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Periodic Solution Problems for a Class of Hebbian-Type Networks with Time-Varying Delays

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Abstract: By using Gronwall's inequality and coincidence degree theory, the sufficient conditions of the globally exponential stability and existence are given for a Hebbian-type network with time-varying delays. The periodic behavior phenomenon is one of the hot topics in network systems research, from which we can discover the symmetric characteristics of certain neurons. The main theorems in the present paper are illustrated using a numerical example.

Keywords: periodic solution; networks; Hebbian-type; existence; stability

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Citation: Xu, M.; Yin, H.; Du, B. Periodic Solution Problems for a Class of Hebbian-Type Networks with Time-Varying Delays. *Symmetry* **2023**, *15*, 1985. https://doi.org/10.3390/sym15111985

Academic Editors: Sergei D. Odintsov and Abraham A. Ungar

Received: 5 October 2023
Revised: 19 October 2023
Accepted: 26 October 2023
Published: 27 October 2023

1. Introduction

In the past few decades, Hopfield-type neural networks has been investigated by engineers, physicists, computer scientists and mathematicians. Gopalsamy [1] considered a second-order Hopfield-type neural network with constant lags and an unsupervised Hebbian-type learning algorithm as follows:

$$\begin{cases} u_i'(t) & = -a_i u_i(t) + \sum_{j=1}^n b_{ij} f_j(u_j(t-\gamma_j)) + \sum_{j=1}^n \sum_{k=1}^n T_{ijk} f_j(u_j(t-\gamma_j)) \\ & \quad f_k(u_k(t-\gamma_k)) + D_i \sum_{j=1}^n m_{ij}(t) p_j + I_i, \\ m_{ij}'(t) & = -\alpha_i m_{ij}(t) + \beta_i f_i(u_i(t)) p_j. \end{cases} \tag{1}$$

All coefficients of system (1) are constants. However, in nature, dynamic systems are inevitably influenced by external environments, and variable coefficient systems are more capable of depicting real-world situations. Hence, we study a neural networks with variable coefficients and lags as follows:

$$\begin{cases} u_i'(t) & = -a_i(t) u_i(t) + \sum_{j=1}^n b_{ij}(t) f_j(u_j(t-\gamma_j(t))) + \sum_{j=1}^n \sum_{k=1}^n T_{ijk}(t) \\ & \quad f_j(u_j(t-\gamma_j(t))) f_k(u_k(t-\gamma_k(t))) + D_i(t) \sum_{j=1}^n m_{ij}(t) p_j + I_i(t), \\ m_{ij}'(t) & = -\alpha_i(t) m_{ij}(t) + \beta_i(t) f_i(u_i(t)) p_j, \end{cases} \tag{2}$$

where $i = 1, 2, \cdots, n$, $u_i(t)$ is the state of the system; $a_i(t) > 0$ is the feedback rate of the system; $m_{ij}(t)$ represents the synaptic vector; $D_i(t)$ represents the acceptance rate of the input signals; $b_{ij}(t)$ and $T_{ijk}(t)$ are synaptic weights; $\beta_i(t)$ and $\alpha_i(t) > 0$ are disposable; p_j is learning the signal vector; $\gamma_j(t) > 0$ is the time-varying lag with $\gamma_j'(t) < 1$; $I_i(t)$ is the external input signal vector; and $f_j(\cdot)$ is the neuronal activation function. Let

$$v_i(t) = \sum_{j=1}^n m_{ij}(t) p_j \text{ and } \sum_{j=1}^n p_j^2 = c > 0.$$

We rewrite (2) into

$$\begin{cases} u_i'(t) & = -a_i(t) u_i(t) + \sum_{j=1}^n b_{ij}(t) f_j(u_j(t-\gamma_j(t))) \\ & \quad + \sum_{j=1}^n \sum_{k=1}^n T_{ijk}(t) f_j(u_j(t-\gamma_j(t))) f_k(u_k(t-\gamma_k(t))) + D_i(t) v_i(t) + I_i(t), \\ v_i'(t) & = -\alpha_i(t) v_i(t) + \beta_i(t) c f_i(u_i(t)), \end{cases} \tag{3}$$

where $t \geq 0$. The initial conditions of system (3) are given by

$$\begin{cases} u_i(s) = \tilde{\phi}_i(s), \; s \in [-\gamma, 0], \; i = 1, 2, \cdots, n, \\ v_i(s)) = \tilde{\psi}_i(s), \; s \in [-\gamma, 0], \; i = 1, 2, \cdots, n, \end{cases} \tag{4}$$

where $\tilde{\phi}_i(\cdot)$ and $\tilde{\psi}_i(\cdot)$ are continuous and bounded functions on $[-\gamma, 0]$, and γ is defined by (5).

From the models proposed by Amari [2,3], we obtain system (3) which belongs to a higher-order network system. A high-order neural network system can simulate complex neuronal changes, and many scholars have been dedicated to this research. A method for defining an unknown nonlinear system was presented by Gonzalez, Basin and Vargas [4]. Huang and Cao [5] studied bifurcation problems regarding different lags of high-order fractional networks. Liu et al. [6] obtained fixed-time combined control for high-order multi-agent networks with variable failures. For more results regarding high-order neural networks, please see, e.g., references [7–11].

