## Article

# Study on the Criteria for Starlikeness in Integral Operators Involving Bessel Functions 

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#### Abstract

The study presented in this paper follows a line of research familiar for Geometric Function Theory, which consists in defining new integral operators and conducting studies for revealing certain geometric properties of those integral operators such as univalence, starlikness, or convexity. The present research focuses on the Bessel function of the first kind and order $v$ unveiling the conditions for this function to be univalent and further using its univalent form in order to define a new integral operator on the space of holomorphic functions. For particular values of the parameters implicated in the definition of the new integral operator involving the Bessel function of the first kind, the well-known Alexander, Libera, and Bernardi integral operators can be obtained. In the first part of the study, necessary and sufficient conditions are obtained for the Bessel function of the first kind and order $v$ to be a starlike function or starlike of order $\alpha \in[0,1)$. The renowned prolific method of differential subordination due to Sanford S. Miller and Petru T. Mocanu is employed in the reasoning. In the second part of the study, the outcome of the first part is applied in order to introduce the new integral operator involving the form of the Bessel function of the first kind, which is starlike. Further investigations disclose the necessary and sufficient conditions for this new integral operator to be starlike or starlike of order $\frac{1}{2}$.


Keywords: holomorphic function; starlike function; univalent function; Bessel function of the first kind; Alexander integral operator; Libera integral operator; Bernardi integral operator; differential subordination; special functions

MSC: 30C45; 30C80; 33C10

## 1. Introduction

Complex analysis is essentially the study of functions involving a complex variable. The foundations of this theory were established at the middle of the 19th century. The investigation of the necessary and sufficient conditions of univalence for various types of functions constitutes a basic challenge in the theory of functions of a complex variable. Numerous of these univalence conditions reflect geometric properties such as starlikeness, starlikeness of a particular order, convexity, convexity of a particular order, and other similar properties. Another basic concern in the theory of functions of a complex variable is the study of different types of operators, of which integral operators are one of the main topics of interest, as suggested by the recent review on operators used in geometric function theory [1]. The first integral operator was introduced by J. Alexander in 1915 [2], which marked the beginning of the research on integral operators. Another renowned integral operator was introduced by R.J. Libera in 1965 [3]. It has been established that the Libera integral operator preserves a number of classes of univalent functions, including the class of starlike and convex functions, respectively. S.D. Bernardi generalized this operator and introduced the Bernardi integral operator in 1969 [4], another prominent
operator that was proved to preserve the same classes of univalent functions like the Libera integral operator. All of these operators will be mentioned in the study that is presented in this work. The importance of the integral operators in the study regarding geometric function theory is highlighted in [1]. This article provides a brief overview of the historical evolution over more than a century of some of the most well-known integral and related operators in geometric function theory. Previously established results could be more easily re-obtained and intriguing new conclusions about the geometric properties of different classes of analytic functions emerged nicely with the use of integral operators.

A notable tool for establishing univalence conditions and geometric properties of different types of operators including integral operators is provided by the theory of differential subordination established by Sanford S. Miller and Petru T. Mocanu in two papers published in 1978 [5] and 1981 [6] and consolidated in the following years, as shown in the monograph that presents all the basic aspects of this theory [7].

The study exhibited in the present paper develops this line of investigation by using means of the method of differential subordination for establishing geometric properties concerning the outstanding Bessel function of the first kind and further involving it in the definition of a new generalized integral operator.

Researchers have improved our knowledge of the properties and behavior of univalent functions in various contexts by examining the interactions between integral operators and special functions. It is significant that researchers working on the topic nowadays are striving to develop new theoretical approaches and strategies that integrate observational findings with a range of real-world uses.

Bessel functions are significant special functions that are used in the mathematical models of a variety of physical phenomena. Because it results from the Laplace equation when there is cylindrical symmetry, the Bessel equation is significant in mathematical physics. In his work [8], G.N. Watson provided an overview of all the features and uses of Bessel functions. Special functions are significant tools in geometric function theory. Perhaps the most famous application of certain special functions is their use in the proof given by Louis de Branges in 1985 [9] of the famous Bieberbach conjecture established in 1916.

The present investigation on the Bessel function of the first kind was inspired by the known geometric properties of this function initially established by Á. Baricz [10-12] and further investigated in more recent works like [13-16].

Additional motivation for the definition of the new generalized integral operator involving the Bessel function of the first kind is provided by the research, which involved other generalized integral operators $[17,18]$, and by the compelling results recently made available concerning the geometric properties of integral operators defined pertaining to the Bessel function [19,20].

After exposing the research challenge and outlining the purpose for the study, we introduce the basic concepts and notations accustomed to geometric function theory which give the environment for the proposed investigation.

The class of holomorphic functions in $U=\{z \in \mathbb{C}:|z|<1\}$ is denoted as the notation $\mathcal{H}(U)$. Other designations linked to the unit disc $U$ are $\bar{U}=\{z \in \mathbb{C}:|z| \leq 1\}$ and $\partial U=\{z \in \mathbb{C}:|z|=1\}$.

