and form a complete basis such that any other state of the harmonic oscillator may be written in terms of them.

We can define the annihilation and creation operators by the formulas

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} + \frac{i}{\sqrt{\hbar\omega m}}\hat{p}\right), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\hat{x} - \frac{i}{\sqrt{\hbar\omega m}}\hat{p}\right), \quad [\hat{a}, \hat{a}^\dagger] = 1,$$

respectively. The time dependent-coherent states $|z,t\rangle$ are eigenstates of the annihilation operator $\hat{a}$,

$$\hat{a}|z,t\rangle = z(t)|z,t\rangle, \quad z(t) = ze^{-i\omega t},$$

where the eigenvalue of the operator $\hat{a}$ is a complex number $z(t)$, which is a function of time t. The coherent states may be written as

$$|z,t\rangle = e^{-i\omega t/2}e^{-|z(t)|^2/2}\sum_{n=0}^{\infty}\frac{z^n(t)}{\sqrt{n!}}|n\rangle, \quad \langle z,t \mid z,t\rangle = 1. \tag{17}$$

In the coordinate representation, we have

$$\alpha(t,x;z) = \langle x \mid z,t\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-i\frac{\omega t}{2} - \left(\sqrt{\frac{m\omega}{2\hbar}}x - z(t)\right)^2 + \frac{z(t)^2}{2} - \frac{|z|^2}{2}\right]. \tag{18}$$

The real and imaginary parts of the quantum number z characterize the mean values of position and momentum operators:

$$\langle\hat{x}(t)\rangle = \sqrt{\frac{2\hbar}{m\omega}}\mathrm{Re}(z(t)), \quad \langle\hat{p}(t)\rangle = \sqrt{2m\hbar\omega}\mathrm{Im}(z(t)).$$

Equation (15) admits four integrals of motion in the class of the first-order linear differential operators:

$$\begin{aligned}
iX_1 &= (\hbar\omega)^{-1}\hat{p}_0, \quad \hat{p}_0 = i\hbar\partial_t,\\
-iX_2 &= -i\sqrt{\frac{m\omega\hbar}{2}}\left(e^{i\omega t}\hat{a} - e^{-i\omega t}\hat{a}^\dagger\right) = \cos(\omega t)\hat{p} + m\omega x\sin(\omega t),\\
-iX_3 &= -\sqrt{\frac{m\omega\hbar}{2}}\left(e^{i\omega t}\hat{a} + e^{-i\omega t}\hat{a}^\dagger\right) = \sin(\omega t)\hat{p} - m\omega x\cos(\omega t),\\
-iX_4 &= m\omega\hbar.
\end{aligned} \tag{19}$$

These operators form the Lie algebra $\mathfrak{g}_{osc}$ with non-zero commutation relations

$$[X_1, X_2] = -X_3, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = -X_4. \tag{20}$$

The algebra $\mathfrak{g}_{osc}$ with the commutation relations (20) is called the oscillatory Lie algebra. In the next section, we will construct a special irreducible λ-representation of this Lie algebra, which is necessary for solving the Equation (15) in terms of the noncommutative integration method.

4. λ-Representation of the Oscillatory Lie Algebra

Let $\{e_a\}$ be some fixed basis of the Lie algebra $\mathfrak{g}_{osc}$, $a = 1,\ldots,4$, and let $[\cdot,\cdot]$ be the commutator in $\mathfrak{g}_{osc}$,

$$[e_1, e_2] = -e_3, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = -e_4.$$

An arbitrary element $X \in \mathfrak{g}_{osc}$ is determined by its components X^a with respect to the chosen basis, $X = X^a e_a$. In turn, an arbitrary element $f \in \mathfrak{g}^*_{osc}$ of the dual space $\mathfrak{g}^*$ is determined by the components of f_a with respect to the basis $\{e^b\}$ dual to the basis $\{e_a\}, f = f_a e^a$, $\langle e^b, e_a\rangle = \delta^b_a$.