The periodic phenomenon widely exists in nature, and its research can deepen the understanding of the changing patterns of neurons, promoting the development and utilization of network systems. Over the past decades, there are many profound results for the periodic solution research of high-order network systems. A stochastic Cohen–Grossberg neural network with variable lags has been studied by Wu, Yang and Ren [12]. Zhang and Liu [13] dealt with global exponential stability and the existence of a periodic solution for a BAM neural network with multiple lags on time scales. From the Lyapunov functional method and some inequality techniques, Luo, Jiang and Wang [14] studied the anti-periodic solutions of a Clifford-valued high-order neural network with proportional lags. The almost periodic solution problem for a quaternion-valued neural networks has been investigated by Li and Xiang [15]. For more results for high-dimensional dynamic systems and networks systems, please see, e.g., references [16–20] and related references.

Affected by the existing research results, in the present paper, we will study the periodic solution for a Hebbian-type network with time-varying lags. The main contributions of this paper are as follows:

(1) There is not much research on the periodic solution research of system (3), and this study expands its research scope.
(2) On the basis of fully considering the variable delays and coefficients, this article constructs a new function, which can conveniently obtain the stability of system (3).

The remaining parts of this article are arranged as follows. Section 2 gives some preparations. Section 3 gives some existence results of a periodic solution for the system (3). Section 4 gives the stability results of a periodic solution for system (3). In Section 5, we give an example to illustrate the correctness of Theorems 1 and 2. The conclusions are given in Section 6.

In the whole paper, the following notations are listed:

$$|f|_0 = \max_{t \in [0,T]} |f(t)|, \; \bar{f} = \frac{1}{T} \int_0^T f(t)dt, \; \chi_j = \max_{t \in [0,T]} \frac{1}{1 - \tau_j'(t)}, \; \gamma = \max_{t \in [0,T]} \gamma_j(t), \; j = 1, 2, \cdots, n. \tag{5}$$

Furthermore, the following assumptions hold.

(H_1) In system (3), for $i, j, k = 1, 2, \cdots, n$, $a_i(\cdot), b_{ij}(\cdot), T_{ijk}(\cdot), \gamma_j(\cdot), \alpha_i(\cdot), \beta_i(\cdot), D_i(\cdot), I_i(\cdot)$ are $T-$periodic continuous functions.

(H_2) There is constant $l_j \geq 0$ such that

$$|f_j(y)| \leq l_j, \; j = 1, 2, \cdots, n, \; \forall y \in \mathbb{R}.$$

(H_3) There is constant $m_j \geq 0$ such that

$$0 \leq \frac{f_j(y) - f_j(z)}{y - z} \leq m_j,\ f_j(y) = 0,\ j = 1, 2, \cdots, n,\ \forall y, z \in \mathbb{R}.$$

2. Preliminaries

Let Δ and Π be two Banach spaces. Let $\mathcal{L} : D(\mathcal{L}) \subset \Delta \to \Pi$ be a Fredholm operator with index zero, which means that $Im\mathcal{L}$ is closed in Π and $dimKer\mathcal{L} = codimIm\mathcal{L} < +\infty$. Let projectors $P : \Delta \to \Delta$, $Q : \Pi \to \Pi$ such that $ImP = Ker\mathcal{L}$, $Im\mathcal{L} = KerQ$. Furthermore, $\mathcal{L}_{D(\mathcal{L}) \cap KerP} : (I - P)\Delta \to Im\mathcal{L}$ is invertible. Denote using K_p the inverse of $\mathcal{L}_P$.

Let Γ be an open bounded subset of Δ. Let the operator $\mathcal{N} : \bar{\Gamma} \to \Pi$ be $\mathcal{L}$-compact in $\bar{\Gamma}$ which means that $Q\mathcal{N}(\bar{\Gamma})$ is bounded and the operator $K_p(I - Q)\mathcal{N}(\bar{\Gamma})$ is relatively compact. The following lemma is the famous Mawhin's continuation theorem.

Lemma 1 ([21])**.** *Assume that Δ and Π are two Banach spaces, and $\mathcal{L} : D(\mathcal{L}) \subset \Delta \to \Pi$, is a Fredholm operator with index zero. Furthermore, $\Gamma \subset \Delta$ is an open bounded set and $\mathcal{N} : \bar{\Gamma} \to \Pi$ is L-compact on $\bar{\Gamma}$. If all the following conditions hold:*

(1) $\mathcal{L}x \neq \lambda \mathcal{N}x, \forall x \in \partial\Gamma \cap D(\mathcal{L}), \forall \lambda \in (0, 1)$,
(2) $\mathcal{N}x \notin Im\mathcal{L}, \forall x \in \partial\Gamma \cap Ker\mathcal{L}$,
(3) $deg\{Q\mathcal{N}, \Gamma \cap Ker\mathcal{L}, 0\} \neq 0$.