The functions belonging to the classes

$$
\mathcal{H}[a, n]=\left\{f \in \mathcal{H}(U): f(z)=a+a_{n} z^{n}+a_{n+1} z^{n+1}+\cdots, z \in U\right\}
$$

and

$$
\mathcal{A}_{n}=\left\{f \in \mathcal{H}(U): f(z)=z+a_{n+1} z^{n+1}+\cdots, z \in U\right\} \text { with } \mathcal{A}_{1}=\mathcal{A}
$$

contribute to this study.
The study additionally involves functions from the fundamental class $\mathcal{S}$, comprising functions $f \in \mathcal{A}$ that are univalent in $U$ and are ruled by the requirements $f(0)=0$, $f^{\prime}(0)=1$.

The class

$$
\mathcal{S}^{*}(\alpha)=\left\{f \in \mathcal{A}: \operatorname{Re} \frac{z f^{\prime}(z)}{f(z)}>\alpha\right\}
$$

with $\alpha<1$ identifies starlike functions of order $\alpha$. When $\alpha=0$ the class of starlike functions is obtained, written $\mathcal{S}^{*}$.

The class

$$
\mathcal{K}(\alpha)=\left\{f \in \mathcal{A}: \operatorname{Re}\left(\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}+1\right)>\alpha\right\}
$$

with $\alpha<1$ identifies convex functions of order $\alpha$. When $\alpha=0$ the class of convex functions is obtained, written $\mathcal{K}$.

Finally, the Carathéodory class of functions

$$
\mathcal{P}=\{p \in \mathcal{H}(U): p(0)=1, \operatorname{Re} p(z)>0, z \in U\}
$$

is also referred to in the study.
Remark 1. A duality theorem ([21], $p$. 76) is valid for the classes $\mathcal{S}^{*}(\alpha)$ and $\mathcal{S}^{*}$ that is interpreted as $\mathcal{S}^{*}(\alpha) \subset \mathcal{S}^{*}$ when $\alpha \in[0,1)$, i.e., functions that are starlike of order $\alpha$ are univalent. However, if $\alpha<0$, then functions $f \in \mathcal{S}^{*}(\alpha)$ are referred to as starlike of negative order, and such functions are not always univalent.

The fundamental ideas of the differential subordination theory are given in the following.
Definition 1 ([5-7]). Let $f$ and $F$ be members of $\mathcal{H}(U)$. The function $f$ is said to be subordinate to $F$, written $f \prec F, f(z) \prec F(z)$, if there exists a function $w$ analytic in $U$, with $w(0)=0$ and $|w(z)|<1, z \in U$ and such that $f(z)=F(w(z))$. If $F$ is univalent, then $f \prec F$ if and only if $f(0)=F(0)$ and $f(U) \subset F(U)$.

Definition 2 ([7]). Let $Q$ denote the set of functions q that are holomorphic and injective on $\bar{U} \backslash E(q)$, where

$$
E(q)=\left\{\zeta \in \partial U: \lim _{z \rightarrow \zeta} q(z)=\infty\right\}
$$

and $q^{\prime}(\zeta) \neq 0$, for $\zeta \in \partial U \backslash E(q)$. The set $E(q)$ is called the exception set.
Remark 2 ([7]). If $q \in Q$, the domain $\Delta=q(U)$ is simply connected. Functions $q_{1}(z)=z$ and $q_{2}(z)=\frac{1+z}{1-z}$ are in the set $Q$.

Definition 3 ([7]). Let $\psi(r, s, t ; z): \mathbb{C}^{3} \times U \rightarrow \mathbb{C}$ and let $h$ be univalent in $U$. If $p$ is analytic and satisfies the (second-order) differential subordination

$$
\begin{equation*}
\psi\left(p(z), z p^{\prime}(z), z^{2} p^{\prime \prime}(z) ; z\right) \prec h(z), \tag{1}
\end{equation*}
$$

then $p$ is called a solution of the differential subordination. The univalent function $q$ is called a dominant of the solutions of the differential subordination or more simply a dominant, if $p \prec q$ for all $p$ satisfying (1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants $q$ of (1) is said to be the best dominant of (1).

Definition 4 ([7]). Let $\Omega \subset \mathbb{C}$, let $q \in Q$ and $n \in \mathbb{N}, n \geq 1$. Denote by $\Psi_{n}[\Omega, q]$ the class of functions $\psi: \mathbb{C}^{3} \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition,

$$
\begin{equation*}
\psi(r, s, t ; z) \notin \Omega \tag{2}
\end{equation*}
$$

whenever

$$
r=q(\zeta), s=m \zeta q^{\prime}(\zeta), \operatorname{Re}\left(\frac{t}{s}+1\right) \geq m \cdot \operatorname{Re}\left[\frac{\zeta q^{\prime \prime}(\zeta)}{q^{\prime}(\zeta)}+1\right], \zeta \in \partial U \backslash E(q), z \in U, m \geq n
$$

The set $\Psi_{n}[\Omega, q]$ is called the class of admissible functions and condition (2) is referred to as an admissibility condition.

