symmetry

MDPI

Article

Control Problem Related to a 2D Parabolic–Elliptic Chemo-Repulsion System with Nonlinear Production

Exequiel Mallea-Zepeda [1,*] and Luis Medina [2]

[1] Departamento de Matemáticas, Universidad Católica del Norte, Av. Angamos 0610, Antofagasta 1240000, Chile
[2] Departamento de Matemáticas, Facultad de Ciencias Básicas, Universidad de Antofagasta, Av. Angamos 601, Antofagasta 1240000, Chile; luis.medina@uantof.cl
* Correspondence: emallea@ucn.cl

Abstract: In this work, we analyze a bilinear optimal control problem related to a 2D parabolic–elliptic chemo-repulsion system with a nonlinear chemical signal production term. We prove the existence of global optimal solutions with bilinear control, and applying a generic result on the existence of Lagrange multipliers in Banach spaces, we obtain first-order necessary optimality conditions and derive an optimality system for a local optimal solution.

Keywords: chemo-repulsion system; optimal control problem; optimality conditions

check for updates

Citation: Mallea-Zepeda, E.; Medina, L. Control Problem Related to a 2D Parabolic–Elliptic Chemo-Repulsion System with Nonlinear Production. *Symmetry* **2023**, *15*, 1949. https://doi.org/10.3390/sym15101949

Academic Editors: Calin Ciufudean and Junesang Choi

Received: 21 September 2023
Revised: 15 October 2023
Accepted: 19 October 2023
Published: 21 October 2023

1. Introduction

In recent years, it has become extremely important to analyze the interaction of living organisms with the environment in which they reside. Frequently, the form of interaction involves the movement of living organisms in response to an external stimulus; the movement generated in response to such stimulus is called taxis. The process that leads to taxis is divided into three steps [1]: first the cell detects the extracellular signal through certain receptors found on its surface; then, it processes the signal; and finally, it modifies its mobile behavior. There exist different types of taxis, which depend on the nature of the stimulus (see, for instance, [2]), one of them is *chemotaxis*.

The phenomenon of chemotaxis is understood as the alteration of the mobile behavior of living organisms generated by the presence of certain chemical substances found in the environment where they reside. In 1970, professors Keller and Segel proposed a mathematical model describing the chemotactic aggregation of cellular molds that preferentially move toward regions containing high concentrations of a chemical secreted by the amoebas themselves (see [3]). This phenomenon is called chemo-attraction. In contrast, the opposite phenomenon is called chemo-repulsion, if the regions that have a high chemical concentration generate a repulsive effect on the organisms. The most classic mathematical model in the framework of chemotactic movements is the Keller–Segel system [3,4], which is given by the following coupled nonlinear system of partial differential equations:

$$\left\{\begin{array}{rcl} \partial_t u &=& \alpha_u \Delta u - \nabla(\chi u \nabla v) \quad \text{in} \quad Q, \\ \partial_t v &=& \alpha_v \Delta v - \beta v + h(u) \quad \text{in} \quad Q, \\ u(0,x) &=& u_0(x),\ v(0,x) = v_0(x) \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} &=& \dfrac{\partial v}{\partial \mathbf{n}} \quad \text{on} \quad (0,T)\times\partial\Omega, \end{array}\right. \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n = 2,3$, is a bounded domain with smooth boundary $\partial\Omega$, $(0,T)$ is a time interval with $0 < T < \infty$, $Q := (0,T)\times\Omega$ is the time–space region, and the vector $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$. The unknown functions are the cell density $u := u(t,x) \geq 0$ and a chemical concentration $v := v(t,x) \geq 0$. The cell flux and chemical are given, respectively, by $\chi u \nabla v - \alpha_u \nabla u$ and $\alpha_v \nabla v$, where $\alpha_u, \alpha_v > 0$ and $\chi \neq 0$ are real

constants. Therefore, the cells perform a biased random walk in the direction of the chemical gradient, and the chemical diffuses (it is produced by the cells, and it degrades) [5]. The term $\chi u \nabla v$ models the transport of cells. If the parameter $\chi > 0$, the transport is towards regions with high concentrations of chemical substance (chemo-attraction), and if $\chi < 0$, the transport is towards regions with lower concentrations of chemical (chemo-repulsion). The term $-\beta v + h(u)$ models the consumption–production rate of the chemical, where β is a real parameter that measures the self-degradation of the chemical, and the function $h(u)$ is the cell production term; this function must be non-negative when $u \geq 0$.

The main focus of this paper is to carry out a theoretical study of a bilinear optimal control problem related to the parabolic–elliptic system associated to problem (1), considering nonlinear chemical signal production and a proliferation/degradation coefficient acting on a control subdomain ω. Specifically, we consider a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega \in C^{2,1}$ and a time interval $(0, T)$, with $0 < T < \infty$. Then, we analyze an optimal control problem related to the following parabolic–elliptic system of partial differential equations in the time–space region $Q := (0, T) \times \Omega$:

$$\begin{cases} \partial_t u - \Delta u &= \nabla \cdot (u \nabla v), \\ -\Delta v + v &= u^p + f v \, 1_\omega, \end{cases} \tag{2}$$

where $\omega \subset \Omega$ is the control subdomain, f denotes a bilinear control that acts on the subdomain ω, and 1_ω is the characteristic function of ω. In general, bilinear control problems are a subclass of nonlinear control systems, in which the nonlinear term is constructed by multiplying the state and control variables. The control lies in a nonempty, closed, and convex set $\mathcal{F}$. Notice that when $f \geq 0$ we inject chemical substance in the subdomain ω, and when $f \leq 0$ we extract chemical substance in ω; thus, we can interpret the control function f as a proliferation/degradation coefficient acting on the subregion ω, which from the biological point of view makes sense. The term u^p, for $p > 1$, is the nonlinear chemical signal production term.

We complete the system (2) with the initial condition for the cell density

$$u(0, x) = u_0(x) \geq 0 \quad \text{in} \quad \Omega, \tag{3}$$

and the Neumann boundary conditions

$$\frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega. \tag{4}$$

Studies on the existence of solutions related to system (2)–(4) can be consulted in [6–15]. In particular, the case when $f \equiv 0$ and $p = 1$ has been analyzed by Mock in [6,7], in which the author proved the existence and uniqueness of global-in-time classical solutions and that the respective solutions are uniformly bounded and converge at an exponential rate to steady-state. The parabolic–parabolic system related to problem (2)–(4) has been studied in [8–14]. In [8,9] considering linear production and $f \equiv 0$ were studied, where Ciéslak et al. [8] proved the existence of a unique smooth classical solution in two-dimensional domains as well as the existence of weak solutions in 2D and 3D domains. In [9], the author delimits his analysis to a n-dimensional convex domain ($n \geq 3$) and changes the chemotactic term $\nabla \cdot (u \nabla v)$ by $\nabla \cdot (g(u) \nabla v)$, where $g(u)$ is an adequate smooth function. With this modification, the author proved the existence and uniqueness of global-in-time classical solutions and that the pair solution (u, v) converges to $\frac{1}{|\Omega|}(\int_\Omega u_0, \int_\Omega u_0)$, as t goes to ∞. Moreover, in [10,11], the authors proved for a quadratic production term ($p = 2$), the existence of weak solutions in 3D domains and global-in-time strong solutions assuming a regularity criterion in spaces of dimension 1 and 2; furthermore, they analyzed some numerical schemes to approximate the weak solutions. In [12], for $p = 1$ and $f \not\equiv 0$, the authors proved the existence and uniqueness of strong solutions in 2D domains, and deduced that the solution (u, v) does not blow-up at a finite time. The same authors in [13] extend the results obtained in [12] to 3D domains, and presented results on the existence

of weak solutions and established a regularity criterion to obtain global-in-time strong solutions. In [14], the author consider, in a two-dimensional domain, the nonlinear case for $p \in (1,2]$ and $f \not\equiv 0$ and proved the existence and uniqueness of strong solutions. The existence and uniqueness of strong solutions for problem (2)–(4), considering $p \in (1,2)$ and $f \in L^q(\omega)$ for $2 < q < \infty$, has been proved by Ancoma-Huarachi et al. [15]. For the stationary case and linear production term, we can refer to a recent study developed by Lorca et al. [16]. The case of nonlinear production is interesting to analyze, because when saturation effects at large (or short) densities are taken into account, the signal production through the cell no longer shows dependence on the population density in a linear manner (see, for instance, [17]). It is important to mention that there is a close relationship between chemotactic phenomena and the dynamics of symmetric pattern formation. Indeed, motile cells of Escherichia coli aggregate to form stable patterns of remarkable regularity when grown at a single point on certain substrates and central to this self-organization is chemotaxis (see [18] for more details).

Optimal control problems related with chemotaxis systems can be consulted in [12–14,16,19–24]. All these works proved the existence of at least one global optimal solution and derived an optimality system, in particular obtaining first-order necessary optimality conditions. In [21], the authors studied an optimal control problem with state equations driven by a chemo-attractive Navier–Stokes evolution system in 3D domains. They stated first-order necessary optimality conditions, previously proving that the state is differentiable with respect to the control variable. Rodríguez-Bellido et al. [22] analyzed a distributed optimal control problem related to a stationary chemotaxis model coupled with the Navier–Stokes equations. Also, they derived an optimality system through a family of penalized problems, because the application control-to-state is multivalued. In [24], the authors considered a 2D chemotaxis model with a logistic source and proved the existence of weak solutions for the dynamical equation, the existence of global optimal solutions, and they derive an optimality system using a generic result on the existence of Lagrange multipliers. The works [12–14] are dedicated to study control problems related to chemo-repulsion models and considered the parabolic–parabolic system associated with problem (2)–(4). In [12], the authors explored a bilinear optimal control problem in 2D domains, proved the existence of global optimal solutions, and derived an optimality system. The same authors in [13] studied the 3D version of [12]. Guillén-González et al. [14] extended the results of [13] for the superlinear production term, that is, for $p \in (1,2]$. Recently, in [16] a bilinear optimal control problem related to a stationary version system of (2)–(4) has been studied for n-dimensional domains, with $n = 1,2,3$. In this work the authors proved the existence of global optimal solutions and derived first-order necessary optimality conditions for local optimal solutions. As far as we know, optimal control problems related with system (2)–(4) have not been considered in the literature.

