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New Monotonic Properties for Solutions of a Class of Functional Differential Equations and Their Applications

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Abstract: This paper delves into the enhancement of asymptotic and oscillatory behaviors in solutions to even-order neutral differential equations with multiple delays. The main objective is to establish improved inequalities to advance the understanding of oscillation theory for these equations. The paper's approach is centered on improving the understanding of the intricate relationship between solutions and their corresponding functions. This is achieved by harnessing the modified monotonic properties of positive solutions, which provide valuable insights into oscillation behavior. Furthermore, leveraging the symmetry between positive and negative solutions, we derived criteria that ensure oscillation for all solutions, with a specific emphasis on excluding only positive solutions. To illustrate the significance of our findings, we provide an illustrative example.

Keywords: differential equation; oscillatory; non-oscillatory; neutral delay; even order

MSC: 34C10; 34K11

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1. Introduction

The central focus of this research revolves around a comprehensive examination of the oscillatory characteristics exhibited by the solutions to an even-order quasilinear differential equation (DE), denoted as

$$\left(r(\ell)\left(\mathrm{y}^{(n-1)}(\ell)\right)^{\alpha}\right)' + \sum_{i=1}^{v} q_i(\ell)\chi^{\alpha}(\sigma_i(\ell)) = 0,\ \ell \geq \ell_0, \tag{1}$$

where $\mathrm{y}(\ell) = \chi(\ell) + \mathrm{p}(\ell)\chi(\mathfrak{y}(\ell))$. In this paper, we assume that

(H$_1$) $n \geq 4$, α is the ratio of two positive odd integers;

(H$_2$) $r, \mathfrak{y}, \sigma_i \in C^1([\ell_0, \infty))$, and $q(\ell) \in C([\ell_0, \infty))$;

(H$_3$) $\mathfrak{y}(\ell) \leq \ell$, $\sigma_i(\ell) \leq \ell$, $\sigma_i'(\ell) > 0$, and $\lim_{\ell\to\infty} \mathfrak{y}(\ell) = \lim_{\ell\to\infty} \sigma_i(\ell) = \infty$;

(H$_4$) $r(\ell) > 0$, $r'(\ell) \geq 0$, $0 \leq \mathrm{p}(\ell) < \mathrm{p}_0$ and $q(\ell) \geq 0$;

(H$_5$) $\pi_0(\ell_0) < \infty$, where

$$\pi_0(\ell) := \int_{\ell}^{\infty} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s,$$

and

$$\pi_i(\ell) := \int_{\ell}^{\infty} \pi_{i-1}(s)\mathrm{d}s,\ i = 1, 2, \ldots, n-2.$$

A function $\chi \in C([\ell_\chi, \infty))$, where $\ell_\chi \geqslant \ell_0$, is said to be a solution of Equation (1) which has the property $r(\mathrm{y}^{(n-1)})^{\alpha} \in C^1[\ell_\chi, \infty)$, and it satisfies Equation (1) for all

$\chi \in [\ell_\chi, \infty)$. We consider only those solutions χ of Equation (1), which exist on some half-line $[\ell_\chi, \infty)$ and satisfy the condition

$$\sup\{|\chi(\ell)| : \ell \geqslant L\} > 0, \text{ for all } L \geq \ell_\chi.$$

A solution to Equation (1) is termed oscillatory if it does not tend towards either eventual positivity or eventual negativity. Otherwise, it is classified as non-oscillatory. Equation (1) is considered oscillatory when all of its solutions exhibit oscillatory behavior.

A neutral DE is a specialized type of DE in which the rate of change of a function depends not only on its current state but also on past values, introducing a time delay. These equations find significant relevance in various fields such as biology, physics, and engineering, where systems exhibit delayed responses. Neutral DEs provide a crucial mathematical framework for modeling dynamic systems with memory and are instrumental in analyzing real-world phenomena with time-delay effects. Understanding their solutions and behavior is vital for accurately describing and predicting the dynamics of such systems, making them indispensable in scientific and engineering applications, see [1–5].

Oscillation solutions, which refer to the periodic behavior of solutions oscillating around a specific function or value, are commonly observed in physical systems, like mechanical systems, electrical circuits, and biological oscillators. The oscillation theorem is an essential result in the theory of DEs that describes the oscillatory nature of solutions. Its wide-ranging applications span various fields, such as physics, engineering, and economics. The implications of this theorem have a broad range of applications, spanning diverse fields. These include its relevance in analyzing oscillatory systems, like pendulums and vibrating strings, as well as its utility in examining population dynamics and the spread of infectious diseases. Furthermore, the oscillation theorem bears notable importance in the domains of control theory and signal processing. Here, it assumes a pivotal role in evaluating the stability and performance of feedback systems, as demonstrated in [6–8].

Even-order quasilinear DEs represent a significant class of mathematical models that find wide-ranging applications in science and engineering. These equations, often characterized by terms involving the function and its derivatives of the same order, offer a versatile framework for studying complex phenomena. Their utility extends to various fields, including physics, biology, and control theory. Specifically, they are employed to analyze dynamic systems with even-order dynamics, such as mechanical systems, electrical circuits, and heat transfer problems. This versatility makes the study of even-order quasilinear DEs a vital endeavor, as it provides essential insights into the behavior of numerous real-world systems.

In the field of mathematical research, there has been a notable surge in interest in the investigation of delay DEs in unconventional contexts. This keen academic interest is apparent in the body of work such as [9–11]. Similarly, refs. [12–15], have directed their efforts toward understanding neutral DEs. Furthermore, Moaaz et al. extended this analytical exploration to encompass even-order equations in their publications, such as [16,17].

Many investigations have explored the complex topic of oscillations in even-order neutral DEs. These investigations have proposed diverse methodologies aimed at establishing criteria for identifying oscillatory behavior in these equations. It is worth highlighting that this topic has received extensive attention, particularly in the canonical scenario denoted by the integral expression:

$$\int_{\ell_0}^{\infty} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s = \infty, \tag{2}$$

as evidenced by the comprehensive body of prior scholarly works, including references such as [18–20].

We will now discuss several essential findings from previous research papers that have significantly advanced the study of even-order differential equations.

Koplatadze [21], Wei [22], and Koplatadze et al. [23] investigated the oscillation criterion of equation

$$\chi''(\ell) + q(\ell)\chi(\sigma(\ell)) = 0, \ell \geq \ell_0,$$

and obtained sufficient conditions for it to be oscillatory. Similarly, Bai [24] and Karpuz et al. [25] discussed the oscillation criterion for the equation

$$\chi^{(n)}(\ell) + q(\ell)\chi(\sigma(\ell)) = 0, \ \ell \geq \ell_0,$$

and derived sufficient conditions for all solutions to be oscillatory. Additionally, Baculikova [26] conducted an investigation into the monotonic properties of non-oscillatory solutions concerning the linear equation:

$$\left(r(\ell)\chi'(\ell)\right)' + q(\ell)\chi(\mathfrak{y}(\ell)) = 0.$$

This investigation considered both delay and advanced scenarios. Furthermore, Ramos et al. [27] introduced a novel oscillation criterion for solutions to the fourth-order quasilinear neutral DE:

$$\left(r(\ell)\left(z'''(\ell)\right)^{\alpha}\right)' + q(\ell)\chi^{\alpha}(\sigma(\ell)) = 0,$$

subject to a non-canonical condition:

$$\int_{\ell_0}^{\infty} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s < \infty.$$

Furthermore, the research conducted by Han et al. [28], and Li et al. [29] involved an investigation into the asymptotic properties of positive solutions pertaining to the even-order neutral DE, defined as:

$$\left(r(\ell)(\chi(\ell) + \mathrm{p}(\ell)\chi(\mathfrak{y}(\ell)))^{(n-1)}\right)' + q(\ell)\chi(\sigma(\ell)) = 0.$$

Lastly, the study carried out by Xing et al. [30] explored various oscillation theorems for the equation:

$$\left(r(\ell)\left((\chi(\ell) + \mathrm{p}(\ell)\chi(\mathfrak{y}(\ell)))^{(n-1)}\right)^{\alpha}\right)' + q(\ell)\chi^{\alpha}(\sigma(\ell)) = 0,$$

subject Condition (2).

Li and Rogovchenko [31] explored the asymptotic behavior of solutions to higher-order quasilinear neutral DEs, represented as follows

$$\left(r(\ell)\left((\chi(\ell) + \mathrm{p}(\ell)\chi(\mathfrak{y}(\ell)))^{(n-1)}\right)^{\alpha}\right)' + q(\ell)\chi^{\beta}(\sigma(\ell)) = 0,$$

with a particular focus on both even- and odd-order equations featuring diverse argument patterns, including alternating delayed and advanced characteristics.

The exploration of asymptotic and oscillatory properties in neutral DEs (NDEs) relies on the intricate relationship between the solution χ and its corresponding function y. In the typical context of second-order equations, the standard association is often defined as:

$$\chi(\ell) > (1 - \mathrm{p}(\ell))\mathrm{y}(\ell).$$

This expression is widely utilized. Conversely, in the case of positive, decreasing solutions within non-canonical settings, the relevant relationship takes the following form:

$$\chi(\ell) > \left(1 - \mathrm{p}(\ell)\frac{\pi_0(\mathfrak{y}(\ell))}{\pi_0(\ell)}\right)\mathrm{y}(\ell).$$

This relationship has been demonstrated in previous studies, such as [32,33].

Moaaz et al. [34] developed new inequalities that improve the understanding of the asymptotic and oscillatory behaviors of solutions for fourth-order neutral DEs of the form

$$\left(r(\ell)\left(z'''(\ell)\right)\right)' + q(\ell)\chi(\sigma(\ell)) = 0,$$

specifically in the noncanonical case.

Lemma 1 ([34]). *If χ represents an eventually positive solution of Equation (1), then eventually we have*

$$\chi(\ell) > \sum_{r=0}^{\kappa}\left(\prod_{l=0}^{2r}\mathrm{p}\left(\mathfrak{y}^{[l]}(\ell)\right)\right)\left[\frac{\mathrm{y}\left(\mathfrak{y}^{[2r]}(\ell)\right)}{\mathrm{p}\left(\mathfrak{y}^{[2r]}(\ell)\right)} - \mathrm{y}\left(\mathfrak{y}^{[2r+1]}(\ell)\right)\right], \tag{3}$$

for any $\kappa \geq 0$.

In this study, our primary objective is to build upon earlier work [34], which applied a similar approach to fourth-order equations. Our research is primarily motivated by the desire to extend this methodology, pushing the boundaries of our understanding by encompassing higher-order equations in our current investigation. This expansion marks a significant advancement in the scope of our research, opening up new avenues for exploration and discovery in the field.

2. Auxiliary Results

In this section, we will introduce several essential lemmas that serve as the foundational building blocks for establishing our main results. To streamline our notation, we will define the following expressions:

$$\mathrm{U}^{[0]}(\ell) = \mathrm{U}(\ell) \text{ and } \mathrm{U}^{[j]}(\ell) = \mathrm{U}\left(\mathrm{U}^{[j-1]}(\ell)\right), \text{ for } j = 1, 2, \ldots, \kappa,$$

$$\sigma(\ell) := \min\{\sigma_i(\ell),\ i = 1, 2, \ldots, v\},$$

$$\widetilde{\sigma}(\ell) := \max\{\sigma_i(\ell),\ i = 1, 2, \ldots, v\},$$

$$\mathrm{p}_1(\ell;\kappa) := \sum_{r=0}^{\kappa}\left(\prod_{l=0}^{2r}\mathrm{p}\left(\mathfrak{y}^{[l]}(\ell)\right)\right)\left[\frac{1}{\mathrm{p}\left(\mathfrak{y}^{[2r]}(\ell)\right)} - 1\right]\left(\frac{\mathfrak{y}^{[2r]}(\ell)}{\ell}\right)^{(n-2)/\epsilon},$$

$$\mathrm{p}_2(\ell;\kappa) := \sum_{r=0}^{\kappa}\left(\prod_{l=0}^{2r}\mathrm{p}\left(\mathfrak{y}^{[l]}(\ell)\right)\right)\left[\frac{1}{\mathrm{p}\left(\mathfrak{y}^{[2r]}(\ell)\right)} - \frac{\pi_{n-2}\left(\mathfrak{y}^{[2r+1]}(\ell)\right)}{\pi_{n-2}\left(\mathfrak{y}^{[2r]}(\ell)\right)}\right],$$

$$\widehat{\mathrm{p}}_2(\ell;\kappa) := \sum_{r=0}^{\kappa}\left(\prod_{l=0}^{2r}\mathrm{p}\left(\mathfrak{y}^{[l]}(\ell)\right)\right)\left[\frac{1}{\mathrm{p}\left(\mathfrak{y}^{[2r]}(\ell)\right)} - \frac{\pi_{n-2}\left(\mathfrak{y}^{[2r+1]}(\ell)\right)}{\pi_{n-2}\left(\mathfrak{y}^{[2r]}(\ell)\right)}\right]\frac{\pi_{n-2}^{k_m}\left(\mathfrak{y}^{[2r]}(\ell)\right)}{\pi_{n-2}^{k_m}(\ell)},$$

$$\mathrm{Q}_0(\ell) := \sum_{i=1}^{v} q_i(\ell)(1 - \mathrm{p}(\sigma_i(\ell)))^{\alpha},$$

$$\mathrm{Q}_j(\ell) := \sum_{i=1}^{v} q_i(\ell)\mathrm{p}_j^{\alpha}(\sigma_i(\ell),\kappa),\ j = 1, 2,$$

and

$$\widehat{\mathrm{Q}}_2(\ell) := \sum_{i=1}^{v} q_i(\ell)\widehat{\mathrm{p}}_2^{\alpha}(\sigma_i(\ell),\kappa).$$

Lemma 2 ([35]). *Let $w \in \mathbf{C}^m([\ell_0,\infty),(0,\infty))$, $w^{(i)}(\ell) > 0$ for $i = 1, 2, \ldots, m$, and $w^{(m+1)}(\ell) \leq 0$, eventually. Then,*

$$\frac{w(\ell)}{w'(\ell)} \geq \frac{\epsilon}{m}\ell,$$

for every $\epsilon \in (0,1)$.