Remark 3 ([7]). In the special case when $\psi: \mathbb{C}^{2} \times U \rightarrow \mathbb{C}$, the admissibility condition becomes:

$$
\begin{equation*}
\psi(r, s ; z) \notin \Omega \tag{3}
\end{equation*}
$$

whenever

$$
r=q(\zeta), s=m \zeta q^{\prime}(\zeta), \zeta \in \partial U \backslash E(q), z \in U, m \geq n
$$

If $q(U)=\Delta=\{w \in \mathbb{C}: \operatorname{Re} w>\alpha, \alpha>0\}$, and $q(0)=a \in \Delta, \operatorname{Re} a>0$, then the class of functions $\Psi_{n}[\Omega, q]$ is denoted by $\Psi_{n}[\Omega, q] \equiv \Psi_{n}\{a\}$. In this case, the admissibility condition (2) can be written as:

$$
\begin{equation*}
\psi(\rho i, \sigma, \mu+i \delta ; z) \notin \Omega \tag{4}
\end{equation*}
$$

$\rho, \sigma, \mu, \delta \in \mathbb{R}, \sigma \leq-\frac{n}{2} \cdot \frac{|a-i \rho|^{2}}{\operatorname{Rea}}, \sigma+\mu \leq 0, z \in U, n \geq 1$.
When $a=1$, then $\Psi_{n}[\Omega, q] \equiv \Psi_{n}\{1\}$ and (2) is written as:

$$
\begin{equation*}
\psi(\rho i, \sigma, \mu+i \delta ; z) \notin \Omega \tag{5}
\end{equation*}
$$

$\rho, \sigma, \mu, \delta \in \mathbb{R}, \sigma \leq-\frac{n}{2} \cdot\left(1+\rho^{2}\right), \sigma+\mu \leq 0, z \in U, n \geq 1$.
The following well-known result in the theory of differential subordinations is necessary for the proofs of the results presented in the next section.

Lemma 1 ([7]). Let $p \in \mathcal{H}[a, n]$. If $\psi \in \Psi_{n}\{a\}$ then

$$
\operatorname{Re} \psi\left(p(z), z p^{\prime}(z), z^{2} p^{\prime \prime}(z) ; z\right)>0, z \in U
$$

implies

$$
\operatorname{Re} p(z)>0, z \in U
$$

The main tool of the investigation is presented next.
Definition 5 ([11]). Consider the second-order differential equation,

$$
z^{2} \frac{d^{2} y(z)}{d z^{2}}+z \frac{d y(z)}{d z}+\left(z^{2}-v^{2}\right) y(z)=0
$$

which is called Bessel's equation, where $v \in \mathbb{R}$ or $v \in \mathbb{C}$ and $z \in \mathbb{R}$ or $z \in \mathbb{C}$. The particular solutions of this equation are called Bessel functions.

The Bessel function of the first kind and order $v$ is given by:

$$
\begin{gather*}
I_{v}(z)=\left(\frac{z}{2}\right)^{v} \sum_{p=0}^{\infty} \frac{(-1)^{p}\left(\frac{z}{2}\right)^{2 p}}{p!\Gamma(v+p+1)}=  \tag{6}\\
\left(\frac{z}{2}\right)^{v} \frac{1}{\Gamma(v+1)}-\left(\frac{z}{2}\right)^{v+2} \frac{1}{\Gamma(v+2)}+\left(\frac{z}{2}\right)^{v+4} \frac{1}{2 \Gamma(v+3)}+\cdots, v \geq 0
\end{gather*}
$$

where $\Gamma$ is Euler's gamma function with $\Gamma(z+1)=z \Gamma(z), \Gamma(n+1)=n!, \Gamma(1)=1$.

Remark 4. For $v=1$, the Bessel function of the first kind and order 1 is:

$$
\begin{equation*}
I_{1}(z)=\frac{z}{2}-\left(\frac{z}{2}\right)^{3} \cdot \frac{1}{2}+\left(\frac{z}{2}\right)^{5} \cdot \frac{1}{12}+\cdots, I_{1}(0)=0, I_{1}^{\prime}(0)=\frac{1}{2} \neq 0 . \tag{7}
\end{equation*}
$$

The outcome of the present investigation is divided into two sections. In Section 2 of the paper, the study provides necessary and sufficient conditions such that the function $I_{v}(z)$ given by (6) to be starlike or starlike of order $\alpha \in[0,1)$. A theorem and three corollaries deal with this matter. In Section 3, the discussion focuses on defining a new generalized integral operator that has as particular cases the classical integral operators Alexander, Libera, and Bernardi, using the univalent form of the Bessel function of the first kind $I_{1}(z)$ given by (7). Moreover, the new generalized integral operator is examined in order to obtain theorems and corollaries that reveal necessary and sufficient conditions for its starlikeness of certain orders. Section 4 provides the conclusions of the research and potential uses for the results obtained.