This paper is organized as follows: In Section 2 we fix some notations, introduce the function spaces that will be used through the work, give the concept of strong solutions of problem (2)–(4), present a result concerning to the existence and uniqueness of global-in-time strong solutions of (2)–(4), and establish two regularity (parabolic and elliptic) results for the Neumann heat problem that will be used to achieve our results. Finally, in Section 3 we analyze the bilinear optimal control problem and obtain several important results, which include the existence of global optimal solutions, the derivation of the an optimality system for a local optimal solution, via a result on existence of Lagrange multipliers in Banach spaces, and we obtain some extra regularity properties of the Lagrange multipliers.

2. Preliminaries

We dedicate this section to establish some notations, definitions, and preliminary results that will be used throughout this work. We will use the classical Lebesgue spaces $L^s := L^s(\Omega)$, for $1 \leq s \leq \infty$, with norm denoted by $\|\cdot\|_{L^s}$. In particular, for $s = 2$, the L^2-norm and the respective L^2-inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Moreover, we use the Sobolev spaces $W^{m,s} := W^{m,s}(\Omega) = \{u \in L^s : \|\partial^\alpha u\|_{L^s}, \forall |\alpha| \leq m\}$, with norm

denoted by $\|\cdot\|_{W^{m,s}}$. When $s = 2$, we denote it by $H^m := W^{m,2}$ and the respective norm by $\|\cdot\|_{H^m}$. Also, we will use the space $W^{m,s}_{\mathbf{n}} := \{u \in W^{m,s} : \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\}$ for $m > 1 + \frac{1}{s}$, with norm denoted by $\|\cdot\|_{W^{m,s}_{\mathbf{n}}}$. Moreover, if X is a generic Banach space, we will denote it by $L^s(X)$ to space of functions $u : [0,T] \to X$ that are integrable in the Bochner sense, and its norm will be denoted by $\|\cdot\|_{L^s(X)}$. For simplicity, we will denote $L^s(Q) := L^s(L^s)$ and its norm by $\|\cdot\|_{L^s(Q)}$. Also, $C(X)$ denotes the space of continuous functions u from $[0,T]$ onto X and its respective norm by $\|\cdot\|_{C(X)}$. In contrast, X' denotes the topological dual space of a Banach space X, and the respective duality for a pair X and X' by $\langle\cdot,\cdot\rangle_{X'}$ or simply by $\langle\cdot,\cdot\rangle$ unless this leads to ambiguity. Finally, as usual, $C, K, C_1, K_1, \ldots$, denote positive constants independent of state variables u and v, but its value may change from line to line.

We are interested in studying a bilinear optimal control problem related with the strong solutions of problem (2)–(4). The following definition establishes the concept of strong solutions of system (2)–(4); more details can be consulted in [15].

Definition 1 (Strong solutions)**.** *Let $f \in L^q(\omega)$, for $2 < q < \infty$, $u_0 \in H^1$ with $u_0 \geq 0$ a.e. in Ω. We say that a pair (u,v) is a strong solution of system (2)–(4) in $(0,T)$, if $u \geq 0$, $v \geq 0$ a.e. in Q,*

$$u \in S_u \quad := \quad \{u \in L^\infty(H^1) \cap L^2(H^2_{\mathbf{n}}) : \partial_t u \in L^2(Q)\}, \tag{5}$$

$$v \in S_v \quad := \quad L^q(W^{2,q}_{\mathbf{n}}), \tag{6}$$

the pair (u,v) satisfies point-wisely a.e. $(t,x) \in Q$ the system

$$\begin{cases} \partial_t u - \Delta u &= \nabla \cdot (u \nabla v), \\ -\Delta v + v &= u^p + f v \, 1_\omega, \end{cases}$$

and the initial and boundary conditions (3) and (4) are satisfied, respectively.

Some properties that can be extracted directly from system (2)–(4) and that are key to obtaining the existence of strong solutions are the following:

- System (2)–(4) is conservative in u. Integrating $(2)_1$ in the spatial variable, we have

$$\frac{d}{dt}\left(\int_\Omega u\right) = 0; \text{ hence, } \int_\Omega u(t) = \int_\omega u_0 := m_0 \ \ \forall t > 0. \tag{7}$$

- Integrating $(2)_2$ in Ω, we have

$$\int_\Omega v = \int_\Omega u^p + \int_\omega f v. \tag{8}$$

Now, we present a result related to the existence and uniqueness of strong solutions to problem (2)–(4). This result is valid only when $p \in (1,2)$. For this reason, we restrict our analysis to $1 < p < 2$ (see [15] for more details).

Theorem 1 (Strong solutions ([15], Theorem 2.7))**.** *Assume that $p \in (1,2)$. Let $u_0 \in H^1$ with $u_0 \geq 0$ in Ω and $f \in L^q(\omega)$ for $2 < q < \infty$. Suppose that there exists a constant $\beta > 0$ such that $\|f\|_{L^q(\omega)}$ is small enough satisfying*

$$\|f\|_{L^q(\omega)} < \beta \leq \widehat{K}, \tag{9}$$

where $\widehat{K} := \widehat{K}(|\Omega|, q, p) > 0$ is a constant. Then, there exists a unique pair of functions (u,v) that is a strong solution of system (1) and (2) in the sense of Definition 1. Moreover, there exists a positive constant $K := K(m_0, T, \|f\|_{L^q(\omega)}, \widehat{K})$ such that

$$\|u\|_{S_u} + \|v\|_{S_v} \leq K. \tag{10}$$

Remark 1. *The constant $\widehat{K}$, given in Theorem 1, is mainly related to the Sobolev embeddings $H^1 \hookrightarrow L^s$ for $1 \leq s < \infty$, and $W^{2,q} \hookrightarrow L^\infty$ for $2 < q < \infty$ and the continuous injection $L^q \hookrightarrow L^2$.*

Throughout this paper, we frequently use the following equivalent norms in the spaces H^1 and H^2 (see, for instance, [25]):

$$\|u\|_{H^1}^2 \simeq \left(\|\nabla u\|^2 + \left(\int_\Omega u \right)^2 \right) \ \forall u \in H^1, \tag{11}$$

$$\|u\|_{H^2}^2 \simeq \left(\|\Delta u\|^2 + \left(\int_\Omega u \right)^2 \right) \ \forall u \in H^2_{\mathbf{n}}, \tag{12}$$

and the classical 2D interpolation inequality

$$\|u\|_{L^4} \leq C\|u\|^{1/2}\|u\|_{H^1}^{1/2} \ \forall u \in H^1. \tag{13}$$

Moreover, we will apply the following results concerning parabolic and elliptic regularity for the Neumann heat problem:

Theorem 2 (Parabolic-regularity ([26], Theorem 10.22))**.** *Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, $u_0 \in \widehat{W}^{2-2/s,s}$ and $g \in L^s(Q)$, for $s \in (1, \infty)$ with $s \neq 3$. Then, there exists a unique strong solution u of problem*

$$\begin{cases} \partial_t u - \Delta u &= g \ \text{in} \ Q, \\ u(0,x) &= u_0(x) \ \text{in} \ \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} &= 0 \ \text{on} \ (0,T) \times \partial\Omega, \end{cases}$$

such that

$$u \in L^\infty(\widehat{W}^{2-2/s,s}) \cap L^s(W^{2,s}), \partial_t u \in L^s(Q).$$

Moreover, there exists a constant $C := C(|\Omega|, T) > 0$ such that

$$\|u\|_{L^\infty(\widehat{W}^{2-2/s,s})} + \|u\|_{L^s(W^{2,s})} + \|\partial_t u\|_{L^s(Q)} \leq C\Big(\|u_0\|_{\widehat{W}^{2-2/s,s}} + \|g\|_{L^s(Q)} \Big).$$

Here, the space $\widehat{W}^{2-2/s,s} := W^{2-2/s,s}$ for $s < 3$ and $\widehat{W}^{2-2/s,s} := W^{2-2/s,s}_{\mathbf{n}}$ for $s > 3$.

Theorem 3 (Elliptic-regularity ([27], Theorem 2.4.2.7))**.** *Let $\Omega \in C^{1,1}$ be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, and $h \in L^s$ with $1 < s < \infty$. Then, the elliptic system*

$$\begin{cases} -\Delta u + u &= h \ \text{in} \ \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} &= 0 \ \text{on} \ \partial\Omega, \end{cases}$$

admits a unique solution u in the class $W^{2,s}$. Moreover, there exists a positive constant $C := C(|\Omega|)$ such that

$$\|u\|_{W^{2,s}} \leq C\|h\|_{L^s}.$$

3. The Bilinear Optimal Control Problem

In this section, we study a bilinear optimal control problem related with the strong solutions of the chemo-repulsion system (2)–(4). Firstly, we establish the statement of the bilinear control problem under analysis. Indeed, we assume that the controls set is $\mathcal{F}$, which is a nonempty, closed, and convex subset of $\mathcal{B}(\widehat{K})$, where $\mathcal{B}(\widehat{K}) \subset L^q(\omega)$, for $2 < q < \infty$, is the open ball

$$\mathcal{B}(\widehat{K}) := \{ f \in L^q(\omega) \ : \ \|f\|_{L^q(\omega)} < \beta \leq \widehat{K} \}, \tag{14}$$

where β and $\widehat{K}$ are the constants given in (9) (see Theorem 1 above) and $\omega \subset \Omega$ is the control domain. We consider the initial data $u_0 \in H^1$ with $u_0 \geq 0$ and the function $f \in \mathcal{F}$ that describes a bilinear control acting on the chemical Equation $(2)_2$.

