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Bi-Unitary Superperfect Polynomials over $\mathbb{F}_2$ with at Most Two Irreducible Factors

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Abstract: A divisor B of a nonzero polynomial A, defined over the prime field of two elements, is unitary (resp. bi-unitary) if $gcd(B, A/B) = 1$ (resp. $gcd_u(B, A/B) = 1$), where $gcd_u(B, A/B)$ denotes the greatest common unitary divisor of B and A/B. We denote by $\sigma^{**}(A)$ the sum of all bi-unitary monic divisors of A. A polynomial A is called a bi-unitary superperfect polynomial over $\mathbb{F}_2$ if the sum of all bi-unitary monic divisors of $\sigma^{**}(A)$ equals A. In this paper, we give all bi-unitary superperfect polynomials divisible by one or two irreducible polynomials over $\mathbb{F}_2$. We prove the nonexistence of odd bi-unitary superperfect polynomials over $\mathbb{F}_2$.

Keywords: sum of divisors; bi-unitary divisors; polynomials; finite fields.

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1. Introduction

Let n and k be positive integers, and let $\sigma(n)$ (resp. $\sigma^*(n)$) denote the sum of positive (resp. unitary) divisors of the integer n. A divisor d of n is unitary if d and n/d are coprime. We call the number n a $k-$superperfect number if $\sigma^k(n) = \underbrace{\sigma(\sigma(...(\sigma(n))))}_{k\text{-times}} = 2n$. When $k = 1$, n is called a perfect number. An integer $M = 2^p - 1$, where p is a prime number, is called a Mersenne number. It is also well known that an even integer n is perfect if and only if $n = M(M+1)/2$ for some Mersenne prime number M. Suryanarayana [1] considered $k-$superperfect numbers in the case $k = 2$. Numbers of the form 2^{p-1} (p is prime) are 2-superperfect if $2^{p-1} - 1$ is a Mersenne prime. It is not known if there are odd k-superperfect numbers. Sitaramaiah and Subbarao [2] studied the unitary superperfect numbers, with the integers n satisfying $\sigma^{*2}(n) = \sigma^*(\sigma^*(n)) = 2n$. They found all unitary superperfect numbers below 10^8. The first unitary superperfect numbers are 2, 9, 165, and 238. A positive integer n has a bi-unitary divisor, d, if the greatest common unitary divisor of d and n/d is equal to 1. The arithmetic function $\sigma^{**}(n)$ denotes the sum of positive bi-unitary divisors of the integer n. Wall [3] proved that there are only three bi-unitary perfect numbers ($\sigma^{**}(n) = 2n$), namely, 6, 60, and 90. Yamada [4] proved that 2 and 9 are the only bi-unitary superperfect numbers, that is, $\sigma^{**2}(n) = 2n$ if and only if $n \in \{2, 9\}$.

Here, let A be a nonzero polynomial over the prime field $\mathbb{F}_2$. We say that A is a splitting polynomial if it can be factored completely into linear factors over $\mathbb{F}_2$. A divisor B of A is unitary (resp. bi-unitary) if $gcd(B, A/B) = 1$ (resp. $gcd_u(B, A/B) = 1$), where $gcd_u(A, A/B)$ denotes the greatest common unitary divisor of B and A/B. We denote by σ the sum of the monic divisors B of A, that is, $\sigma(A) = \sum_{B|A} B$. $\sigma^*(A)$ (resp. $\sigma^{**}(A)$) represents the sum of all unitary (resp. bi-unitary) monic divisors of A. Note that all the functions σ, σ^*, and σ^{**} are multiplicative and degree-preserving.

We say that a polynomial A is an even polynomial if it has a linear factor in $\mathbb{F}_2[x]$; otherwise, it is an odd polynomial. A polynomial M of the form $1 + x^a(x+1)^b$ is called Mersenne. The first five Mersenne polynomials over $\mathbb{F}_2$ are $M_1 = 1 + x + x^2$, $M_2 = 1 + x + x^3$,

$M_3 = 1 + x^2 + x^3$, $M_4 = 1 + x + x^2 + x^3 + x^4$, and $M_5 = 1 + x^3 + x^4$. Note that all these polynomials are irreducible, so we call them Mersenne primes.

Notations: We use the following notations throughout the article:

- $\mathbb{N}$ (resp. $\mathbb{N}^*$) represents the set of non-negative (resp. positive) integers.
- $\deg(A)$ denotes the degree of the polynomial A.
- $\overline{A}$ is the polynomial obtained from A with x replaced by $x+1$, that is, $\overline{A}(x) = A(x+1)$.
- P and Q are distinct irreducible non-constant polynomials.
- P_i and Q_j are distinct odd irreducible non-constant polynomials.

Let $\omega(A)$ denote the number of distinct irreducible monic polynomials that divide A. The notion of a perfect polynomial over $\mathbb{F}_2$ was introduced first by Canaday [5]. A polynomial A is perfect if $\sigma(A) = A$. Canaday studied the case of even perfect polynomials with $\omega(A) \leq 3$. In the past few years, Gallardo and Rahavandrainy [6–8] showed the non-existence of odd perfect polynomials over $\mathbb{F}_2$ with either $\omega(A) = 3$ or with $\omega(A) \leq 9$ in the case where all exponents of the irreducible factors of A are equal to 2. A polynomial A is said to be a unitary (resp. a bi-unitary) perfect if $\sigma^*(A) = A$ (resp. $\sigma^{**}(A) = A$). Furthermore, A is called a unitary (resp. a bi-unitary) superperfect if $\sigma^{*2}(A) = \sigma^*(\sigma^*(A)) = A$ (resp. $\sigma^{**2}(A) = \sigma^{**}(\sigma^{**}(A)) = A$).