## 2. Univalence Results for Bessel Function of the First Kind and Order v

In the first part of the investigation, necessary and sufficient conditions are obtained for the Bessel function of the first kind and order $v$ to be univalent.

The first result proved for the general form given by (6) uses a differential subordination in order to obtain information on the real part of an expression that is assessed in order to establish the starlikeness of the function involved.

Theorem 1. Let $I_{v}(z)$ be the function given by (6). Consider the function $g \in \mathcal{K}, g(z)=\frac{1+(1-2 \mu) z}{1-z}$, $z \in U, 0 \leq \mu<1$ and let $H: \mathcal{A}_{n} \rightarrow \mathcal{A}_{n}$ be given by

$$
\begin{equation*}
H(z)=2 \Gamma(v+1)\left(\frac{z}{2}\right)^{1-v} I_{v}(z), \quad z \in U \tag{8}
\end{equation*}
$$

If the differential subordination

$$
\begin{equation*}
\frac{z H^{\prime}(z)}{H(z)} \prec \frac{1+(1-2 \mu) z}{1-z}, \quad z \in U, 0 \leq \mu<1, \tag{9}
\end{equation*}
$$

holds, then

$$
\operatorname{Re} \frac{z I_{v}^{\prime}(z)}{I_{v}(z)}>v-1+\mu, \quad z \in U
$$

Proof. For proving that $g(z)=\frac{1+(1-2 \mu) z}{1-z}$ is convex in $U$, we calculate $g^{\prime}(z)=\frac{2(1-\mu)}{(1-z)^{2}}$, $g^{\prime \prime}(z)=\frac{4(1-\mu)}{(1-z)^{3}}$ and we evaluate

$$
\operatorname{Re}\left(\frac{z g^{\prime \prime}(z)}{g^{\prime}(z)}+1\right)=\operatorname{Re} \frac{1+z}{1-z}>0, z \in U
$$

Hence, $g \in \mathcal{K}$.
Further, we know that $g(U)=\{z \in \mathbb{C}: \operatorname{Re} z>\mu, 0 \leq \mu<1\}$ is a convex domain. By differentiating (8), we have

$$
\begin{equation*}
\frac{z H^{\prime}(z)}{H(z)}=1-v+\frac{z I_{v}^{\prime}(z)}{I_{v}(z)}, z \in U \tag{10}
\end{equation*}
$$

Using (10) in (9), following simple calculations, we write:

$$
\begin{equation*}
\frac{z I_{v}^{\prime}(z)}{I_{v}(z)} \prec v-1+\frac{1+(1-2 \mu) z}{1-z}, \quad z \in U . \tag{11}
\end{equation*}
$$

Since $g \in \mathcal{K}$ and $g(U)$ is a convex domain, differential subordination (11) is equivalent to:

$$
\begin{equation*}
\operatorname{Re} \frac{z I_{v}^{\prime}(z)}{I_{v}(z)}>v-1+\mu \tag{12}
\end{equation*}
$$

Remark 5. If we consider $v \geq 1$ and $0 \leq \mu<1$, we have that $v-1+\mu \geq 0$. However, relation (12) which characterizes starlikeness does not imply that $I_{v}(z) \in \mathcal{S}^{*}(v-1+\mu)$ since $I_{v}^{\prime}(0)=0$ which contradicts the necessary condition for starlikeness requiring $I_{v}^{\prime}(0) \neq 0$.

Example 1. Consider $v=2$. Using (6), we obtain:

$$
I_{2}(z)=\left(\frac{z}{2}\right)^{2} \cdot \frac{1}{\Gamma(3)}-\left(\frac{z}{2}\right)^{4} \frac{1}{\Gamma(4)}+\cdots=\frac{z^{2}}{4} \cdot \frac{1}{2!}-\frac{z^{4}}{16} \cdot \frac{1}{3!}+\cdots=\frac{z^{2}}{8}-\frac{z^{4}}{96}+\cdots
$$

Using (8), we have:

$$
H(z)=2 \Gamma(3)\left(\frac{z}{2}\right)^{-1} I_{2}(z)=z-\frac{z^{4}}{8} \cdot \frac{\Gamma(3)}{\Gamma(4)}+\cdots=z-\frac{z^{4}}{8} \cdot \frac{2!}{3!}+\cdots=z-\frac{z^{4}}{24}+\cdots
$$