Furthermore, we consider the Banach spaces

$$\mathbb{X} := S_u \times S_v \times L^q(\omega) \text{ and } \mathbb{Y} := L^2(Q) \times L^q(Q) \times H^1(\Omega),$$

the functional $J : \mathbb{X} \to \mathbb{R}$ defined by

$$J(u,v,f) := \frac{\alpha_u}{2}\int_0^T \|u-u_d\|^2 + \frac{\alpha_v}{2}\int_0^T \|v-v_d\|^2 + \frac{\alpha_f}{q}\|f\|^q_{L^q(\omega)} \tag{15}$$

and the operator $R := (R_1, R_2, R_3) : \mathbb{X} \to \mathbb{Y}$, where $R_i : \mathbb{X} \to \mathbb{Y}$, for $i = 1,2,3$, are defined at each point $s := (u,v,f) \in \mathbb{X}$ by

$$\begin{cases} R_1(s) &= \partial_t u - \Delta u - \nabla \cdot (u \nabla v), \\ R_2(s) &= -\Delta v + v - u^p - f v 1_\omega, \\ R_3(s) &= u(0) - u_0. \end{cases} \tag{16}$$

In the functional J, defined in (15), the pair (u_d, v_d) belongs to $\in L^2(Q) \times L^2(Q)$ and represents the desired states. The real numbers α_u, α_v and α_f are non-negative (nonzero simultaneously) and measure the cost of the states (u,v) and the control f, respectively. The functional J describes the deviation of the cell density u from the desired cell density u_d and the deviation of the chemical concentration v from the desired chemical v_d with the cost of the control measured in the L^q-norm.

Then, taking $S := S_u \times S_v \times \mathcal{F}$ we formulate the following bilinear optimal control problem:

$$\min_{s \in S} J(s) \text{ subject to } R(s) = \mathbf{0}. \tag{17}$$

Notice that $S \subset \mathbb{X}$ is a closed and convex set and that the functional J is weakly lower semi-continuous on S. The set of the admissible solutions of control problem (17) is given by

$$\mathcal{S}_{ad} := \{s = (u,v,f) \in S \,:\, R(s) = \mathbf{0}\},$$

which, by virtue of Theorem 1, is a nonempty set.

It is important to mention that in problem (17), the choice of bilinear type control is due to fact that the solution (u,v) of system (2)–(4) must be non-negative. If the control were of distributed type as an external sink, the positivity of v could not be warranted, because it would be conditioned by the sign of control f. In fact, f must be a non-negative function, but this is a very strong restriction, since the set of controls is reduced. Thus, the bilinear control makes sense.

We are interested in proving the existence of global optimal solutions to problem (17) and deriving the so-called first-order necessary optimality conditions for any local optimal solution of control problem (17). In the following definitions, we present the concepts of global optimal solutions and local optimal solutions of problem (17), respectively.

Definition 2 (Global optimal solutions)**.** *A triplet $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f})$ is called a global optimal solution of control problem* (17) *if*

$$J(\tilde{s}) = \min_{s \in \mathcal{S}_{ad}} J(s). \tag{18}$$

Definition 3 (Local optimal solutions)**.** *We say that a triplet $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ is a local optimal solution of problem* (17) *if there exists $\varepsilon > 0$ such that for any $s = (u,v,f) \in \mathcal{S}_{ad}$ satisfying*

$$\|\tilde{u} - u\|_{S_u} + \|\tilde{v} - v\|_{S_v} + \|\tilde{f} - f\|_{L^q(\omega)} \leq \varepsilon,$$

then $J(\tilde{s}) \leq J(s)$.

3.1. Existence of Optimal Solutions

In this subsection, we will prove the existence of at least one global optimal solution $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ for control problem (17). Specifically, we will prove the following result:

Theorem 4 (Existence of global optimal solutions)**.** *Consider the assumptions of Theorem 1. Then, the optimal control problem (17) has at least one global optimal solution $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$.*

Proof. Since $f \in \mathcal{B}(\widehat{K})$ (hence, in particular, $\|f\|_{L^q(\omega)} < \beta \leq \widehat{K}$), from Theorem 1 we deduce that the admissible set $\mathcal{S}_{ad}$ is nonempty. Moreover, considering that the functional J is bounded from below, we deduce that there exists a minimizing sequence $\{s_m\}_{m\geq 1} := \{(u_m, v_m, f_m)\}_{m\geq 1} \subset \mathcal{S}_{ad}$ such that

$$\lim_{m\to\infty} J(s_m) = \inf_{s\in\mathcal{S}_{ad}} J(s).$$

Now, from the definition of J and because the control set $\mathcal{F}$ is bounded in $L^q(\omega)$, we determine that the sequence

$$\{f_m\}_{m\geq 1} \text{ is bounded in } L^q(\omega). \tag{19}$$

On the other hand, by definition of the admissible set $\mathcal{S}_{ad}$, for each $m \in \mathbb{N}$, the triplet (u_m, v_m, f_m) satisfies system (2)–(4). Thus, from estimate (9) we conclude that there exists a positive constant C, independent of m, such that

$$\|u_m\|_{S_u} + \|v_m\|_{S_v} \leq C. \tag{20}$$

Therefore, because of (19)–(20) and the fact that the control set $\mathcal{F}$ is a closed and convex subset of $L^q(\omega)$, by Mazur's lemma (see [28]), which is weakly closed in $L^q(\omega)$, we deduce that there exists a limit element $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in S_u \times S_v \times \mathcal{F}$ and a subsequence of $\{s_m\}_{m\geq 1}$, which, for simplicity, is still denoted by $\{s_m\}_{m\geq 1}$, such that the following convergences hold, as $m \to \infty$:

$$\begin{cases} u_m & \to & \tilde{u} \text{ weakly in } L^2(H^2_{\mathbf{n}}) \text{ and weakly}^* \text{ in } L^\infty(H^1), \\ v_m & \to & \tilde{v} \text{ weakly in } L^q(W^{2,q}_{\mathbf{n}}), \\ \partial_t u_m & \to & \partial_t \tilde{u} \text{ weakly in } L^2(Q), \\ f_m & \to & \tilde{f} \text{ weakly in } L^q(\omega) \text{ with } \tilde{f} \in \mathcal{F}. \end{cases} \tag{21}$$

In particular, following the arguments given in [14], we determine that u_m converges strongly to $\tilde{u}$ in $L^4(Q)$, which implies that

$$u_m^p \to \tilde{u}^p \text{ weakly in } L^2(Q). \tag{22}$$

Furthermore, from $(21)_1$, $(21)_3$, the Aubin–Lions lemma (see [29], Theorem 5.1), and ([30], Corollary 4) we have

$$u_m \to \tilde{u} \text{ strongly in } C(L^2) \cap L^2(H^1). \tag{23}$$

Therefore, considering the convergences (21)–(23) and following a standard argument (see, for instance, [14]), we can pass to the limit in system (2)–(4) writing by (u_m, v_m, f_m), as m goes to ∞; and thus, we deduce that $(\tilde{u}, \tilde{v}, \tilde{f})$ is a solution of (2)–(4). Therefore, the limit element $(\tilde{u}, \tilde{v}, \tilde{f})$ belongs to $\mathcal{S}_{ad}$ and

$$\lim_{m\to\infty} J(u_m, v_m, f_m) = \inf_{(u,v,f)\in\mathcal{S}_{ad}} J(u, v, f) \leq J(\tilde{u}, \tilde{v}, \tilde{f}). \tag{24}$$

Also, considering that the cost functional J is weakly lower semi-continuous on $\mathcal{S}_{ad}$, we have

$$J(\tilde{u}, \tilde{v}, \tilde{f}) \leq \liminf_{m\to\infty} J(u_m, v_m, f_m);$$

which together with (24) implies (18). Therefore, the triplet $(\tilde{u}, \tilde{v}, \tilde{f})$ is a global optimal solution of problem (17). □

3.2. Optimality System

In this subsection we will obtain first-order necessary optimality conditions and derive an optimality system for a local optimal solution $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f})$ of control problem (17), using a generic result on the existence of Lagrange multipliers in Banach spaces. This result, concerning on the existence of Lagrange multipliers, was established by Zowe and Kurcyusz in 1979 (see [31]).

The following results related to the differentiability of the functional J, and the operator R can be easily deduced.

Lemma 1. *The cost functional $J : \mathbb{X} \to \mathbb{R}$ is Fréchet-differentiable and the Fréchet derivative of J at the point $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathbb{X}$ in the direction $r = (U, V, F) \in \mathbb{X}$ is given by*

$$J'(\tilde{s})[r] = \alpha_u \int_0^T \int_\Omega (\tilde{u} - u_d)U + \alpha_v \int_0^T \int_\Omega (\tilde{v} - v_d)V + \alpha_f \int_\omega \operatorname{sgn}(\tilde{f})|\tilde{f}|^{q-1}F. \tag{25}$$

Lemma 2. *The operator $R : \mathbb{X} \to \mathbb{Y}$, defined in (16), is continuously Fréchet-differentiable and its Fréchet derivative at the point $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathbb{X}$, in the direction $r = (U, V, F) \in \mathbb{X}$, is the linear and continuous operator $R'(\tilde{s})[r] := (R_1'(\tilde{s})[r], R_2'(\tilde{s})[r], R_3'(\tilde{s})[r])$ defined by*

$$\begin{cases} R_1'(\tilde{s})[r] &= \partial_t U - \Delta U - \nabla \cdot (U \nabla \tilde{v}) - \nabla \cdot (\tilde{u} \nabla V), \\ R_2'(\tilde{s})[r] &= -\Delta V + V - p\tilde{u}^{p-1}U - \tilde{f}V\,1_\omega - F\tilde{v}, \\ R_3'(\tilde{s})[r] &= U(0). \end{cases} \tag{26}$$

By adapting the abstract sense given in [31], we have the following definition:

Definition 4. *An admissible element $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f})$ is a regular point for the optimal control problem (17) if for each triplet $(f_u, f_v, U_0) \in \mathbb{Y}$ there exists $r = (U, V, F) \in S_u \times S_v \times \mathcal{C}(\tilde{f})$ such that*

$$R'(\tilde{s})[r] = (f_u, f_v, U_0). \tag{27}$$

Here, $\mathcal{C}(\tilde{f}) := \{\delta(f - \tilde{f}) \,:\, \delta \geq 0,\, f \in \mathcal{F}\}$ is the conical hull of $\tilde{f}$ in $\mathcal{F}$.

Our aim is to prove the existence of Lagrange multipliers, which is guaranteed if a local optimal solution of control problem (17) is a regular point. The following result goes in that direction.