Note that the function σ^{**2} is degree-preserving but not multiplicative, and this is the main challenge in this work. Thus, working on bi-unitary superperfect polynomials over $\mathbb{F}_2$ is not an easy task especially when A is divisible by more than two irreducible factors.

In this paper, we prove the non-existence of odd bi-unitary superperfect polynomials A when A is divisible by at least two irreducible factors (Corollary 4). We give a complete classification for all bi-unitary superperfect polynomials over $\mathbb{F}_2$ that are divisible by at most two distinct irreducible factors (Theorems 1 and 2). Bi-unitary superperfect polynomials over $\mathbb{F}_2$ that are neither unitary perfect nor bi-unitary perfect are found. The polynomials $x^4(x+1)^4, x^9(x+1)^9, x^9(x+1)^{13}$, and $x^2(x+1)^{2^d-1}$, d is a positive integer, are examples of bi-unitary superperfect polynomials that are neither unitary perfect nor bi-unitary perfect.

Our main results are given in the following theorems:

Theorem 1. *Let A be a bi-unitary superperfect over $\mathbb{F}_2$ such that $\omega(A) = 1$; then, $A, \overline{A} \in \{x^2, x^{2^d-1}\}$, where $d \in \mathbb{N}^*$.*

Theorem 2. *Let A be a bi-unitary superperfect over $\mathbb{F}_2$ such that $\omega(A) = 2$; then, $A, \overline{A} \in \{x^2(x+1)^2, x^4(x+1)^4, x^9(x+1)^9, x^9(x+1)^{13}, x^2(x+1)^{2^d-1}, x^{2^{d_1}-1}(x+1)^{2^{d_2}-1}\}$, where $d, d_1, d_2 \in \mathbb{N}^*$.*

2. Previous Work

Many researchers studied the unitary perfect polynomials over $\mathbb{F}_2$. In their works [7,8], the authors listed the unitary perfect polynomials over $\mathbb{F}_2$, where $\omega(A)$ does not exceed 4. They listed others that are divisible by $x(x+1)M$, where M is a Mersenne polynomial, raised to certain powers. They proved that the only unitary perfect polynomials over $\mathbb{F}_2$ of the form $A = x^a(x+1)^b \prod_{i=1} M_i$ and $h_i = 2^{n_i}$, $n_i \in \mathbb{N}^*$ are those of the form B^{2n} or $\overline{B}^{2n}$, where

$$B \in \begin{cases} x^3(x+1)^3M_1^2, x^3(x+1)^2M_1, x^5(x+1)^4M_4 & \text{if } \omega(A) \leq 3, \\ x^7(x+1)^4M_2M_3, x^5(x+1)^6M_1^2M_4, x^5(x+1)^5M_4M_5, x^7(x+1)^7M_2^2M_3^2 & \text{if } \omega(A) = 4, \\ x^7(x+1)^6M_2^1M_2M_3, x^7(x+1)^5M_2M_3M_5 & \text{if } \omega(A) = 5. \end{cases}$$

In [9], Beard found many bi-unitary perfect polynomials over $\mathbb{F}_{p^d}$, some of which are neither perfect nor unitary perfect. Beard showed that the only bi-unitary perfect polynomials over $\mathbb{F}_2$ with exactly two prime factors are $x^2(x+1)^2$ and $x^{2^{n-1}}(x+1)^{2^{n-1}}$, for any $n \in \mathbb{N}^*$ (Theorem 5 in [9]). He conjectured a characterization of the bi-unitary perfect polynomials, which splits over $\mathbb{F}_p$ when $p > 2$. Beard also gave examples of non-splitting bi-unitary perfect polynomials over $\mathbb{F}_p$ when $p \in \{2, 3, 5\}$. Rahavandrainy [10] gave all bi-unitary perfect polynomials over the prime field $\mathbb{F}_2$, with at most four irreducible factors (Lemmas 7 and 8).

Gallardo and Rahavandrainy [11] classified some unitary superperfect polynomials with a small number of prime divisors under some conditions on the number of prime factors of $\sigma^*(A)$. They proved that $A \in \mathbb{F}_2[x]$ is a unitary superperfect polynomial if

$$A \in \begin{cases} x^{2^n}(x+1)^{2^m}, x^{3\cdot 2^n}(x+1)^{3\cdot 2^m}, x^3(x+1)^5, x(x+1)^5, x^7(x+1)^7 & \text{if } \omega(A) = 2, \\ x^2(x+1)^3 M_1, x^3(x+1)^3 M_1^a, x(x+1)^5 M_1^a, x(x+1)^5(x^3+x^2+1) & \text{if } \omega(A) = 3. \end{cases}$$

For some $m, n \in \mathbb{N}^*$ and $a \in \{1, 2\}$.

3. Preliminaries

The following two lemmas are helpful.

