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Delay Differential Equations with Several Sublinear Neutral Terms: Investigation of Oscillatory Behavior

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Abstract: In this work, new oscillation criteria are established for a second-order differential equation with several sublinear neutral terms and in the canonical case. To determine the oscillation conditions, we followed the Riccati approach and also compared the studied equation with a first-order delay equation. Obtaining the oscillation conditions required deducing some new relationships linking the solution to the corresponding function as well as its derivatives. The paper addresses some interesting analytical points in the study of the oscillation of equations with several sublinear neutral terms. These new findings complement some well-known findings in the literature. Furthermore, an example is provided to show the importance of the results.

Keywords: functional differential equation; second order; several sublinear neutral terms; oscillatory behavior; Riccati approach; comparison principles

MSC: 34C10; 34K11

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Citation: Muhsin, W.; Moaaz, O.; Askar, S.S.; Alshamrani, A.M.; Elabbasy, E.M. Delay Differential Equations with Several Sublinear Neutral Terms: Investigation of Oscillatory Behavior. *Symmetry* **2023**, *15*, 2105. https://doi.org/10.3390/sym15122105

Academic Editor: Cheon-Seoung Ryoo

Received: 22 July 2023

Revised: 4 November 2023

Accepted: 15 November 2023

Published: 23 November 2023

1. Introduction

In this study, we focus on the oscillatory behavior of second-order differential equations with several sublinear neutral terms

$$\left(r(\mathrm{s})z'(\mathrm{s})\right)' + p(\mathrm{s})x^{\gamma}(\sigma(\mathrm{s})) = 0, \tag{1}$$

where $z(\mathrm{s}) = x(\mathrm{s}) + \sum_{i=1}^{k} m_i(\mathrm{s})x^{\alpha_i}(\tau_i(\mathrm{s}))$, $\mathrm{s} \geq \mathrm{s}_0 > 0$, and $k \geq 1$ is an integer. Throughout the paper, we assume the following:

(H$_1$) $0 < \alpha_i \leq 1$ for $i = 1, 2, \ldots, k$, and α_i and γ are ratios of odd positive integers;

(H$_2$) $r, m_i, p : [\mathrm{s}_0, \infty) \to \mathbb{R}^+$ are continuous functions and $\lim_{\mathrm{s}\to\infty} m_i(\mathrm{s}) = 0$ for $i = 1, 2, \ldots, k$;

(H$_3$) $\tau_i, \sigma : [\mathrm{s}_0, \infty) \to \mathbb{R}$ are continuous functions with $\tau_i(\mathrm{s}) < \mathrm{s}$, $\sigma(\mathrm{s}) < \mathrm{s}$, $\sigma'(\mathrm{s}) > 0$, $\lim_{\mathrm{s}\to\infty} \tau_i(\mathrm{s}) = \infty$, and $\lim_{\mathrm{s}\to\infty} \sigma(\mathrm{s}) = \infty$.

Equation (1) is said to be in the canonical case when

$$\int_{\mathrm{s}_0}^{\infty} \frac{1}{r(\nu)} \mathrm{d}\nu = \infty. \tag{2}$$

Under a solution of (1), we refer to a nontrivial function $x \in \mathbf{C}([s_a,\infty),\mathbb{R})$, $s_a \geq s_0$, which has the properties z and $rz' \in \mathbf{C}^1([s_a,\infty),\mathbb{R})$ and satisfies (1) on $[s_a,\infty)$. We consider only those solutions x of (1) which satisfy

$$\sup\{|x(s)| : s \geq S\} > 0 \text{ for all } S \geq s_a.$$

A solution x of (1) is said to be nonoscillatory if it is ultimately positive or ultimately negative; otherwise, it is said to be oscillatory. Equation (1) is called oscillatory if all of its solutions oscillate.

It is difficult to find a closed-form solution to the nonlinear differential equations that are obtained when modeling many phenomena. This has made studying the properties of these equations an interest of many researchers. The qualitative theory of differential equations has received great attention, and one of its most important branches is the theory of oscillation, which is concerned with studying the oscillatory and asymptotic behavior of solutions; see [1,2].

In 1836, Sturm [3] posed the oscillation problems for a class of second-order ordinary linear differential equations when he studied thermal conductivity. Later on, precisely, in the first half of the twentieth century, an innovative article [4] by Fite was published on the oscillation theory of differential equations. Since then, many studies have been carried out investigating the oscillatory behavior of different types of differential equations of different orders.