The Lie algebra $\mathfrak{g}_{osc}$ admits two Casimir functions

$$K_1(f) = 2f_1f_4 + f_2^2 + f_3^2, \quad K_2(f) = f_4, \quad f \in \mathfrak{g}_{osc}^*.$$

Nondegenerate orbits of the coadjoint representation (K-orbits) pass through the parametrized covector $\lambda = (j_1, 0, 0, j_2)$,

$$\mathcal{O}_\lambda = \{K_1(f) = 2j_1j_2, \quad K_2(f) = j_2, \quad \neg(f_2 = f_3 = f_4 = 0)\}.$$

Denote by $G_{osc} = \exp \mathfrak{g}_{osc}$ the local Lie group of the Lie algebra $\mathfrak{g}_{osc}$. Let us introduce canonical coordinates of the second kind, $x = (x_1, x_2, x_3, x_4)$, on the group G_{osc} as

$$g(x_1, x_2, x_3, x_4) = e^{x_4e_4}e^{x_3e_3}e^{x_2e_2}e^{x_1e_1} \in G_{osc}. \tag{21}$$

The group composition law in the coordinates (21) has the form

$$\begin{aligned}
&g = g(x_1, x_2, x_3, x_4), \quad \tilde{g} = \tilde{g}(y_1, y_2, y_3, y_4),\\
&g\tilde{g} = (g\tilde{g})\Big[x_1 + y_1, x_2 + y_2\cos x_1 + y_3 \sin x_1, x_3 + y_3 \cos x_1 - y_2 \sin x_1,\\
&x_4 + y_4 + x_2(y_1 \sin x_1 - y_3 \cos x_1) + y_2y_3 \sin^2 x_1 + \frac{y_2^2 - y_3^2}{4}\sin(2x_1)\Big].
\end{aligned}$$

The Lie group G_{osc} acts on itself by the left L_g and right R_g shifts. The left-invariant vector fields $\xi_a(g) = (L_g)_*e_a$ on the group G_{osc} in local coordinates (21) are

$$\begin{aligned}
&\xi_1 = \partial_1, \quad \xi_4 = \partial_4,\\
&\xi_2 = \cos x_1\partial_2 - \sin x_1\partial_3 + x_2 \sin x_1\partial_4,\\
&\xi_3 = \sin x_1\partial_2 + \cos x_1\partial_3 - x_2 \cos x_1\partial_4.
\end{aligned} \tag{22}$$

The right-invariant vector fields $\eta_a(g) = -(R_g)_*e_a$ are in turn defined by the expressions

$$\begin{aligned}
&\eta_3 = -\partial_1, \quad \eta_4 = -\partial_4,\\
&\eta_1 = -\partial_1 - x_3\partial_2 + x_2\partial_3 + \frac{1}{2}\left(x_3^2 - x_2^2\right)\partial_4,\\
&\eta_2 = -\partial_2 + x_3\partial_4.
\end{aligned}$$

The Lie group G_{osc} is unimodular and the Haar measure coincides with the Lebesgue measure $d\mu(g) = dx_1dx_2dx_3dx_4$. Suppose that the coordinates x_i values take values in $\mathbb{R}^1$.

There exists a three-dimensional complex subalgebra $\mathfrak{h} = \operatorname{span}\{e_1, e_2 + ie_3, e_4\}$ of the complex extension $\mathfrak{g}_{\mathbb{C}}^{osc}$ of the algebra $\mathfrak{g}_{osc}$ subject to the functional $\lambda(j)$, so that $\langle\lambda(j), [\mathfrak{h}, \mathfrak{h}]\rangle = 0$. This subalgebra is a complex polarization corresponding to the linear functional $\lambda(j)$. This polarization corresponds to the canonical transition

$$\begin{aligned}
&f_1(q, p, \lambda) = ipq + j_1, \quad f_2(q, p, \lambda) = -\frac{i}{2}p + j_2q,\\
&f_3(p, q, \lambda) = \frac{1}{2}p - ij_2q, \quad f_4(p, q, \lambda) = j_2.
\end{aligned}$$

The λ-representation operators are of the form

$$\begin{aligned}
&\ell_1(q, \partial_q, \lambda) = i\left[q\partial_q - \frac{1}{\hbar}\left(\frac{j_2}{2}q^2 - j_1\right)\right], \quad \ell_2(q, \partial_q, \lambda) = -i\left(\partial_q - \frac{j_2}{\hbar}q\right),\\
&\ell_3(q, \partial_q, \lambda) = \partial_q, \quad \ell_4(q, \partial_q, \lambda) = \frac{i}{\hbar}j_2, \quad Q \in \mathbb{C},\\
&K_2(-i\hbar\ell) = -(\hbar - 2j_1)j_2, \quad K_2(-i\hbar\ell) = j_2.
\end{aligned} \tag{23}$$