Lemma 1. *Let A be a polynomial in $\mathbb{F}_2[x]$; then, $\sigma^*\left(A^{2^n}\right) = (\sigma^*(A))^{2^n}$ and n is a non-negative integer.*

Proof. The result follows since σ^* is multiplicative and $\sigma^*(p^{2^n}) = 1 + p^{2^n} = (1+p)^{2^n} = (\sigma^*(p))^{2^n}$. □

Lemma 2. *If A is a unitary superperfect polynomial over $\mathbb{F}_2$, then A^{2^n} is also a unitary superperfect polynomial over $\mathbb{F}_2$ for all non-negative integers n.*

Proof. Let A be a unitary superperfect, and let $B = \sigma^*(A)$. By Lemma 1, we have $\sigma^{*2}\left(A^{2^n}\right) = \sigma^*\left(\sigma^*\left(A^{2^n}\right)\right) = \sigma^*\left(B^{2^n}\right) = (\sigma^*(B))^{2^n} = (\sigma^*(\sigma^*(A)))^{2^n} = A^{2^n}$. □

Lemma 3 (Lemma 2.4 in [11])**.** *Let A be a polynomial in $\mathbb{F}_2[x]$.*

(1) If P is an odd prime factor of A, then $x(x+1)$ divides $\sigma^*(A)$.
(2) If $x(x+1)$ divides A, then $x(x+1)$ divides $\sigma^*(A)$.
(3) If A is unitary superperfect that has an odd prime factor, then $x(x+1)$ divides A.

The following results are needed, and they are a result of Beard's [9] and Rahavandrainy's [10] works.

Lemma 4 (Theorem 1 and its Corollary in [9])**.** *If A is a non-constant bi-unitary perfect polynomial, then $x(x+1)$ divides A and $\omega(A) \geq 2$.*

Proposition 1 (Lemma 2.2 in [10])**.**

(1) $\sigma^{**}(P^{2a+1}) = \sigma(P^{2a+1})$.
(2) $\sigma^{**}(P^{2a}) = (1 + P^{a+1})\sigma(P^{a-1}) = (1+P)\sigma(P^a)\sigma(P^{a-1})$.

The table in Section 7 shows some values of $\sigma^{**}(A)$ when A is a power of the first five Merssene primes.

Corollary 1. *If a is a positive integer, then*

(1) 1+x divides $\sigma^{**}(x^a)$.
(2) x divides $\sigma^{**}((1+x)^a)$.

Proof. An immediate result of Proposition 1. □

Corollary 2 (Corollary 2.3 in [10]). *Let $T \in \mathbb{F}_2[x]$ be irreducible. Then,*

(i) *If $a \in \{4r, 4r+2\}$, where $2r-1$ or $2r+1$ is of the form $2^\alpha u - 1$, u odd, then $\sigma^{**}(P^a) = (1+P)^{2^\alpha} \cdot \sigma(P^{2r}) \cdot (\sigma(P^{u-1}))^{2^\alpha}$, $\gcd(\sigma(P^{2r}), \sigma(P^{u-1})) = 1$.*

(ii) *If $a = 2^\alpha u - 1$ is odd, with u odd, then $\sigma^{**}(P^a) = (1+P)^{2^\alpha - 1} \cdot (\sigma(P^{u-1}))^{2^\alpha}$.*

The proof of the below proposition follows from Proposition 1 and the binomial formula.

Proposition 2. *Let the polynomial M_i be the Mersenne prime and Q_j be an irreducible polynomial over $\mathbb{F}_2$, and let $a, c \in \mathbb{N}^*$. If $\alpha_j \in \mathbb{N}$, then*

(1) $x(x+1)$ *divides* $\sigma^{**}(M_i^c)$.

(2) $\sigma^{**}(M_1^c) = x^a (x+1)^a \prod_j Q_j^{\alpha_j}$.

(3) $\sigma^{**}(M_2^c) = x^a (x+1)^{2a} \prod_j Q_j^{\alpha_j}$.

(4) $\sigma^{**}(M_3^c) = x^{2a} (x+1)^a \prod_j Q_j^{\alpha_j}$.

(5) $\sigma^{**}(M_4^c) = x^a (x+1)^{3a} \prod_j Q_j^{\alpha_j}$.

(6) $\sigma^{**}(M_5^c) = x^{3a} (x+1)^a \prod_j Q_j^{\alpha_j}$.

Proposition 3 (Corollary 2.4 in [10]).

(1) *$\sigma^{**}(x^a)$ splits over $\mathbb{F}_2$ if and only if $a = 2$ or $a = 2^d - 1$, for some $d \in \mathbb{N}^*$.*

(2) *$\sigma^{**}(P^c)$ splits over $\mathbb{F}_2$ if and only if P is Mersenne and $c = 2$ or $c = 2^d - 1$ for some $d \in \mathbb{N}^*$.*

Lemma 5 summarizes key results taken from Canaday's paper [5].

Lemma 5. *Let T be irreducible in $\mathbb{F}_2[x]$ and let $n, m \in \mathbb{N}$.*

(i) *If T is a Mersenne prime and if $T = T^*$, then $T \in \{M_1, M_4\}$.*

(ii) *If $\sigma(x^{2n}) = PQ$ and $P = \sigma((x+1)^{2m})$, then $2n = 8$, $2m = 2$, $P = M_1$, and $Q = P(x^3) = 1 + x^3 + x^6$.*

(iii) *If any irreducible factor of $\sigma(x^{2n})$ is a Mersenne prime, then $2n \leq 6$.*

(iv) *If $\sigma(x^{2n})$ is a Mersenne prime, then $2n \in \{2, 4\}$.*

Lemma 6 (Lemma 2.6 in [12]). *Let $m \in \mathbb{N}^*$ and M be a Mersenne prime. Then, $\sigma(x^{2m})$, $\sigma((x+1)^{2m})$, and $\sigma(M^{2m})$ are all odd and square-free.*

4. Bi-Unitary Superperfect Polynomials

Recall that A is a bi-unitary superperfect polynomial in $\mathbb{F}_2[x]$ if $\sigma^{**2}(A) = \sigma^{**}(\sigma^{**}(A)) = A$. The polynomial $A = x^4(1+x)^4$ is a bi-unitary superperfect polynomial over $\mathbb{F}_2$.