Then equation $\mathcal{L}x = \mathcal{N}x$ has a solution on $\bar{\Gamma} \cap D(\mathcal{L})$.

Definition 1. *Let $w^*(\cdot) = (u_1^*(\cdot), \cdots, u_n^*(\cdot), v_1^*(\cdot), \cdots, v_n^*(\cdot))^T$ be a periodic solution of (3) with initial conditions $\phi^* \in C([-\gamma, 0], \mathbb{R}^{2n})$. and $w(\cdot) = (u_1(\cdot), \cdots, u_n(\cdot), v_1(\cdot), \cdots, v_n(\cdot))^T$ be a solution of (3) with initial conditions $\phi \in C([-\gamma, 0], \mathbb{R}^{2n})$. The periodic solution $w^*(\cdot)$ is called globally exponentially stable, if there are $\alpha > 0$ and $\mathcal{M} \geq 1$ such that*

$$\sum_{k=1}^{2n} |w_k(t) - w_k^*(t)| \leq \mathcal{M}||\phi - \phi^*||e^{\alpha t},\ t > 0,$$

where $||\phi - \phi^*|| = \sum_{k=1}^{2n} \max_{s \in [-\gamma, 0]} |\phi_k(s) - \phi_k^*(s)|$.

3. Existence of Periodic Solution

Theorem 1. *Suppose that (H_1)–(H_3) satisfy. There exists for system (3) at least one periodic solution, provided that*

$$\nu_1 = \min_{1 \leq i \leq n} \left\{ \bar{a}_i - \sum_{j=1}^{n} |\bar{b}_{ji}| m_i - \sum_{j=1}^{n} \sum_{k=1}^{n} |\bar{T}_{kji}| l_j m_i - |\bar{\beta}_i| c m_i \right\} > 0,\ \nu_2 = \min_{1 \leq i \leq n} \left\{ \bar{\alpha}_i - |\bar{D}_i| \right\} > 0. \tag{6}$$

Proof. Let

$$\Delta = \{w(\cdot) \in C(\mathbb{R}, \mathbb{R}^{2n}) : w(t + T) = w(t)\},$$

where $w(\cdot) = (u_1(\cdot), u_2(\cdot), \cdots, u_n(\cdot), v_1(\cdot), v_2(\cdot), \cdots, v_n(\cdot))^T$. Define the norm of Δ by $||w|| = \sum_{i=1}^{n} (|u_i|_0 + |v_i|_0)$ for $w \in \Delta$. Obviously, Δ is a Banach space. Define projectors P and Q by, respectively,

$$P : \Delta \to \mathrm{Ker}\mathcal{L},\ (Pw)(t) = \frac{1}{T} \int_0^T w(s) ds$$

and

$$Q : \Delta \to \Delta / \mathrm{Im}\mathcal{L},\ (Pw)(t) = \frac{1}{T} \int_0^T w(s) ds.$$

Set

$$(\mathcal{L}w)(t) : Dom(\mathcal{L}) \subset \Delta \to \Delta,\ (\mathcal{L}w)(\cdot) = (u_1'(\cdot), u_2'(\cdot), \cdots, u_n'(\cdot), v_1'(\cdot), v_2'(\cdot), \cdots, v_n'(\cdot))^T \tag{7}$$

and $\mathcal{N} : \Delta \to \Delta$,

$$(\mathcal{N}w)(\cdot) = ((\mathcal{N}u_1)(\cdot), (\mathcal{N}u_2)(\cdot), \cdots, (\mathcal{N}u_n)(\cdot), (\mathcal{N}v_1)(\cdot), (\mathcal{N}v_2)(\cdot), \cdots, (\mathcal{N}v_n)(\cdot))^T, \tag{8}$$

where

$$\begin{aligned}(\mathcal{N}u_i)(t) = &-a_i(t)u_i(t) + \sum_{j=1}^{n} b_{ij}(t)f_j(u_j(t-\gamma_j(t))) \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk}(t)f_j(u_j(t-\gamma_j(t)))f_k(u_k(t-\gamma_k(t))) + D_i(t)v_i(t) + I_i(t),\end{aligned}$$

$$(\mathcal{N}v_i)(t) = -\alpha_i(t)v_i(t) + \beta_i(t)cf_i(u_i(t)).$$

Obviously, the operator $\mathcal{L}$ is a Fredholm operator with index zero. Consider the following operator equation:

$$(\mathcal{L}w)(t) = \lambda(\mathcal{N}w)(t),\ t \in \mathbb{R},\ \lambda \in (0,1), \tag{9}$$