Proposition 1. *Suppose that the assumptions of Theorem 4 hold. Then, an element $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ is a regular point for the optimal control problem (17).*

Proof. Let $\tilde{s} \in \mathcal{S}_{ad}$ be a fixed element and $(f_u, f_v, U_0) \in \mathbb{Y}$. Notice that 0 belongs to the conical hull $\mathcal{C}(\tilde{f})$; hence, it is suffices to prove the existence of a pair $(U, V) \in S_u \times S_v$ such that

$$\begin{cases} \partial_t U - \Delta U - \nabla \cdot (U \nabla \tilde{v}) - \nabla \cdot (\tilde{u} \nabla V) &= f_u \text{ in } Q, \\ -\Delta V + V - p\tilde{u}^{p-1} - \tilde{f}V\,1_\omega &= f_v \text{ in } Q, \\ U(0) &= U_0 \text{ in } \Omega, \\ \dfrac{\partial U}{\partial \mathbf{n}} &= \dfrac{\partial V}{\partial \mathbf{n}} = 0 \text{ on } (0,T) \times \partial\Omega. \end{cases} \tag{28}$$

Now, we define the linear operator $S : (\overline{U}, \overline{V}) \in W_u \times W_v \mapsto (U, V) \in S_u \times S_v \hookrightarrow W_u \times W_v$, where (U, V) is the solution of the problem

$$\begin{cases} \partial_t U - \Delta U &= \nabla \cdot (\tilde{u} \nabla V) + \nabla(\overline{U} \nabla V) + f_u \text{ in } Q, \\ -\Delta V + V &= p\tilde{u}^{p-1}\overline{U} + \tilde{f}\overline{V}\,1_\omega + f_v \text{ in } Q, \end{cases} \tag{29}$$

endowed with the respective initial and boundary conditions $(28)_3$ and $(28)_4$. The *weak spaces* W_u and W_v are defined as follows:

$$W_u := C(L^2) \cap L^{\frac{2q}{q-2}}(H^1) \text{ and } W_v := L^q(L^\infty), \text{ with } 2 < q < \infty.$$

Following [15], we can prove easily that operator S is well-defined from $W_u \times W_v$ to $S_u \times S_v$ and completely continuous from $W_u \times W_v$ onto itself (see [15], Lemma 3.2). Also, from ([15], Lemma 3.1) we determine that the space $S_u \times S_v$ is compactly embedded in $W_u \times W_v$.

On the other hand, we consider the set

$$S_\alpha := \{(U,V) \in S_u \times S_v \,:\, (U,V) = \alpha S(U,V) \text{ for some } \alpha \in [0,1]\}.$$

The set S_α is bounded in $S_u \times S_v$, independently of the parameter $\alpha \in [0,1]$. Indeed, let $(U,V) \in S_\alpha$ and $\alpha \in (0,1]$ (the case $\alpha = 0$ is clear). Then, since operator S is well-defined from $W_u \times W_v$ to $S_u \times S_v$, we deduce that $(U,V) \in S_u \times S_v$ and satisfies point-wisely a.e. in Q the following problem:

$$\begin{cases} \partial_t U - \Delta U &= \nabla \cdot (\tilde{u}\nabla V) + \alpha \nabla \cdot (U \nabla \tilde{v}) + \alpha f_u, \\ -\Delta V + V &= \alpha p \tilde{u}^{p-1} U + \alpha \tilde{f} V 1_\omega + \alpha f_v, \end{cases} \tag{30}$$

endowed with corresponding initial and boundary conditions. Then, testing $(30)_1$ by U and considering that $\nabla \cdot (\tilde{u}\nabla V) = \tilde{u}\Delta V + \nabla \tilde{u} \cdot \nabla V$, we have

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \|\nabla U\|^2 \leq |(\tilde{u}\Delta V, U)| + |(\nabla \tilde{u} \cdot \nabla V, U)| + \alpha|(U\nabla \tilde{v}, \nabla U)| + \alpha|(f_u, U)|. \tag{31}$$

Now, we will bound the terms on the right-hand side of (31). Applying the Hölder and Young inequality and taking into account the 2D interpolation inequality (13) and that $\alpha \leq 1$, we can obtain

$$\begin{aligned} |(\tilde{u}\Delta V, U)| &\leq \|\tilde{u}\|_{L^4}\|\Delta V\|\|U\|_{L^4} \leq \varepsilon\|\Delta V\|^2 + C\|\tilde{u}\|_{L^4}^2\|U\|\|U\|_{H^1} \\ &\leq \varepsilon(\|\Delta V\|^2 + \|U\|_{H^1}^2) + C\|\tilde{u}\|_{L^4}^4\|U\|^2, && (32) \\ |(\nabla \tilde{u} \cdot \nabla V, U)| &\leq \|\nabla V\|\|\nabla \tilde{u}\|_{L^4}\|U\|_{L^4} \\ &\leq \varepsilon(\|\nabla V\|^2 + \|U\|_{H^1}^2) + C\|\nabla \tilde{u}\|_{L^4}^4\|U\|^2, && (33) \\ \alpha|(U\nabla \tilde{v}, \nabla U)| &\leq \|U\|\|\nabla \tilde{v}\|_{L^\infty}\|\nabla U\| \\ &\leq \varepsilon\|\nabla U\|^2 + C\|\nabla \tilde{v}\|_{L^\infty}^2\|U\|^2, && (34) \\ \alpha|(f_u, U)| &\leq \|f_u\|\|U\| \leq \|f_u\|^2 + \|U\|^2, && (35) \end{aligned}$$

where $\varepsilon > 0$ is arbitrary. Then, replacing (32)–(35) in (31) and adding to both sides $\|U\|^2$ in order to complete the H^1-norm, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|U\|^2 + \|U\|_{H^1}^2 &\leq \varepsilon\Big(\|\nabla U\|^2 + \|U\|_{H^1}^2 + \|\nabla V\|^2 + \|\Delta V\|^2\Big) \\ &\quad + C\Big(\|\tilde{u}\|_{L^4}^4 + \|\nabla \tilde{u}\|_{L^4}^4 + \|\nabla \tilde{v}\|_{L^\infty}^2 + 1\Big)\|U\|^2 + \|f_u\|^2. \end{aligned} \tag{36}$$

Also, testing Equation $(30)_2$ by $(V - \Delta V)$ we have

$$\begin{aligned} &\|V\|_{H^1}^2 + \|\Delta V\|^2 + \|\nabla V\|^2 \\ &\leq \alpha p|(\tilde{u}^{p-1}U, V)| + \alpha|(\tilde{f}V 1_\omega, V)| + \alpha|(f_v, V)| \\ &\quad + \alpha p|(\tilde{u}^{p-1}U, \Delta V)| + \alpha|(\tilde{f}V 1_\omega, \Delta V)| + \alpha|(f_v, \Delta V)|. \end{aligned} \tag{37}$$

Thus, working in a similar way as we did to obtain the estimate (36), we arrive at

$$\begin{aligned}\frac{1}{2}\Big(\|V\|_{H^1}^2+\|V\|_{H^2}^2\Big) &\leq \varepsilon\Big(\|U\|_{H^1}^2+\|V\|^2+\|\Delta V\|^2\Big)+\frac{1}{4}\Big(\|V\|^2+\|\Delta V\|^2\Big)\\ &\quad +C\|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)}\|U\|^2+2K_1^2\|\tilde{f}\|_{L^q(\omega)}^2\|V\|_{H^1}^2+C\|f_v\|_{L^q}^2. \quad (38)\end{aligned}$$

Here, the positive constant $K_1 := K_1(|\Omega|)$ is given by the Sobolev embedding $H^1 \hookrightarrow L^s$, for $s \in (2,\infty)$. This injection is necessary to estimate the terms $\alpha|(\tilde{f}V\,1_\omega, V)|$ and $\alpha|(\tilde{f}V\,1_\omega, \Delta V)|$. Indeed, from the Hölder inequality we have the estimate $\alpha|(\tilde{f}V\,1_\omega, V)| \leq \|\tilde{f}\|_{L^q}\|V\|_{L^s}\|V\|$, with $\frac{1}{q}+\frac{1}{s}=\frac{1}{2}$. Thus, using $H^1 \hookrightarrow L^s$, we determine that there exists a constant $K_1 > 0$ such that $\|V\|_{L^s} \leq K_1\|V\|_{H^1}$; consequently, we deduce that $\alpha|(\tilde{f}V\,1_\omega, V)| \leq K_1\|\tilde{f}\|_{L^q}\|V\|_{H^1}\|V\|$. Similarly, we can obtain that $\alpha|(\tilde{f}V\,1_\omega, \Delta V)| \leq K_1\|\tilde{f}\|_{L^q}\|V\|_{H^1}\|\Delta V\|$.

Now, adding inequalities (36) and (38), and choosing $\varepsilon > 0$ suitably, we can obtain the following estimate

$$\begin{aligned}\frac{d}{dt}\|U\|^2+C\|V\|_{H^1}^2+\frac{1}{2}\|V\|_{H^2}^2 &\leq C\Big(\|\tilde{u}\|_{L^4}^4+\|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)}+\|\nabla\tilde{u}\|_{L^4}^4+\|\nabla\tilde{v}\|_{L^\infty}^2+1\Big)\|U\|^2\\ &\quad +4K_1^2\|\tilde{f}\|_{L^q(\omega)}^2\|V\|_{H^1}^2+C\Big(\|f_u\|^2+\|f_v\|_{L^q}^2\Big)\\ &\leq C\Big(\|\tilde{u}\|_{L^4}^4+\|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)}+\|\nabla\tilde{u}\|_{L^4}^4+\|\nabla\tilde{v}\|_{L^\infty}^2+1\Big)\|U\|^2\\ &\quad +4K_1^2\|\tilde{f}\|_{L^q(\omega)}^2\|V\|_{H^2}^2+C\Big(\|f_u\|^2+\|f_v\|_{L^q}^2\Big),\end{aligned}$$

which implies

$$\begin{aligned}\frac{d}{dt}\|U\|^2 &+ C\|V\|_{H^1}^2+\left(\frac{1-8K_1^2\|\tilde{f}\|_{L^q(\omega)}^2}{2}\right)\|V\|_{H^2}^2\\ &\leq C\Big(\|\tilde{u}\|_{L^4}^4+\|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)}+\|\nabla\tilde{u}\|_{L^4}^4+\|\nabla\tilde{v}\|_{L^\infty}^2+1\Big)\|U\|^2\\ &\quad +C\Big(\|f_u\|^2+\|f_v\|_{L^q}^2\Big). \quad (39)\end{aligned}$$

From assumption (9) given in Theorem 1 we deduce that $\|\tilde{f}\|_{L^q(\omega)} < \frac{1}{2\sqrt{2}K_1}$; thus, we conclude that $1-8K_1^2\|\tilde{f}\|_{L^q(\omega)}^2 > 0$. Hence, from (39) and Gronwall's lemma we determine that U is bounded in $L^\infty(L^2)\cap L^2(H^1)$. Moreover, integrating (39) in $(0,T)$ we obtain that $V \in L^2(H^2)$.