The function space

$$\mathscr{F}^{\lambda} = \operatorname{span}\left\{ \varphi_n(q) = q^n \exp\left(\frac{j_2}{4\hbar} q^2\right) \mid n = 0, 1, 2, \dots \right\}$$

is invariant under the λ-representation operators and is a Hilbert space with respect to the scalar product (4) with the measure

$$d\mu_{j_2}(q) = \exp\left[-\frac{j_2}{4\hbar}(q-\overline{q})^2\right] = \exp\left[-\frac{j_2}{4\hbar}(q-\overline{q})^2\right].$$

The functions of the space $\mathscr{F}^{\lambda}$ are entire analytic functions of the complex variable q. The generalized Dirac function in the space $\mathscr{F}^{\lambda}$,

$$\psi(q) = \int_Q \psi(q')\delta_{j_2}(q, \overline{q'}) d\mu_{j_2}(q'), \quad \psi \in \mathscr{F}^{\lambda}$$

is defined by the expression

$$\delta_{j_2}(q, \overline{q'}) = -\frac{j_2}{2\pi\hbar} \sum_{n=0}^{\infty} \frac{(-j_2/2\hbar)^n}{n!} \varphi_n(q)\overline{\varphi_n(q')} = -\frac{j_2}{2\pi\hbar} \exp\left[\frac{j_2}{4\hbar}\left(q - \overline{q'}\right)^2\right].$$

By integrating the system of Equation (7), we obtain

$$\begin{aligned} &\mathscr{D}^{\lambda}_{qq'}\left(g^{-1}\right) = U^{\lambda}(q,g)\delta_{j_2}\left(qg^{-1}, \overline{q'}\right), \\ &U^{\lambda}(q,g) = \exp\left\{-i\frac{j_1}{\hbar}x_1 - i\frac{j_2}{\hbar}x_4 + \frac{j_2}{4\hbar}\left[\left(1 - e^{-2ix_1}\right)q^2 + 2\left(x_2 - 2iqe^{-ix_1}\right)x_2\right]\right\}, \end{aligned} \quad (24)$$

where $qg^{-1} = q\exp(-ix_1) + ix_2 - x_3$ is the action of the group G_{osc} on the complex manifold Q, which is given by the generators

$$\alpha_1(q) = iq\partial_q, \quad \alpha_2(q) = -i\partial_q, \quad \alpha_3(q) = \partial_q,$$

so we have

$$\xi^{\mu}_X(g)\frac{\partial(qg)^a}{\partial g^{\mu}} = \alpha^a_X(qg), \quad qe = q, \quad X \in \mathfrak{g}.$$

The representation (9) becomes an induced representation of the Lie group G_{osc} and, according to (24), has the form

$$\begin{aligned} &(T^{\lambda}_{g^{-1}}\psi)(q) = U^{\lambda}(q,g)\psi(qg^{-1}), \\ &U^{\lambda}(q,\tilde{g}g) = U^{\lambda}(q,g)U^{\lambda}(qg^{-1},\tilde{g}), \quad U^{\lambda}(q,e) = 1. \end{aligned}$$

It can be shown that any λ-representation of the Lie algebra in the class of the first-order linear partial differential operators leads to the induced representation of the Lie group constructed in the framework of the Kirillov orbit method (see Refs. [6,7,9]). The relations (10) are satisfied with respect to the measure

$$d\mu(\lambda) = \frac{j_2}{(2\pi\hbar)^3} dj_1 dj_2, \quad \int_J (\cdot) d\mu(\lambda) = -\int_{-\infty}^{\infty} dj_1 \int_{-\infty}^{\infty} (\cdot) j_2 dj_2.$$

The direct Fourier transform (12) in the space $L_2(G, d\mu(g))$ has the form

$$\Psi(g) = \int_Q \psi(q'; q, \lambda)\mathscr{D}^{\lambda}_{qq'}\left(g^{-1}\right) d\mu_{j_2}(q') d\mu_{j_2}(q) d\mu(\lambda), \quad \psi \in \mathscr{F}^{\lambda}. \quad (25)$$