$\mathcal{L}$ and $\mathcal{N}$ can be found in (7) and (8), respectively. We first prove that the solution of the operator system (9) is bounded. Using (9), we obtain

$$\begin{aligned}u_i'(t) = &-\lambda a_i(t)u_i(t) + \lambda\sum_{j=1}^{n} b_{ij}(t)f_j(u_j(t-\gamma_j(t))) \\ &+ \lambda\sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk}(t)f_j(u_j(t-\gamma_j(t)))f_k(u_k(t-\gamma_k(t))) + \lambda D_i(t)v_i(t) + \lambda I_i(t),\end{aligned} \tag{10}$$

$$v_i'(t) = -\lambda\alpha_i(t)v_i(t) + \lambda\beta_i(t)cf_i(u_i(t)). \tag{11}$$

From (11), (H_1) and (H_2), we obtain

$$v_i(t) = \int_0^T \frac{e^{-\int_s^t \lambda\alpha_i(u)du}}{1-e^{-\int_0^T \lambda\alpha_i(u)du}}\lambda\beta_i(s)cf_i(u_i(s))ds$$

and

$$|v_i(t)| \le \frac{|\beta_i|_0 c l_i}{1-e^{-\int_0^T \alpha_i(u)du}} = p_i. \tag{12}$$

From (10), (12), (H_1) and (H_2), we have

$$\begin{aligned}u_i(t) = &\int_0^T \frac{e^{-\int_s^t \lambda a_i(u)du}}{1-e^{-\int_0^T \lambda a_i(u)du}}\Bigg(\lambda\sum_{j=1}^{n} b_{ij}(s)f_j(u_j(s-\gamma_j(s))) \\ &+ \lambda\sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk}(s)f_j(u_j(s-\gamma_j(s)))f_k(u_k(s-\gamma_k(s))) + \lambda D_i(s)v_i(s) + \lambda I_i(s)\Bigg)ds\end{aligned}$$

and

$$|u_i(t)| \le \frac{1}{1-e^{-\int_0^T a_i(u)du}}\left(\sum_{j=1}^{n} |b_{ij}l_j + \sum_{j=1}^{n}\sum_{k=1}^{n} |T_{ijk}|_0 l_j l_k + |D_i|_0 p_i + |I_i|_0\right) = q_i.$$

Obviously, p_i and q_i do not depend on λ. Let

$$\widetilde{M} = \sum_{i=1}^{n}(p_i + q_i) + K.$$

Here, we select $K > 0$ that is sufficiently large such that

$$\nu\widetilde{M} - \sum_{i=1}^{n} |\bar{I}_i| > 0,$$

where $\nu = \min\{\nu_1, \nu_2\}$ is defined by (6). Let

$$\Xi = \left\{ (u,v)^T \in \Delta : ||(u,v)^T|| < \widetilde{M} \right\},$$

where $u = (u_1(\cdot), u_2(\cdot), \cdots, u_n(\cdot))^T$, $v = (v_1(\cdot), v_2(\cdot), \cdots, v_n(\cdot))^T$. It is easy to see that condition (1) in Lemma 1 satisfies. For $(u,v)^T \in \partial\Xi \cap \mathbb{R}^{2n}$, then u_i and v_i are constants with $\sum_{i=1}^{n}(|u_i| + |v_i|) = \widetilde{M}$. From (6), ($H_2$), and ($H_3$), we obtain

$$QN(u,v)^T = \begin{pmatrix} \sum_{j=1}^{n} \bar{b}_{1j} f_j(u_j) - \bar{a}_1 u_1 \\ + \sum_{j=1}^{n} \sum_{k=1}^{n} \bar{T}_{1jk} f_j(u_j) f_k(u_k) + \bar{D}_1 v_1 + \bar{I}_1 \\ \vdots \\ \sum_{j=1}^{n} \bar{b}_{nj} f_j(u_j) - \bar{a}_n u_n \\ + \sum_{j=1}^{n} \sum_{k=1}^{n} \bar{T}_{njk} f_j(u_j) f_k(u_k) + \bar{D}_n v_n + \bar{I}_n \\ -\bar{\alpha}_1 v_1 + \bar{\beta}_1 c f_1(u_1) \\ \vdots \\ -\bar{\alpha}_n v_n + \bar{\beta}_n c f_n(u_n) \end{pmatrix}$$

and

$$\begin{aligned} ||Q\mathcal{N}(u,v)^T|| &= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \bar{b}_{ij} f_j(u_j) - \bar{a}_i u_i + \sum_{j=1}^{n} \sum_{k=1}^{n} \bar{T}_{ijk} f_j(u_j) f_k(u_k) + \bar{D}_i v_i + \bar{I}_i \right| \\ &+ \sum_{i=1}^{n} \left| -\bar{\alpha}_i v_i + \bar{\beta}_i c f_i(u_i) \right| \\ &\geq \sum_{i=1}^{n} \bar{a}_i |u_i| - \sum_{i=1}^{n} \sum_{j=1}^{n} |\bar{b}_{ij}| m_j |u_j| - \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} |\bar{T}_{ijk}| l_j m_k |u_k| \\ &- \sum_{i=1}^{n} |\bar{D}_i||v_i| - |\bar{I}_i| + \sum_{i=1}^{n} \bar{\alpha}_i |v_i| - \sum_{i=1}^{n} |\bar{\beta}_i| c m_i |u_i| \\ &= \sum_{i=1}^{n} \left(\bar{a}_i - \sum_{j=1}^{n} |\bar{b}_{ji}| m_i - \sum_{j=1}^{n} \sum_{k=1}^{n} |\bar{T}_{kji}| l_j m_i - |\bar{\beta}_i| c m_i \right) |u_i| \\ &+ \sum_{i=1}^{n} \left(\bar{\alpha}_i - |\bar{D}_i| \right) |v_i| - \sum_{i=1}^{n} |\bar{I}_i| \\ &\geq \nu \sum_{i=1}^{n} (|u_i| + |v_i|) - \sum_{i=1}^{n} |\bar{I}_i| \\ &= \nu\widetilde{M} - \sum_{i=1}^{n} |\bar{I}_i| > 0. \end{aligned}$$