It remains to prove that the pair (U,V) is bounded in $S_u \times S_v$. Indeed, notice that due to $\tilde{u} \in L^\infty(H^1)\cap L^2(H^2_{\mathbf{n}})$, from Sobolev embeddings we deduce that $\tilde{u} \in L^s(Q)$ for any $s \in [1,\infty)$. Also, since $p \in (1,2)$ we have $\tilde{u}^{p-1} \in L^s(Q)$, and since $U \in L^2(L^2)\cap L^2(H^1) \hookrightarrow L^4(Q)$, we determine that $\alpha p\tilde{u}^{p-1}U$ belongs to $L^q(Q)$ for any $q \in (2,\infty)$. Moreover, using that $V \in L^2(H^2_{\mathbf{n}})$ (in particular, from Sobolev embeddings, $V \in L^2(L^\infty)$), $(\tilde{f}, f_v) \in L^q(\omega)\times L^q(\Omega)$, we deduce that

$$\alpha p\tilde{u}^{p-1}(t,\cdot)U(t,\cdot)+\alpha\tilde{f}V(t,\cdot)\,1_\omega+\alpha f_v \in L^q \text{ for any } t \in (0,T).$$

Therefore, applying Theorem 3 (for $s=q>2$) we conclude that $V(t,\cdot) \in W^{2,q}_{\mathbf{n}}$ for any time $t \in (0,T)$, satisfies the elliptic problem

$$\begin{cases} -\Delta V+V &= \alpha p\tilde{u}^{p-1}U+\alpha\tilde{f}V\,1_\omega+\alpha f_v \text{ in } \Omega,\\ \dfrac{\partial V}{\partial \mathbf{n}} &= 0 \text{ on } \partial\Omega,\end{cases}$$

and the following estimate holds

$$\begin{aligned}\|V\|_{W^{2,q}} &\leq \alpha C\Big(p\|\tilde{u}^{p-1}U\|_{L^q}+\|\tilde{f}\|_{L^q(\omega)}\|V\|_{L^\infty}+\|f_v\|_{L^q}\Big)\\ &\leq C\Big(p\|\tilde{u}^{p-1}U\|_{L^q}+\|f_v\|_{L^q}\Big)+K_2\|\tilde{f}\|_{L^q(\omega)}\|V\|_{W^{2,q}_{\mathbf{n}}},\end{aligned}$$

where $K_2 := K_2(|\Omega|) > 0$ is a constant given by the Sobolev injection $W^{2,q}_{\mathbf{n}} \hookrightarrow L^\infty$. Thus, we have

$$\Big(1-K_2\|\tilde{f}\|_{L^q(\omega)}\Big)\|V\|_{W^{2,q}_{\mathbf{n}}} \leq C\Big(\|\tilde{u}^{p-1}U\|_{L^q}+\|f_v\|_{L^q}\Big). \tag{40}$$

Moreover, from Theorem 1 we deduce that $K_2\|\tilde{f}\|_{L^q(\omega)} < 1$; hence, from (40) we conclude that

$$\|V\|^q_{W^{2,q}_{\mathbf{n}}} \leq \frac{C}{\Big(1-K_2\|\tilde{f}\|_{L^q(\omega)}\Big)^q}\Big(\|\tilde{u}^{p-1}U\|_{L^q}+\|f_v\|_{L^q}\Big)^q. \tag{41}$$

Since $\tilde{u}^{p-1}U \in L^q(Q)$, after integrating (41) in time, we deduce $V \in L^q(W^{2,q}_{\mathbf{n}}) = S_v$.

On the other hand, testing $(28)_1$ by $-\Delta U$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla U\|+\|\Delta U\|^2 &\leq |(\tilde{u}\Delta V,\Delta U)|+|(\nabla\tilde{u}\cdot\nabla V,\Delta U)|+\alpha|(U\Delta\tilde{v},\Delta U)|\\ &\quad+\alpha|(\nabla U\cdot\nabla\tilde{v},\Delta U)|+\alpha|(f_u,\Delta U)|,\end{aligned}$$

and working in a similar way as we did to obtain the estimate (36), we can obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla U\|+\|\Delta U\|^2 &\leq \varepsilon\|\Delta U\|^2+C\Big(\|\nabla\tilde{v}\|^2_{L^\infty}+\|\Delta\tilde{v}\|^2_{L^q}\Big)\|U\|^2_{H^1}\\ &\quad+C\|\tilde{u}\|^2_{L^s}\|\Delta V\|^2_{L^q}+C\|\nabla\tilde{u}\|^2\|\nabla V\|^2_{L^\infty}+C\|f_u\|^2,\end{aligned} \tag{42}$$

where $\varepsilon > 0$ is arbitrary and $s \in (2,\infty)$ is chosen in such a way that $\frac{1}{s}+\frac{1}{q}=\frac{1}{2}$. Also, integrating the U-equation $(28)_1$ in the spatial variable we obtain

$$\frac{d}{dt}\int_\Omega U = \alpha\int_\Omega f_u,$$

which implies the following inequalities

$$\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega U\Big)^2 \leq C\Big(\int_\Omega f_u\Big)^2+C\Big(\int_\Omega U\Big)^2 \leq C|\Omega|^2\Big(\|f_u\|^2+\|U\|^2\Big), \tag{43}$$

$$\Big|\int_\Omega U(t)\Big|^2 = \Big|\int_\Omega U_0+\alpha\int_0^t\int_\Omega f_u\Big|^2 \leq C. \tag{44}$$

Adding estimates (42)–(44) and taking into account the equivalent norms given in (11)–(12) we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|U\|^2_{H^1}+\|U\|^2_{H^2} &\leq \varepsilon\|\Delta U\|^2+C\Big(\|\nabla\tilde{v}\|^2_{L^\infty}+\|\Delta\tilde{v}\|^2_{L^q}+|\Omega|^2\Big)\|U\|^2_{H^1}\\ &\quad+C\|\tilde{u}\|^2_{L^s}\|\Delta V\|^2_{L^q}+C\|\nabla\tilde{u}\|^2\|\nabla V\|^2_{L^\infty}+C\|f_u\|^2+C;\end{aligned}$$

which, for $\epsilon > 0$, suitably implies

$$\begin{aligned}\frac{d}{dt}\|U\|^2_{H^1}+C\|U\|^2_{H^2} &\leq C\Big(\|\nabla\tilde{v}\|^2_{L^\infty}+\|\Delta\tilde{v}\|^2_{L^q}+|\Omega|^2\Big)\|U\|^2_{H^1}\\ &\quad+\|\tilde{u}\|^2_{L^s}\|\Delta V\|^2_{L^q}+C\|\nabla\tilde{u}\|^2\|\nabla V\|^2_{L^\infty}+C\|f_u\|^2+C.\end{aligned} \tag{45}$$

Then, from (45), Gronwall's lemma, and taking into account that $\tilde{u} \in L^\infty(H^1)\cap L^2(H^2_{\mathbf{n}})$ (hence $\tilde{u} \in L^\infty(L^s)$, for any $s \in (2,\infty)$, and $\nabla\tilde{u} \in L^\infty(L^2)$), we conclude that $U \in L^\infty(H^1)\cap$

$L^2(H^2_{\mathbf{n}})$; thus, using that $\Delta U \in L^2(Q)$, $\tilde{u}, U \in L^\infty(H^1) \cap L^2(H^2_{\mathbf{n}})$ and $\tilde{v}, V \in L^q(W^{2,q}_{\mathbf{n}})$, we deduce that $\partial_t U = \Delta U + \nabla \cdot (\tilde{u}\nabla V) + \alpha \nabla \cdot (U \nabla \tilde{v}) + \alpha f_u \in L^2(Q)$. Then,

$$\begin{aligned} \|\partial_t U\|_{L^2(Q)} &\leq \|\Delta U + \nabla \cdot (\tilde{u}\nabla V) + \alpha \nabla \cdot (U\nabla \tilde{v}) + \alpha f_u\|_{L^2(Q)} \\ &\leq \|\Delta U\|_{L^2(Q)} + \|\nabla \cdot (\tilde{u}\nabla V)\|_{L^2(Q)} + \|\nabla \cdot (U\nabla \tilde{v})\|_{L^2(Q)} + \|f_u\|_{L^2(Q)} \\ &\leq C. \end{aligned}$$

Therefore, $U \in S_u$.

Consequently, we deduce that the operator S and the set S_α satisfy the conditions of the Leray–Schauder fixed-point theorem. Thus, there exists a pair $(U, V) \in S_u \times S_v$ such that $S(U, V) = (U, V)$, which is a solution of system (28). Moreover, following a classical comparison argument we can deduce the solution (U, V) of problem (28) is unique. □

We are now in a position to prove the existence of Lagrange multipliers for the optimal control problem (17).