In delay differential equations, history (previous memory) affects how a system evolves at any given time. A differential model's dynamics are significantly enhanced by including such time delays. This means that mathematical models for many real problems result in differential equations based on ancient history rather than the present; therefore, it is necessary to investigate both the qualitative and quantitative aspects of how this class of differential equations behaves. Since then, many academics have shown interest in oscillation theory in functional differential equations. We refer the reader to the monographs [5,6].

The importance of studying the behavior of solutions to differential equations with delay is seen from their application in various sciences. For example, in physics, Norkin [7] showed that the oscillation of contacts of electromagnetic switches could be characterized by the oscillation of solutions of the second-order delay differential equation with a damping term. In the field of medicine, several models have been presented, the most important of which is the red blood cell survival model, where the presence of a delay function in the equation represents the time it takes for the bone marrow to form red blood cells; a model for diagnosing diabetes [8] and lung expansion in patients with COVID-19; and many other models that explain epidemiological fluctuations.

A differential equation with a neutral delay (NDDE) is one in which both portions of the equation contain the highest derivative of the function, whether it is known or unknown, the part with the delay and the part without the delay. It should be mentioned that the NDDEs and their special cases have applications that are intriguing in real-life settings. In modeling networks that use lossless transmission lines (as, for example, in high-speed computers, where lossless transmission lines are used to connect what are known as switching circuits), technology-neutral equations are included. In the field of biology, when considering the birth of a person or population problems, many changes do not occur immediately, both in the interpretation of the human body's capacity for self-balancing and in the design of robots containing two legs [9]. Such phenomena, when modeled as a differential equation, require later times because such cases are difficult to express in a simple ordinary differential equation.

This study deals with the oscillatory properties of second-order differential equations with many sublinear neutral terms. The study of the oscillatory behavior of solutions to differential equations with delay addresses several points: classifying positive solutions according to the sign of their derivatives; finding relationships (which appear in the form of inequalities) between these derivatives; and estimating the relationship between the solution with delay and without it. We used two methods to obtain our results, namely, the method of comparison with the oscillation in differential equations with first-order delay and the Riccati method.

Literature Review

Recently, there has been a remarkable development in the study of the oscillatory behavior of solutions of functional differential equations of different orders and of different types, such as equations with delay, neutral and advanced equations, as well as equations that include a middle term that includes damping. Second-order delay equations have been the subject of interest and development in [10–12]. Refs. [13,14] dealt with the oscillation of advanced equations, while neutral equations received a large share of interest, for example, we mention [15–18]. For information, techniques, and results on the oscillation of even-order equations, see [19–22]. On the other hand, odd-order equations have also received great attention and extensive investigation; see [23–29].

For many years, many articles have investigated the qualitative and asymptotic behavior of the NDDEs, and a large number of important results for such equations have already been found.

Regarding the oscillation of second-order NDDEs, by refining the standard integral averaging technique, Hasanbulli and Rogovchenko [30] obtained oscillation conditions for the equation

$$\left(r(\mathrm{s})(x(\mathrm{s})+m(\mathrm{s})x^{\alpha}(\mathrm{s}-\tau))'\right)'+p(\mathrm{s})f(x(\mathrm{s}),x(\sigma(\mathrm{s})))=0,\quad \mathrm{s}\geq\mathrm{s}_0\geq 0,$$

where r, $\sigma\in\mathbf{C}^1([\mathrm{s}_0,+\infty),(0,+\infty))$, m, $p\in\mathbf{C}([\mathrm{s}_0,+\infty),\mathbb{R})$, $\tau\geq 0$ is a constant, and $f\in\mathbf{C}(\mathbb{R}^2,\mathbb{R})$.

Many authors have discussed the oscillatory behavior of the following NDDEs or special classes of them:

$$\left(r(\mathrm{s})\left((x(\mathrm{s})+m(\mathrm{s})x^{\alpha}(\tau(\mathrm{s})))'\right)^{\beta}\right)'+p(\mathrm{s})x^{\gamma}(\sigma(\mathrm{s}))=0,\quad \mathrm{s}\geq\mathrm{s}_0\geq 0, \tag{3}$$

where α, β, and γ are ratios of positive odd integers; $0<\alpha\leq 1$ and $\gamma\geq\beta$.

Baculíková et al. [31] studied the oscillation behavior of (3) with $\alpha=1$, and in the canonical case (2). They provided comparison theorems that contrast the second-order Equation (3) with a first-order differential equation. Moreover, they assumed that $\tau\circ\sigma=\sigma\circ\tau$ and $0\leq m(\mathrm{s})\leq m_o<\infty$. Unlike many researchers who have studied the oscillation of (3) when $0\leq m(\mathrm{s})<1$; see, e.g., Refs. [32,33].