Consequently,

$$Q\mathcal{N}(u,v)^T \neq \mathbf{0} \text{ for } (u,v)^T \in \partial\Xi \cap \mathbb{R}^{2n}$$

which means that condition (2) in Lemma 1 satifies. Define $\Phi(u,v,\mu) : Ker\mathcal{L} \times [0,1]$ by

$$\Phi(u,v,\mu) = -\mu(\bar{a}_1 u_1, \cdots, \bar{a}_1 u_n, \bar{\alpha}_1 v_1, \cdots, \bar{\alpha}_1 v_n)^T + (1-\mu) Q\mathcal{N}(u,v)^T.$$

For $(u,v)^T \in \partial\Xi \cap Ker\mathcal{L}$, $(u,v)^T$ is a $2n$-dimensional constant vector. Thus,

$$\begin{aligned}
||\Phi(u,v,\mu)|| = & \sum_{i=1}^{n} \bigg| -\bar{a}_i u_i - (1-\mu)\bigg(\sum_{j=1}^{n} \bar{b}_{ij} f_j(u_j) + \sum_{j=1}^{n}\sum_{k=1}^{n} \bar{T}_{ijk} f_j(u_j) f_k(u_k) + \bar{D}_i v_i + \bar{I}_i \bigg) \bigg| \\
& + \sum_{i=1}^{n} \bigg| -\bar{\alpha}_i v_i - (1-\mu)\bar{\beta}_i c f_i(u_i) \bigg| \\
& \geq \sum_{i=1}^{n} \bar{a}_i |u_i| - \sum_{i=1}^{n}\sum_{j=1}^{n} |\bar{b}_{ij}| m_j |u_j| - \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} |\bar{T}_{ijk}| l_j m_k |u_k| \\
& - \sum_{i=1}^{n} |\bar{D}_i||v_i| - |\bar{I}_i| + \sum_{i=1}^{n} \bar{\alpha}_i |v_i| - \sum_{i=1}^{n} |\bar{\beta}_i| c m_i |u_i| \\
& \geq \nu \widetilde{M} - \sum_{i=1}^{n} |\bar{I}_i| > 0
\end{aligned}$$

and

$$\Phi(u,v,\mu) \neq \mathbf{0} \text{ for } (u,v)^T \in \partial\Xi \cap Ker\mathcal{L}.$$

Hence,

$$\begin{aligned}
& \deg(Q\mathcal{N}(u,v)^T, \Xi \cap Ker\mathcal{L}, \mathbf{0}) \\
& = \deg(-\mu(\bar{a}_1 u_1, \cdots, \bar{a}_1 u_n, \bar{\alpha}_1 v_1, \cdots, \bar{\alpha}_1 v_n)^T, \Xi \cap Ker\mathcal{L}, \mathbf{0}) \\
& \neq 0.
\end{aligned}$$

Therefore, all conditions of Lemma 1 satisfy and there exists a periodic solution for system (3). □

Remark 1. *Usually, the Lyapunov–Kravsovskii functional method is the main method for studying the stability of equations, and a large number of stability results are obtained using this method. But constructing a suitable Lyapunov function is very difficult. In this article, we use the basic theory of differential equations and Gronwall's inequality to obtain the dynamical behaviors of system (3). The method presented in this article is easier to understand and the proof process is not complicated.*

4. Globally Exponential Stability

Theorem 2. *Assume that all conditions of Theorem 1 satisfy. Then, the periodic solution of system (3) is globally exponentially stable, if*

$$\xi > \rho_2, \tag{13}$$

where

$$\xi = \min_{1\leq i\leq n}\{a_i(t), \alpha_i(t)\}, \rho_2 = \max_{1\leq i\leq n}\bigg\{ \sum_{j=1}^{n} |b_{ij}|_0 m_j \chi_j + \sum_{j=1}^{n}\sum_{k=1}^{n} |T_{ikj}|_0 l_k m_j \chi_j, |D_i|_0 + |\beta_i|_0 c m_i \bigg\}.$$