The following polynomials are considered over $\mathbb{F}_2$:

$$
\begin{array}{lll}
C = 1 + x + x^4, & B_1 = x^3(x+1)^4 M_1, & B_2 = x^3(x+1)^5 M_1^2, \\
B_3 = x^4(x+1)^4 M_1^2, & B_4 = x^6(x+1)^6 M_1^2, & B_5 = x^4(x+1)^5 M_1^3, \\
B_6 = x^7(x+1)^8 M_5, & B_7 = x^7(x+1)^9 M_5^2, & B_8 = x^8(x+1)^8 M_4 M_5, \\
B_9 = x^8(x+1)^9 M_4 M_5^2, & B_{10} = x^7(x+1)^{10} M_1^2 M_5, & B_{11} = x^7(x+1)^{13} M_2^2 M_3^2, \\
B_{12} = x^9(x+1)^9 M_4^2 M_5^2, & B_{13} = x^{14}(x+1)^{14} M_2^2 M_3^2, & R_1 = x^4(x+1)^5 M_1^4 C, \\
R_2 = x^4(x+1)^5 M_1^5 C^2. & &
\end{array}
$$

The proof of the following lemmas follow directly.

Proposition 4. *If A is a bi-unitary perfect polynomial over $\mathbb{F}_2$, then A is also a bi-unitary superperfect polynomial.*

Proposition 5. *If A is a bi-unitary superperfect polynomial over $\mathbb{F}_2$, then $B = \sigma^{**}(A)$ is also a bi-unitary superperfect polynomial.*

Rahavandrainy (Lemma 2.6 in [10]) proved that if A is a bi-unitary perfect polynomial over $\mathbb{F}_2$, where $A = A_1A_2$ such that $gcd(A_1, A_2) = 1$, then A_1 is a bi-unitary perfect polynomial if and only if A_2 is a bi-unitary perfect polynomial. Rahavandrainy's previous result is not valid in the case of bi-unitary superperfect polynomials because the bi-unitary superperfect polynomial $A = x^2(1+x)^2(1+x+x^2)^2$ is a counterexample over $\mathbb{F}_2$. In fact, $A_1 = x^2(1+x)^2$ is a bi-unitary superperfect, but $A_2 = (1+x+x^2)^2$ is not a bi-unitary superperfect.

Lemma 7 (Theorem 1.1 in [10])**.** *Let $A \in \mathbb{F}_2[x]$ be a bi-unitary perfect polynomial such that $\omega(A) = 3$. Then, $A, \overline{A} \in \{B_j : j \leq 7\}$.*

Lemma 8 (Theorem 1.2 in [10])**.** *Let $A \in \mathbb{F}_2[x]$ be a bi-unitary perfect polynomial such that $\omega(A) = 4$. Then $A, \overline{A} \in \{B_j : 8 \leq j \leq 13\} \bigcup \{R_1, R_2\}$.*

Proposition 6. *If $A(x)$ is a bi-unitary superperfect polynomial over $\mathbb{F}_2$, then so is $\overline{A}(x)$.*

Lemma 9. *$x(x+1)$ divides $\sigma^{**}(P^a)$, a is a positive integer.*

Proof. Since P is odd, then $P(0) = P(1) = 1$. If $a = 2n+1$, then $\sigma^{**}(P^{2n+1})(0) = 1 + \underbrace{P(0) + \ldots + P^{2n+1}(0)}_{(2n+1)\text{-times}} = 1 + 2n + 1 = 0$. If $a = 2n$, then $1 + P^{n+1}(0) = 0$. Thus, x divides $\sigma^{**}(P^a)$ for every $a \in \mathbb{N}$. Similarly, $x+1$ divides $\sigma^{**}(P^a)$. Hence, $x(x+1)$ divides $\sigma^{**}(P^a)$. □

Lemma 10. *Let A be a polynomial in $\mathbb{F}_2[x]$.*

(1) If P is an odd prime factor of A, then $x(x+1)$ divides $\sigma^{**}(A)$.
(2) If $x(x+1)$ divides A, then $x(x+1)$ divides $\sigma^{**}(A)$.

Proof.

(1) We write $A = P^aB$, where $a \in \mathbb{N}^*$ and $B \in \mathbb{F}_2[x]$ such that $gcd(P, B) = 1$. However, $1 + P$ divides $\sigma^{**}(A)$, and the result follows since $x(x+1)$ divides $1 + P$.
(2) In a similar manner, we write $A = x^a(x+1)^bB$, where $a, b \in \mathbb{N}^*$.