Agarwal et al. [34] obtained sufficient conditions for the oscillation of all solutions of Equation (3) with $\gamma=1$. In 2016, Agarwal et al. [11] proved several oscillation results for (3) with $\alpha=1$ and $\beta=\gamma$. They obtained several types of criteria that guarantee the oscillation of all solutions using generalized Riccati substitution, which does not require assumptions:

$$m'(\mathrm{s})\geq 0 \text{ and } \sigma(\mathrm{s})\leq\tau(\mathrm{s})=\mathrm{s}-\tau_0 \text{ for } \mathrm{s}\geq\mathrm{s}_0.$$

Their oscillation criteria for Equation (3) complemented the results reported by Hasanbulli and Rogovchenko [30].

Unlike the canonical case, there is a possibility of decreasing positive solutions to second-order equations in the noncanonical case, and this requires another condition to exclude it. Li et al. [35] studied the oscillatory characteristics of the second-order NDDE

$$\left(r(\mathrm{s})\left(\left(x(\mathrm{s})+\sum_{i=1}^{k}m_i(\mathrm{s})x(\tau_i(\mathrm{s}))\right)'\right)^{\alpha}\right)'+p(\mathrm{s})x^{\alpha}(\sigma(\mathrm{s}))=0,\quad \mathrm{s}\geq\mathrm{s}_0\geq 0,$$

when

$$\int_{s_0}^{\infty} r^{-1}(\nu)\mathrm{d}\nu < \infty.$$

Despite the many studies that have been conducted and are still continuing to study differential equations with a delay of the second order, few results have been obtained about the oscillation of differential equations of the second order with a sublinear neutral term.

Recently, by using some elementary inequalities, Dzurina et al. [36] derived oscillation findings for Equation (1) in the canonical case that extended those reported in [37]. The findings in [36] apply to a number of classes of differential equations of the neutral type.

In [38], Moaaz et al. investigated the oscillatory properties of a neutral delay differential equation with several delays

$$\left(r(s)\left[(x(s) + m(s)x(\tau(s)))'\right]^{\alpha}\right)' + \sum_{i=1}^{k} p_i(s)x^{\alpha}(\sigma_i(s)) = 0.$$

They introduced new monotonic characteristics to the solutions of this equation, as these properties are recursive in nature. Also, they discovered a new condition, $\delta > 1/2$, which ensures that all solutions oscillate. Additionally, if this criterion was not met, they utilized an iterative strategy to improve it.

Moaaz et al. [39] derived new monotonic features of the second-order NDDE

$$\left(r(s)(z'(s))^{\gamma}\right)' + p(s)x^{\gamma}(\sigma(s)) = 0.$$

They then used these features to obtain optimized oscillation parameters. Moreover, they set criteria that ensured the oscillation of solutions of the fourth-order NDDE

$$\left(r(s)\left(z'''(s)\right)^{\gamma}\right)' + p(s)x^{\gamma}(\sigma(s)) = 0.$$

The novelty of the paper lies in obtaining new criteria that guarantee that all solutions to the studied equation are oscillatory. Our recent findings complement those verified in [40–42], and generalize [37] that dealt with an equation that had only one sublinear neutral term. This manuscript is considered an extension of some of our previous work, which was concerned with developing the study of oscillation for second-order differential equations; see [43–45].

2. Main Results

To facilitate a clearer presentation of the main findings, we define the following functions:

$$m(s) = \max_{0 \le i \le k} m_i(s),$$

$$R(s, s_1) := \int_{s_1}^{s} \frac{1}{r(\nu)}\mathrm{d}\nu,$$

$$\widetilde{R}(s) := R(s, s_1) + \int_{s_1}^{s} \frac{\sigma^{\gamma}(\nu)}{\nu^{\gamma}}\beta^{\gamma} p(\nu)\rho(\nu)R^2(\nu, s_1)\mathrm{d}\nu,$$

$$\widehat{R}(s) := \exp\left(-\int_{\sigma(s)}^{s} \frac{1}{\widetilde{R}(\nu)r(\nu)}\mathrm{d}\nu\right),$$

and

$$\rho(s) := \begin{cases} \lambda_1 & \text{if } \gamma > 1; \\ 1 & \text{if } \gamma = 1; \\ \lambda_2 R^{\gamma-1}(s, s_1) & \text{if } \gamma < 1, \end{cases}$$

where λ_1, λ_2, and $\beta \in (0, 1)$.

2.1. Auxiliary Lemmas

In this section, we introduce some important lemmas that will contribute to substantiating the main results.