Proof. Using Theorem 1, (3) has a periodic solution $w^*(t) = (u^*(t), v^*(t))^T$, where $u^*(\cdot) = (u_1^*(\cdot), u_2^*(\cdot), \cdots, u_n^*(\cdot))^T$, $v^*(\cdot) = (v_1^*(\cdot), v_2^*(\cdot), \cdots, v_n^*(\cdot))^T$. Let $w(t) = (u(t), v(t))^T$ be any solution of (3). Let

$$\tilde{w}_i(t) = w_i(t) - w_i^*(t),\ \tilde{u}_i(t) = u_i(t) - u_i^*(t),\ \tilde{v}_i(t) = v_i(t) - v_i^*(t),$$

$$g_j(\tilde{u}_j(t)) = f_j(\tilde{u}_j(t) + u_j^*(t)) - f_j(u_j^*(t)).$$

Obviously $g_j(\tilde{u}_j(t))$ satisfies (H_3). System (3) can be rewritten as follows:

$$\begin{aligned}\tilde{u}_i'(t) = -a_i(t)\tilde{u}_i(t) &+ \sum_{j=1}^{n} b_{ij}(t)g_j(\tilde{u}_j(t-\gamma_j(t))) \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk}(t)g_j(\tilde{u}_j(t-\gamma_j(t)))g_k(\tilde{u}_k(t-\gamma_k(t))) + D_i(t)\tilde{v}_i(t),\end{aligned} \tag{14}$$

$$\tilde{v}_i'(t) = -\alpha_i(t)\tilde{v}_i(t) + \beta_i(t)cg_i(\tilde{u}_i(t)). \tag{15}$$

For $t \geq 0$, using (14) we obtain

$$\begin{aligned}\tilde{u}_i(t) = e^{-\int_0^t a_i(r)dr}\tilde{u}_i(0) + e^{-\int_0^t a_i(r)dr}\int_0^t \Bigg(\sum_{j=1}^{n} b_{ij}(r)g_j(\tilde{u}_j(r-\gamma_j(r))) \\ + \sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk}(r)g_j(\tilde{u}_j(r-\gamma_j(r)))g_k(\tilde{u}_k(r-\gamma_k(r))) + D_i(r)\tilde{v}_i(r)\Bigg)e^{\int_0^r a_i(u)du}dr.\end{aligned} \tag{16}$$

Furthermore, for $t \geq 0$, using (15) we obtain

$$\tilde{v}_i(t) = e^{-\int_0^t \alpha_i(r)dr}\tilde{v}_i(0) + e^{-\int_0^t \alpha_i(r)dr}\int_0^t \Big(\beta_i(r)cg_i(\tilde{u}_i(r))\Big)e^{\int_0^r \alpha_i(u)du}dr. \tag{17}$$

Using (14), (H_2), and (H_3), we have

$$\begin{aligned}|\tilde{u}_i(t)| \leq{}& e^{-\int_0^t a_i(r)dr}\Bigg[|\tilde{u}_i(0)| + \int_0^t \Bigg(\sum_{j=1}^{n}|b_{ij}|_0 m_j|\tilde{u}_j(r-\gamma_j(r))| \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{n}|T_{ijk}|_0 l_j m_k|\tilde{u}_k(r-\gamma_k(r))| + |D_i|_0|\tilde{v}_i(r)|\Bigg)e^{\int_0^r a_i(u)du}dr\Bigg] \\ \leq{}& e^{-\xi t}|\tilde{u}_i(0)| + e^{-\xi t}\sum_{j=1}^{n}|b_{ij}|_0 m_j\chi_j\int_{-|\gamma_j|_0}^{t-\gamma_j(t)}|\tilde{u}_j(r)|e^{|a_i|_0(r+|\gamma_j|_0)}dr \\ &+ e^{-\xi t}\sum_{j=1}^{n}\sum_{k=1}^{n}|T_{ikj}|_0 l_k m_j\chi_j\int_{-|\gamma_j|_0}^{t-\gamma_j(t)}|\tilde{u}_j(r)|e^{|a_i|_0(r+|\gamma_j|_0)}dr \\ &+ e^{-\xi t}|D_i|_0\int_0^t|\tilde{v}_i(r)|e^{|a_i|_0 r}dr \\ \leq{}& e^{-\xi t}\Bigg[|\tilde{u}_i(0)| + \Bigg(\sum_{j=1}^{n}|b_{ij}|_0 m_j\chi_j e^{|a_i|_0|\gamma_j|_0} + \sum_{j=1}^{n}\sum_{k=1}^{n}|T_{ikj}|_0 l_k m_j\chi_j e^{|a_i|_0|\gamma_j|_0}\Bigg)\max_{-\tau\leq s\leq 0}|\tilde{u}_j(s)|\Bigg] \\ &+ e^{-\xi t}\Bigg(\sum_{j=1}^{n}|b_{ij}|_0 m_j\chi_j + \sum_{j=1}^{n}\sum_{k=1}^{n}|T_{ikj}|_0 l_k m_j\chi_j\Bigg)\int_0^t|\tilde{u}_i(r)|e^{|a_i|_0 r}dr \\ &+ e^{-\xi t}|D_i|_0\int_0^t|\tilde{v}_i(r)|e^{|a_i|_0 r}dr.\end{aligned} \tag{18}$$