For an invariant second-order differential equation on the group G_{osc} written as

$$H(-i\hbar\eta)\Psi(g) = 0, H(f) = A^{ab}f_af_b + B^af_a + C, \tag{26}$$

where A^{ab}, B^a, and C are constants, the general solution is sought in the form (25) in the framework of NIM. Then, we have the reduced equation for the function $\psi(q';q,\lambda)$,

$$H(-i\hbar\ell(q',\partial_q,\lambda))\psi(q';q,\lambda) = 0, \tag{27}$$

which is an ordinary differential equation with respect to the independent variable q'. Equation (27) will be called the Equation (26) in the λ-*representation*, and the transition from (26) to (27) will be called the *non-commutative reduction* of Equation (26).

5. The Schrödinger Equation on the Oscillatory Lie Group

Let us show that the Schrödinger Equation (15) describing the quantum harmonic oscillator is equivalent to the following system of equations on the Lie group G_{osc} :

$$K_1(-i\hbar\xi)\Psi(g) = 0, \tag{28}$$
$$K_2(-i\hbar\xi)\Psi(g) = \hbar m\Psi(g), \tag{29}$$
$$\eta_3\Psi(g) = 0. \tag{30}$$

Indeed, the general solution to the Equations (29) and (30) can be written as

$$\Psi(g) = \psi\left(\frac{x_1}{\omega}, \sqrt{\frac{\hbar}{\omega}}x_2\right)e^{imx_4}.$$

Substituting $\Psi(g)$ into the second Equation (28), we obtain the Schrödinger equation for the function $\psi(t,x)$ in the form

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi, \quad \psi = \psi(t,x), \quad x_1 = \omega t, \quad x_2 = \sqrt{\frac{\omega}{\hbar}}x. \tag{31}$$

Thus, we have reduced the Schrödinger equation to the system of Equations (29) and (30) on the Lie group G_{osc}, for which the set of basic left-invariant vector fields (22) is a set of non-commuting integrals of motion forming the Lie algebra $\mathfrak{g}_{osc}$.

Let us integrate the system (29) and (30) using the NIM. We are looking for a solution to this system in the form (25). Then, we obtain the non-commutative reduced system of equations for the function $\psi(q';q,\lambda)$ as

$$\left(j_1 - \frac{\hbar}{2}\right)\psi(q';q,\lambda) = 0, \quad (j_2 + \hbar m)\psi(q';q,\lambda) = 0, \quad \partial_{q'}\psi(q';q,\lambda) = 0. \tag{32}$$

The system (32) says that the quantum harmonic oscillator corresponds to the orbit of the coadjoint representation $\mathcal{O}_\sigma$ of the group G_{osc}, which passes through the parameterized covector $\sigma = (1/2, 0, 0, -m)$, and the function $\psi(q';q,\lambda)$ describing the quantum harmonic oscillator in terms of the λ-representation does not depend on variable q'. From (32), we have

$$\psi(q';q,\lambda) = \psi(q)\delta\left(j_1 - \frac{\hbar}{2}\right)\delta(j_2 + \hbar m). \tag{33}$$

Substituting (33) into (25) yields the general solution

$$\Psi(g) = \int_Q \psi(q)\mathscr{D}^{\lambda}_{qq'}\left(g^{-1}\right)d\mu_\lambda(q)d\mu_\lambda(q')\Big|_{j_1=\hbar/2,\, j_2=-\hbar m}$$
$$= \int_Q \psi(q)\mathscr{D}_q\left(x_1, x_2, x_4; \frac{\hbar}{2}, -\hbar m\right)d\mu_{-m\hbar}(q),$$
$$\mathscr{D}_q(x_1, x_2, x_4; j_1; j_2) = \int_Q \mathscr{D}^{\lambda}_{qq'}\left(g^{-1}\right)d\mu_{j_2}(q) = U^\lambda(q, g). \tag{34}$$