Theorem 5. *Suppose that assumptions of Theorem 4 hold. Let $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ be a local optimal solution of the control problem (17). Then, there exists a triplet of the Lagrange multipliers $(\varphi, \psi, \chi) \in L^2(Q) \times (L^q(Q))' \times (H^1(\Omega))'$ such that for all $(U, V, F) \in S_u \times S_v \times \mathcal{C}(\tilde{f})$ one has*

$$\begin{aligned} \alpha_u \int_0^T \int_\Omega (\tilde{u} - u_d)U + \alpha_v \int_0^T \int_\Omega (\tilde{v} - v_d)V + \alpha_f \int_\omega \operatorname{sgn}(\tilde{f})|\tilde{f}|^{q-1}F \\ - \int_0^T \int_\Omega (\partial_t U - \Delta U - \nabla \cdot (U\nabla \tilde{v}) - \nabla \cdot (\tilde{u}\nabla V))\varphi \\ - \int_0^T \int_\Omega \Big(-\Delta V + V - p\tilde{u}^{p-1}U - \tilde{f}V\,1_\omega\Big)\psi - \int_\Omega U(0)\chi + \int_0^T \int_\omega F\tilde{v}\varphi \geq 0. \end{aligned} \tag{46}$$

Proof. From Proposition 1 we determine that $\tilde{s} \in \mathcal{S}_{ad}$ is a regular point. Then, from ([31], Theorem 3.1) we deduce that there exist Lagrange multipliers $(\varphi, \psi, \chi) \in L^2(Q) \times (L^q(Q))' \times (H^1(\Omega))'$ such that the following variational inequality holds

$$J'(\tilde{s})[r] - \langle R_1'(\tilde{s})[r], \varphi\rangle - \langle R_2'(\tilde{s})[r], \psi\rangle_{(L^q)'} - \langle R_3'(\tilde{s})[r], \chi\rangle_{(H^1)'} \geq 0 \tag{47}$$

for all $r = (U, V, F) \in S_u \times S_v \times \mathcal{C}(\tilde{f})$. Therefore, inequality (46) follows from (25), (26), and (47). □

From Theorem 5 we can derive an optimality system for the control problem (17); for this purpose we will consider the following linear subspace of S_u:

$$\widehat{S}_u := \{u \in S_u \,:\, u(0) = 0\}. \tag{48}$$

The choice of the space $\widehat{S}_u$ permits us to focus our analysis on the Lagrange multipliers φ and ψ.

Corollary 1. *Under assumptions of Theorem 4, let $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ be a local optimal solution of control problem (17). Then, the Lagrange multipliers $(\varphi, \psi) \in L^2(Q) \times (L^q(Q))'$, provided by Theorem 5, satisfy the following variational formulation*

$$\begin{aligned} &\int_0^T \int_\Omega (\partial_t U - \Delta U - \nabla \cdot (U\nabla \tilde{v}))\varphi - p\int_0^T \int_\Omega \tilde{u}^{p-1}U\psi \\ &= \alpha_u \int_0^T \int_\Omega (\tilde{u} - u_d)U \quad \forall U \in \widehat{S}_u, \end{aligned} \tag{49}$$

$$\begin{aligned} &\int_0^T \int_\Omega (-\Delta V + V)\psi - \int_0^T \int_\omega \tilde{f}V\psi - \int_0^T \int_\Omega \nabla \cdot (\tilde{u}\nabla V)\varphi \\ &= \alpha_v \int_0^T \int_\Omega (\tilde{v} - v_d)V \quad \forall V \in S_v \end{aligned} \tag{50}$$

and the optimality condition

$$\int_\omega \left(\alpha_f \mathrm{sgn}(\tilde{f})|\tilde{f}|^{q-1} + \int_0^T \tilde{v}\varphi\right)(f-\tilde{f}) \geq 0 \ \ \forall f \in \mathcal{F}. \tag{51}$$

Proof. Notice that $\widehat{S}_u \times S_v$ is a vector space. Hence, (49) can be obtained by taking $(V,F) = (0,0)$ into (46). Similarly, taking $(U,F) = (0,0)$ in (46) we deduce (50). Finally, taking $(U,V) = (0,0)$ in (46) we obtain

$$\alpha_f \int_\omega \mathrm{sgn}(\tilde{f})|\tilde{f}|^{q-1}F + \int_0^T \int_\omega F\tilde{v}\varphi \geq 0 \ \ \forall F \in \mathcal{C}(\tilde{f}),$$

which implies

$$\int_\omega \left(\alpha_f \mathrm{sgn}(\tilde{f})|\tilde{f}|^{q-1} + \int_0^T \tilde{v}\varphi\right)F \geq 0 \ \ \forall F \in \mathcal{C}(\tilde{f}). \tag{52}$$

Therefore, by choosing $F = (f-\tilde{f}) \in \mathcal{C}(\tilde{f})$ in (52) we deduce inequality (51). □

Finally, we will derive an optimality system for a local optimal solution $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f})$ of control problem (17). Firstly, we must improve the regularity of the Lagrange multipliers obtained in Theorem 5. The following result goes in that direction.

Theorem 6. *Suppose that assumptions of Theorem 4 hold. Let $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ be a local optimal solution of control problem (17). If $\|\tilde{u}\|_{L^{4(p-1)}}$ and $\|\tilde{f}\|_{L^q(\omega)}$ are small enough such that*

$$\|\tilde{u}\|^{4(p-1)}_{L^{4(p-1)}} + \|\tilde{f}\|^2_{L^q(\omega)} < \frac{1}{\min\{C, K_1^2, \widehat{K}_2^2\}}, \tag{53}$$

where C, K_1, and $\widehat{K}_2$ are positive constants that depend on $|\Omega|$. Then, the Lagrange multipliers $(\varphi, \psi) \in L^2(Q) \times (L^q(Q))'$, provided by Theorem 5, have the following strong regularity:

$$\begin{cases} \varphi \in S_\varphi & := \ \{\varphi \in L^\infty(H^1) \cap L^2(H^2_{\mathbf{n}}) \ : \ \partial_t \varphi \in L^2(Q)\}, \\ \psi \in S_\psi & := \ L^r(W^{2,r}), \text{ for any } r \in (1,2). \end{cases} \tag{54}$$

Proof. Notice that the pair of functions $(\varphi, \psi) \in L^2(Q) \times (L^q(Q))'$, obtained in Theorem 5, corresponds with the concept of a very weak solution of the following adjoint system

$$\begin{cases} -\partial_t \varphi - \Delta\varphi + \nabla\varphi \cdot \nabla\tilde{v} - p\tilde{u}^{p-1}\psi &= \alpha_u(\tilde{u} - u_d) \ \text{ in } \ Q, \\ -\Delta\psi - \nabla \cdot (\tilde{u}\nabla\varphi) + \psi - \tilde{f}\psi 1_\omega &= \alpha_v(\tilde{v} - v_d) \ \text{ in } \ Q, \\ \varphi(T) &= 0 \ \text{ in } \ \Omega, \\ \dfrac{\partial\varphi}{\partial\mathbf{n}} &= \dfrac{\partial\psi}{\partial\mathbf{n}} = 0 \ \text{ on } \ (0,T) \times \partial\Omega. \end{cases} \tag{55}$$

Thus, first we will analyze the regularity of the solutions of problem (55) and then we will improve the regularity of the pair of the Lagrange multipliers (φ, ψ). Indeed, let $\tau := T - t$, with $t \in (0,T)$ and $\widehat{\varphi}(\tau) = \varphi(t)$. Then, system (55) is equivalent to the following forward problem

$$\begin{cases} \partial_\tau \widehat{\varphi} - \Delta\widehat{\varphi} + \nabla\widehat{\varphi} \cdot \nabla\tilde{v} - p\tilde{u}^{p-1}\psi &= \alpha_u(\tilde{u} - u_d) \ \text{ in } \ Q, \\ -\Delta\psi - \nabla \cdot (\tilde{u}\nabla\widehat{\varphi}) + \psi - \tilde{f}\psi 1_\omega &= \alpha_v(\tilde{v} - v_d) \ \text{ in } \ Q, \\ \widehat{\varphi}(0) &= 0 \ \text{ in } \ \Omega, \\ \dfrac{\partial\widehat{\varphi}}{\partial\mathbf{n}} &= \dfrac{\partial\psi}{\partial\mathbf{n}} = 0 \ \text{ on } \ (0,T) \times \partial\Omega. \end{cases} \tag{56}$$

Since system (56) is a linear problem, we argue in a formal sense, proving that any regular enough solution is bounded in the space $S_\varphi \times S_\psi$ (a rigorous proof can be performed

using the Leray–Schauder fixed point theorem, similar to what was used for the proof of Proposition 1).

Testing $(56)_1$ by $\widehat{\varphi} - \Delta\widehat{\varphi}$, applying the Hölder and Young inequalities, and taking into account the 2D interpolation inequality (13) for the L^4-norm, we can obtain the following estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|\widehat{\varphi}\|_{H^1}^2 + \|\nabla\widehat{\varphi}\|^2 + \|\Delta\widehat{\varphi}\|^2 \\
&\leq \varepsilon\Big(\|\Delta\widehat{\varphi}\|^2 + \|\nabla\widehat{\varphi}\|^2 + \|\psi\|_{H^1}^2\Big) + C\Big(\|\tilde{u}^{p-1}\|_{L^4}^2 + \|\nabla\tilde{v}\|_{L^\infty}^2 + 1\Big)\|\widehat{\varphi}\|_{H^1}^2 \\
&\quad + C\|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)}\|\psi\|_{H^1}^2 + C\|\tilde{u} - u_d\|^2,
\end{aligned} \tag{57}$$

where $\varepsilon > 0$ is arbitrary. Similarly, testing the ψ-equation (56) by ψ we can arrive at

$$\begin{aligned}
\|\psi\|_{H^1}^2 \quad \leq \quad & \varepsilon\Big(\|\psi\|^2 + \|\nabla\psi\|^2 + \|\widehat{\varphi}\|_{H^2}^2\Big) + C\|\tilde{u}\|_{L^4}^4\|\nabla\widehat{\varphi}\|^2 \\
& + K_1^2\|\tilde{f}\|_{L^q(\omega)}^2\|\psi\|_{H^1}^2 + C\|\tilde{v} - v_d\|^2,
\end{aligned} \tag{58}$$

where $\varepsilon > 0$ is arbitrary and the constant $K_1 := K_1(|\Omega|) > 0$ is given by the Sobolev embedding $H^1 \hookrightarrow L^s$ for $s \in (2, \infty)$.