Lemma 3 ([36]). *Let* $f \in C^m([\ell_0,\infty),\mathbb{R}^+)$. *Assume that* $f^{(m)}(\ell)$ *has a fixed sign and is not identically zero on* $[\ell_0,\infty)$ *and that there exists* $\ell_1 \geq \ell_0$, *such that* $f^{(m-1)}(\ell)f^{(m)}(\ell) \leq 0$ *for all* $\ell_1 \geq \ell_0$. *If* $\lim_{\ell\to\infty} f(\ell) \neq 0$, *then, for every* $\epsilon \in (0,1)$, *there exists* $\ell_\delta \in [\ell_1,\infty)$, *such that*

$$f(\ell) \geq \frac{\epsilon}{(m-1)!}\ell^{m-1}\left|f^{(m-1)}(\ell)\right|,$$

for $\ell \in [\ell_\delta,\infty)$.

Lemma 4 ([37]). *Assuming* $\chi > 0$ *is a solution of Equation (1), we have that* $r\left(\mathrm{y}^{(n-1)}\right)^\alpha$ *is a decreasing function, and* y *fulfills one of the subsequent scenarios:*

$$\begin{aligned}
&(\mathrm{C}_1)\ \mathrm{y}^{(r)} > 0 \text{ for } r = 0,1,n-1 \text{ and } \mathrm{y}^{(n)} < 0;\\
&(\mathrm{C}_2)\ \mathrm{y}^{(r)} > 0 \text{ for } r = 0,1,n-2 \text{ and } \mathrm{y}^{(n-1)} < 0;\\
&(\mathrm{C}_3)\ (-1)^r\mathrm{y}^{(r)} > 0 \text{ for } r = 0,1,2,\ldots,n-1,
\end{aligned}$$

eventually.

Notation 1. *The symbol* Ω_i *is defined as the collection of all solutions that eventually become positivity, with their respective functions satisfying condition* (C_i) *for* $i = 1,2,3$.

3. Properties of Asymptotic and Monotonic Behaviors

We establish asymptotic and monotonic properties for the solutions of the neutral DE (1), in this section.

3.1. Category Ω_2

Lemma 5. *Assume that* $\chi \in \Omega_2$. *Then, eventually,*

$(\Upsilon_{1,1})$ $\mathrm{y} \geq \frac{\epsilon}{n-2}\ell\mathrm{y}'$;

$(\Upsilon_{1,2})$ $\mathrm{y} \geq \frac{\epsilon}{(n-2)!}\ell^{n-2}\mathrm{y}^{(n-2)}$ *for all* $\epsilon \in (0,1)$;

$(\Upsilon_{1,3})$ $\mathrm{y}^{(n-2)} \geq -r^{1/\alpha}\pi_0\mathrm{y}^{(n-1)}$;

$(\Upsilon_{1,4})$ $\mathrm{y}^{(n-2)}/\pi_0$ *is increasing;*

$(\Upsilon_{1,5})$ $\chi \geq \mathrm{p}_1(\ell;\kappa)\mathrm{y}$;

$(\Upsilon_{1,6})$ $\left(r\left(\mathrm{y}^{(n-1)}\right)^\alpha\right)' \leq -\mathrm{y}^\alpha(\sigma)\mathrm{Q}_1$.

Proof. $(\Upsilon_{1,1})$ By employing Lemma 2 with the substitutions $m = n-2$ and $w = \mathrm{y}$, the resulting inequality is

$$\mathrm{y} \geq \frac{\epsilon}{n-2}\ell\mathrm{y}'.$$

$(\Upsilon_{1,2})$ By employing Lemma 3 with the substitutions $m = n-1$ and $f = \mathrm{y}$, the resulting inequality is

$$\mathrm{y} \geq \frac{\epsilon_0}{(n-2)!}\ell^{n-2}\mathrm{y}^{(n-2)},$$

for all $\epsilon_0 \in (0,1)$.

$(\Upsilon_{1,3})$ Because $r^{1/\alpha}\mathrm{y}^{(n-1)}$ is decreasing, we deduce that

$$\begin{aligned}
\mathrm{y}^{(n-2)}(\ell) &\geq -\int_\ell^\infty r^{1/\alpha}(s)\mathrm{y}^{(n-1)}(s)\frac{1}{r^{1/\alpha}(s)}\mathrm{d}s\\
&\geq -r^{1/\alpha}\pi_0\mathrm{y}^{(n-1)}.
\end{aligned}$$

$(\Upsilon_{1,4})$ From $(\Upsilon_{1,3})$, we obtain

$$\left(\frac{\mathrm{y}^{(n-2)}}{\pi_0}\right)' = \frac{1}{r^{1/\alpha}\pi_0^2}\left(r^{1/\alpha}\pi_0\mathrm{y}^{(n-1)} + \mathrm{y}^{(n-2)}\right) \geq 0.$$

$(\Upsilon_{1,5})$ From Lemma 1, Equation (3) holds. After considering the properties of solutions in the class Ω_2, it can be deduced that $\mathrm{y}\left(\mathfrak{y}^{[2r]}\right) \geq \mathrm{y}\left(\mathfrak{y}^{[2r+1]}\right)$ for $i = 1, 2, \ldots$. Therefore, Equation (3) can betransformed into

$$\chi > \sum_{r=0}^{\kappa}\left(\prod_{l=0}^{2r}\mathrm{p}\left(\mathfrak{y}^{[l]}\right)\right)\left[\frac{1}{\mathrm{p}\left(\mathfrak{y}^{[2r]}\right)} - 1\right]\mathrm{y}\left(\mathfrak{y}^{[2r]}\right).$$

By utilizing $(\Upsilon_{1,1})$, we obtain

$$\mathrm{y}\left(\mathfrak{y}^{[2r]}\right) \geq \left(\frac{\mathfrak{y}^{[2r]}}{\ell}\right)^{(n-2)/\epsilon}\mathrm{y},$$

which, with (4), gives

$$\begin{aligned}\chi &> \sum_{r=0}^{\kappa}\left(\prod_{l=0}^{2r}\mathrm{p}\left(\mathfrak{y}^{[l]}\right)\right)\left[\frac{1}{\mathrm{p}\left(\mathfrak{y}^{[2r]}\right)} - 1\right]\left(\frac{\mathfrak{y}^{[2r]}}{\ell}\right)^{(n-2)/\epsilon}\mathrm{y} \\ &= \mathrm{p}_1(\ell;\kappa)\mathrm{y}.\end{aligned}$$

$(\Upsilon_{1,6})$ When combined with $(\Upsilon_{1,5})$, Equation (1) can be expressed as follows:

$$\begin{aligned}\left(r\left(\mathrm{y}^{(n-1)}\right)^{\alpha}\right)' &= -\sum_{i=1}^{v} q_i\chi^{\alpha}(\sigma_i) \\ &\leq -\sum_{i=1}^{v} q_i\mathrm{p}_1^{\alpha}(\sigma_i;\kappa)\mathrm{y}^{\alpha}(\sigma_i) \\ &\leq -\mathrm{y}^{\alpha}(\sigma)\mathrm{Q}_1.\end{aligned}$$

□

Lemma 6. *Assuming that $\chi \in \Omega_2$ and there exist $\delta > 0$ and $\ell_1 \geq \ell_0$, such that*

$$\frac{1}{\alpha}r^{1/\alpha}(\ell)\pi_0^{1+\alpha}(\ell)\left(\sigma^{n-2}(\ell)\right)^{\alpha}\mathrm{Q}_1(\ell) \geq ((n-2)!)^{\alpha}\delta, \tag{4}$$

we can deduce the following for $\ell \geq \ell_1$:

$(\Upsilon_{2,1})$ $\lim_{\ell\to\infty}\mathrm{y}^{(n-2)}(\ell) = 0$;

$(\Upsilon_{2,2})$ $\mathrm{y}^{(n-2)}/\pi_0^{\beta_0}$ *is decreasing;*

$(\Upsilon_{2,3})$ $\lim_{\ell\to\infty}\mathrm{y}^{(n-2)}(\ell)/\pi_0^{\beta_0}(\ell) = 0$;

$(\Upsilon_{2,4})$ $\mathrm{y}^{(n-2)}/\pi_0^{1-\beta_0}$ *is increasing; for $\ell \geq \ell_0$, where $\beta_0 = \epsilon\delta^{1/\alpha}$, $\epsilon \in (0,1)$ and $\alpha \leq 1$.*

Proof. $(\Upsilon_{2,1})$: Since $\chi \in \Omega_2$, we can conclude that $(\Upsilon_{1,1})$—$(\Upsilon_{1,6})$ in Lemma 5 are satisfied for all $\ell \geq \ell_1$, *with* ℓ_1 being large enough. When considering $\mathrm{y}^{(n-2)}$ as a positive and decreasing function, it can be deduced that $\lim_{\ell\to\infty}\mathrm{y}^{(n-2)}(\ell) = c_1 \geq 0$. Our claim is that $c_1 = 0$. To support this claim, suppose that $c_1 \neq 0$. Consequently, there exists a point where

$\mathrm{y}^{(n-2)} \geq c_1 > 0$ eventually. Utilizing this information along with $(\mathrm{Y}_{1,2})$, we derive the following inequality

$$\begin{aligned} \mathrm{y} &\geq \frac{\epsilon}{(n-2)!}\ell^{n-2}\mathrm{y}^{(n-2)} \\ &\geq \frac{\epsilon c_1}{(n-2)!}\ell^{n-2}. \end{aligned}$$

Consequently, from $(\mathrm{Y}_{1,6})$, we can deduce that

$$\begin{aligned} \left(r\left(\mathrm{y}^{(n-1)}\right)^{\alpha}\right)' &\leq -\mathrm{y}^{\alpha}(\sigma)\mathrm{Q}_1 \\ &\leq -\left(\frac{\epsilon c_1}{(n-2)!}\sigma^{n-2}\right)^{\alpha}\mathrm{Q}_1 \\ &\leq -\epsilon^{\alpha}c_1^{\alpha}\frac{\left(\sigma^{n-2}\right)^{\alpha}}{\left((n-2)!\right)^{\alpha}}\mathrm{Q}_1, \end{aligned}$$

which with Equation (4) gives

$$\begin{aligned} \left(r\left(\mathrm{y}^{(n-1)}\right)^{\alpha}\right)' &\leq -\alpha c_1^{\alpha}\epsilon^{\alpha}\delta\frac{1}{r^{1/\alpha}\pi_0^{1+\alpha}} \\ &\leq -\alpha c_1^{\alpha}\beta_0^{\alpha}\frac{1}{r^{1/\alpha}\pi_0^{1+\alpha}}. \end{aligned}$$

After integrating the preceding inequality from ℓ_2 to ℓ, the result is as follows

$$\begin{aligned} r(\ell)\left(\mathrm{y}^{(n-1)}(\ell)\right)^{\alpha} &\leq r(\ell_2)\left(\mathrm{y}^{(n-1)}(\ell_2)\right)^{\alpha} - \alpha c_1^{\alpha}\beta_0^{\alpha}\int_{\ell_2}^{\ell}\frac{1}{r^{1/\alpha}(s)\pi_0^{1+\alpha}(s)}\mathrm{d}s \\ &\leq \beta_0^{\alpha}c_1^{\alpha}\left(\frac{1}{\pi_0^{\alpha}(\ell_2)} - \frac{1}{\pi_0^{\alpha}(\ell)}\right). \end{aligned} \tag{5}$$

Since $\pi_0^{-\alpha}(\ell) \to \infty$ as $\ell \to \infty$, there is a $\ell_3 \geq \ell_2$, such that $\pi_0^{-\alpha}(\ell) - \pi_0^{-\alpha}(\ell_2) \geq \mu_0\pi_0^{-\alpha}(\ell)$ for all $\mu_0 \in (0,1)$. Hence, Equation (5) becomes

$$\mathrm{y}^{(n-1)} \leq -c_1\mu_0^{1/\alpha}\beta_0\frac{1}{r^{1/\alpha}\pi_0},$$

for all $\ell \geq \ell_3$. Integrating the last inequality from ℓ_3 to ℓ, we find

$$\begin{aligned} \mathrm{y}^{(n-2)}(\ell) &\leq \mathrm{y}^{(n-2)}(\ell_3) - c_1\mu_0^{1/\alpha}\beta_0\int_{\ell_3}^{\ell}\frac{1}{r^{1/\alpha}(s)\pi_0(s)}\mathrm{d}s \\ &\leq \mathrm{y}^{(n-2)}(\ell_3) - c_1\mu_0^{1/\alpha}\beta_0\ln\frac{\pi_0(\ell_3)}{\pi_0(\ell)} \to -\infty \text{ as } \ell \to \infty, \end{aligned}$$

which is a contradiction. Then, $c_1 = 0$.