Equation (25) gives the general solution to the system of Equations (29) and (30). According to Equation (31), the general solution to the Schrödinger equation is obtained from (34) by setting $x_1 = \omega t$, $x_2 = x\sqrt{\omega/\hbar}$, $x_3 = x_4 = 0$. It is convenient to represent the general solution to the Schrödinger equation as follows. Let us introduce a set of functions

$$\mathscr{D}(t, x \mid u; \mu) = \omega\frac{\sqrt{\hbar m}}{2\pi}\left(\frac{\omega m}{\pi\hbar}\right)^{1/4}\mathscr{D}_{u\sqrt{\omega\hbar}}\left(\omega t, \sqrt{\frac{\omega}{\hbar}}x, 0; \mu\hbar; -m\hbar\right), \tag{35}$$

which satisfies the completeness and orthogonality conditions:

$$\int_{\mathbb{R}^2}\overline{\mathscr{D}(t, x \mid \tilde{u}; \tilde{\mu})}\mathscr{D}(t, x \mid u; \mu)dtdx = \delta(u, \overline{\tilde{u}})\delta(\mu - \tilde{\mu}),$$
$$\int_{-\infty}^{\infty}d\mu\int_{\mathbb{C}^1}d\mu(u)\overline{\mathscr{D}(\tilde{t}, \tilde{x} \mid u; \mu)}\mathscr{D}(t, x \mid u; \mu) = \delta(t - \tilde{t})\delta(x - \tilde{x}),$$
$$\delta(u, \overline{\tilde{u}}) = \frac{m\omega\hbar}{2\pi}\exp\left[-\frac{m\omega\hbar}{4}\left(u - \overline{\tilde{u}}\right)^2\right], \quad d\mu(u) = \exp\left[\frac{m\omega\hbar}{4}(u - \overline{u})^2\right].$$

Then the general solution of the Schrödinger equation, according to (34), is written as

$$\psi(t, x) = \int_{\mathbb{C}^1}\overline{\varphi(u)}\mathscr{D}(t, x \mid u; 1/2)du, \tag{36}$$
$$u = q/\sqrt{\omega\hbar} \in \mathbb{C}^1, \quad \varphi \in \mathscr{F}^{\lambda_\omega}, \quad \lambda_\omega = \left(\frac{1}{2}, 0, 0, -m\omega\hbar\right). \tag{37}$$

Moreover, for the solution norm (36), we have

$$\|\psi\|^2 = \int_{-\infty}^{\infty}|\psi(t, x)|^2dx = \frac{\omega}{2\pi}\int_{-\infty}^{\infty}|\varphi(u)|^2d\mu(u) = \frac{\omega}{2\pi}\|\varphi\|^2_Q.$$

As a result, using the NIM, we have found a general solution (36) to the Schrödinger equation for the quantum harmonic oscillator. We say that this solution describes the H-state of the harmonic oscillator. Let us show that for a given solution (36), stationary solutions are obtained, which are determined from the equation

$$\hat{p}_0\psi(t, x) = E\psi(t, x). \tag{38}$$

Substituting (36) into (38) by the function $\varphi(u)$, we obtain the equation

$$-i\omega\hbar\ell_1(u, \partial_u, \lambda_\omega)\varphi(u) = E\varphi(u).$$

From here, we obtain

$$\varphi(u) = u^{\frac{E}{\omega\hbar}-\frac{1}{2}}e^{-\frac{m\omega\hbar}{4}u^2}.$$

The function $\varphi(u)$ belongs to the space $\mathscr{F}^{\lambda_\omega}$ iff $E/\omega\hbar - 1/2 = n$, and n is an integer. This condition results in the well-known spectrum of the quantum harmonic oscillator: $E_n = \hbar\omega(n + 1/2)$. The corresponding wave functions on the manifold Q coincide with the basis functions $\varphi_n(q)$ up to a normalization factor:

$$\varphi_n(u) = C_n u^n e^{-\frac{m\omega\hbar}{4}u^2},$$
$$C_n = (-i)^n \left(\frac{\omega}{2\pi}\|\varphi_n\|_Q^2\right)^{-1/2} = \sqrt{\frac{\hbar m}{2^n n!}}(-m\omega\hbar)^{n/2}. \tag{39}$$

Then, (36) provides the well-known expression for the wave functions of the harmonic oscillator in terms of the Hermite polynomials (16) as

$$\psi_n(t,x) = \int_{\mathbb{C}^1} \overline{\varphi_n(u)}\mathscr{D}(t,x \mid u;1/2)d\mu(u).$$

Thus, Fock's states $|n\rangle$ of the harmonic oscillator in the λ-representation (39) generate the space $\mathscr{F}^{\lambda_\omega}$ in which the λ-representation of the oscillatory group acts.