Now, summing estimates (57) and (58) and then adding $\|\widehat{\varphi}\|^2$ on both sides of the resulting inequality, with the aim of completing the H^2-norm, it is possible to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{d\tau}\|\widehat{\varphi}\|_{H^1}^2 + \|\widehat{\varphi}\|_{H^2}^2 + \|\psi\|_{H^1}^2 \\
&\leq \varepsilon\Big(\|\Delta\widehat{\varphi}\|^2 + \|\nabla\widehat{\varphi}\|^2 + \|\widehat{\varphi}\|_{H^2}^2 + \|\psi\|^2 + \|\nabla\psi\|^2 + \|\psi\|_{H^1}^2\Big) \\
&\quad + C\Big(\|\tilde{u}^{p-1}\|_{L^4}^2 + \|\tilde{u}\|_{L^4}^4 + \|\nabla\tilde{v}\|_{L^\infty}^2 + 1\Big)\|\widehat{\varphi}\|_{H^1}^2 \\
&\quad + \Big(C\|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)} + K_1^2\|\tilde{f}\|_{L^q(\omega)}^2\Big)\|\psi\|_{H^1}^2 + C\Big(\|\tilde{u} - u_d\|^2 + \|\tilde{v} - v_d\|^2\Big).
\end{aligned}$$

Thus, choosing $\varepsilon > 0$ suitably in the last inequality, we can obtain

$$\begin{aligned}
&\frac{d}{d\tau}\|\widehat{\varphi}\|_{H^1}^2 + C\|\widehat{\varphi}\|_{H^2}^2 + \Big(1 - \Big(C\|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)} + K_1^2\|\tilde{f}\|_{L^q(\omega)}^2\Big)\Big)\|\psi\|_{H^1}^2 \\
&\leq C\Big(\|\tilde{u}^{p-1}\|_{L^4}^2 + \|\tilde{u}\|_{L^4}^4 + \|\nabla\tilde{v}\|_{L^\infty}^2 + 1\Big)\|\widehat{\varphi}\|_{H^1}^2 + C\Big(\|\tilde{u} - u_d\|^2 + \|\tilde{v} - v_d\|^2\Big).
\end{aligned} \tag{59}$$

Notice that assumption (53) implies that $1 - \Big(C\|\tilde{u}\|_{L^{4(p-1)}}^{4(p-1)} + K_1^2\|\tilde{f}\|_{L^q(\omega)}^2\Big) > 0$. Hence, from (59), Gronwall's lemma, and taking into account that the terms $\|\tilde{u}^{p-1}\|_{L^4}^2$, $\|\tilde{u}\|_{L^4}^4$, $\|\nabla\tilde{v}\|_{L^\infty}^2$, $\|\tilde{u} - u_d\|^2$, $\|\tilde{v} - v_d\|^2$ are integrable in time, we deduce that $\widehat{\varphi} \in L^\infty(H^1) \cap L^2(H_{\mathbf{n}}^2)$. Similarly, integrating in time (59) for $\tau \in (0, T)$, we determine that $\psi \in L^2(H^1)$.

Now, using that $(\tilde{u}, \tilde{v}) \in \widehat{S}_u \times S_v$ (in particular $\tilde{u}^{p-1} \in L^\infty(L^s)$, for any $s \in (1, \infty)$, and $\nabla\tilde{v} \in L^q(L^\infty)$) and that $\nabla\widehat{\varphi} \in L^\infty(L^2)$, we deduce that

$$p\tilde{u}^{p-1}\psi + \alpha_u(\tilde{u} - u_d) - \nabla\widehat{\varphi} \cdot \nabla\tilde{v} \in L^2(Q).$$

Hence, applying Theorem 2 to $(56)_1$ (for $s = 2$), we deduce that $\widehat{\varphi} \in S_\varphi$, which implies that $\varphi \in S_\varphi$. Moreover, using that $\tilde{u} \in L^s(Q)$, for any $s \in (1, \infty)$, and $\Delta\widehat{\varphi} \in L^2(Q)$, we deduce that $\nabla \cdot (\tilde{u}\nabla\widehat{\varphi}) + \tilde{f}\psi\, 1_\omega + \alpha_v(\tilde{v} - v_d) \in L^r(Q)$, for any $r \in (1, 2)$. Thus,

$$\nabla \cdot (\tilde{u}(\tau, \cdot)\nabla\widehat{\varphi}(\tau, \cdot)) + \tilde{f}\psi(\tau, \cdot)\, 1_\omega + \alpha_v(\tilde{v}(\tau, \cdot) - v_d(\tau, \cdot)) \in L^r(\Omega),$$

for any $r \in (1,2)$ and any time $\tau \in (0,T)$. Then, applying elliptic regularity to (56), for $s = 2$, (see Theorem 3) we conclude that $\psi \in W^{2,r}$ and satisfies the estimate

$$\begin{aligned} \|\psi\|_{W^{2,r}} &\leq C\Big(\|\nabla \cdot (\tilde{u}\nabla\widehat{\varphi})\|_{L^r} + \|\tilde{v} - v_d\|_{L^r} + \|\tilde{f}\|_{L^q(\omega)}\|\psi\|_{L^\infty}\Big) \\ &\leq C(\|\nabla \cdot (\tilde{u}\nabla\widehat{\varphi})\|_{L^r} + \|\tilde{v} - v_d\|) + \widehat{K}_2\|\tilde{f}\|_{L^q(\omega)}\|\psi\|_{W^{2,r}}, \end{aligned}$$

where $\widehat{K}_2 := \widehat{K}_2(|\Omega|)$ is a constant given by the embedding $W^{2,r} \hookrightarrow L^\infty$. Thus, we obtain

$$\Big(1 - \widehat{K}_2\|\tilde{f}\|_{L^q(\omega)}\Big) \leq C(\|\nabla \cdot (\tilde{u}\nabla\widehat{\varphi})\|_{L^r} + \|\tilde{v} - v_d\|). \tag{60}$$

From assumption (53) we deduce that $\widehat{K}_2\|\tilde{f}\|_{L^q(\omega)} < 1$; then, from (60) we have

$$\|\psi\|^r_{W^{2,r}} \leq \frac{C}{\Big(1 - \widehat{K}_2\|\tilde{f}\|_{L^q(\omega)}\Big)^r}(\|\nabla \cdot (\tilde{u}\nabla\widehat{\varphi})\|_{L^r} + \|\tilde{v} - v_d\|)^r. \tag{61}$$

Therefore, by integrating (61) in time we conclude that $\psi \in L^r(W^{2,r}) = S_\psi$ for any $r \in (1,2)$. Moreover, following a classical comparison argument we can deduce the uniqueness of the pair (φ, ψ) solving the adjoint system (55).

It remains to be proven that the pair of Lagrange multipliers, provided by Theorem 5, have strong regularity (54). Indeed, let $(\widetilde{\varphi}, \widetilde{\psi}) \in S_\varphi \times S_\psi$, the unique solution of adjoint problem (55), and $(U, V) \in S_u \times S_v$, the unique solution of the linear problem (28), be used with data $f_u := (\varphi - \widetilde{\varphi}) \in L^2(Q)$ and $f_v := \operatorname{sgn}(\psi - \widetilde{\psi})|\psi - \widetilde{\psi}|^{\frac{1}{q-1}} \in L^q(Q)$. We recall that $\psi \in (L^q(Q))'$; that is, $\psi \in L^{\frac{q}{q-1}}(Q)$, thus $f_v = \operatorname{sgn}(\psi - \widetilde{\psi})|\psi - \widetilde{\psi}|^{\frac{1}{q-1}} \in L^q(Q)$ make sense. Then, in order to prove that the pair (φ, ψ) have the regularity (54), it suffices to identify (φ, ψ) with $(\widetilde{\varphi}, \widetilde{\psi})$. Now, writing (55) for $(\widetilde{\varphi}, \widetilde{\psi})$ instead of (φ, ψ), then testing the first equation by U and the second by V, after integrating by parts, we can obtain

$$\int_0^T \int_\Omega (\partial_t U - \Delta U - \nabla \cdot (U\nabla\tilde{v}))\widetilde{\varphi} - p\int_0^T \int_\Omega \tilde{u}^{p-1}U\widetilde{\psi} = \alpha_u \int_0^T \int_\Omega (\tilde{u} - u_d)U, \tag{62}$$

$$\int_0^T \int_\Omega (-\Delta V + V)\widetilde{\psi} - \int_0^T \int_\omega \tilde{f}V\widetilde{\psi} - \int_0^T \int_\Omega \nabla \cdot (\tilde{u}\nabla V) = \alpha_v \int_0^T \int_\Omega (\tilde{v} - v_d)V. \tag{63}$$

Taking the difference between (49) and (62) and between (50) and (63), and adding the respective equalities, we deduce

$$\begin{aligned} &\int_0^T \int_\Omega (\partial_t U - \Delta U - \nabla \cdot (U\nabla\tilde{v}) - \nabla \cdot (\tilde{u}\nabla V))(\varphi - \widetilde{\varphi}) \\ &+ \int_0^T \int_\Omega \Big(-\Delta V + V - p\tilde{u}^{p-1}U\Big)(\psi - \widetilde{\psi}) - \int_0^T \int_\omega \tilde{f}V(\psi - \widetilde{\psi}) = 0. \end{aligned} \tag{64}$$

Thus, considering that the element (U, V) is the unique solution of system (28) for $(\varphi - \widetilde{\varphi}) \in L^2(Q)$ and $\operatorname{sgn}(\psi - \widetilde{\psi})|\psi - \widetilde{\psi}|^{\frac{1}{q-1}} \in L^q(Q)$, from (64) we conclude that

$$\|\varphi - \widetilde{\varphi}\|^2_{L^2(Q)} + \|\psi - \widetilde{\psi}\|^{\frac{q}{q-1}}_{L^{\frac{q}{q-1}}(Q)} = 0.$$

Therefore, $(\varphi, \psi) = (\widetilde{\varphi}, \widetilde{\psi}) \in L^2(\Omega) \times (L^q(Q))'$. Consequently, the Lagrange multiplier (φ, ψ), provided by Theorem 5, has the strong regularity (54). □

Theorem 6 allows us derive an optimality system for the control problem (17).