Lemma 1 (Lemma 3 [46])**.** *Let condition (H_3) hold. If the function g satisfies $g > 0$, $g' > 0$ and $g'' \leq 0$ for $\mathrm{s} \geq \mathrm{s}_0$, so that there exists $\mathrm{s}_v \geq \mathrm{s}_0$ such that*

$$g(\sigma(\mathrm{s})) \geq \frac{v}{\mathrm{s}}\sigma(\mathrm{s})g(\mathrm{s}), \tag{4}$$

for all $v \in (0,1)$.

Lemma 2 (Lemma 2.3 [47])**.** *Let $\chi(u) = Lu - Mu^{\frac{(\alpha+1)}{\alpha}}$, where L and $M > 0$ are constants, and α is a quotient of odd natural numbers. Then, χ at $\theta = (\alpha L/((\alpha+1)M))^{\alpha}$ reaches its maximum value on $\mathbb{R}$ and*

$$\max_{u\in\mathbb{R}} \chi = \chi(\theta) = \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{L^{\alpha+1}}{M^{\alpha}}. \tag{5}$$

Lemma 3. *Assume that x is a positive solution of Equation (1) such that*

$$\int_{\mathrm{s}_0}^{\infty} \frac{1}{r(u)} \int_{u}^{\infty} p(\nu)\mathrm{d}\nu\mathrm{d}u = \infty. \tag{6}$$

Then, for $\mathrm{s} \geq \mathrm{s}_0$,

- **(i)** $z(\mathrm{s}) > 0$, $z'(\mathrm{s}) > 0$ *and* $(r(\mathrm{s})z'(\mathrm{s}))' \leq 0$;
- **(ii)** $\lim_{\mathrm{s}\to\infty} z(\mathrm{s}) = \infty$;
- **(iii)** $z(\mathrm{s})/R(\mathrm{s},\mathrm{s}_1)$ *is decreasing.*

Proof. Let x be a positive solution of (1) on $[\mathrm{s}_0,\infty)$. Then, $x(\mathrm{s}) > 0$ for $\mathrm{s} \geq \mathrm{s}_1 \geq \mathrm{s}_0$.

We obtain (i) directly from the definition of z as well as Equation (1). Thus,

$$\begin{aligned} z(\mathrm{s}) &\geq r(\mathrm{s})z'(\mathrm{s}) \int_{\mathrm{s}_0}^{\infty} \frac{1}{r(\nu)}\mathrm{d}\nu \\ &\geq r(\mathrm{s})z'(\mathrm{s})R(\mathrm{s},\mathrm{s}_1). \end{aligned}$$

From the last inequality, $z(\mathrm{s})/R(\mathrm{s},\mathrm{s}_1)$ is a decreasing function. We claim that (6) ensures $z(\mathrm{s}) \to \infty$ as $\mathrm{s} \to \infty$. Actually, as we know there is a constant $c > 0$, since $z(\mathrm{s})$ is a positive increasing function, such that

$$z(\mathrm{s}) \geq 2c > 0. \tag{7}$$

Additionally, it comes from $z(\mathrm{s})$ that

$$\begin{aligned} x(\mathrm{s}) &= z(\mathrm{s}) - \sum_{i=1}^{k} m_i(\mathrm{s})x^{\alpha_i}(\tau_i(\mathrm{s})) \\ &\geq z(\mathrm{s}) - \sum_{i=1}^{k} m_i(\mathrm{s})z^{\alpha_i}(\tau_i(\mathrm{s})) \\ &\geq z(\mathrm{s}) - \sum_{i=1}^{k} m_i(\mathrm{s})(\alpha_i z(\mathrm{s}) + (1-\alpha_i)) \\ &= z(\mathrm{s})\left(1 - \sum_{i=1}^{k}\alpha_i m_i(\mathrm{s}) - \frac{1}{z(\mathrm{s})}\sum_{i=1}^{k}(1-\alpha_i)m_i(\mathrm{s})\right), \end{aligned}$$

where the inequality (2) was used, with $b = 1$. Thus, we have

$$x(\mathrm{s}) \geq z(\mathrm{s})\left(1 - m(\mathrm{s})\sum_{i=1}^{k}\alpha_i - \frac{m(\mathrm{s})}{z(\mathrm{s})}\sum_{i=1}^{k}(1-\alpha_i)\right). \tag{8}$$

Substituting (7) into (8), we obtain

$$x(\mathrm{s}) \geq 2c\left(1 - m(\mathrm{s})\sum_{i=1}^{k}\alpha_i - \frac{m(\mathrm{s})}{c}\sum_{i=1}^{k}(1-\alpha_i)\right).$$