Comparing (18) and (35), we obtain the relationship between the H-states and the harmonic oscillator coherent states in the form

$$\mathscr{D}(t,x \mid u;1/2) = \frac{\omega}{2\pi}\sqrt{\hbar m}\,\alpha\left(t,x;i\sqrt{\frac{m\omega\hbar}{2}}e^{-i\omega t}u\right)\exp\left[-\frac{m\omega\hbar}{4}\left(u^2-|u|^2\right)\right].$$

From here, we can see that the H-solution (36) is related to coherent states of the harmonic oscillator, but it differs from the latter by a constant factor. In bracket notation, the solution (36) can be represented as

$$|\psi(t)\rangle = \int_{\mathbb{C}^1} du\overline{\varphi(u)}|u,t\rangle,$$
$$|u,t\rangle = \frac{\omega}{2\pi}\sqrt{\hbar m}\exp\left[-\frac{m\omega\hbar}{4}\left(u^2-|u|^2\right)\right]|z,t\rangle, \quad \langle u,t \mid u,t\rangle = \frac{\omega}{2\pi}\delta(u,\overline{u}), \tag{40}$$
$$z(t) = i\sqrt{\frac{m\omega\hbar}{2}}e^{-i\omega t}u.$$

Here $|z,t\rangle$ is a coherent state with a wave function (18), and the wave function (36) corresponds to the state $|u,t\rangle$. Accordingly, for mean values, one can obtain

$$\langle\hat{x}(t)\rangle_Q = \langle u,t \mid \hat{x}(t) \mid u,t\rangle = -m\left(\frac{\omega\hbar}{2\pi}\right)^2\exp\left[-\frac{m\omega\hbar}{2}\left(u^2-|u|^2\right)\right]\mathrm{Im}\left(e^{-i\omega t}u\right),$$
$$\langle\hat{p}(t)\rangle_Q = \langle u,t \mid \hat{p}(t) \mid u,t\rangle = \hbar m\left(\frac{\omega}{2\pi}\right)^2\sqrt{2m\hbar\omega}\exp\left[-\frac{m\omega\hbar}{2}\left(u^2-|u|^2\right)\right]\mathrm{Re}\left(e^{-i\omega t}u\right).$$

From (17), it is easy to write out the expansion of H-states $|u,t\rangle$ in terms of Fock's states:

$$|u,t\rangle = \frac{\omega}{2\pi}\sqrt{\hbar m}\exp\left[-\frac{i}{2}\omega t-\frac{m\omega\hbar}{4}u^2\right]\sum_{n=0}^{\infty}\left(\frac{m\omega\hbar}{2}\right)^n\frac{(-1)^{n/2}u^n}{\sqrt{n!}}e^{-in\omega t}|n\rangle.$$

Thus, as the result of applying the NIM to the system of Equations (29) and (30), we have obtained the H-states (40) of the harmonic oscillator, which, up to a normalization factor, coincide with known coherent states $|\,z,t\rangle$.

6. Conclusions

In this paper, we have shown that the oscillatory Lie algebra $\mathfrak{g}_{osc}$ naturally arises as the Lie algebra formed by the symmetry operators (19) of the Schrödinger equation, (15) and the Schrödinger equation itself for the harmonic oscillator is equivalent to a system of right-invariant equations on the corresponding Lie group G_{osc}. As a result of the noncommutative integration of this system, a complete set of solutions (36) (H-solutions) is found. Moreover, the quantum harmonic oscillator corresponds to the only non-degenerate orbit $\mathcal{O}_\sigma$ of the adjoint representation of the Lie group G_{osc}. It is shown that the Fock states of the harmonic oscillator in the λ-representation form a Hilbert space $\mathscr{F}^{\lambda_\omega}$, which is invariant under the operators of the λ-representation (23) constructed along the given orbit. It turns out that

the H-solutions are eigenvalues for the annihilation operator $\hat{a}$, and therefore, they differ from the known coherent states of the harmonic oscillator by a factor that does not depend on t and x (see Equation (40)) but depends on the complex quantum number u.

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Abbreviations

The following abbreviations are used in this manuscript:

NIM noncommutative integration method

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