Corollary 2. *Under conditions of Theorem 6, let $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ be a local optimal solution of optimal control problem (17). Then the pair of Lagrange multipliers $(\varphi, \psi) \in S_\varphi \times \psi$ satisfies the following optimality system*

$$\left\{\begin{array}{rcl} -\partial_t\varphi - \Delta\varphi + \nabla\varphi\cdot\nabla\tilde{v} - p\tilde{u}^{p-1}\psi & = & \alpha_u(\tilde{u}-u_d) \ \ a.e.\ (t,x)\in Q, \\ -\Delta\psi - \nabla\cdot(\tilde{u}\nabla\varphi) + \psi - \tilde{f}\psi 1_\omega & = & \alpha_v(\tilde{v}-v_d) \ \ a.e.\ (t,x)\in Q, \\ \varphi(T) & = & 0 \ \ in \ \ \Omega, \\ \dfrac{\partial\varphi}{\partial\mathbf{n}} & = & \dfrac{\partial\psi}{\partial\mathbf{n}} = 0 \ \ on \ \ (0,T)\times\partial\Omega, \\ \displaystyle\int_\omega\left(\alpha_f \mathrm{sgn}(\tilde{f})|\tilde{f}|^{q-1} + \int_0^T \tilde{v}\varphi\right)(f-\tilde{f}) & \geq 0 & \forall f\in\mathcal{F}. \end{array}\right. \tag{65}$$

Remark 2. *The optimality system (65) can serve as the basis for computing approximations to optimal solutions numerically of control problem (17).*

4. Some Comments

It is important to note that a bilinear optimal control problem related with the chemo-repulsion system (2)–(4) has been studied. The choice of bilinear control is due to the fact that the solutions (u, v) of (2)–(4) must be non-negative. If we had worked with the typical distributed control, we would have had to condition the sign of the control f. In fact, f should have been a non-negative function. The latter is very restrictive for the set of controls, since in theory this set should be the largest possible set. When working with bilinear control, we do not need to impose sign on the control. Furthermore, from a biological point of view, this makes sense, since we manage to manipulate the behavior of cells and chemistry by injecting or extracting chemical substance in a subregion $\omega \subset \Omega$.

Furthermore, this paper concludes with the derivation of an optimality system for local optimal solutions of control problem (17). The optimality system can serve as the basis for computing approximations to optimal solutions numerically of control problem (17). Therefore, we provide a starting point for anyone interested in carrying out a numerical study of control problems similar to problem (17).

5. Conclusions

In this article, we have studied an optimal control problem for a 2D parabolic–elliptic chemo-repulsion model with a nonlinear chemical signal production term. We controlled the system, applying a bilinear control on a subdomain $\omega \subset \Omega$, which acts on the chemical Equation $(2)_2$ as the degradation/proliferation coefficient. We proved the existence of at least one global optimal solution and derive first-order necessary optimality conditions for a local optimal solution, applying a generic result on the existence of Lagrange multipliers. Also, for the Lagrange multipliers obtained, we improve their regularity, which allows us conclude that they satisfy point-wisely an adjoint system related to primal problem (2)–(4).

Author Contributions: Investigation, E.M.-Z. and L.M.; visualization, L.M.; conceptualization, E.M.-Z.; methodology, E.M.-Z.; writing–original draft preparation, E.M.-Z. and L.M.; writing–review and editing, E.M.-Z. and L.M.; supervision, E.M.-Z.; validation, L.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: E. Mallea-Zepeda has been supported by ANID-Chile throughout Fondecyt de Iniciación 11200208. L. Medina has been partially supported by MathAmSud code AMSUD220015, MINEDUC-UA code ANT1999. Also, the authors would like to thank the anonymous referees for their kind and helpful remarks and comments.

Conflicts of Interest: The author declare no conflicts of interest.

References

1. Morales-Rodrigo, C. On some models describing cellular movement: The macroscopic scale. *Boletín Soc. Española Matemática Apl.* **2009**, *2*, 83–109.
2. Othmer, H.; Stevens, A. Aggregation, blowup and collapse: The ABC'S of taxis in reinforced random walks. *SIAM J. Appl. Math.* **1997**, *57*, 1044–1081.
3. Keller, E.F.; Segel, L.A. Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.* **1970**, *26*, 399–415. [CrossRef] [PubMed]
4. Keller, E.F.; Segel, L.A. Model for chemotaxis. *J. Theor. Biol.* **1971**, *30*, 225–234. [CrossRef] [PubMed]
5. Di Francesco, D.; Lorz, A.; Markowich, P. Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: Global existence and asymptotic behavior. *Discret. Contin. Dyn. Syst.* **2010**, *28*, 1437–1453. [CrossRef]
6. Mock, M.S. An initial value problem from semiconductor device theory. *SIAM J. Math. Anal.* **1974**, *5*, 597–612. [CrossRef]
7. Mock, M.S. Asymptotic behavior of solutions of transport equations for semiconductor devices. *J. Math. Anal. Appl.* **1975**, *49*, 215–225. [CrossRef]
8. Ciéslak, T.; Laurençot, P.; Morales-Rodrigo, C. Global existence and convergence to steady states in a chemorepulsion system. In *Parabolic and Navier-Stokes Equations, Part 1*; 81m Part 1; Banach Center Publications, Institute of Mathematics of the Polish Academy of Sciences: Warsaw, Poland, 2008; pp. 105–117.
9. Tao, Y. Global dynamics in a higher-dimensional repulsion chemotaxis model with nonlinear sensitivity. *Discret. Contin. Dyn. Syst. B* **2013**, *18*, 2705–2722. [CrossRef]
10. Guillén-González, F.; Rodríguez-Bellido, M.A.; Rueda-Gómez, D.A. Study of a chemo-repulsion model with quadratic production. Part I: Analysis of the continuous problem and time-discrete numerical schemes. *Comput. Math. Appl.* **2020**, *80*, 692–713. [CrossRef]
11. Guillén-González, F.; Rodríguez-Bellido, M.A.; Rueda-Gómez, D.A. Study of a chemo-repulsion model with quadratic production. Part II: Analysis of an unconditional energy-stable fully discrete scheme. *Comput. Math. Appl* **2020**, *80*, 636–652. [CrossRef]
12. Guillén-González, F.; Mallea-Zepeda, E.; Rodríguez-Bellido, M.A. Optimal bilinear control problem related to a chemo-repulsion system in 2D domains. *ESAIM Control Optim. Calc. Var.* **2020**, *26*, 29. [CrossRef]
13. Guillén-González, F.; Mallea-Zepeda, E.; Rodríguez-Bellido, M.A. A regularity criterion for a 3D chemo-repulsion system and its application to a bilinear optimal control problem. *SIAM J. Control Optim.* **2020**, *56*, 1457–1490. [CrossRef]
14. Guillén-González, F.; Mallea-Zepeda, E.; Villamizar-Roa E.J. On a bi-dimensional chemo-repulsion model with nonlinear production and a related optimal control problem. *Acta Appl. Math.* **2020**, *170*, 963–979. [CrossRef]
15. Ancoma-Huarachi, A.; Malle-Zepeda, E. A parabolic-elliptic chemo-repulsion system in 2D domains with nonlinear production. *Hacet. J. Math. Stat.* **2023**, 1–15. [CrossRef]
16. Lorca, S.; Mallea-Zepeda, E.; Villamizar-Roa, E.J. Stationary solutions to a chemo-repulsion system and a related optimal bilinear control problem. *Bull. Braz. Math. Soc. New Ser.* **2023**, *54*, 31. [CrossRef]
17. De Leenheer, P.; Gopalakrishnan, J.; Zuhr, E. Nonnegativity of exact and numerical solutions of some chemotactic models. *Comput. Math. Appl.* **2013**, *66*, 356–375. [CrossRef]
18. Budrene, E.O.; Berg, H.C. Dynamics of formation of symmetrical patterns by chemotactic bacteria. *Nature* **1995**, *376*, 49–53. [CrossRef] [PubMed]
19. De Araujo, A.L.A.; De Magalhães, P.M.D. Existence of solutions and optimal control for a model of tissue invasion by solid tumours. *J. Math. Anal. Appl.* **2015**, *421*, 842–877. [CrossRef]
20. Fister, K.R.; Mccarthy, C.M. Optimal control of a chemotaxis system. *Quart. Appl. Math.* **2003**, *61*, 193–211. [CrossRef]
21. López-Ríos, J.; Villamizar-Roa, E.J. An optimal control problem related to a 3D-chemotaxis-Navier-Stokes model. *ESAIM Control Optim. Calc. Var.* **2021**, *27*, 58. [CrossRef]
22. Rodríguez-Bellido, M.A.; Rueda-Gómez, D.A.; Villamizar-Roa, E.J. On a distributed control problem for a coupled chemotaxis-fluid model. *Discret. Contin. Dyn. Syst. B* **2018**, *23*, 557–572. [CrossRef]
23. Ryu, S.-U.; Tagi, A. Optimal control of Keller-Segel equations. *J. Math. Anal. Appl.* **2001**, *256*, 45–66. [CrossRef]
24. Braz e Silva, P.; Guillén-González, F.; Perusato, C.F.; Rodríguez-Bellido, M.A. Bilinear optimal control of the Keller-Segel logistic model in 2D-domains. *Appl. Math. Optim.* **2023**, *87*, 20. [CrossRef]
25. Nečas, J. *Direct Methods in the Theory of Elliptic Equations*; Springer: Berlin/Heidelberg, Germany, 2012.
26. Feireisl, E.; Novotný, A. *Singular Limits in Thermodynamics of Viscous Fluids. Advances in Mathematical Fluid Mechanics*; Birkhäuser Verlag: Basel, Switzerland, 2009.
27. Grisvard, P. *Elliptic Problems in Nonsmooth Domains*; Pitman Advanced Publishing Program: Boston, MA, USA, 1985.
28. Brézis, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*; Springer: New York, NY, USA, 2011.
29. Lions, J.L. *Quelques Métodes de Résolution des Problèmes aux Limites non Linéares*; Dunod: Paris, France, 1969.
30. Simon, J. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl.* **1987**, *146*, 65–96. [CrossRef]
31. Zowe, J.; Kurcyusz, S. Regularity and stability for the mathematical programming problem in Banach spaces. *Appl. Math. Optim.* **1979**, *5*, 49–62. [CrossRef]