Taking (H_2) into account, we have

$$x(\mathrm{s}) \geq c > 0, \quad \mathrm{s} \geq \mathrm{s}_1. \tag{9}$$

Integrating (1) from s to ∞ and employing (9) in the obtained inequality, we obtain

$$z'(\mathrm{s}) \geq \frac{c^\gamma}{r(\mathrm{s})}\int_{\mathrm{s}}^{\infty} p(\nu)\mathrm{d}\nu.$$

By further integrating the previous inequality from s_1 to s, we can show that

$$z(\mathrm{s}) \geq z(\mathrm{s}_1) + c^\gamma \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{1}{r(u)}\int_{u}^{\infty} p(\nu)\mathrm{d}\nu\mathrm{d}u,$$

which, with (6), suggests that $z(\mathrm{s}) \to \infty$ for $\mathrm{s} \to \infty$. Hence, the proof of the lemma is complete. □

2.2. Oscillation Theorems

In this section, we present our new oscillatory criteria based on the results in the previous section. Now, using the comparison principle, we obtain the following theorem:

Theorem 1. *Assume condition (6) holds. If*

$$\limsup_{\mathrm{s}\to\infty}\int_{\sigma(\mathrm{s})}^{\mathrm{s}} p(\nu)\widetilde{R}^\gamma(\sigma(\nu))\mathrm{d}\nu > 1 \tag{10}$$

or

$$\liminf_{\mathrm{s}\to\infty}\int_{\sigma(\mathrm{s})}^{\mathrm{s}} p(\nu)\widetilde{R}^\gamma(\sigma(\nu))\mathrm{d}\nu > \frac{1}{\mathrm{e}}, \tag{11}$$

for every $\lambda_1, \lambda_2, \beta \in (0,1)$, then every solution of (1) is oscillatory.

Proof. Suppose that (1) has a nonoscillatory solution x on $[\mathrm{s}_0, \infty)$. Taking (H_2) and the characteristics of $z(\mathrm{s})$, one can see that

$$m(\mathrm{s})\sum_{i=1}^{k}\alpha_i + \frac{m(\mathrm{s})}{z(\mathrm{s})}\sum_{i=1}^{k}(1-\alpha_i) < \xi,$$

for any $\xi \in (0,1)$. As a result of the previous inequality and (8) that

$$x(\mathrm{s}) \geq \beta z(\mathrm{s}), \tag{12}$$

where $\beta = (1-\xi) \in (0,1)$. Substituting (12) into (1), we obtain

$$\left(r(\mathrm{s})z'(\mathrm{s})\right)' + p(\mathrm{s})\beta^\gamma z^\gamma(\sigma(\mathrm{s})) \leq 0.$$

That is

$$\left(r(\mathrm{s})z'(\mathrm{s})\right)' \leq -p(\mathrm{s})\beta^\gamma z^\gamma(\sigma(\mathrm{s})). \tag{13}$$

However, it is a result of monotony $r(s)z'(s)$ that

$$z(s) = z(s_1) + \int_{s_1}^{s} \frac{1}{r(\nu)} r(\nu) z'(\nu) d\nu \geq R(s, s_1) r(s) z'(s). \tag{14}$$

Simple calculations demonstrate that

$$\left(z(s) - R(s, s_1) r(s) z'(s)\right)' = -R(s, s_1) \left(r(s) z'(s)\right)'. \tag{15}$$

Thus,

$$-R(s, s_1) \left(r(s) z'(s)\right)' \geq R(s, s_1) p(s) \beta^{\gamma} z^{\gamma}(\sigma(s)). \tag{16}$$

Combining (15) and (16), we have

$$\left(z(s) - R(s, s_1) r(s) z'(s)\right)' \geq R(s, s_1) p(s) \beta^{\gamma} z^{\gamma}(\sigma(s)).$$

From (4), we obtain

$$\begin{aligned} \left(z(s) - R(s, s_1) r(s) z'(s)\right)' &\geq \upsilon^{\gamma} \frac{\sigma^{\gamma}(s)}{s^{\gamma}} R(s, s_1) p(s) \beta^{\gamma} z^{\gamma}(s) \\ &\geq \upsilon^{\gamma} \frac{\sigma^{\gamma}(s)}{s^{\gamma}} R(s, s_1) p(s) \beta^{\gamma} z^{\gamma-1}(s) z(s) \\ &\geq \upsilon^{\gamma} \frac{\sigma^{\gamma}(s)}{s^{\gamma}} R^2(s, s_1) p(s) \beta^{\gamma} z^{\gamma-1}(s) r(s) z'(s). \end{aligned} \tag{17}$$