Now, considering the results proved in Theorem 1, we state the following:
Let $I_{2}=\frac{z^{2}}{8}-\frac{z^{4}}{96}+\cdots$. Consider the function $g \in \mathcal{K}, g(z)=\frac{1+(1-2 \mu) z}{1-z}, z \in U, 0 \leq \mu<1$ and let $H: \mathcal{A}_{n} \rightarrow \mathcal{A}_{n}$ be given by

$$
H(z)=z-\frac{z^{4}}{24}+\cdots, z \in U
$$

If the differential subordination,

$$
\frac{z H^{\prime}(z)}{H(z)} \prec \frac{1+(1-2 \mu) z}{1-z}, \quad z \in U, 0 \leq \mu<1,
$$

holds, then

$$
\operatorname{Re} \frac{z I_{2}^{\prime}(z)}{I_{2}(z)}>1+\mu, \quad z \in U .
$$

We remark that even if $1+\mu>0$, the conclusion remains that $I_{2}(z) \notin \mathcal{S}^{*}$ because $I_{2}^{\prime}(0)=0$, hence the necessary condition for starlikeness requiring $I_{2}^{\prime}(0) \neq 0$ is not satisfied.

In order to obtain the necessary and sufficient conditions for starlikeness of order $\alpha \in[0,1)$, the function $I_{v}(z)$ should be considered with $v=1$, obtaining the form given by $(7)$, and then $I_{1}^{\prime}(0)=\frac{1}{2} \neq 0$. With this restriction, the following corollary can be stated and proved:

Corollary 1. Let $I_{1}(z)$ be given by (7). Consider the function $g \in \mathcal{K}, g(z)=\frac{1+(1-2 \mu) z}{1-z}, z \in U$, $0 \leq \mu<1$ and let $H: \mathcal{A}_{n} \rightarrow \mathcal{A}_{n}$ be given by $H(z)=I_{1}(z), z \in U$.

If the differential subordination,

$$
\frac{z H^{\prime}(z)}{H(z)} \prec \frac{1+(1-2 \mu) z}{1-z}, \quad z \in U,
$$

holds, then

$$
\begin{equation*}
\operatorname{Re} \frac{z I_{1}^{\prime}(z)}{I_{1}(z)}>\mu, \quad z \in U, 0 \leq \mu<1 \tag{13}
\end{equation*}
$$

Moreover, letting $p(z)=\frac{z I_{1}^{\prime}(z)}{I_{1}(z)}, z \in U$, since $p(0)=1$, relation (13) gives that $\frac{z I_{1}^{\prime}(z)}{I_{1}(z)}$ is a Carathéodory function written $\frac{z I_{1}^{\prime}(z)}{I_{1}(z)} \in \mathcal{P}$.

Proof. Indeed, since $I_{1}(0)=0, I_{1}^{\prime}(0) \neq 0$ and $0 \leq \mu<1$, then relation (13) implies that $I_{1}(z) \in \mathcal{S}^{*}(\mu)$. Since $\mu \in[0,1)$, considering Remark 1, from the duality theorem we conclude that $\mathcal{S}^{*}(\mu) \subset \mathcal{S}^{*}$, hence $I_{1}(z) \in \mathcal{S}$.

Another interesting corollary of Theorem 1 can be obtained considering $v=1$, $\mu=\frac{1}{2} \in(0,1)$, and the property of the operator $I_{1}(z)$ to be starlike of order $\frac{1}{2}$ is revealed.

Corollary 2. Let $I_{1}(z)$ be given by (7). Consider the function $g \in \mathcal{K}, g(z)=\frac{1}{1-z}, z \in U$, and let $H: \mathcal{A}_{n} \rightarrow \mathcal{A}_{n}$ be given by $H(z)=I_{1}(z), z \in U$.

If the differential subordination,

$$
\frac{z H^{\prime}(z)}{H(z)} \prec \frac{1}{1-z^{\prime}}, \quad z \in U,
$$

holds, then

$$
\operatorname{Re} \frac{z I_{1}^{\prime}(z)}{I_{1}(z)}>\frac{1}{2}, \text { i.e., } I_{1}(z) \in \mathcal{S}^{*}\left(\frac{1}{2}\right) \subset \mathcal{S}^{*}
$$

hence $I_{1}(z) \in \mathcal{S}$.
The property of starlikeness can be emphasized for $I_{1}(z)$ given by (7) if $\mu=0$.
Corollary 3. Let $I_{1}(z)$ be given by (7). Consider the function $g \in \mathcal{K}, g(z)=\frac{1+z}{1-z}$ and let $H(z)=I_{1}(z), H: \mathcal{A}_{n} \rightarrow \mathcal{A}_{n}$.

If the differential subordination

$$
\frac{z H^{\prime}(z)}{H(z)} \prec \frac{1+z}{1-z}, \quad z \in U,
$$

holds, then

$$
\operatorname{Re} \frac{z I_{1}^{\prime}(z)}{I_{1}(z)}>0, \quad z \in U, \text { i.e., } I_{1}(z) \in \mathcal{S}^{*}
$$

hence $I_{1}(z) \in \mathcal{S}$.
Remark 6. Considering the results given by Corollaries 1-3, the conclusion is that the Bessel function of first kind and order $v$ is starlike or starlike of order $\alpha, \alpha \in[0,1)$ only for $v=1$.