□

Corollary 3. *If $A \in \mathbb{F}_2[x]$ and $\omega(A) \geq 2$, then $x(x+1)$ divides $\sigma^{**}(A)$.*

Proof. Let $\omega(A) \geq 2$. If $x(x+1)$ divides A, then Corollary 1 is completed. If $x(x+1)$ does not divide A, then A is divisible by an irreducible polynomial $P \notin \{x, 1+x\}$, and the result follows using Lemma 9. □

Corollary 4. *Let A be a polynomial in $\mathbb{F}_2[x]$ with $\omega(A) \geq 2$. If A is a bi-unitary superperfect, then $x(x+1)$ divides A.*

Proof. Let $A = \sigma^{**2}(A) = \sigma^{**}(B)$, where $B = \sigma^{**}(A)$. Since $\omega(A) \geq 2$, then either P or $x(x+1)$ divides A. In both cases, $x(x+1)$ divides $\sigma^{**}(A) = B$ (Lemma 10). Thus, $x(x+1)$ divides $\sigma^{**}(B) = \sigma^{**2}(A)$. □

The below corollary follows directly from Corollary 4.

Corollary 5. *If $A = P^a Q^b$ and $a, b \in \mathbb{N}^*$. is a bi-unitary superperfect polynomial over $\mathbb{F}_2$, then $A = x^a(x+1)^b$.*

The following lemma is similar to Proposition 3.

Lemma 11. *Let $a, b \in \mathbb{N}^*$, then*

(1) *If a is even; then, $\sigma^{**2}(x^a)$ and $\sigma^{**2}((x+1)^a)$ splits over $\mathbb{F}_2$ if and only if $a \in \{2, 4, 10, 12\}$.*

(2) *If a is odd, then $\sigma^{**2}(x^a)$ and $\sigma^{**2}((x+1)^a)$ splits over $\mathbb{F}_2$ if and only if $a \in \left\{5, 9, 13, 2^d - 1\right\}$ for some $d \in \mathbb{N}^*$.*

Proof.

(1) If $\sigma^{**}(x^a)$ splits, $a = 2$ (Proposition 3) and $\sigma^{**2}(x^a) = (x+1)^2$. Suppose that $\sigma^{**}(x^a)$ does not split with $a = 4r, 2r - 1 = 2^\alpha u - 1$, (resp. $a = 4r + 2, 2r + 1 = 2^\alpha u - 1$), u is odd, $r \geq 1$. However, $\sigma^{**2}(x^a) = \sigma^{**}\left((1+x)^{2^\alpha} \cdot \sigma(x^{2r}) \cdot \left(\sigma(x^{u-1})\right)^{2^\alpha}\right)$; thus, $\sigma^{**}\left((1+x)^{2^\alpha}\right)$ must split. Hence, $\alpha = 1$, and since $\sigma(x^{2r})$ is odd and square-free (Lemma 6), then $\sigma(x^{2r})$ has a Mersenne factor. Thus, $2r \leq 6$ and, hence, $u \leq 3$.

(2) Assume $a = 2^\alpha u - 1$, with u is odd. If $\sigma^{**}(x^a)$ splits, then $a = 2^d - 1$, d is positive (Proposition 3). If $\sigma^{**}(x^a)$ does not split, then $a \neq 2^d - 1$ and since $\sigma^{**2}(x^a) = x^{2^\alpha - 1} \cdot \sigma^{**}\left(\left(\sigma(x^{u-1})\right)^{2^\alpha}\right)$ splits, $u > 1$. Again, using Lemma 6, $\sigma(x^{2r})$ has a Mersenne factor. Thus, $u - 1 \leq 6$ and, hence, $u \in \{3, 5, 7\}$. For $u = 3$, $\sigma^{**2}(x^a) = x^{2^\alpha - 1} \cdot \sigma^{**}\left(\left(\sigma(x^2)\right)^{2^\alpha}\right) = x^{2^\alpha - 1} \cdot \sigma^{**}\left(M_1^{2^\alpha}\right)$. Hence, $\alpha = 1$ and the same result is obtained when $u \in \{5, 7\}$.

The same proof is performed for $\sigma^{**2}((x+1)^a)$, and the proof is complete. □

Lemma 12. *Let a and b have the form $2^n - 1$, where $n \in \mathbb{N}^*$, and let the polynomial $A = 1 + x^a(x+1)^b$ be Mersenne prime over $\mathbb{F}_2$; then, $\sigma^{**2}(A) = x^b(x+1)^a$.*

Proof. Let $a = 2^{n_1} - 1$ and $b = 2^{n_2} - 1$; then,

$$\begin{aligned}
\sigma^{**2}(A) &= \sigma^{**2}\left(1 + x^a(x+1)^b\right) \\
&= \sigma^{**}(\sigma(1 + x^a(x+1)^b) \\
&= \sigma^{**}\left(x^a(x+1)^b\right) \\
&= x^b(x+1)^a.
\end{aligned}$$

□

5. Proof of Theorem 1

We consider the polynomial $A = P^a$ and $a \in \mathbb{N}^*$. We prove that $\sigma^{**}(A)$ cannot have more than one prime factor when A is a prime power.

Proposition 7. *If $A \in \{x, x+1\}$ and $\sigma^{**2}(A^a)$ splits over $\mathbb{F}_2$, then A is a bi-unitary superperfect polynomial.*

Proof. Follows from part (1) of Lemma 11. □

Proposition 8. *Assume P is odd, then $A = P^\alpha \in \mathbb{F}_2[x]$ is not a bi-unitary superperfect polynomial.*

Proof. Assume $A = P^a$ is a bi-unitary superperfect. Since P divides A, then $x(x+1)$ divides $\sigma^{**}(A)$, and using Lemma 10, we have that $x(x+1)$ divides $\sigma^{**2}(A) = P^a$, a contradiction. $\square$

In particular, if M is a Mersenne prime polynomial over $\mathbb{F}_2$, then M^c (c is a positive integer) is never a bi-unitary superperfect polynomial.