$(\mathrm{Y}_{2,2})$ From (4), $(\mathrm{Y}_{1,2})$ and $(\mathrm{Y}_{1,6})$, we obtain

$$\left(r\left(\mathrm{y}^{(n-1)}\right)^{\alpha}\right)' \leq -\frac{\alpha\beta_0^{\alpha}}{r^{1/\alpha}\pi_0^{1+\alpha}}\left(\mathrm{y}^{(n-2)}(\sigma)\right)^{\alpha}.$$

After integrating the previous inequality from ℓ_1 to ℓ, and considering the condition $\mathrm{y}^{(n-1)} < 0$, we obtain the following result

$$r(\ell)\left(\mathrm{y}^{(n-1)}(\ell)\right)^{\alpha} \leq r(\ell_1)\left(\mathrm{y}^{(n-1)}(\ell_1)\right)^{\alpha} + \frac{\beta_0^{\alpha}}{\pi_0^{\alpha}(\ell_1)}\left(\mathrm{y}^{(n-2)}(\ell)\right)^{\alpha} - \frac{\beta_0^{\alpha}}{\pi_0^{\alpha}(\ell)}\left(\mathrm{y}^{(n-2)}(\ell)\right)^{\alpha}.$$

Because $\mathrm{y}^{(n-2)}(\ell) \to 0$ as $\ell \to \infty$ there is a $\ell_2 \geq \ell_1$, such that

$$r(\ell_1)\left(\mathrm{y}^{(n-1)}(\ell_1)\right)^{\alpha} + \frac{\beta_0^{\alpha}}{\pi_0^{\alpha}(\ell_1)}\left(\mathrm{y}^{(n-2)}(\ell)\right)^{\alpha} \leq 0,$$

for $\ell \geq \ell_2$. Therefore, we have

$$r\left(\mathrm{y}^{(n-1)}\right)^{\alpha} \leq -\frac{\beta_0^{\alpha}}{\pi_0^{\alpha}}\left(\mathrm{y}^{(n-2)}\right)^{\alpha},$$

alternatively,

$$r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_0 + \beta_0\mathrm{y}^{(n-2)} \leq 0. \tag{6}$$

Thus,

$$\left(\frac{\mathrm{y}^{(n-2)}}{\pi_0^{\beta_0}}\right)' = \frac{r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_0 + \beta_0\mathrm{y}^{(n-2)}}{r^{1/\alpha}\pi_0^{1+\beta_0}} \leq 0.$$

$(\mathrm{Y}_{2,3})$ Given that $\mathrm{y}^{(n-2)}/\pi_0^{\beta_0}$ represents a positive and decreasing function,

$$\lim_{\ell\to\infty} \mathrm{y}^{(n-2)}(\ell)/\pi_0^{\beta_0}(\ell) = c_2 \geq 0.$$

Our claim asserts that $c_2 = 0$. To support this claim, assume the contrary, where $c_2 \neq 0$. In such a scenario, it follows that $\mathrm{y}^{(n-2)}/\pi_0^{\beta_0} \geq c_2 > 0$ eventually. Let us introduce the function

$$w = \frac{\mathrm{y}^{(n-2)} + \pi_0 r^{1/\alpha}\mathrm{y}^{(n-1)}}{\pi_0^{\beta_0}}.$$

Considering the expression $(\mathrm{Y}_{1,3})$, it is observed that $w > 0$ and

$$\begin{aligned}
w' &= \frac{\mathrm{y}^{(n-1)} + \pi_0\left(r^{1/\alpha}\mathrm{y}^{(n-1)}\right)' - \mathrm{y}^{(n-1)}}{\pi_0^{\beta_0}(\ell)} + \beta_0\frac{\mathrm{y}^{(n-2)} + \pi_0 r^{1/\alpha}\mathrm{y}^{(n-1)}}{r^{1/\alpha}\pi_0^{1+\beta_0}} \\
&= \frac{\left(r^{1/\alpha}\mathrm{y}^{(n-1)}\right)'}{\pi_0^{\beta_0-1}} + \beta_0\frac{\mathrm{y}^{(n-2)}}{r^{1/\alpha}\pi_0^{1+\beta_0}} + \beta_0\frac{\mathrm{y}^{(n-1)}}{\pi_0^{\beta_0}} \\
&= \frac{1}{\alpha}\frac{\left(r\left(\mathrm{y}^{(n-1)}\right)^{\alpha}\right)'\left(r^{1/\alpha}\mathrm{y}^{(n-1)}\right)^{1-\alpha}}{\pi_0^{\beta_0-1}} \\
&\quad + \beta_0\frac{\mathrm{y}^{(n-2)}}{r^{1/\alpha}\pi_0^{1+\beta_0}} + \beta_0\frac{\mathrm{y}^{(n-1)}}{\pi_0^{\beta_0}}.
\end{aligned}$$

Using $(\mathrm{Y}_{1,2})$, $(\mathrm{Y}_{1,6})$, and Equation (4), we obtain

$$\left(r\left(\mathrm{y}^{(n-1)}\right)^{\alpha}\right)' \leq -\alpha\beta_0^{\alpha}\frac{1}{r^{1/\alpha}\pi_0^{1+\alpha}}\left(\mathrm{y}^{(n-2)}(\sigma)\right)^{\alpha}. \tag{7}$$

From Equation (6), we know that

$$r^{1/\alpha}\mathrm{y}^{(n-1)} \leq -\beta_0\frac{\mathrm{y}^{(n-2)}}{\pi_0},$$

and

$$\left(r^{1/\alpha}\mathrm{y}^{(n-1)}\right)^{1-\alpha} \geq \left(\beta_0\frac{\mathrm{y}^{(n-2)}}{\pi_0}\right)^{1-\alpha}. \tag{8}$$

Using (7) and (8), we obtain

$$\begin{aligned} w' &\leq -\frac{\beta_0^{\alpha}}{\pi_0^{\beta_0-1}}\frac{1}{r^{1/\alpha}\pi_0^{1+\alpha}}\left(\mathrm{y}^{(n-2)}(\sigma)\right)^{\alpha}\left(\beta_0\frac{\mathrm{y}^{(n-2)}}{\pi_0}\right)^{1-\alpha} \\ &\quad +\beta_0\frac{\mathrm{y}^{(n-2)}}{r^{1/\alpha}\pi_0^{1+\beta_0}}+\beta_0\frac{\mathrm{y}^{(n-1)}}{\pi_0^{\beta_0}}. \end{aligned}$$

Since $\mathrm{y}^{(n-1)} < 0$, $\sigma(\ell) \leq \ell$, we obtain $\mathrm{y}^{(n-2)}(\sigma(\ell)) \geq \mathrm{y}^{(n-2)}(\ell)$, and then

$$\begin{aligned} w' &\leq -\frac{\beta_0^{\alpha}}{\pi_0^{\beta_0-1}}\frac{1}{r^{1/\alpha}\pi_0^{1+\alpha}}\left(\mathrm{y}^{(n-2)}\right)^{\alpha}\left(\beta_0\frac{\mathrm{y}^{(n-2)}}{\pi_0}\right)^{1-\alpha} \\ &\quad +\beta_0\frac{\mathrm{y}^{(n-2)}}{r^{1/\alpha}\pi_0^{1+\beta_0}}+\beta_0\frac{\mathrm{y}^{(n-1)}}{\pi_0^{\beta_0}} \\ &\leq -\beta_0\frac{\mathrm{y}^{(n-2)}}{r^{1/\alpha}\pi_0^{1+\beta_0}}+\beta_0\frac{\mathrm{y}^{(n-2)}}{r^{1/\alpha}\pi_0^{1+\beta_0}}+\beta_0\frac{\mathrm{y}^{(n-1)}}{\pi_0^{\beta_0}} \\ &\leq \beta_0\frac{\mathrm{y}^{(n-1)}}{\pi_0^{\beta_0}}. \end{aligned}$$

Using the fact that $\mathrm{y}^{(n-2)}/\pi_0^{\beta_0} \geq c_2$, and Equation (6), we have

$$\begin{aligned} w' &\leq \beta_0\frac{\mathrm{y}^{(n-1)}}{\pi_0^{\beta_0}} \leq \beta_0\frac{1}{\pi_0^{\beta_0}}\left(\frac{-\beta_0\mathrm{y}^{(n-2)}}{r^{1/\alpha}\pi_0}\right) \\ &\leq -\frac{\mathrm{y}^{(n-2)}}{\pi_0^{\beta_0}}\frac{\beta_0^2}{r^{1/\alpha}\pi_0} \leq \frac{-c_2\beta_0^2}{r^{1/\alpha}\pi_0} < 0. \end{aligned}$$

The function w tends toward a constant non-negative value due to its consistent positive and decreasing nature. After integrating the preceding inequality from ℓ_3 to ∞, the result is as follows

$$-w(\ell_3) \leq -\beta_0^2 c_2 \lim_{\ell\to\infty} \ln\frac{\pi_0(\ell_3)}{\pi_0(\ell)},$$

or equivalently

$$w(\ell_3) \geq \beta_0^2 c_2 \lim_{\ell\to\infty} \ln\frac{\pi_0(\ell_3)}{\pi_0(\ell)} \to \infty,$$

which is a contradiction, and we obtain $c_2 = 0$.