For the second part of this investigation, only this function will be considered as given by (7).

## 3. A New Generalized Integral Operator Involving Bessel Function of the First Kind

By applying the findings shown in Section 2, we now define a new integral operator that generalizes the well-known integral operators introduced by Alexander [2], Libera [3] and Bernardi [4]. For this purpose, we use the univalent form of the Bessel function of the first kind and order 1.

Definition 6. Let $I_{1}(z)$ be given by (7). We define the integral operator $F: \mathcal{A}_{n} \rightarrow \mathcal{A}_{n}$ as:

$$
\begin{equation*}
F\left(I_{1}(z)\right)=: F(z)=\frac{(\alpha+\beta+\gamma)}{z^{\alpha+\beta+\gamma-1}} \int_{0}^{1}\left[I_{1}(t)\right]^{\alpha+\beta} t^{\gamma-1} d t \tag{14}
\end{equation*}
$$

with $\alpha+\beta+\gamma-1 \geq 0, \gamma \geq 0$.

## Remark 7.

(i) For $\alpha+\beta=1, \gamma \geq 0$, the integral operator given by (14) becomes $B: \mathcal{A}_{n} \rightarrow \mathcal{A}_{n}$ given by

$$
\begin{equation*}
B(z)=\frac{2(1+\gamma)}{z^{\gamma}} \int_{0}^{1} I_{1}(t) t^{\gamma-1} d t \tag{15}
\end{equation*}
$$

the Bernardi integral operator [4].
(ii) For $\alpha+\beta=1, \gamma=1$, the integral operator given by (14) becomes $L: \mathcal{A}_{n} \rightarrow \mathcal{A}_{n}$ given by

$$
\begin{equation*}
L(z)=\frac{4}{z} \int_{0}^{1} I_{1}(t) d t \tag{16}
\end{equation*}
$$

the Libera integral operator [3].
(iii) For $\alpha+\beta=1, \gamma=0$, the integral operator given by (14) becomes Alexander integral operator [2] given by

$$
\begin{equation*}
A(z)=2 \int_{0}^{1} \frac{I_{1}(t)}{t} d t \tag{17}
\end{equation*}
$$

The next theorem proves that the generalized integral operator given by (14) is starlike. The proof uses the outcome of Corollary 1.

Theorem 2. Let $I_{1}(z)$ be given by (7) satisfying:

$$
\begin{equation*}
\operatorname{Re} \frac{z I_{1}^{\prime}(z)}{I_{1}(z)}>\mu, \quad z \in U, 0 \leq \mu<1 \tag{18}
\end{equation*}
$$

Then

$$
\begin{equation*}
\operatorname{Re} \frac{z F^{\prime}(z)}{F(z)}>0, \quad z \in U, \text { i.e., } F(z) \in \mathcal{S}^{*} \tag{19}
\end{equation*}
$$

where $F(z)$ is given by (14).
Proof. Using (14), we write

$$
\begin{equation*}
z^{\alpha+\beta+\gamma-1} F(z)=(\alpha+\beta+\gamma) 2^{\alpha+\beta} \int_{0}^{z}\left[I_{1}(t)\right]^{\alpha+\beta} t^{\gamma-1} d t, \quad z \in U . \tag{20}
\end{equation*}
$$

By differentiating (20), after a few calculations, it yields

$$
\begin{equation*}
F(z)\left[\alpha+\beta+\gamma-1+\frac{z F^{\prime}(z)}{F(z)}\right]=(\alpha+\beta+\gamma) 2^{\alpha+\beta}\left[I_{1}(z)\right]^{\alpha+\beta} z^{1-\alpha-\beta} \tag{21}
\end{equation*}
$$

Let

$$
\begin{equation*}
p(z)=\frac{z F^{\prime}(z)}{F(z)}, z \in U \tag{22}
\end{equation*}
$$

Using $F(z)$ given by (14), we conclude that $p(0)=1$. Replacing (22) in (21), yields

$$
\begin{equation*}
F(z)[\alpha+\beta+\gamma-1+p(z)]=(\alpha+\beta+\gamma) 2^{\alpha+\beta}\left[I_{1}(z)\right]^{\alpha+\beta} z^{1-\alpha-\beta} \tag{23}
\end{equation*}
$$

By differentiating (23), we get:

$$
\begin{equation*}
\frac{z F^{\prime}(z)}{F(z)}+\frac{z p^{\prime}(z)}{\alpha+\beta+\gamma-1+p(z)}=(\alpha+\beta) \frac{z I_{1}^{\prime}(z)}{I_{1}(z)}+(1-\alpha-\beta) . \tag{24}
\end{equation*}
$$