Corollary 6. *Let $a \in \mathbb{N}^*$ and let $A = P^a$ be a bi-unitary superperfect polynomial over $\mathbb{F}_2$; then, $P \in \{x, x+1\}$.*

It is clear from the preceding two corollaries that a bi-unitary superperfect polynomial must be even.

Lemma 13. *Let A be a polynomial over $\mathbb{F}_2$ with $\omega(A) = 1$; then, A is a bi-unitary superperfect polynomial if and only if $A, \overline{A} \in \{x^2, x^{2^d-1}\}$, where $d \in \mathbb{N}^*$.*

Proof. Using Corollary 6, $A = x^\alpha$ or $(x+1)^\alpha$. Assume $A = x^\alpha$ and $\alpha = 2m$; then, $\sigma^{**2}(A) = \sigma^{**}\left((x^{m+1}+1)\dfrac{x^m-1}{x-1}\right)$. Both $x^{m+1}+1$ and x^m+1 split over $\mathbb{F}_2$ only when $m = 1$. Thus, $\sigma^{**2}(A) = \sigma^{**}(x^2+1) = x^2$. If $\alpha = 2m+1$, then $\sigma^{**2}(A) = \sigma^{**}\left(\dfrac{x^{2(m+1)}-1}{x-1}\right)$. The expression $x^{2(m+1)}+1$ splits over $\mathbb{F}_2$ when $2m+2 = 2^d$, $d \in \mathbb{N}^*$. Then, $\sigma^{**2}(A) = \sigma^{**}\left(\dfrac{x^{2^d}-1}{x-1}\right) = A = x^{2^d-1}$. The sufficient condition follows via direct computation, and the result follows since if A is a bi-unitary superperfect, then so is $\overline{A}$. $\square$

6. Proof of Theorem 2

We consider the polynomial $A = P^aQ^b$ and $a, b \in \mathbb{N}^*$. Note that $A = x^2(1+x)^2$ and $A = x^{2^\alpha-1}(1+x)^{2^\alpha-1}$ are bi-unitary superperfect polynomials over $\mathbb{F}_2$, as shown Proposition 4 and Theorem 5 in [9].

Proposition 9 (Lemma 3.1 in [10]). *If the polynomial $\sigma^{**}(x^a(x+1)^b)$ does not split, then $(a \geq 3$ or $b \geq 3)$ and $(a \neq 2^n - 1$ or $b \neq 2^m - 1$ for any $n, m \geq 1)$.*

Lemma 14. *Let $a, b, d \in \mathbb{N}^*$. The polynomial $A = x^a(x+1)^b$ is a bi-unitary superperfect over $\mathbb{F}_2$ if and only if one of the following is true.*

(1) *If a and b are odd and $\sigma^{**}(x^a(x+1)^b)$ splits, then a and b are of the form 2^d-1.*
(2) *If a and b are odd and $\sigma^{**}(x^a(x+1)^b)$ does not split, then $(a,b) \in \{(9,9),(9,13),(13,9)\}$.*
(3) *If a and b are even, then $a = b \in \{2,4\}$.*
(4) *If a and b are of opposite parity, then $(a,b) \in \left\{\left(2, 2^d-1\right), \left(2^d-1, 2\right)\right\}$.*

Proof.

(1) If $a = 2m+1$ and $b = 2n+1$, then $\sigma^{**2}(A) = \sigma^{**}\left(\sigma^{**}(x^a)(1+x)^b\right)$. However, $\sigma^{**}(x^{2m+1})$ and $\sigma^{**}(x+1)^{2n+1}$ split over $\mathbb{F}_2$ when $2m+1$ and $2n+1$ are of the form 2^d-1 (Proposition 3).
(2) If $a = 2^\alpha u - 1$ and $b = 2^\beta v - 1$, u, v are odd. We have $u > 1$ and $v > 1$ since $\sigma^{**}(x^a(x+1)^b)$ does not split. $\sigma^{**}(x^a(x+1)^b) = \sigma^{**}\left((1+x)^{2^\alpha-1}(\sigma(x^{u-1}))^{2^\alpha} x^{2^\beta-1}\sigma\left((x+1)^{v-1}\right)^{2^\beta}\right)$. Using Proposition 9 ($u-1 \geq 3$ and $\alpha = 1$) or ($v-1 \geq 3$ and $\beta = 1$). Furthermore, $\sigma(x^{u-1})$ and $\sigma\left((x+1)^{v-1}\right)$ does not split since $\sigma^{**}(x^a(x+1)^b)$ does not split. Thus, there exist Merssene primes M (resp. M') that divides $\sigma(x^{u-1})$ (resp. $\sigma\left((x+1)^{v-1}\right)$.

Hence, $(u-1 \leq 6)$ or $(v-1 \leq 6)$, and we have that $u, v \in \{5, 7\}$. If $u = v = 5$, then $a = b = 9$. If $u = 5$ and $v = 7$, then $a = 9$ and $b = 13$. If $u = v = 7$, then $a = b = 13$ is dismissed.