$(\mathrm{Y}_{2,4})$ Now, we have

$$\begin{aligned} &\left(r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_0+\mathrm{y}^{(n-2)}\right)' \\ =\ &\left(r^{1/\alpha}\mathrm{y}^{(n-1)}\right)'\pi_0-\mathrm{y}^{(n-1)}+\mathrm{y}^{(n-1)} \\ =\ &\left(r^{1/\alpha}\mathrm{y}^{(n-1)}\right)'\pi_0 \\ =\ &\frac{1}{\alpha}\left(r\left(\mathrm{y}^{(n-1)}\right)^{\alpha}\right)'\left(r^{1/\alpha}\mathrm{y}^{(n-1)}\right)^{1-\alpha}\pi_0, \end{aligned}$$

which with (7) and (8), we obtain

$$\begin{aligned}
&\left(r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_0+\mathrm{y}^{(n-2)}\right)'\\
\leq\ & -\beta_0^{\alpha}\frac{1}{r^{1/\alpha}\pi_0^{1+\alpha}}\left(\mathrm{y}^{(n-2)}(\sigma)\right)^{\alpha}\left(\beta_0\frac{\mathrm{y}^{(n-2)}}{\pi_0}\right)^{1-\alpha}\pi_0\\
\leq\ & -\beta_0^{\alpha}\frac{1}{r^{1/\alpha}\pi_0^{\alpha}}\left(\mathrm{y}^{(n-2)}\right)^{\alpha}\left(\beta_0\frac{\mathrm{y}^{(n-2)}}{\pi_0}\right)^{1-\alpha}\\
\leq\ & \frac{-\beta_0}{r^{1/\alpha}\pi_0}\mathrm{y}^{(n-2)}.
\end{aligned}$$

When we integrate the previous inequality from ℓ to ∞, we arrive at

$$-r^{1/\alpha}(\ell)\mathrm{y}^{(n-1)}(\ell)\pi_0(\ell)-\mathrm{y}^{(n-2)}(\ell)\leq-\beta_0\int_{\ell}^{\infty}\frac{1}{r^{1/\alpha}(s)\pi_0(s)}\mathrm{y}^{(n-2)}(s)\mathrm{d}s,$$

or equivalently

$$\begin{aligned}
r^{1/\alpha}(\ell)\mathrm{y}^{(n-1)}(\ell)\pi_0(\ell)+\mathrm{y}^{(n-2)}(\ell) &\geq \beta_0\int_{\ell}^{\infty}\frac{1}{r^{1/\alpha}(s)\pi_0(s)}\mathrm{y}^{(n-2)}(s)\mathrm{d}s\\
&\geq \beta_0\frac{\mathrm{y}^{(n-2)}(\ell)}{\pi_0(\ell)}\int_{\ell}^{\infty}\frac{1}{r^{1/\alpha}(s)}\mathrm{d}s\\
&\geq \beta_0\mathrm{y}^{(n-2)}(\ell).
\end{aligned}$$

That is,

$$r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_0+(1-\beta_0)\mathrm{y}^{(n-2)}\geq 0.$$

Thus,

$$\left(\frac{\mathrm{y}^{(n-2)}}{\pi_0^{1-\beta_0}}\right)'=\frac{\pi_0 r^{1/\alpha}\mathrm{y}^{(n-1)}+(1-\beta_0)\mathrm{y}^{(n-2)}}{r^{1/\alpha}\pi_0^{2-\beta_0}}\geq 0. \tag{9}$$

□

If the condition $\beta_0\leq 1/2$ holds, we have the opportunity to improve the characteristics stated in Lemma 6. This improvement is demonstrated in the subsequent lemma.

Lemma 7. *Assuming that $\chi\in\Omega_2$ and Condition (4) is satisfied. If the following limit is satisfied:*

$$\lim_{\ell\to\infty}\frac{\pi_0(\sigma(\ell))}{\pi_0(\ell)}=\lambda<\infty, \tag{10}$$

and there exists an increasing sequence $\{\beta_j\}_{j=1}^{m}$ defined as follows:

$$\beta_j:=\beta_0\frac{\lambda^{\beta_{j-1}}}{\left(1-\beta_{j-1}\right)^{1/\alpha}},$$

where $\alpha\leq 1$, $\beta_0=\epsilon\delta^{1/\alpha}$, $\beta_{m-1}\leq 1/2$ and $\beta_m,\epsilon\in(0,1)$, then eventually,

$(\mathrm{Y}_{3,1})$ $\mathrm{y}^{(n-2)}/\pi_0^{\beta_m}$ *is decreasing;*

$(\mathrm{Y}_{3,2})$ $\lim_{\ell\to\infty}\mathrm{y}^{(n-2)}(\ell)/\pi_0^{\beta m}(\ell)=0$.

Proof. $(\mathrm{Y}_{3,1})$ Since $\chi\in\Omega_2$, we can conclude that $(\mathrm{Y}_{1,1})$–$(\mathrm{Y}_{1,5})$ in Lemma 5 are satisfied for all $\ell\geq\ell_1$, *with* ℓ_1 being large enough. Furthermore, from Lemma 6, we have that $(\mathrm{Y}_{2,1})$–$(\mathrm{Y}_{2,4})$ hold.

Now, assume that $\beta_0 \leq 1/2$, and

$$\beta_1 := \beta_0 \frac{\lambda^{\beta_0}}{(1-\beta_0)^{1/\alpha}}.$$

Next, we will prove $(Y_{3,1})$ and $(Y_{3,2})$ at $m = 1$. Following the proof in Lemma 6, we obtain the inequality:

$$\left(r\left(y^{(n-1)}\right)^{\alpha}\right)' \leq -\alpha\beta_0^{\alpha}\frac{1}{r^{1/\alpha}\pi_0^{1+\alpha}}\left(y^{(n-2)}(\sigma)\right)^{\alpha},$$

By integrating the final inequality from ℓ_1 to ℓ, and employing $(Y_{2,2})$ and condition (10), we can derivethe following expression

$$\begin{aligned}
& r(\ell)\left(y^{(n-1)}(\ell)\right)^{\alpha} \\
\leq\ & r(\ell_1)\left(y^{(n-1)}(\ell_1)\right)^{\alpha} - \alpha\beta_0^{\alpha}\int_{\ell_1}^{\ell}\frac{1}{r^{1/\alpha}(s)\pi_0^{1+\alpha}(s)}\left(y^{(n-2)}(\sigma(s))\right)^{\alpha}ds \\
\leq\ & r(\ell_1)\left(y^{(n-1)}(\ell_1)\right)^{\alpha} - \alpha\beta_0^{\alpha}\int_{\ell_1}^{\ell}\frac{1}{r^{1/\alpha}(s)\pi_0^{1+\alpha}(s)}\pi_0^{\alpha\beta_0}(\sigma(s))\left(\frac{y^{(n-2)}(s)}{\pi_0^{\beta_0}(s)}\right)^{\alpha}ds \\
\leq\ & r(\ell_1)\left(y^{(n-1)}(\ell_1)\right)^{\alpha} - \alpha\beta_0^{\alpha}\left(\frac{y^{(n-2)}(\ell)}{\pi_0^{\beta_0}(\ell)}\right)^{\alpha}\int_{\ell_1}^{\ell}\frac{\pi_0^{-1-\alpha+\alpha\beta_0}(s)}{r^{1/\alpha}(s)}\frac{\pi_0^{\alpha\beta_0}(\sigma(s))}{\pi_0^{\alpha\beta_0}(s)}ds \\
\leq\ & r(\ell_1)\left(y^{(n-1)}(\ell_1)\right)^{\alpha} - \alpha\beta_0^{\alpha}\lambda^{\alpha\beta_0}\left(\frac{y^{(n-2)}(\ell)}{\pi_0^{\beta_0}(\ell)}\right)^{\alpha}\int_{\ell_1}^{\ell}\frac{\pi_0^{-1-\alpha+\alpha\beta_0}(s)}{r^{1/\alpha}(s)}ds \\
\leq\ & r(\ell_1)\left(y^{(n-1)}(\ell_1)\right)^{\alpha} - \frac{\beta_0^{\alpha}\lambda^{\alpha\beta_0}}{1-\beta_0}\left(\frac{y^{(n-2)}(\ell)}{\pi_0^{\beta_0}(\ell)}\right)^{\alpha}\left(\frac{1}{\pi_0^{\alpha(1-\beta_0)}(\ell)} - \frac{1}{\pi_0^{\alpha(1-\beta_0)}(\ell_1)}\right) \\
\leq\ & r(\ell_1)\left(y^{(n-1)}(\ell_1)\right)^{\alpha} + \beta_1^{\alpha}\frac{1}{\pi_0^{\alpha(1-\beta_0)}(\ell_1)}\left(\frac{y^{(n-2)}(\ell)}{\pi_0^{\beta_0}(\ell)}\right)^{\alpha} - \beta_1^{\alpha}\left(\frac{y^{(n-2)}(\ell)}{\pi_0(\ell)}\right)^{\alpha}.
\end{aligned}$$

Using the fact that $y^{(n-2)}(\ell)/\pi_0^{\beta_0}(\ell) \to 0$ as $\ell \to \infty$, we have that

$$r(\ell_1)\left(y^{(n-1)}(\ell_1)\right)^{\alpha} + \beta_1^{\alpha}\frac{1}{\pi_0^{\alpha(1-\beta_0)}(\ell_1)}\left(\frac{y^{(n-2)}(\ell)}{\pi_0^{\beta_0}(\ell)}\right)^{\alpha} \leq 0.$$

Therefore,

$$r\left(y^{(n-1)}\right)^{\alpha} \leq -\beta_1^{\alpha}\left(\frac{y^{(n-2)}}{\pi_0}\right)^{\alpha},$$

or equivalently

$$r^{1/\alpha}y^{(n-1)}\pi_0 + \beta_1 y^{(n-2)} \leq 0,$$

and then

$$\left(\frac{y^{(n-2)}}{\pi_0^{\beta_1}}\right)' = \frac{\pi_0 r^{1/\alpha}y^{(n-1)} + \beta_1 y^{(n-2)}}{r^{1/\alpha}\pi_0^{1+\beta_1}} \leq 0.$$

By employing the method previously utilized, we can demonstrate that

$$\lim_{\ell\to\infty}\frac{y^{(n-2)}(\ell)}{\pi_0^{\beta_1}(\ell)} = 0,$$

and

$$\left(\frac{\mathrm{y}^{(n-2)}}{\pi_0^{1-\beta_1}}\right)' \geq 0.$$

Moreover, if $\beta_{k-1} < \beta_k \leq 1/2$, then we can establish that

$$r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_0 + \beta_k \mathrm{y}^{(n-2)} \leq 0, \tag{11}$$

and

$$\lim_{\ell\to\infty} \frac{\mathrm{y}^{(n-2)}(\ell)}{\pi_0^{\beta_k}(\ell)} = 0,$$

for $k = 2, 3, \ldots, m$. This, in turn, concludes the proof of the Lemma. □

Theorem 1. *Assuming that condition (4) is satisfied. If*

$$\beta_0 > 1/2, \tag{12}$$

then $\Omega_2 = \varnothing$. *Here,* β_0 *is defined according to Lemma 6.*

Proof. Assume that $\chi \in \Omega_2$ leads to a contradiction. According to Lemma 6, the functions $\mathrm{y}^{(n-2)}/\pi_0^{\beta_0}$ and $\mathrm{y}^{(n-2)}/\pi_0^{1-\beta_0}$ are decreasing and increasing for $\ell \geq \ell_1$, respectively. As a result, we can conclude that

$$\beta_0 \leq 1/2.$$

This conclusion contradicts the initial assumption. Hence, the proof is considered complete. □

Theorem 2. *Let us assume that conditions (4) and (10) are satisfied. Suppose there exists a positive integer value m, such that the following inequality holds*

$$\liminf_{\ell\to\infty} \int_{\sigma(\ell)}^{\ell} \pi_0(s)\pi_0^{\alpha-1}(\sigma(s))\left(\sigma^{n-2}(s)\right)^{\alpha} \mathrm{Q}_1(s)\mathrm{d}s > \frac{\alpha\beta_m^{\alpha-1}(1-\beta_m)((n-2)!)^{\alpha}}{\mathrm{e}}, \tag{13}$$

then $\Omega_2 = \varnothing$, *where* $\alpha \leq 1$.

Proof. Let us assume the opposite scenario, where $\chi \in \Omega_2$. According to the information provided in Lemma 7, it follows that both $(\mathrm{Y}_{3,1})$ and $(\mathrm{Y}_{3,2})$ hold.
We can now establish the function in the following manner

$$w = r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_0 + \mathrm{y}^{(n-2)}.$$

From $(\mathrm{Y}_{1,3})$, we can deduce that $w > 0$ when $\ell \geq \ell_1$. Furthermore, according to $(\mathrm{Y}_{3,1})$, we can conclude that

$$r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_0 \leq -\beta_m \mathrm{y}^{(n-2)}.$$

Next, based on the definition of w, we can derive that

$$\begin{aligned} w(\ell) &= r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_0 + \beta_m \mathrm{y}^{(n-2)} - \beta_m \mathrm{y}^{(n-2)} + \mathrm{y}^{(n-2)} \\ &\leq (1-\beta_m)\mathrm{y}^{(n-2)}. \end{aligned} \tag{14}$$

By employing Lemma 5, we can determine that $(\Upsilon_{1,1})$–$(\Upsilon_{1,5})$ hold. From $(\Upsilon_{1,2})$ and $(\Upsilon_{1,6})$, we obtain

$$\begin{aligned} w' &= \left(r^{1/\alpha}\mathrm{y}^{(n-1)}\right)'\pi_0 \\ &\leq \frac{1}{\alpha}\left(r\left(\mathrm{y}^{(n-1)}\right)^{\alpha}\right)'\left(r^{1/\alpha}\mathrm{y}^{(n-1)}\right)^{1-\alpha}\pi_0 \\ &\leq -\frac{1}{\alpha}\left(\sum_{i=1}^{v} q_i \mathrm{p}_1^{\alpha}(\sigma_i;\kappa)\right)\mathrm{y}^{\alpha}(\sigma)\left(r^{1/\alpha}\mathrm{y}^{(n-1)}\right)^{1-\alpha}\pi_0 \\ &\leq -\frac{1}{\alpha}\mathrm{Q}_1\mathrm{y}^{\alpha}(\sigma)\left(\beta_m\frac{\mathrm{y}^{(n-2)}}{\pi_0}\right)^{1-\alpha}\pi_0 \\ &\leq -\frac{1}{\alpha}\beta_m^{1-\alpha}\mathrm{Q}_1\pi_0\mathrm{y}^{\alpha}(\sigma)\left(\frac{\mathrm{y}^{(n-2)}}{\pi_0}\right)^{1-\alpha} \\ &\leq -\frac{1}{\alpha}\beta_m^{1-\alpha}\mathrm{Q}_1\pi_0\left(\frac{\epsilon}{(n-2)!}\sigma^{n-2}\right)^{\alpha}\left(\mathrm{y}^{(n-2)}(\sigma)\right)^{\alpha}\left(\frac{\mathrm{y}^{(n-2)}}{\pi_0}\right)^{1-\alpha}. \end{aligned}$$