From (15) and (H_3), we have

$$|\tilde{v}_i(t)| \leq e^{-\xi t}|\tilde{v}_i(0)| + e^{-\xi t}|\beta_i|_0 cm_i\int_0^t|\tilde{u}_i(r)|e^{|a_i|_0 r}dr. \tag{19}$$

From (18) and (19), we obtain

$$\begin{aligned}\delta(t) &= \sum_{i=1}^{n}(|\tilde{u}_i(t)| + |\tilde{v}_i(t)|)\\ &\le \rho_1||\phi - \phi^*||e^{-\xi t} + \rho_2\int_0^t \delta(r)e^{\xi(r-t)}dr,\end{aligned}$$

where

$$\rho_1 = 2 + \max_{1\le n}\left(\sum_{j=1}^{n}|b_{ij}|_0 m_j\chi_j e^{|a_i|_0|\gamma_j|_0} + \sum_{j=1}^{n}\sum_{k=1}^{n}|T_{ikj}|_0 l_k m_j \chi_j e^{|a_i|_0|\gamma_j|_0}\right),$$

ρ_2 is defined by (13). Using Gronwall's inequality, we obtain

$$\delta(t) = \sum_{i=1}^{n}(|\tilde{u}_i(t)| + |\tilde{v}_i(t)|) \le \rho_1||\phi - \phi^*||e^{-(\xi-\rho_2)t}.$$

Due to Definition 1, there exists a globally exponentially stable periodic solution for system (3). □

Remark 2. *Usually, the Lyapunov–Kravsovskii functional method is the main method for studying the stability of equations, and a large number of stability results are obtained using this method. But constructing a suitable Lyapunov function is very difficult. In this article, we use the basic theory of differential equations and Gronwall's inequality to obtain the dynamical behaviors of system (3). The method presented in this article is easier to understand and the proof process is not complicated.*

5. Example

Consider the following Hebbian-type networks:

$$\begin{aligned}
u_1'(t) &= -(12-\sin t)u_1(t) + \sin t f_1(u_1(1-\frac{1}{2}\sin t)) + \sin t f_2(u_2(1-\frac{1}{2}\sin t))\\
&\quad + \cos t f_1(u_1(1-\frac{1}{2}\sin t))f_1(u_1(1-\frac{1}{2}\sin t)) + \sin t f_1(u_1(1-\frac{1}{2}\sin t)) + \sin t f_2(u_2(1-\frac{1}{2}\sin t))\\
&\quad + \cos t f_2(u_2(1-\frac{1}{2}\sin t))f_1(u_1(1-\frac{1}{2}\sin t)) + \cos t f_2(u_2(1-\frac{1}{2}\sin t))f_2(u_2(1-\frac{1}{2}\sin t)) + (\frac{1}{5}\cos t)v_1(t),\\
v_1'(t) &= -(10-\cos t)v_1(t) + \frac{1}{2}\sin t f_1(u_1(t)),\\
u_2'(t) &= -(12-\sin t)u_2(t) + \sin t f_1(u_1(1-\frac{1}{2}\sin t)) + \sin t f_2(u_2(1-\frac{1}{2}\sin t))\\
&\quad + \cos t f_1(u_1(1-\frac{1}{2}\sin t))f_1(u_1(1-\frac{1}{2}\sin t)) + \sin t f_1(u_1(1-\frac{1}{2}\sin t))dt + \sin t f_2(u_2(1-\frac{1}{2}\sin t))\\
&\quad + \cos t f_2(u_2(1-\frac{1}{2}\sin t))f_1(u_1(1-\frac{1}{2}\sin t)) + \cos t f_2(u_2(1-\frac{1}{2}\sin t))f_2(u_2(1-\frac{1}{2}\sin t)) + (\frac{1}{5}\cos t)v_2(t),\\
v_2'(t) &= -(10-\cos t)v_2(t) + \frac{1}{2}\sin t f_2(u_2(t)),
\end{aligned} \tag{20}$$

where

$$i,j,k = 1,2,\ a_i(t) = 12-\sin t,\ \alpha_i(t) = 10-\cos t,\ f_1(x) = f_2(x) = \frac{1}{10}\tanh x,$$

$$\gamma_j(t) = 1-\frac{1}{2}\sin t,\ I_i(t) = 0,\ b_{ij}(t) = \sin t,\ D_i(t) = \frac{1}{5}\cos t,\ c = 1,$$

$$\beta_i(t) = \frac{1}{2}\sin t,\ T_{ijk} = \cos t.$$

After simple calculations, we have

$$l_j = m_j = \frac{1}{10}, \ \tau = \frac{3}{2}, \ \chi_j = 2, \ |b_{ij}|_0 = |T_{ijk}|_0 = 1, \ |D_i|_0 = \frac{1}{5}, \ |\beta_i|_0 = \frac{1}{2},$$