(3) If a, b even, then $a \in \{4r, 4r+2\}$ such that $2r-1, 2r+1$ is of the form $2^{\alpha}u - 1$, where u is odd and $b \in \{4r', 4r'+2\}$ such that $2r'-1, 2r'+1$ is of the form 2^{β} vs. -1, v odd. Thus,

$$\sigma^{**}(A) = (1+x)^{2^{\alpha}-1}\sigma(x^{2r})\Big(\sigma(x^{u-1})\Big)^{2^{\alpha}} x^{2^{\beta}-1}\sigma\Big((x+1)^{2r\prime}\Big)\Big(\sigma\Big((x+1)^{v-1}\Big)\Big)^{2^{\beta}}.$$

If $\sigma(x^{2r})$, $\sigma\Big((x+1)^{2r\prime}\Big)$, $\sigma(x^{u-1})$, and $\sigma\Big((x+1)^{v-1}\Big)$ are Mersenne, then $2r, 2r', u-1, v-1 \in \{2, 4\}$. Thus, $a = b = 4$. If $\sigma(x^{2r})$, $\sigma(x^{u-1})$, $\sigma\Big((x+1)^{2r\prime}\Big)$ and $\sigma\Big((x+1)^{v-1}\Big)$ are not Mersenne, then $r, r', u-1, v-1 > 2$ and $\omega(\sigma^{**2}(A)) > 2$, a contradiction. For $a = b = 2$, A is bi-unitary perfect; hence, A is a bi-unitary superperfect.

(4) Now, let $a = 2m+1$ and $b = 2n$. Since $\sigma^{**}\Big((x+1)^{2n}\Big)$ splits over $\mathbb{F}_2$ only when $n = 1$, then $\sigma^{**2}(A) = \sigma^{**}\big(\sigma^{**}(x^{2m+1})\sigma^{**}\big((x+1)^2\big)\big)$. However, $\sigma^{**}(x^{2m+1})$ splits over $\mathbb{F}_2$ if $2m+1$ is of the form $2^d - 1$. If $a = 2m$ and $b = 2n+1$, then $a = 2$ and $b = 2^d - 1$. The sufficient condition can be easily verified.

□

The proof of Theorem 2 is now complete.

7. Some Values of $\sigma^{**}(A)$ and $\sigma^{**2}(A)$

For convenience of readers, we list the below table that consists of the values of $\sigma^{**}(A)$ and $\sigma^{**2}(A)$ for $A \in \{x^a, (x+1)^a, M_i^b\}$, where $1 \leq a \leq 13$, $1 \leq b \leq 7$. We consider the polynomials $C_1 = x^4 + x + 1$, $C_2 = x^6 + x^5 + x^4 + x^2 + 1$, $C_3 = x^6 + x^5 + x^4 + x + 1$, and $C_4 = x^{10} + x^9 + x^8 + x^7 + x^2 + x + 1$.

Table 1. $A \in \{x^a, (x+1)^a, M^a\}$.

A	a	σ^{**}	σ^{**2}
x^a	1	x	$x+1$
	2	x^2	$(x+1)^2$
	3	x^3	$(x+1)^3$
	4	x^2M_1	$x(x+1)^3$
	5	xM_1^2	$x^2(x+1)^3$
	6	x^4M_1	$x(x+1)^3M_1$
	7	x^7	$(x+1)^7$
	8	x^4M_5	$x^3(x+1)^3M_1$
	9	xM_5^2	$x^6(x+1)^3$
	10	$x^2M_1^2M_5$	$x^5(x+1)^5$
	11	$x^3M_1^4$	$x^2(x+1)^5C_1$
	12	$x^2M_1^2M_2M_3$	$x^5(x+1)^7$
	13	$xM_2^2M_3^2$	$x^6(x+1)^7$

Table 1. *Cont.*

A	a	σ^{**}	σ^{**2}
....	....	....	....
$(1+x)^a$	1	x	$x+1$
	2	x^2	$(x+1)^2$
	3	x^3	$(x+1)^3$
	4	x^2M_1	$x(x+1)^3$
	5	xM_1^2	$x^2(x+1)^3$
	6	x^4M_1	$x(x+1)^3M_1$
	7	x^7	$(x+1)^7$
	8	x^4M_5	$x^3(x+1)^3M_1$
	9	xM_5^2	$x^6(x+1)^3$
	10	$x^2M_1^2M_5$	$x^5(x+1)^5$
	11	$x^3M_1^4$	$x^2(x+1)^5C_1$
	12	$x^2M_1^2M_2M_3$	$x^5(x+1)^7$
	13	$xM_2^2M_3^2$	$x^6(x+1)^7$
....	....	....	....
M_1^a	1	$x(x+1)$	$x(x+1)$
	2	$x^2(x+1)^2$	$x^2(x+1)^2$
	3	$x^3(x+1)^3$	$x^3(x+1)^3$
	4	$x^2(x+1)^2C_1$	$x^3(x+1)^3M_1$
	5	$x(x+1)C_1^2$	$x^3(x+1)^3M_1^2$
	6	$x^4(x+1)^4C_1$	$x^3(x+1)^3M_1^3$
	7	$x^7(x+1)^7$	$x^7(x+1)^7$
....	....	....	....
M_2^a	1	$x(x+1)^2$	$x^2(x+1)$
	2	$x^2(x+1)^4$	$x^2(x+1)^2M_1$
	3	$x^3(x+1)^6$	$x^4(x+1)^3M_1$
	4	$x^2(x+1)^4M_1M_5$	$x^6(x+1)^4M_1$
	5	$x(x+1)^2M_1^2M_5^2$	$x^{10}(x+1)^5$
	6	$x^4(x+1)^8M_1M_5$	$x^8(x+1)^4M_1M_5$
	7	$x^7(x+1)^{14}$	$x^8(x+1)^7M_2M_3$
M_3^a	1	$x^2(x+1)$	$x(x+1)^2$
	2	$x^4(x+1)^2$	$x^2(x+1)^2M_1$
	3	$x^6(x+1)^3$	$x^3(x+1)^4M_1$
	4	$x^4(x+1)^2M_1M_4$	$x^4(x+1)^6M_1$
	5	$x^2(x+1)M_1^2M_4^2$	$x^5(x+1)^{10}$
	6	$x^8(x+1)^4M_1M_4$	$x^4(x+1)^8M_1M_4$
	7	$x^{14}(x+1)^7$	$x^7(x+1)^8M_2M_3$
....	....	....	....
M_4^a	1	$x(x+1)^3$	$x^3(x+1)$
	2	$x^2(x+1)^6$	$x^4(x+1)^2M_1$
	3	$x^3(x+1)^9$	$x(x+1)^3(M_5)^2$
	4	$x^2(x+1)^6M_1C_2$	$x^7(x+1)^4M_1M_2$
	5	$x(x+1)^3M_1^2C_2^2$	$x^9(x+1)^5M_2^2$
	6	$x^4(x+1)^{12}M_1C_2$	$x^5(x+1)^4M_1^3M_2^2M_3$
	7	$x^7(x+1)^{21}$	$x(x+1)^7$ C_4^2