Using $(\Upsilon_{1,4})$ in Lemma 5, we note that $\mathrm{y}^{(n-2)}(\ell)/\pi_0(\ell)$ is increasing, then

$$\frac{\mathrm{y}^{(n-2)}(\sigma)}{\pi_0(\sigma)} \leq \frac{\mathrm{y}^{(n-2)}}{\pi_0},$$

and

$$\left(\frac{\mathrm{y}^{(n-2)}(\sigma)}{\pi_0(\sigma)}\right)^{1-\alpha} \leq \left(\frac{\mathrm{y}^{(n-2)}}{\pi_0}\right)^{1-\alpha}.$$

Therefore,

$$w' \leq -\frac{1}{\alpha}\frac{\beta_m^{1-\alpha}\epsilon^{\alpha}}{((n-2)!)^{\alpha}}\mathrm{Q}_1\frac{\pi_0}{\pi_0^{1-\alpha}(\sigma)}\left(\sigma^{n-2}\right)^{\alpha}\mathrm{y}^{(n-2)}(\sigma).$$

From Equation (14), we obtain the following

$$w' + \frac{1}{\alpha}\frac{\epsilon^{\alpha}\beta_m^{1-\alpha}}{((n-2)!)^{\alpha}(1-\beta_m)}\frac{\pi_0}{\pi_0^{1-\alpha}(\sigma)}\left(\sigma^{n-2}\right)^{\alpha}\mathrm{Q}_1 w(\sigma) \leq 0. \tag{15}$$

Therefore, w represents a constructive solution to the differential inequality (15). Nevertheless, according to the findings presented in Theorem 2.1.1 from [8], condition (13) ensures the oscillatory nature of Equation (15). Consequently, this contradiction serves as conclusive evidence for proving the theorem. □

3.2. Category Ω_3

Lemma 8. *Suppose that $\chi \in \Omega_3$. Under this assumption, the following conditions hold for sufficiently large values of ℓ :*

$(\Upsilon_{4,1})$ *The expression y/π_{n-2} is monotonically increasing;*

$(\Upsilon_{4,2})$ *For all $i = 0, 1, 2, \ldots, n-2$, we have $(-1)^{i+1}\mathrm{y}^{(n-i-2)} \leq r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_i$.*

Proof. $(\mathrm{Y}_{4,1})$ Suppose that $\chi \in \Omega_3$. From Equation (1), we have $r\left(\mathrm{y}^{(n-1)}\right)^{\alpha}$ is decreasing, and, hence,

$$\begin{aligned} r^{1/\alpha}(\ell)\mathrm{y}^{(n-1)}(\ell)\int_{\ell}^{\infty}\frac{1}{r^{1/\alpha}(s)}\mathrm{d}s &\geq \int_{\ell}^{\infty}\frac{1}{r^{1/\alpha}(s)}r^{1/\alpha}(s)\mathrm{y}^{(n-1)}(s)\mathrm{d}s \\ &= \lim_{\ell\to\infty}\mathrm{y}^{(n-2)}(\ell)-\mathrm{y}^{(n-2)}(\ell). \end{aligned} \tag{16}$$

Given that $\mathrm{y}^{(n-2)}$ represents a positive and decreasing function, it exhibits convergence toward a non-negative constant, as $\ell \to \infty$. Consequently, the Equation (16) can be represented as

$$-\mathrm{y}^{(n-2)} \leq r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_0,$$

which implies that

$$\left(\frac{\mathrm{y}^{(n-2)}}{\pi_0}\right)' = \frac{r^{1/\alpha}\pi_0\mathrm{y}^{(n-1)}+\mathrm{y}^{(n-2)}}{r^{1/\alpha}\pi_0^2} \geq 0,$$

which leads to

$$-\mathrm{y}^{(n-3)}(\ell) = \int_{\ell}^{\infty}\frac{\mathrm{y}^{(n-2)}(s)}{\pi_0(s)}\pi_0(s)\mathrm{d}s \geq \frac{\mathrm{y}^{(n-2)}(\ell)}{\pi_0(\ell)}\pi_1(\ell).$$

This implies

$$\left(\frac{\mathrm{y}^{(n-3)}}{\pi_1}\right)' = \frac{\pi_1\mathrm{y}^{(n-2)}+\mathrm{y}^{(n-3)}\pi_0}{\pi_1^2} \leq 0.$$

Additionally, we iterate the aforementioned procedure $(n-4)$ times to yield

$$\left(\frac{\mathrm{y}'}{\pi_{n-3}}\right)' \leq 0.$$

Now

$$-\mathrm{y}(\ell) = \int_{\ell}^{\infty}\frac{\mathrm{y}'(s)}{\pi_{n-3}(s)}\pi_{n-3}(s)\mathrm{d}s \leq \frac{\mathrm{y}'(\ell)}{\pi_{n-3}(\ell)}\pi_{n-2}(\ell).$$

This implies

$$\left(\frac{\mathrm{y}}{\pi_{n-2}}\right)' = \frac{\pi_{n-2}\mathrm{y}'+\mathrm{y}\pi_{n-3}}{\pi_{n-2}^2} \geq 0.$$

$(\mathrm{Y}_{4,1})$ Assume that $\chi \in \Omega_3$. Then, we obtain

$$r^{1/\alpha}(\ell)\mathrm{y}^{(n-1)}(\ell)\pi_0(\ell) \geq \int_{\ell}^{\infty}\frac{r^{1/\alpha}(s)\mathrm{y}^{(n-1)}(s)}{r^{1/\alpha}(s)}\mathrm{d}s \geq -\mathrm{y}^{(n-2)}(\ell),$$

or equivalently

$$\mathrm{y}^{(n-2)} \geq -r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_0.$$

After integrating the final inequality from ℓ to ∞, the result is expressed as

$$\begin{aligned} -\mathrm{y}^{(n-3)}(\ell) &\geq -\int_{\ell}^{\infty} r^{1/\alpha}(s)\mathrm{y}^{(n-1)}(s)\pi_0(s)\mathrm{d}s \\ &\geq -r^{1/\alpha}(\ell)\mathrm{y}^{(n-1)}(\ell)\int_{\ell}^{\infty}\pi_0(s)\mathrm{d}s \\ &\geq -r^{1/\alpha}(\ell)\mathrm{y}^{(n-1)}(\ell)\pi_1(s), \end{aligned}$$

or equivalently

$$\mathrm{y}^{(n-3)} \leq r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_1.$$

After integrating the final inequality from ℓ to ∞, the result is expressed as

$$\begin{aligned} -\mathrm{y}^{(n-4)}(\ell) &\leq \int_{\ell}^{\infty} r^{1/\alpha}(s)\mathrm{y}^{(n-1)}(s)\pi_1(s)\mathrm{d}s \\ &\leq r^{1/\alpha}(\ell)\mathrm{y}^{(n-1)}(\ell)\int_{\ell}^{\infty}\pi_1(s)\mathrm{d}s \\ &\leq r^{1/\alpha}(\ell)\mathrm{y}^{(n-1)}(\ell)\pi_2(\ell), \end{aligned}$$

or equivalently

$$\mathrm{y}^{(n-4)} \geq -r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_2.$$

Through iterative integration of the preceding inequality from ℓ to ∞, we deduce that

$$(-1)^{i+1}\mathrm{y}^{(n-i-2)} \leq r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_i,$$

for $i = 0, 1, 2, \ldots, n-2$. The lemma's proof is now complete. □

Lemma 9. *If $\chi \in \Omega_3$, then eventually*

$(\mathrm{Y}_{5,1})$ $\chi > \mathrm{p}_2(\ell,\kappa)\mathrm{y}$;

$(\mathrm{Y}_{5,2})$ $\left(r\left(\mathrm{y}^{(n-1)}\right)^{\alpha}\right)' \leq -\mathrm{Q}_2\mathrm{y}^{\alpha}(\widetilde{\sigma})$.

Proof. $(\mathrm{Y}_{5,1})$ From Lemma 1, we have Equation (3) holds. From $(\mathrm{Y}_{4,1})$, we conclude that

$$\mathrm{y}\left(\mathfrak{y}^{[2r+1]}\right) \leq \frac{\pi_{n-2}\left(\mathfrak{y}^{[2r+1]}\right)}{\pi_{n-2}\left(\mathfrak{y}^{[2r]}\right)}\mathrm{y}\left(\mathfrak{y}^{[2r]}\right),$$

which, with Equation (3), gives

$$\chi > \sum_{r=0}^{\kappa}\left(\prod_{l=0}^{2r}\mathrm{p}\left(\mathfrak{y}^{[l]}\right)\right)\left[\frac{1}{\mathrm{p}\left(\mathfrak{y}^{[2r]}\right)} - \frac{\pi_{n-2}\left(\mathfrak{y}^{[2r+1]}\right)}{\pi_{n-2}\left(\mathfrak{y}^{[2r]}\right)}\right]\mathrm{y}\left(\mathfrak{y}^{[2r]}\right). \tag{17}$$

Since y is decreasing, then Equation (17) becomes

$$\begin{aligned} \chi &> \sum_{r=0}^{\kappa}\left(\prod_{l=0}^{2r}\mathrm{p}\left(\mathfrak{y}^{[l]}\right)\right)\left[\frac{1}{\mathrm{p}\left(\mathfrak{y}^{[2r]}\right)} - \frac{\pi_{n-2}\left(\mathfrak{y}^{[2r+1]}\right)}{\pi_{n-2}\left(\mathfrak{y}^{[2r]}\right)}\right]\mathrm{y} \\ &= \mathrm{p}_2(\ell,\kappa)\mathrm{y}(\ell). \end{aligned}$$

$(\mathrm{Y}_{5,2})$ Equation (1) with $(\mathrm{Y}_{5,1})$ becomes

$$\begin{aligned} \left(r\left(\mathrm{y}^{(n-1)}\right)^{\alpha}\right)' &= \sum_{i=1}^{v} -q_i\chi^{\alpha}(\sigma_i) \\ &\leq -\sum_{i=1}^{v} q_i\mathrm{p}_2^{\alpha}(\sigma_i,\kappa)\mathrm{y}^{\alpha}(\sigma_i) \\ &\leq -\mathrm{y}^{\alpha}(\widetilde{\sigma})\mathrm{Q}_2. \end{aligned}$$

Therefore, the Lemma's proof has been successfully concluded. □

Lemma 10. *Assume that $\chi \in \Omega_3$. If*

$$\int_{\ell_0}^{\infty}\left(\frac{1}{r(u)}\int_{\ell_0}^{u}\mathrm{Q}_2(s)\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u = \infty, \tag{18}$$

and there exists a $k_0 \in (0,1)$*, such that*

$$\frac{1}{\alpha}\pi_{n-2}^{\alpha+1}(\ell)\pi_{n-3}^{-1}(\ell)\mathrm{Q}_2(\ell) \geq k_0^{\alpha}. \tag{19}$$

Then

$(\mathrm{Y}_{6,1})$ $\lim_{\ell\to\infty} \mathrm{y}(\ell) = 0$;

$(\mathrm{Y}_{6,2})$ $\mathrm{y}(\ell)/\pi_{n-2}^{k_0}(\ell)$ *is decreasing;*

$(\mathrm{Y}_{6,3})$ $\lim_{\ell\to\infty} \mathrm{y}(\ell)/\pi_{n-2}^{\beta_0}(\ell) = 0$;

Proof. $(\mathrm{Y}_{6,1})$ Assume that $\chi \in \Omega_3$. Since y is positive and decreasing, we have that $\lim_{\ell\to\infty} \mathrm{y}(\ell) = c_3 \geq 0$. Assume the contrary, that $c_3 > 0$. Then there is a $\ell_2 \geq \ell_1$ with $\mathrm{y} \geq c_3$ for $\ell \geq \ell_2$. Then from $(\mathrm{Y}_{5,2})$, we obtain

$$\begin{aligned}\left(r\left(\mathrm{y}^{(n-1)}\right)^{\alpha}\right)' &\leq -\mathrm{y}^{\alpha}(\widetilde{\sigma})\mathrm{Q}_2\\ &\leq -c_3^{\alpha}\mathrm{Q}_2.\end{aligned}$$

Integrating this inequality twice from ℓ_2 to ℓ, we obtain

$$r(\ell)\left(\mathrm{y}^{(n-1)}(\ell)\right)^{\alpha} - r(\ell_2)\left(\mathrm{y}^{(n-1)}(\ell_2)\right)^{\alpha} \leq -c_3^{\alpha}\int_{\ell_2}^{\ell}\mathrm{Q}_2(s)\mathrm{d}s.$$

Using case (C_3), we have $\mathrm{y}^{(n-1)} < 0$ for $\ell \geq \ell_1$. Then, $r(\ell_2)\left(\mathrm{y}^{(n-1)}(\ell_2)\right)^{\alpha} < 0$, and so

$$\mathrm{y}^{(n-1)}(\ell) \leq -\frac{c_3}{r^{1/\alpha}(\ell)}\int_{\ell_2}^{\ell}\mathrm{Q}_2(s)\mathrm{d}s,$$

and then

$$\mathrm{y}^{(n-2)}(\ell) \leq \mathrm{y}^{(n-2)}(\ell_2) - c_3\int_{\ell_2}^{\ell}\left(\frac{1}{r(u)}\int_{\ell_2}^{u}\mathrm{Q}_2(s)\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u \to -\infty \text{ as } \ell\to\infty.$$

This contradicts the positivity of $\mathrm{y}^{(n-2)}$. Therefore, $c_3 = 0$.