Now, since $z(s)$ is positive and increasing, we have that $z(s) \geq z(s_2) \geq \mu > 0$ for $s \geq s_2 \geq s_1$. Moreover, since $r(s)z'(s)$ is positive and decreasing, we see that $r(s)z'(s) \leq r(s_2)z'(s_2) = \mu^*$ for $s \geq s_2$, and hence

$$z(s) \leq z(s_2) + \mu^* R(s, s_1). \tag{18}$$

Since $R(\infty, s_1) = \infty$, there exist constant $\chi > 0$ and $s_\chi > s_\upsilon$ such that $R(s, s_1) > \chi$ for all $s \geq s_\chi$. Hence, from (18), we find

$$z(s) \leq N R(s, s_2),$$

where $N := \left(\frac{1}{\chi} z(s_2) + \mu^*\right)$. Then, we can pick $s_2 \geq s_\chi$ sufficiently large such that

$$z^{\gamma-1}(s) \geq \begin{cases} \lambda_1 & \text{if } \gamma > 1; \\ 1 & \text{if } \gamma = 1; \\ \lambda_2 R^{\gamma-1}(s, s_1) & \text{if } \gamma < 1, \end{cases} \tag{19}$$

for $s \geq s_2$, where $\lambda_1 = \mu^{\gamma-1}$ and $\lambda_2 = N^{\gamma-1}$. Hence,

$$z^{\gamma-1}(s) \geq \rho(s),$$

for some $\lambda_1, \lambda_2 \in (0, 1)$. Combining (17) with (19), we arrive at

$$\left(z(s) - R(s, s_1) r(s) z'(s)\right)' \geq \upsilon^{\gamma} \frac{\sigma^{\gamma}(s)}{s^{\gamma}} R^2(s, s_1) p(s) \beta^{\gamma} \rho(s) r(s) z'(s).$$

Integrating the inequality from s_1 to s, we obtain

$$z(s) \geq R(s, s_1) r(s) z'(s) + \upsilon^{\gamma} \int_{s_1}^{s} \frac{\sigma^{\gamma}(\nu)}{\nu^{\gamma}} \beta^{\gamma} p(\nu) \rho(\nu) R^2(\nu, s_1) r(\nu) z'(\nu) d\nu.$$

In view of the monotonicity of $r(\mathrm{s})z'(\mathrm{s})$ and (14), gives

$$z(\mathrm{s}) \geq r(\mathrm{s})z'(\mathrm{s})\left(R(\mathrm{s},\mathrm{s}_1) + \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{\sigma^\gamma(\nu)}{\nu^\gamma}\beta^\gamma p(\nu)\rho(\nu)R^2(\nu,\mathrm{s}_1)\mathrm{d}\nu\right). \tag{20}$$

Thus, we conclude that

$$z(\sigma(\mathrm{s})) \geq r(\sigma(\mathrm{s}))z'(\sigma(\mathrm{s}))\widetilde{R}(\sigma(\mathrm{s})). \tag{21}$$

Using (21) in (13), it is evident that $y(\mathrm{s}) := r(\mathrm{s})z'(\mathrm{s})$ is a positive solution of the first-order delay differential inequality

$$y'(\mathrm{s}) + \beta^\gamma p(\mathrm{s})\widetilde{R}^\gamma(\sigma(\mathrm{s}))y^\gamma(\sigma(\mathrm{s})) \leq 0.$$

In view of (Theorem 1 [48]), the following associated delay differential equation

$$y'(\mathrm{s}) + \beta^\gamma p(\mathrm{s})\widetilde{R}^\gamma(\sigma(\mathrm{s}))y^\gamma(\sigma(\mathrm{s})) = 0 \tag{22}$$

provides a positive solution. Nevertheless, it is commonly known that either condition (11) or (21) guarantees oscillation of (22). This consequently suggests that (1) cannot have positive solutions. Hence, we complete the proof. □

Now, using the Riccati approach, we obtain the following theorem:

Theorem 2. *Assume condition (2) holds and there exists a function* $\psi \in \mathbf{C}^1([\mathrm{s}_0,\infty),(0,\infty))$ *such that*

$$\limsup_{\mathrm{s}\to\infty} \int_{\mathrm{S}}^{\mathrm{s}} \left(\beta^\gamma \psi(\nu)\rho(\nu)p(v)\widehat{R}(\nu) - \frac{(\psi'_+(\nu))^2 r(\nu)}{4\psi(\nu)}\right)\mathrm{d}\nu = \infty, \tag{23}$$

where $\psi'_+(\nu) = \max\{0,\psi'(\nu)\}$ *and for every* λ_1, $\lambda_2, \beta \in (0,1)$. *Then, every solution of (1) is oscillatory.*