By including (22) into (24), we can see

$$
\begin{equation*}
p(z)+\frac{z p^{\prime}(z)}{\alpha+\beta+\gamma-1+p(z)}=(\alpha+\beta) \frac{z I_{1}^{\prime}(z)}{I_{1}(z)}+(1-\alpha-\beta) . \tag{25}
\end{equation*}
$$

By applying now the condition given by (18) in the hypothesis of this theorem, relation (25) gives:

$$
\operatorname{Re}\left[p(z)+\frac{z p^{\prime}(z)}{\alpha+\beta+\gamma-1+p(z)}\right]>(\alpha+\beta) \mu+(1-\alpha-\beta),
$$

which is equivalent to

$$
\begin{equation*}
\operatorname{Re}\left[p(z)+\frac{z p^{\prime}(z)}{\alpha+\beta+\gamma-1+p(z)}-(\alpha+\beta) \mu-(1-\alpha-\beta)\right]>0 . \tag{26}
\end{equation*}
$$

For obtaining the result claimed by this theorem, we apply Lemma 1. For that, we define the function $\psi: \mathbb{C}^{2} \times U \rightarrow \mathbb{C}$ given by

$$
\begin{equation*}
\psi(r, s ; z)=r+\frac{s}{\alpha+\beta+\gamma-1+r}-(\alpha+\beta) \mu+\alpha+\beta-1, r, s \in \mathbb{C}, z \in U . \tag{27}
\end{equation*}
$$

By replacing in (27) $r=p(z), s=z p^{\prime}(z)$, we write:

$$
\begin{equation*}
\psi\left(p(z), z p^{\prime}(z)\right)=p(z)+\frac{z p^{\prime}(z)}{\alpha+\beta+\gamma-1+p(z)}-(\alpha+\beta) \mu+\alpha+\beta-1 \tag{28}
\end{equation*}
$$

Using (26) in (28) yields:

$$
\operatorname{Re} \psi\left(p(z), z p^{\prime}(z)\right)>0, z \in U .
$$

In order to apply Lemma 1 , we must have $\psi \in \Psi_{n}\{1\}$. Using the admissibility condition given by (5), we evaluate:

$$
\begin{gathered}
\operatorname{Re} \psi(\rho i, \sigma)=\operatorname{Re}\left[\rho i+\frac{\sigma}{\alpha+\beta+\gamma-1+\rho i}-(\alpha+\beta) \mu+\alpha+\beta-1\right]= \\
(\alpha+\beta-1)-(\alpha+\beta) \mu+\operatorname{Re} \frac{\sigma(\alpha+\beta+\gamma-1-\rho i)}{(\alpha+\beta+\gamma-1)^{2}+\rho^{2}}= \\
(\alpha+\beta-1)-(\alpha+\beta) \mu+\frac{\sigma(\alpha+\beta+\gamma-1)}{(\alpha+\beta+\gamma-1)^{2}+\rho^{2}} \leq \\
-\left[(1-\alpha-\beta)+(\alpha+\beta) \mu+\frac{(\alpha+\beta+\gamma-1)}{(\alpha+\beta+\gamma-1)^{2}+\rho^{2}} \cdot \frac{n}{2}\left(1+\rho^{2}\right)\right]<0 .
\end{gathered}
$$

Since $\operatorname{Re} \psi(\rho i, \sigma)<0, z \in U$, using Remark 3, we conclude that $\psi \in \Psi_{n}\{1\}$.
Since $\operatorname{Re} \psi\left(p(z), z p^{\prime}(z)\right)>0$ and $\psi \in \Psi_{n}\{1\}$, by applying Lemma 1 we have that

$$
\begin{equation*}
\operatorname{Re} p(z)>0, z \in U \tag{29}
\end{equation*}
$$

Using (22) in (29), we have

$$
\begin{equation*}
\operatorname{Re} \frac{z F^{\prime}(z)}{F(z)}>0,, z \in U \tag{30}
\end{equation*}
$$

Since $F(0)=0, F^{\prime}(0)=1 \neq 0$, relation (30) implies that $F(z) \in \mathcal{S}^{*}$.
Example 2. Considering $v=1$, using (6), we have:

$$
I_{1}(z)=\frac{z}{2} \cdot \frac{1}{\Gamma(2)}-\left(\frac{z}{2}\right)^{3} \cdot \frac{1}{\Gamma(3)}+\cdots=\frac{z}{2}-\frac{z^{3}}{16}+\cdots
$$

For $\alpha+\beta=1, \gamma=3$ in (14), we obtain:

$$
\begin{gathered}
F(z)=\frac{(1+3)}{z^{1+3-1}} \int_{0}^{1}\left[\frac{z}{2}-\frac{z^{3}}{16}+\cdots\right] t^{2} d t=\frac{4}{z^{3}} \int_{0}^{1}\left[\frac{z^{3}}{2}-\frac{z^{5}}{16}+\cdots\right] d t= \\
\frac{4}{z^{3}}\left[\frac{1}{2} \cdot \frac{z^{4}}{4}-\frac{1}{16} \cdot \frac{z^{6}}{6}+\cdots\right]=\frac{z}{2}-\frac{z^{3}}{24}+\cdots
\end{gathered}
$$