$(\mathrm{Y}_{6,2})$ Integrating $(\mathrm{Y}_{5,2})$ from ℓ_2 to ℓ, and using Equation (19), we obtain

$$\begin{aligned}r(\ell)\left(\mathrm{y}^{(n-1)}(\ell)\right)^{\alpha} &\leq r(\ell_2)\left(\mathrm{y}^{(n-1)}(\ell_2)\right)^{\alpha} - \mathrm{y}^{\alpha}(\widetilde{\sigma}(s))\int_{\ell_2}^{\ell}\mathrm{Q}_2(s)\mathrm{d}s\\ &\leq r(\ell_2)\left(\mathrm{y}^{(n-1)}(\ell_2)\right)^{\alpha} - \mathrm{y}^{\alpha}(\ell)\int_{\ell_2}^{\ell}\mathrm{Q}_2(s)\mathrm{d}s\\ &\leq r(\ell_2)\left(\mathrm{y}^{(n-1)}(\ell_2)\right)^{\alpha} - \mathrm{y}^{\alpha}(\ell)\int_{\ell_2}^{\ell}\alpha k_0^{\alpha}\frac{\pi_{n-3}(s)}{\pi_{n-2}^{\alpha+1}(s)}\mathrm{d}s\\ &\leq r(\ell_2)\left(\mathrm{y}^{(n-1)}(\ell_2)\right)^{\alpha} + k_0^{\alpha}\frac{\mathrm{y}^{\alpha}(\ell)}{\pi_{n-2}^{\alpha}(\ell_2)} - k_0^{\alpha}\frac{\mathrm{y}^{\alpha}(\ell)}{\pi_{n-2}^{\alpha}(\ell)},\end{aligned}$$

which, with $(\mathrm{Y}_{6,1})$, gives

$$r\left(\mathrm{y}^{(n-1)}\right)^{\alpha} \leq -k_0^{\alpha}\frac{\mathrm{y}^{\alpha}}{\pi_{n-2}^{\alpha}},$$

or equivalently

$$r^{1/\alpha}\mathrm{y}^{(n-1)} \leq -k_0\frac{\mathrm{y}}{\pi_{n-2}}. \tag{20}$$

Thus, from $(Y_{4,2})$ at $i = n-3$, we have

$$\frac{y'}{\pi_{n-3}} \leq -k_0 \frac{y}{\pi_{n-2}},$$

or equivalently

$$\pi_{n-2} y' + k_0 \pi_{n-3} y \leq 0. \tag{21}$$

Consequently,

$$\left(\frac{y}{\pi_{n-2}^{k_0}}\right)' = \frac{\pi_{n-2} y' + k_0 \pi_{n-3} y}{\pi_{n-2}^{k_0+1}} \leq 0.$$

$(Y_{6,3})$ Since $y/\pi_{n-2}^{k_0}$ is both positive and decreasing, we can conclude that $\lim_{\ell\to\infty} y(\ell)/\pi_{n-2}^{k_0}(\ell) = c_4 \geq 0$. Now, let's assume the opposite, that is, $c_4 > 0$. In this case, there exists a $\ell_2 \geq \ell_1$ with $y/\pi_{n-2}^{k_0} \geq c_4$ for $\ell \geq \ell_2$. Next, we define:

$$\varphi := \frac{y + r^{1/\alpha} y^{(n-1)} \pi_{n-2}}{\pi_{n-2}^{k_0}}.$$

Then, from $(Y_{4,2})$, $\varphi \geq 0$ for $\ell \geq \ell_2$. Differentiating φ and $(Y_{4,2})$, we find

$$\begin{aligned}
\varphi'(\ell) &= \frac{1}{\pi_{n-2}^{2k_0}(\ell)} \Big[\pi_{n-2}^{k_0}(\ell) \Big(y'(\ell) - r^{1/\alpha}(\ell) y^{(n-1)}(\ell) \pi_{n-3}(\ell) + \Big(r^{1/\alpha}(\ell) \chi^{(n-1)}(\ell) \Big)' \pi_{n-2}(\ell) \Big) \\
&\quad + k_0 \pi_{n-2}^{k_0-1}(\ell) \pi_{n-3}(\ell) \Big(y(\ell) + r^{1/\alpha}(\ell) y^{(n-1)}(\ell) \pi_{n-2}(\ell) \Big) \Big] \\
&\leq \frac{1}{\pi_{n-2}^{k_0+1}(\ell)} \Big[\Big(r^{1/\alpha}(\ell) y^{(n-1)}(\ell) \Big)' \pi_{n-2}^2(\ell) + k_0 \pi_{n-3}(\ell) \Big(y(\ell) + r^{1/\alpha}(\ell) y^{(n-1)}(\ell) \pi_{n-2}(\ell) \Big) \Big] \\
&\leq \frac{1}{\pi_{n-2}^{k_0+1}(\ell)} \Big[\frac{1}{\alpha} \Big(r(\ell) \Big(y^{(n-1)}(\ell) \Big)^{\alpha} \Big)' \Big(r^{1/\alpha}(\ell) y^{(n-1)}(\ell) \Big)^{1-\alpha} \pi_{n-2}^2(\ell) \\
&\quad + k_0 \pi_{n-3}(\ell) \Big(y(\ell) + r^{1/\alpha}(\ell) y^{(n-1)}(\ell) \pi_{n-2}(\ell) \Big) \Big].
\end{aligned}$$

Using $(Y_{5,2})$, we find

$$\begin{aligned}
\varphi' &\leq \frac{1}{\pi_{n-2}^{k_0+1}} \Big[\frac{-1}{\alpha} Q_2 y^{\alpha}(\widetilde{\sigma}) \Big(r^{1/\alpha} y^{(n-1)} \Big)^{1-\alpha} \pi_{n-2}^2 \\
&\quad + k_0 \pi_{n-3} y + k_0 \pi_{n-3} r^{1/\alpha} y^{(n-1)} \pi_{n-2} \Big].
\end{aligned}$$

Since $\alpha \leq 1$, $y^{(n-1)} \leq 0$, and

$$r^{1/\alpha} y^{(n-1)} \leq -k_0 \frac{y}{\pi_{n-2}},$$

also

$$-r^{1/\alpha} y^{(n-1)} \geq k_0 \frac{y}{\pi_{n-2}},$$

which implies that

$$\Big(r^{1/\alpha} y^{(n-1)} \Big)^{1-\alpha} \geq \Big(k_0 \frac{y}{\pi_{n-2}} \Big)^{1-\alpha}.$$

Then

$$\begin{aligned}\varphi' &\leq \frac{1}{\pi_{n-2}^{k_0+1}}\left[\frac{-1}{\alpha}\mathrm{Q}_2\mathrm{y}^{\alpha}\left(k_0\frac{\mathrm{y}}{\pi_{n-2}}\right)^{1-\alpha}\pi_{n-2}^2\right.\\ &\quad\left.+k_0\pi_{n-3}\mathrm{y}+k_0\pi_{n-3}r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_{n-2}\right]\\ &\leq \frac{1}{\pi_{n-2}^{k_0+1}}\left[\frac{-k_0^{1-\alpha}}{\alpha}\mathrm{Q}_2\pi_{n-2}^{\alpha+1}\mathrm{y}\right.\\ &\quad\left.+k_0\pi_{n-3}\mathrm{y}+k_0\pi_{n-3}r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_{n-2}\right].\end{aligned}$$

Using Equation (19), we obtain

$$\begin{aligned}\varphi'(\ell) &\leq \frac{1}{\pi_{n-2}^{k_0+1}(\ell)}\left[-k_0\pi_{n-3}(\ell)\mathrm{y}(\ell)+k_0\pi_{n-3}(\ell)\mathrm{y}(\ell)+k_0\pi_{n-3}(\ell)r^{1/\alpha}(\ell)\mathrm{y}^{(n-1)}(\ell)\pi_{n-2}(\ell)\right]\\ &= \frac{1}{\pi_{n-2}^{k_0}(\ell)}k_0\pi_{n-3}(\ell)r^{1/\alpha}(\ell)\mathrm{y}^{(n-1)}(\ell).\end{aligned}\tag{22}$$

Using the fact that $\mathrm{y}/\pi_{n-2}^{k_0}\geq c_4$ with (20), we obtain

$$r^{1/\alpha}\mathrm{y}^{(n-1)}\leq -k_0\frac{\mathrm{y}}{\pi_{n-2}}\leq -k_0c_4\pi_{n-2}^{k_0-1}.\tag{23}$$

Combining (22) and (23), we obtain

$$\varphi'\leq -k_0^2c_4\frac{\pi_{n-3}}{\pi_{n-2}}<0.$$

By integrating the preceding inequality from ℓ_2 to ℓ, we obtain

$$\varphi(\ell_2)\geq k_0^2c_4\ln\frac{\pi_{n-2}(\ell_2)}{\pi_{n-2}(\ell)}\to\infty \text{ as } \ell\to\infty,$$

a contradiction, and so, $c_4=0$. Consequently, the lemma's proof is now complete. □

Lemma 11. *Let's suppose that $\chi(\ell)\in\Omega_3$ and (18) and (19) are satisfied for a certain value of $k_0\in(0,1)$. If, for every i from 1 to $m-1$, it holds that $k_{i-1}\leq k_i<1$, then*

$(\mathrm{Y}_{7,1,m})$ *$\mathrm{y}/\pi_{n-2}^{k_m}$ is decreasing;*

$(\mathrm{Y}_{7,2,m})$ $\lim_{\ell\to\infty}\mathrm{y}(\ell)/\pi_{n-2}^{k_m}(\ell)=0$;

where

$$k_j=k_0\frac{\lambda_1^{k_{j-1}}}{\left(1-k_{j-1}\right)^{1/\alpha}},\ j=1,2,\ldots,m,\tag{24}$$

and

$$\frac{\pi_{n-2}(\widetilde{\sigma}(\ell))}{\pi_{n-2}(\ell)}\geq\lambda_1,\ \textit{for all}\ \ell\geq\ell_1,\tag{25}$$

for some $\lambda_1\geq 1$.