$$\nu_1 = \min_{1\le i\le n} \left\{ \bar{a}_i - \sum_{j=1}^{n} |\bar{b}_{ji}| m_i - \sum_{j=1}^{n}\sum_{k=1}^{n} |\bar{T}_{kji}| l_j m_i - |\bar{\beta}_i| c m_i \right\} = 12,$$

$$\nu_2 = \min_{1\le i\le n} \left\{ \bar{\alpha}_i - |\bar{D}_i| \right\} = 10, \ \ \xi = \min_{1\le i\le n} \{a_i(t), \alpha_i(t)\} = 9,$$

$$\rho_2 = \max_{1\le i\le n} \left\{ \sum_{j=1}^{n} |b_{ij}|_0 m_j \chi_j + \sum_{j=1}^{n}\sum_{k=1}^{n} |T_{ikj}|_0 l_k m_j \chi_j, |D_i|_0 + |\beta_i|_0 c m_i \right\} = 0.51.$$

We obtain $\nu_1, \nu_2 > 0$ and $\xi > \rho_2$. From selections of $f_1(x)$ and $f_2(x)$, it is obvious that there exists a periodic solution for system (20) $(0,0)^T$. Furthermore, using the above calculations, we confirm that all conditions of Theorem 1 and Theorem 2 satisfy. Hence, there exists a globally asymptotically periodic solution for system (20) $(0,0)^T$. Figures 1 and 2 show the evolution of the solution of system (20).

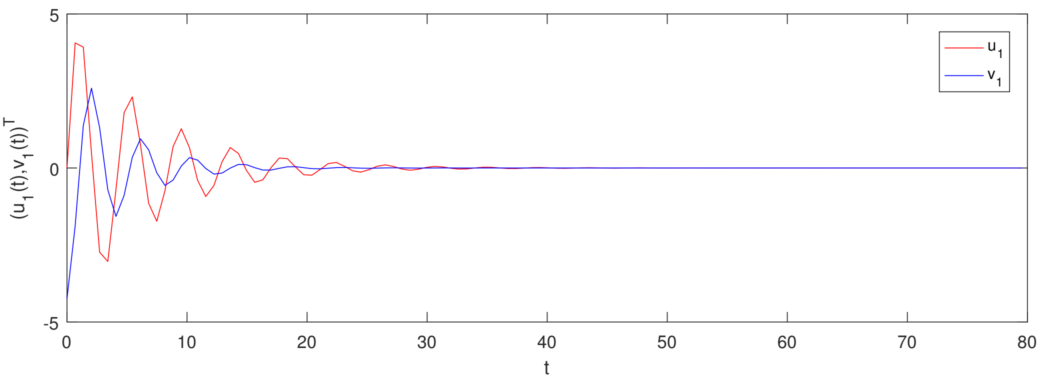

Figure 1. Evolution for the solution $(u_1(t), v_1(t))^T$ of system (20).

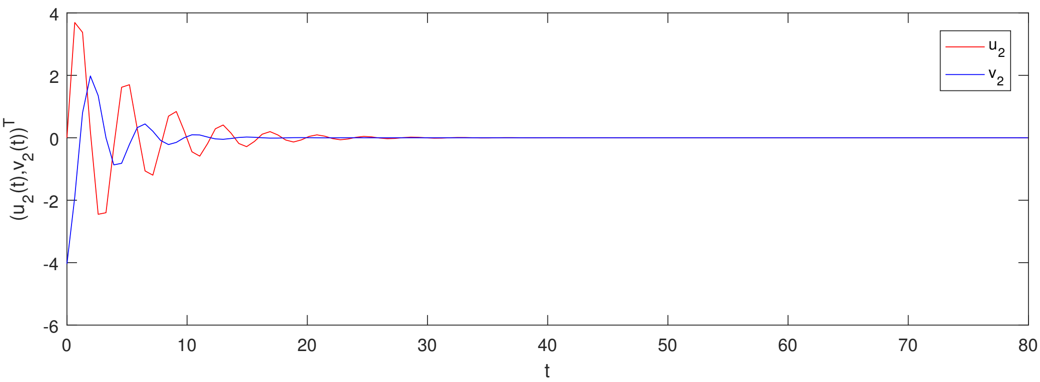

Figure 2. Evolution for the solution $(u_2(t), v_2(t))^T$ of system (20).

6. Conclusions and Discussions

We study a class of Hebbian-type networks with variable lags. More precisely, we obtain some criteria for the existence of a periodic solution. Also, we examine the globally exponential stability analysis of system (3). Finally, we provided an example to show the analytical results. We can further investigate the dynamic behavior of system (3) under different environmental interference factors. One aspect future research is the stochastic Hebbian-type networks.

Author Contributions: Methodology, M.X. and H.Y.; Writing—original draft, B.D. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data used to support the findings of this study are available from the corresponding author upon request.

Acknowledgments: The authors would like to thank the editor and the referees for their valuable comments and suggestions that improved the quality of our paper.

Conflicts of Interest: The author declares that there are no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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