Now, considering the results proved in Theorem 2, we state the following:
Let $I_{1}(z)=\frac{z}{2}-\frac{z^{3}}{16}+\cdots$, satisfying:

$$
\operatorname{Re} \frac{z I_{1}^{\prime}(z)}{I_{1}(z)}>\mu, \quad z \in U, 0 \leq \mu<1
$$

Then,

$$
\operatorname{Re} \frac{z F^{\prime}(z)}{F(z)}>0, \quad z \in U, \text { i.e., } F(z) \in \mathcal{S}^{*},
$$

where $F(z)=\frac{z}{2}-\frac{z^{3}}{24}+\cdots$ is analytic in $U$.
For $\alpha+\beta=1, \gamma \geq 0$, Theorem 2 gives the following corollary for Bernardi integral operator given by (15).

Corollary 4. Let $I_{1}(z)$ be given by (7) satisfying:

$$
\operatorname{Re} \frac{z I_{1}^{\prime}(z)}{I_{1}(z)}>\mu, \quad z \in U, 0 \leq \mu<1
$$

Then,

$$
\operatorname{Re} \frac{z B^{\prime}(z)}{B(z)}>0, \quad z \in U, \text { i.e., } B(z) \in \mathcal{S}^{*},
$$

where $B(z)$ is given by (15).
For $\alpha+\beta=1, \gamma=1$, Theorem 2 gives the following corollary for the Libera integral operator given by (16).

Corollary 5. Let $I_{1}(z)$ be given by (7), satisfying:

$$
\operatorname{Re} \frac{z I_{1}^{\prime}(z)}{I_{1}(z)}>\mu, \quad z \in U, 0 \leq \mu<1
$$

Then,

$$
\operatorname{Re} \frac{z L^{\prime}(z)}{L(z)}>0, \quad z \in U, \text { i.e., } L(z) \in \mathcal{S}^{*}
$$

where $L(z)$ is given by (16).

For $\alpha+\beta=1, \gamma=0$, Theorem 2 gives the following corollary for the Alexander integral operator given by (17).

Corollary 6. Let $I_{1}(z)$ be given by (7), satisfying:

$$
\operatorname{Re} \frac{z I_{1}^{\prime}(z)}{I_{1}(z)}>\mu, \quad z \in U, 0 \leq \mu<1
$$

Then,

$$
\operatorname{Re} \frac{z A^{\prime}(z)}{A(z)}>0, \quad z \in U, \text { i.e., } A(z) \in \mathcal{S}^{*}
$$

where $A(z)$ is given by (17).

## 4. Conclusions

The outcome of the study presented in this paper follows the line of research that results from merging differential subordination theory with the study of different types of operators. For this investigation, a new generalized integral operator is introduced and investigated concerning the geometric properties of the starlikeness of a certain order. The
famous Bessel functions are included in the research by choosing the Bessel function of the first kind, an order $v$. In the first part of the investigation, presented in Section 2, the research reveals in Theorem 1 and Corollaries 1-3 findings on the conditions satisfied by the Bessel function of the first kind and order $v$ in order to be univalent. In the second part of the investigation, exposed in Section 3, the presentation of the results obtained in this investigation starts with the new integral operator introduced in Definition (6) using the univalent form of the Bessel function of the first kind. It is shown that the renowned integral operators, Bernardi, Libera, and Alexander, familiar to studies in geometric function theory, are obtained as particular cases of the new generalized integral operator $F(z)$ given by (14). Theorem 2 provides necessary and sufficient conditions for the generalized integral operator $F(z)$ to be starlike. The results obtained in Corollaries 4-6 following Theorem 2 show that Bernardi, Libera, and Alexander integral operators are starlike. Since those are previously established results in the literature, the aforementioned corollaries confirm the validity of the assertion of Theorem 2.

As future applications of the results presented here, the operator $F(z)$ provided by (14) can be used to define new subclasses of analytical functions with specific geometric properties given by the characteristics of this operator already demonstrated in this article. Results that could motivate the study in this direction have been published in [22]. Furthermore, properties of this operator to preserve other special classes of functions could be investigated as seen for meromorphic functions in [23]. The dual theory of differential superordination can be also applied for obtaining new superordination results involving the new classes defined using the operator $F(z)$ provided by (14) as seen in the recent publications $[24,25]$. Studies concerning the variants of differential subordination and superordiantion theories introduced in recent years as extensions named fuzzy differential subordination and superordination and strong differential subordination and superordination, respectively, could also be conducted on the operator $F(z)$ defined in this paper. Such inspiring results can be seen in $[18,26]$ for fuzzy differential subordination theory and for strong differential subordination and superordination theory in [27,28].

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