Table 1. *Cont.*

A	a	σ^{**}	σ^{**2}
....	....	....	....
$(M_5)^a$	1	$x^3(x+1)$	$x(x+1)^3$
	2	$x^6(x+1)^2$	$x^2(x+1)^4 M_1$
	3	$x^9(x+1)^3$	$x^3(x+1)M_4^2$
	4	$x^6(x+1)^2 M_1 C_3$	$x^4(x+1)^7 M_1 M_3$
	5	$x^3(x+1)M_1^2 C_3^2$	$x^5(x+1)^9 M_3^2$
	6	$x^{12}(x+1)^4 M_1 C_3$	$x^4(x+1)^5 M_1^3 M_2 M_3^2$
	7	$x^{21}(x+1)^7$	$x^7(x+1)\left(\sigma(x^{10})\right)^2$

8. Conclusions

In conclusion, we proved the non-existence of odd bi-unitary superperfect polynomials and provided a classification for bi-unitary superperfect polynomials over $\mathbb{F}_2$ based on their irreducible factors. In particular, we showed that a non-constant bi-unitary superperfect polynomial A over $\mathbb{F}_2$ can be divisible by one irreducible polynomial x or $x+1$ with exponent 2 or $2^n - 1$ for a positive integer n. Furthermore, we showed that the only bi-unitary superperfect polynomials over $\mathbb{F}_2$ with exactly two irreducible factors are of the form $x^a(x+1)^b$ with $a, b \in \{2, 4, 9, 13, 2^d - 1\}$, d is a positive integer.

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References

1. Suryanarayana, D. Super Perfect Numbers. *Elem. Der Math.* **1969**, *24*, 16–17.
2. Sitaramaiah, V.; Subbarao, M. On the Equation $\sigma^{**}(n)) = 2n$. *Util. Math.* **1998**, *53*, 101–124.
3. Wall, C. Bi-unitary perfect numbers. *Proc. Am. Math. Soc.* **1972**, *33*, 39–42. [CrossRef]
4. Yamada, T. 2 and 9 are the only biunitary superperfect numbers. *Ann. Univ. Sci. Bp.* **2018**, *48*, 247–256.
5. Canaday, E.F. The sum of the divisors of a polynomial. *Duke Math. J.* **1941**, *8*, 721–737. [CrossRef]
6. Gallardo, L.H.; Rahavandrainy, O. Odd perfect polynomials over $\mathbb{F}_2$. *J. Théorie Nombres Bordx.* **2007**, *19*, 165–174. [CrossRef]
7. Gallardo, L.H.; Rahavandrainy, O. On even (unitary) perfect polynomials over $\mathbb{F}_2$. *Finite Fields Their Appl.* **2012**, *18*, 920–932. [CrossRef]
8. Gallardo, L.H.; Rahavandrainy, O. All unitary perfect polynomials over $\mathbb{F}_2$ with at most four distinct irreducible factors. *J. Symb. Comput.* **2012**, *47*, 492–502. [CrossRef]
9. Beard, J.T. Bi-unitary perfect polynomials over GF(g). *Ann. Mat. Pura Appl.* **1987**, *149*, 61–68. [CrossRef]
10. Rahavandrainy, O. All bi-unitary perfect polynomials over $\mathbb{F}_2$ with at most four irreducible factors. *arXiv* **2022**, arXiv:2205.08392.
11. Gallardo, L.H.; Rahavandrainy, O. Unitary superperfect binary polynomials. In Proceedings of the Finite Fields: Theory and Applications: Theory and Applications: Ninth International Conference on Finite Fields and Applications, Dublin, Ireland, 13–17 July 2009; American Mathematical Society: Providence , RI, USA , 2010; Volume 518, p. 155.
12. Gallardo, L.H.; Rahavandrainy, O. Characterization of Sporadic perfect polynomials over $\mathbb{F}_2$. *Funct. Approx. Comment. Math.* **2016**, *55*, 7–21. [CrossRef]

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