Proof. Suppose the contrary, that (1) has a nonoscillatory solution x on $[\mathrm{s}_0,\infty)$. The following definition is the Riccati function

$$\omega(\mathrm{s}) = \psi(\mathrm{s})r(\mathrm{s})\frac{z'(\mathrm{s})}{z(\mathrm{s})}, \tag{24}$$

for $\mathrm{s} \geq \mathrm{s}_{0.}$. Then, $\omega(\mathrm{s}) > 0$ for $\mathrm{s} \geq \mathrm{s}_1$. Differentiating (24), we obtain

$$\omega'(\mathrm{s}) = \frac{\psi'(\mathrm{s})}{\psi(\mathrm{s})}\omega(\mathrm{s}) - \psi(\mathrm{s})\frac{(r(\mathrm{s})z'(\mathrm{s}))'}{z(\mathrm{s})} - \frac{1}{\psi(\mathrm{s})r(\mathrm{s})}\omega^2(\mathrm{s}). \tag{25}$$

By virtue of (20), we obtain

$$\frac{z'(\mathrm{s})}{z(\mathrm{s})} \leq \frac{1}{\widetilde{R}(\mathrm{s})r(\mathrm{s})}. \tag{26}$$

Integrating the latter inequality from $\sigma(\mathrm{s})$ to s, we obtain

$$\frac{z(\sigma(\mathrm{s}))}{z((\mathrm{s}))} \geq \exp\left(-\int_{\sigma(\mathrm{s})}^{\mathrm{s}} \frac{1}{\widetilde{R}(\nu)r(\nu)}\mathrm{d}\nu\right).$$

Combining this inequality, (13) and (19), we have

$$\begin{aligned}\frac{(r(s)z'(s))'}{z(s)} &\le -\beta^{\gamma}\rho(s)p(s)\frac{z(\sigma(s))}{z(s)}\\ &\le -\beta^{\gamma}\rho(s)p(s)\exp\left(-\int_{\sigma(s)}^{s}\frac{1}{\widetilde{R}(\nu)r(\nu)}d\nu\right)\\ &= -\beta^{\gamma}\rho(s)p(s)\widehat{R}(s).\end{aligned}$$

By putting

$$L := \frac{\psi'(s)}{\psi(s)} \text{ and } M := \frac{1}{\psi(s)r(s)},$$

into (5) and then applying it in (25), we obtain

$$\omega'(s) \le -\left(\beta^{\gamma}\psi(s)\rho(s)p(s)\widehat{R}(s) - \frac{(\psi'_{+}(s))^2 r(s)}{4\psi(s)}\right).$$

Integrating the latter inequality from S to s, we obtain

$$\int_{S}^{s}\left(\beta^{\gamma}\psi(\nu)\rho(\nu)p(\nu)\widehat{R}(\nu) - \frac{(\psi'_{+}(\nu))^2 r(\nu)}{4\psi(\nu)}\right)d\nu \le \omega(s),$$

which contradicts condition (23). Hence, we complete the proof. □

Example 1. *Consider the following couple of sublinear neutral terms in the following differential equation:*

$$\left(x(s) + \frac{1}{s}x^{1/3}\left(\frac{s}{2}\right) + \frac{1}{s^2}x^{1/5}\left(\frac{s}{3}\right)\right)'' + \frac{a}{s^{4/3}}x^{1/3}\left(\frac{s}{2}\right) = 0,\ s > 0, \tag{27}$$

where $k = 2$, $\alpha_1 = 1/3$, $\alpha_2 = 1/5$, $\gamma = 1/3$, $\tau_1(s) = s/2$, $\tau_2(s) = s/3$, $\sigma(s) = s/2$, $m_1(s) = 1/s$, $m_2(s) = 1/s^2$, *and* $p(s) = a/s^{4/3}$.