Proof. $(\mathrm{Y}_{7,1,m})$: Assume that $\chi\in\Omega_3$. Then, from Theorem 10, we obtain that $(\mathrm{Y}_{6,1})$–$(\mathrm{Y}_{6,3})$ hold. By applying the induction, we establish the validity of $(\mathrm{Y}_{7,1,0})$–$(\mathrm{Y}_{7,3,0})$ based on Lemma 11. Now, let us assume that $(\mathrm{Y}_{7,1,m-1})$–$(\mathrm{Y}_{7,3,m-1})$ are true. When we integrate $(\mathrm{Y}_{5,2})$ from ℓ_2 to ℓ, we obtain the following expression:

$$r(\ell)\left(\mathrm{y}^{(n-1)}(\ell)\right)^{\alpha}\leq r(\ell_2)\left(\mathrm{y}^{(n-1)}(\ell_2)\right)^{\alpha}-\int_{\ell_2}^{\ell}\mathrm{y}^{\alpha}(\widetilde{\sigma}(s))\mathrm{Q}_2(s)\mathrm{d}s.\tag{26}$$

Using $(\mathrm{Y}_{7,1,m-1})$, we obtain

$$\mathrm{y}(\widetilde{\sigma}) \geq \pi_{n-2}^{k_{m-1}}(\widetilde{\sigma})\frac{\mathrm{y}}{\pi_{n-2}^{k_{m-1}}}.$$

Then Equation (26) becomes

$$\begin{aligned} r(\ell)\left(\mathrm{y}^{(n-1)}(\ell)\right)^{\alpha} &\leq r(\ell_2)\left(\mathrm{y}^{(n-1)}(\ell_2)\right)^{\alpha} \\ &\quad - \int_{\ell_2}^{\ell} \pi_{n-2}^{\alpha k_{m-1}}(\widetilde{\sigma}(s))\frac{\mathrm{y}^{\alpha}(s)}{\pi_{n-2}^{\alpha k_{m-1}}(s)}\mathrm{Q}_2(s)\mathrm{d}s. \end{aligned}$$

The fact that $\mathrm{y}/\pi_{n-2}^{k_{m-1}}$ is a decreasing function, in conjunction with this, yields

$$\begin{aligned} r(\ell)\left(\mathrm{y}^{(n-1)}(\ell)\right)^{\alpha} &\leq r(\ell_2)\left(\mathrm{y}^{(n-1)}(\ell_2)\right)^{\alpha} \\ &\quad - \frac{\mathrm{y}^{\alpha}(\ell)}{\pi_{n-2}^{\alpha k_{m-1}}(\ell)}\int_{\ell_2}^{\ell} \pi_{n-2}^{\alpha k_{m-1}}(s)\frac{\pi_{n-2}^{\alpha k_{m-1}}(\widetilde{\sigma}(s))}{\pi_{n-2}^{\alpha k_{m-1}}(s)}\mathrm{Q}_2(s)\mathrm{d}s. \end{aligned}$$

Hence, from (19) and (25), we obtain

$$\begin{aligned} & r(\ell)\left(\mathrm{y}^{(n-1)}(\ell)\right)^{\alpha} \\ \leq\ & r(\ell_2)\left(\mathrm{y}^{(n-1)}(\ell_2)\right)^{\alpha} - \lambda_1^{\alpha k_{m-1}}\frac{\mathrm{y}^{\alpha}(\ell)}{\pi_{n-2}^{\alpha k_{m-1}}(\ell)}\int_{\ell_2}^{\ell}\pi_{n-2}^{\alpha k_{m-1}}(s)\mathrm{Q}_2(s)\mathrm{d}s \\ \leq\ & r(\ell_2)\left(\mathrm{y}^{(n-1)}(\ell_2)\right)^{\alpha} - \alpha k_0^{\alpha}\lambda_1^{\alpha k_{m-1}}\frac{\mathrm{y}^{\alpha}(\ell)}{\pi_{n-2}^{\alpha k_{m-1}}(\ell)}\int_{\ell_2}^{\ell}\frac{\pi_{n-3}(s)}{\pi_{n-2}^{\alpha(1-k_{m-1})+1}(s)}\mathrm{d}s \\ =\ & r(\ell_2)\left(\mathrm{y}^{(n-1)}(\ell_2)\right)^{\alpha} - k_0^{\alpha}\frac{\lambda_1^{\alpha k_{m-1}}}{1-k_{m-1}}\frac{\mathrm{y}^{\alpha}(\ell)}{\pi_{n-2}^{\alpha k_{m-1}}(\ell)}\left(\frac{1}{\pi_{n-2}^{\alpha(1-k_{m-1})}(\ell)} - \frac{1}{\pi_{n-2}^{\alpha(1-k_{m-1})}(\ell_2)}\right) \\ =\ & r(\ell_2)\left(\mathrm{y}^{(n-1)}(\ell_2)\right)^{\alpha} + k_m^{\alpha}\frac{\mathrm{y}^{\alpha}(\ell)}{\pi_{n-2}^{\alpha k_{m-1}}(\ell)}\frac{1}{\pi_{n-2}^{\alpha(1-k_{m-1})}(\ell_2)} - k_m\frac{\mathrm{y}^{\alpha}(\ell)}{\pi_{n-2}^{\alpha}(\ell)}, \end{aligned}$$

this, coupled with the observation that $\lim_{\ell\to\infty}\mathrm{y}(\ell)/\pi_{n-2}^{k_{m-1}}(\ell) = 0$, yields

$$r\left(\mathrm{y}^{(n-1)}\right)^{\alpha} \leq -k_m^{\alpha}\frac{\mathrm{y}^{\alpha}}{\pi_{n-2}^{\alpha}},$$

or equivalently

$$r^{1/\alpha}\mathrm{y}^{(n-1)} \leq -k_m\frac{\mathrm{y}}{\pi_{n-2}}. \tag{27}$$

Hence, based on $(\mathrm{Y}_{4,2})$ at $i = n-3$, we can deduce that

$$\frac{\mathrm{y}'}{\pi_{n-3}} \leq -k_m\frac{\mathrm{y}}{\pi_{n-2}},$$

or equivalently

$$\pi_{n-2}\mathrm{y}' + k_m\pi_{n-3}\mathrm{y} \leq 0. \tag{28}$$

Consequently,

$$\left(\frac{\mathrm{y}}{\pi_{n-2}^{k_m}}\right)' = \frac{1}{\pi_{n-2}^{k_m+1}}\left(\pi_{n-2}\mathrm{y}' + k_m\pi_{n-3}\mathrm{y}\right) \leq 0.$$

Using the same approach employed in demonstrating $(\mathrm{Y}_{6,2})$ as shown in Lemma 10, can ascertain that $\lim_{\ell\to\infty} \mathrm{y}(\ell)/\pi_{n-2}^{k_m}(\ell) = 0$. This conclusion marks the end of the proof. □

Lemma 12. *Assuming that $\chi(\ell) \in \Omega_3$, and conditions (18), and (19) are satisfied for some $k_0 \in (0,1)$. If $k_{i-1} \leq k_i < 1$ for all $i = 1, 2, \ldots, m-1$, then*

$$\chi > \widehat{\mathrm{p}}_2(\ell;\kappa)\mathrm{y}.$$

Proof. Similar to the argument presented in the proof of Lemma 9, we obtain the Equation (17). Considering $(\mathrm{Y}_{7,1,m})$, we deduce that

$$\mathrm{y}\left(\mathfrak{y}^{[2r]}\right) \geq \frac{\pi_{n-2}^{k_m}\left(\mathfrak{y}^{[2r]}\right)}{\pi_{n-2}^{k_m}}\mathrm{y},$$

which with (17) yields

$$\begin{aligned}\chi &> \sum_{r=0}^{\kappa}\left(\prod_{l=0}^{2r}\mathrm{p}\left(\mathfrak{y}^{[l]}\right)\right)\left[\frac{1}{\mathrm{p}\left(\mathfrak{y}^{[2r]}\right)} - \frac{\pi_{n-2}\left(\mathfrak{y}^{[2r+1]}\right)}{\pi_{n-2}\left(\mathfrak{y}^{[2r]}\right)}\right]\frac{\pi_{n-2}^{k_m}\left(\mathfrak{y}^{[2r]}\right)}{\pi_{n-2}^{k_m}}\mathrm{y}\\ &= \widehat{\mathrm{p}}_2(\ell;\kappa)\mathrm{y}(\ell).\end{aligned}$$

□

Theorem 3. *Let's assume that conditions (18) and (19) hold, and that there exists a positive integer m such that*

$$\liminf_{\ell\to\infty}\int_{\sigma(\ell)}^{\ell}\pi_{n-2}(s)\pi_{n-2}^{\alpha-1}(\widetilde{\sigma}(s))\mathrm{Q}_2(s)\mathrm{d}s > \frac{\alpha k_m^{\alpha-1}(1-k_m)}{\mathrm{e}}. \tag{29}$$

Under these conditions, we can conclude that $\Omega_3 = \varnothing$, where $\alpha \leq 1$.

Proof. Let us assume the opposite, i.e., $\chi \in \Omega_3$. According to Lemma 11, both $(\mathrm{Y}_{7,1,m})$ and $(\mathrm{Y}_{7,2,m})$ are satisfied.

Now, we can define the function as follows

$$w = r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_{n-2} + \mathrm{y}.$$

Based on $(\mathrm{Y}_{4,2})$ at $i = n-2$, we find that $w \geq 0$ for $\ell \geq \ell_2$. Additionally, using (27), we obtain

$$r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_{n-2} \leq -k_m\mathrm{y}.$$

Hence, from the definition of w, we can deduce that

$$\begin{aligned}w &= r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_{n-2} + k_m\mathrm{y}^{(n-2)} - k_m\mathrm{y}^{(n-2)} + \mathrm{y}^{(n-3)}\\ &\leq (1-k_m)\mathrm{y}^{(n-2)}.\end{aligned} \tag{30}$$

Thus,

$$w' = \left(r^{1/\alpha}\mathrm{y}^{(n-1)}\right)'\pi_{n-2} - r^{1/\alpha}\mathrm{y}^{(n-1)}\pi_{n-3} + \mathrm{y}'.$$

From $(\mathrm{Y}_{4,2})$ at $i = n-3$, we find

$$\begin{aligned}w' &\leq \left(r^{1/\alpha}\mathrm{y}^{(n-1)}\right)'\pi_{n-2}\\ &= \frac{1}{\alpha}\left(r\left(\mathrm{y}^{(n-1)}\right)^{\alpha}\right)'\left(r^{1/\alpha}\mathrm{y}^{(n-1)}\right)^{1-\alpha}\pi_{n-2}.\end{aligned}$$

Using $(\mathrm{Y}_{5,2})$ and $(\mathrm{Y}_{4,2})$ at $i = n-2$, we deduce that

$$\begin{aligned} w' &\leq \frac{-1}{\alpha}\mathrm{Q}_2\mathrm{y}^{\alpha}(\widetilde{\sigma})\left(r^{1/\alpha}\mathrm{y}^{(n-1)}\right)^{1-\alpha}\pi_{n-2} \\ &\leq \frac{-1}{\alpha}\mathrm{Q}_2\mathrm{y}^{\alpha}(\widetilde{\sigma})\left(-k_m\frac{\mathrm{y}}{\pi_{n-2}}\right)^{1-\alpha}\pi_{n-2} \\ &= \frac{-k_m^{1-\alpha}}{\alpha}\mathrm{Q}_2\mathrm{y}^{\alpha}(\widetilde{\sigma})\left(\frac{\mathrm{y}}{\pi_{n-2}}\right)^{1-\alpha}\pi_{n-2}. \end{aligned}$$

Using $(\mathrm{Y}_{4,1})$ in Lemma 8, we observe that $\mathrm{y}(\ell)/\pi_{n-2}(\ell)$ is increasing, then

$$\frac{\mathrm{y}(\widetilde{\sigma}(\ell))}{\pi_{n-2}(\widetilde{\sigma}(\ell))} \leq \frac{\mathrm{y}(\ell)}{\pi_{n-2}(\ell)},$$

and

$$\left(\frac{\mathrm{y}(\widetilde{\sigma})}{\pi_{n-2}(\widetilde{\sigma})}\right)^{1-\alpha} \leq \left(\frac{\mathrm{y}}{\pi_{n-2}}\right)^{1-\alpha}.$$

Therefore,

$$\begin{aligned} w' &\leq \frac{-k_m^{1-\alpha}}{\alpha}\mathrm{Q}_2\mathrm{y}^{\alpha}(\widetilde{\sigma})\left(\frac{\mathrm{y}(\widetilde{\sigma})}{\pi_{n-2}(\widetilde{\sigma})}\right)^{1-\alpha}\pi_{n-2} \\ &= \frac{-k_m^{1-\alpha}}{\alpha}\pi_{n-2}^{\alpha-1}(\widetilde{\sigma})\pi_{n-2}\mathrm{Q}_2\mathrm{y}(\widetilde{\sigma}), \end{aligned}$$

which, from Equation (30), gives

$$w' + \frac{1}{\alpha}\frac{k_m^{1-\alpha}}{1-k_m}\pi_{n-2}^{\alpha-1}(\widetilde{\sigma})\pi_{n-2}\mathrm{Q}_2 w(\widetilde{\sigma}) \leq 0. \tag{31}$$

Therefore, the positive solution w to the differential inequality can be deduced from Equation (31). Notably, according to the findings in Theorem 2.1.1 in [8], the condition expressed in Equation (29) ensures that Equation (31). This logical contradiction serves as conclusive evidence for proving the Theorem. □

Theorem 4. *Under the assumption that Equations (18) and (19) are satisfied, we consider a positive integer m, such that*

$$\liminf_{\ell\to\infty}\int_{\sigma(\ell)}^{\ell}\pi_{n-2}(s)\pi_{n-2}^{\alpha-1}(\widetilde{\sigma}(s))\widehat{\mathrm{Q}}_2(s)\mathrm{d}s > \frac{\alpha k_m^{\alpha-1}(1-k_m)}{\mathrm{e}}. \tag{32}$$

If the above inequality holds, then $\Omega_3 = \varnothing$.