For our equation it is easy to verify that

$$\lim_{s\to\infty} m_i(s) = 0 \textit{ and } \lim_{s\to\infty}\tau_i(s) = \infty,$$

in addition to

$$R(s, s_1) = s \textit{ and } \widetilde{R}(s) = \left(1 + \frac{a\lambda_2}{2^{1/3}}\right)s.$$

For Theorem 1, condition (10) reduces to

$$\begin{aligned}\limsup_{s\to\infty}\int_{\sigma(s)}^{s} p(\nu)\widetilde{R}^{\gamma}(\sigma(\nu))d\nu &= \limsup_{s\to\infty}\int_{s/2}^{s} a/\nu^{4/3}\left(1 + \frac{a\lambda_2}{2^{1/3}}\right)\left(\frac{\nu}{2}\right)^{1/3}d\nu\\ &> \frac{a}{2^{1/3}}\left(1 + \frac{a}{2^{1/3}}\right)^{1/3}\ln 2 > 1.\end{aligned}$$

Also, condition (11) becomes

$$\begin{aligned}\liminf_{s\to\infty}\int_{\sigma(s)}^{s} p(\nu)\widetilde{R}^{\gamma}(\sigma(\nu))d\nu &= \liminf_{s\to\infty}\int_{s/2}^{s} a/\nu^{4/3}\left(1 + \frac{a\lambda_2}{2^{1/3}}\right)\left(\frac{\nu}{2}\right)^{1/3}d\nu\\ &> \frac{a}{2^{1/3}}\left(1 + \frac{a}{2^{1/3}}\right)^{1/3}\ln 2 > \frac{1}{e}.\end{aligned} \tag{28}$$

Then, Equation (27) is oscillatory if condition (28) is satisfied.
For Theorem 2, we have $\widehat{R}(s) = 2^c$ *where* $c = -1/\left(1 + \frac{a\beta^\gamma \lambda_2}{2^{1/3}}\right)$. *Letting* $\psi(s) = s$, *by applying condition (23). Then, Equation (27) is oscillatory if*

$$
\begin{aligned}
&\limsup_{s\to\infty} \int_S^s \left(\beta^\gamma \psi(\nu)\rho(\nu)p(v)\widehat{R}(\nu) - \frac{(\psi'_+(\nu))^2 r(\nu)}{4\psi(\nu)}\right) d\nu \\
= \;& \limsup_{s\to\infty} \int_S^s \left(\beta^{\frac{1}{3}} \frac{1}{v^{\frac{1}{3}}} \frac{\lambda_2 2^c}{v^{\frac{2}{3}}} \frac{a}{v^{\frac{4}{3}}} - \frac{\left(\frac{1}{3}\right)}{4v^{\frac{1}{3}}}\right) d\nu \\
= \;& \limsup_{s\to\infty} \int_0^t \left(\beta^{\frac{1}{3}} \lambda_2 2^c v \frac{1}{v^{\frac{2}{3}}} \frac{a}{v^{\frac{4}{3}}} - \frac{1}{4v}\right) dv \\
> \;& 2^{-1/\left(1+\frac{a}{2^{1/3}}\right)} a > \frac{1}{4}.
\end{aligned}
$$

Remark 1. *The neutral term in Equation (27) has many sublinear neutral terms, hence the results in [40,41] cannot be applied to Equation (27). Therefore, the results of this work apply to more classes of neutral-type differential equations than the previous results.*

3. Conclusions

In the present study, in the canonical case, we have studied the oscillatory properties of a class of second-order differential equations with several sublinear neutral terms (1). The oscillation of the studied equation is achieved by using two different techniques. Based on them, we introduce new criteria, ensuring that all solutions of the equation under study oscillate. The results obtained complement some of the existing findings in the literature. Moreover, the reported findings are simply applicable to more widespread nonlinear equations,

$$\left(r(s)z'(s)\right)' + p(s)f(x(\sigma(s))) = 0,$$

By including the condition

(H_4) $f \in \mathbf{C}(-\infty,\infty)$, $f'(s) \geq 0$, $sf(s) > 0$, for $s \neq 0$ and $-f(-st) \geq f(st) \geq f(s)f(t)$ for $st > 0$.

Extending the findings to even-order equations and advanced cases would be an intriguing area for future research. Furthermore, it would be interesting to extend this study to higher-order differential equations in both cases, the canonical and non-canonical.

Author Contributions: Conceptualization, W.M., O.M., S.S.A., A.M.A. and E.M.E.; methodology, W.M., O.M., S.S.A., A.M.A. and E.M.E.; investigation, W.M., O.M., S.S.A., A.M.A. and E.M.E.; writing—original draft preparation, W.M., S.S.A., and A.M.A.; writing—review and editing,O.M. and E.M.E. All authors have read and agreed to the published version of the manuscript.

Funding: This project is funded by the Researchers Supporting Program (RSPD2023R533), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: Not applicable.

Acknowledgments: The authors present their appreciation to King Saud University for funding the publication of this research through the Researchers Supporting Program (RSPD2023R533), King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

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