Proof. To demonstrate this, we utilize the relationship

$$\chi > \widehat{\mathrm{p}}_2(\ell;\kappa)\mathrm{y},$$

with respect to Equation (1), employing the identical proof technique used in the previous theorem. □

3.3. Category Ω_1

We know that

$$\mathrm{y} = \chi + \mathrm{p}\chi(\mathfrak{y}),$$

and

$$\chi = \mathrm{y} - \mathrm{p}\chi(\mathfrak{y}) \geq \mathrm{y} - \mathrm{p}\mathrm{y}(\mathfrak{y}).$$

Since $\mathrm{y}' > 0$, then

$$\chi \geq (1 - \mathrm{p})\mathrm{y}.$$

Lemma 13. *If*

$$\liminf_{\ell \to \infty} \int_{\sigma(\ell)}^{\ell} \frac{\left(\sigma^{n-1}(s)\right)^{\alpha}}{r(\sigma(s))} \mathrm{Q}_0(s)\mathrm{d}s > \frac{((n-1)!)^{\alpha}}{\mathrm{e}}, \tag{33}$$

then $\Omega_1 = \varnothing$.

Proof. If we consider the contrary scenario where $\chi \in \Omega_1$, it becomes clear from the information provided by (C_1), that

$$\lim_{\ell \to \infty} \mathrm{y}(\ell) \neq 0.$$

Therefore, it can be deduced from Lemma 3 that, for any $\epsilon \in (0, 1)$,

$$\mathrm{y}(\sigma) \geq \frac{\epsilon}{(n-1)!} \frac{\sigma^{n-1}}{r(\sigma)} \left(r(\sigma) \mathrm{y}^{(n-1)}(\sigma) \right) \tag{34}$$

eventually. Using Equation (34) in Equation (1), we see that

$$\begin{aligned} \left(r \left(\chi^{(n-1)} \right)^{\alpha} \right)' &= -\sum_{i=1}^{v} q_i \chi^{\alpha}(\sigma_i) \\ &\leq -\sum_{i=1}^{v} q_i (1 - \mathrm{p}(\sigma_i))^{\alpha} \mathrm{y}^{\alpha}(\sigma_i) \\ &\leq -\mathrm{y}^{\alpha}(\sigma) \mathrm{Q}_0 \\ &\leq -\mathrm{Q}_0 \left(\frac{\epsilon}{(n-1)!} \frac{\sigma^{n-1}}{r(\sigma)} \right)^{\alpha} \left(r(\sigma) \left(\mathrm{y}^{(n-1)}(\sigma) \right)^{\alpha} \right). \end{aligned}$$

Consider the function $\theta = r \left(\mathrm{y}^{(n-1)} \right)^{\alpha}$. By observing the last inequality, it becomes clear that $\theta(\ell)$ serves as a positive solution to the delay differential inequality, expressed as:

$$\theta' + \frac{\epsilon^{\alpha}}{((n-1)!)^{\alpha}} \mathrm{Q}_0 \frac{\left(\sigma^{n-1}\right)^{\alpha}}{r(\sigma)} \theta(\sigma) \leq 0. \tag{35}$$

Therefore, the positive solution θ to the differential inequality can be deduced from Equation (35). Notably, according to the findings in Theorem 2.1.1 in [8], the condition expressed in Equation (33) ensures that Equation (35) this logical contradiction serves as conclusive evidence for proving the Theorem. □

4. Criteria for Oscillation

This section extends the groundwork laid in the preceding sections to introduce fresh criteria for confirming the oscillatory nature of all solutions within Equation (1). To be more precise, we have pinpointed particular conditions that conclusively exclude the existence of positive solutions in all three scenarios, denoted as (C_1), (C_2), and (C_3). By amalgamating these conditions, as expounded in the subsequent theorems, we can establish robust criteria for ascertaining oscillation.

Theorem 5. *Assume that Equations (12), (29) and (33) hold. Then Equation (1) is oscillatory.*

Theorem 6. *Assume that Equations (13), (29) and (33) hold. Then Equation (1) is oscillatory.*

Theorem 7. *Assume that Equations (12), (32) and (33) hold. Then Equation (1) is oscillatory.*

Theorem 8. *Assume that Equations (13), (32) and (33) hold. Then Equation (1) is oscillatory.*

Example 1. *Consider the NDE*

$$\left(\ell^{4\alpha}\left((\chi(\ell)+\mathrm{p}_0\chi(\mathfrak{y}_0\ell))'''\right)^{\alpha}\right)'+\sum_{i=1}^{v}q_0\ell^{\alpha-1}\chi^{\alpha}(\sigma_i\ell)=0,\ \ell\geq 1, \tag{36}$$

where $0\leq \mathrm{p}_0<1$, $\mathfrak{y}_0$, $\sigma_i\in(0,1)$, $i=1,2,\ldots,v$, *and* $q_0>0$. *By comparing Equation (1) and Equation (36), we see that* $n=4$, $r(\ell)=\ell^{4\alpha}$, $q_i(\ell)=q_0\ell^{\alpha-1}$, $\mathrm{p}(\ell)=\mathrm{p}_0$, $\mathfrak{y}(\ell)=\mathfrak{y}_0\ell$, $\sigma_i(\ell)=\sigma_i\ell$. *It is easy to find that*

$$\pi_0(\ell)=\frac{1}{3\ell^3},\ \pi_1(\ell)=\frac{1}{6\ell^2},\ \pi_2(\ell)=\frac{1}{6\ell},$$

$$\sigma(\ell):=\min\{\sigma_i\ell,\ i=1,2,\ldots,v\}=\sigma_0\ell,$$

$$\widetilde{\sigma}(\ell):=\max\{\sigma_i\ell,\ i=1,2,\ldots,v\}=\widetilde{\sigma}_0\ell,$$

$$\lambda=\frac{\pi_0(\sigma(\ell))}{\pi_0(\ell)}=\frac{1}{\sigma_0^3},\ \lambda_1=\frac{\pi_2(\widetilde{\sigma}(\ell))}{\pi_2(\ell)}=\frac{1}{\widetilde{\sigma}_0},$$

$$\mathrm{p}_1(\ell;\kappa)=(1-\mathrm{p}_0)\sum_{r=0}^{\kappa}\mathrm{p}_0^{2r}\mathfrak{y}^{4r/\epsilon},$$

$$\mathrm{p}_2(\ell;\kappa)=\left[\frac{1}{\mathrm{p}_0}-\frac{1}{\mathfrak{y}_0}\right]\sum_{r=0}^{m}\mathrm{p}^{2r+1},$$

$$\widehat{\mathrm{p}}_2(\ell;\kappa)=\left[\frac{1}{\mathrm{p}_0}-\frac{1}{\mathfrak{y}_0}\right]\sum_{r=0}^{m}\mathrm{p}_0^{2r+1}\frac{1}{\mathfrak{y}_0^{2rk_m}},$$

$$\delta=\frac{vq_0(1-\mathrm{p}_0)}{2\alpha 3^{\alpha+1}}\sigma_0^{2\alpha}\sum_{r=0}^{\kappa}\mathrm{p}_0^{2r}\mathfrak{y}^{4r/\epsilon},$$

$$\beta_0=\epsilon\left(\frac{vq_0(1-\mathrm{p}_0)}{2\alpha 3^{\alpha+1}}\sigma_0^{2\alpha}\sum_{r=0}^{\kappa}\mathrm{p}_0^{2r}\mathfrak{y}^{4r/\epsilon}\right)^{1/\alpha},$$

$$\beta_j:=\beta_0\frac{1}{(1-\beta_{j-1})^{1/\alpha}}\left(\frac{1}{\sigma_0^3}\right)^{\beta_{j-1}},\ j=1,2,\ldots,m,$$

$$k_0=\left(\frac{1}{\alpha}\frac{1}{6^{\alpha}}vq_0\left[\frac{1}{\mathrm{p}_0}-\frac{1}{\mathfrak{y}_0}\right]\sum_{r=0}^{\kappa}\mathrm{p}_0^{2r+1}\right)^{1/\alpha},$$

$$k_j=k_0\frac{1}{(1-k_{j-1})^{1/\alpha}}\left(\frac{1}{\widetilde{\sigma}_0}\right)^{k_{j-1}},\ j=1,2,\ldots,m.$$

Condition Equation (12) leads to

$$q_0>\frac{2^{-\alpha+1}3^{\alpha+1}}{\epsilon^{\alpha}\sigma_0^{2\alpha}(1-\mathrm{p}_0)\sum_{r=0}^{\kappa}\mathrm{p}_0^{2r}\mathfrak{y}^{4r/\epsilon}}, \tag{37}$$

while condition Equation (13) results in

$$q_0>\frac{\alpha 6^{\alpha}\sigma_0^{\alpha-3}\beta_m^{\alpha-1}(1-\beta_m)}{v(1-\mathrm{p}_0)^{\alpha}\left(\sum_{r=0}^{\kappa}\mathrm{p}_0^{2r}\mathfrak{y}^{4r/\epsilon}\right)^{\alpha}\ln\frac{1}{\sigma_0}}\frac{1}{\mathrm{e}}. \tag{38}$$

Condition Equation (29) gives

$$q_0 > \frac{\alpha 6^{\alpha} \widetilde{\sigma}_0^{\alpha-1} k_m^{\alpha-1}(1-k_m)}{v\left[\frac{1}{\mathrm{p}_0}-\frac{1}{\mathfrak{y}_0}\right]^{\alpha}\left(\sum_{r=0}^{m}\mathrm{p}^{2r+1}\right)^{\alpha}\ln\frac{1}{\sigma_0}}\frac{1}{\mathrm{e}}, \tag{39}$$

while condition Equation (32) yields

$$q_0 > \frac{\alpha 6^{\alpha} \widetilde{\sigma}_0^{\alpha-1} k_m^{\alpha-1}(1-k_m)}{v\left[\frac{1}{\mathrm{p}_0}-\frac{1}{\mathfrak{y}_0}\right]^{\alpha}\left(\sum_{r=0}^{\kappa}\mathrm{p}_0^{2r+1}\frac{1}{\mathfrak{y}_0^{2rk_m}}\right)^{\alpha}\ln\frac{1}{\sigma_0}}\frac{1}{\mathrm{e}}. \tag{40}$$

Lastly, condition Equation (33) produces

$$q_0 > \frac{\sigma_0^{\alpha} 6^{\alpha}}{v(1-\mathrm{p}_0)^{\alpha}\ln\frac{1}{\sigma_0}}\frac{1}{\mathrm{e}}. \tag{41}$$

To determine the oscillation of Equation (36), we can apply various theorems.

Theorems 5–8 provide conditions for the oscillatory behavior of Equation (36). Theorem 5 asserts that the satisfaction of Equations (37), (39) and (41) leads to oscillations in Equation (36). Similarly, Theorem 6 indicates that the fulfillment of Equations (38), (39) and (41) results in oscillations in Equation (36). In the same vein, Theorem 7 establishes that if Equations (37), (40) and (41) are met, then Equation (36) exhibits oscillatory behavior. Lastly, Theorem 8 demonstrates that oscillations in Equation (36) occur when Equations (38), (40) and (41) are satisfied.

5. Conclusions

In this study, we delved into the investigation of the oscillatory behavior and monotonic properties of even-order quasilinear neutral differential equations. Our main focus was on a specific type of such equations. Through our research, we were able to establish improved relationships that connect the solution and its corresponding function for two out of the three categories of positive solutions in the equation under study. By leveraging these newly derived relationships, we were able to develop criteria to ascertain that categories Ω_2 and Ω_3 contained no positive solutions. A significant contribution of this work was the introduction of novel criteria to assess the oscillation of Equation (1). These criteria provide a valuable tool for analyzing the oscillatory nature of the equation. Looking ahead, it would be intriguing to extend our findings to explore the behavior of non-linear odd-order neutral DEs, opening up exciting possibilities for future research directions.

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