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A Novel Quintic B-Spline Technique for Numerical Solutions of the Fourth-Order Singular Singularly-Perturbed Problems

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Abstract: Singular singularly-perturbed problems (SSPPs) are a powerful mathematical tool for modelling a variety of real phenomena, such as nuclear reactions, heat explosions, mechanics, and hydrodynamics. In this paper, the numerical solutions to fourth-order singular singularly-perturbed boundary and initial value problems are presented using a novel quintic B-spline (QBS) approximation approach. This method uses a quasi-linearization approach to solve SSPNL initial/boundary value problems. And the non-linear problems are transformed into a sequence of linear problems by applying the quasi-linearization approach. The QBS functions produce more accurate results when compared to other existing approaches because of their local support, symmetry, and partition of unity features. This method can be applied to immediately solve the SSPPs without reducing the order in which they are presented. It has been demonstrated that the suggested numerical approach converges uniformly over the whole domain. The proposed approach is implemented on a few problems to validate the scheme. The computational results are compared, and they illustrate that the proposed approach performs better.

Keywords: singular singularly-perturbed non-linear initial/boundary value problems; uniform convergence; fourth-order Emden–fowler type equation; QBS function; fourth-order BVP and IVP

MSC: 65L11; 34B16; 35G16; 41A15; 65D07; 65M12

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1. Introduction

Fourth-order BVPs are found in a wide variety of applications of practical mathematics, including continuum mechanics, reaction kinetics, fluid mechanics, wave mechanics, statistical mechanics, linear dynamics, rotational dynamics, thermodynamics, hydrokinetics, and geophysics, see [1–5]. The term "singular perturbation" was thought up in the 1940s by kurt Otto Friedriches and Wolfgang R.Wasow. In mathematics, a singular perturbation problem (SPP) is one that has a small parameter that cannot be approximated by setting the parameter value to zero. If a differential equation includes at least one negative or positive shift parameter and the highest-order derivative is multiplied by a tiny parameter, it is said to be singularly perturbed in mathematics.

The approximate solution to any perturbation problem, regardless of whether it be in space or time, can be found. There are two distinct forms of perturbation problem:

regular perturbation and singular perturbation. A regular perturbation problem is one whose perturbation series is a power series in ϵ with a persistent radius of convergence. By just substituting the tiny parameter ϵ with zero throughout in the problem, it is possible to get a satisfactory approximate solution to a regular perturbed problem in almost all applications. This translates into simply using the first term of the expansion, which results in an approximate solution that converges, as ϵ reduces, although potentially slowly, to the exact solution. This method cannot approximate the solution for a singularly perturbed problem. As seen in the above discussion, when a tiny parameter is multiplied with the highest operator, it is said to be singularly perturbed. If the value of the parameter is set to zero, the problem's fundamental structure is altered. In the context of differential equations, boundary conditions cannot be met; when speaking related to algebraic equations, the number of possible solutions is reduced.

In fluid mechanics, a mildly viscous fluid has very distinct characteristics both within and outside of a small boundaries layer. As a result, the fluid displays different spatial scales. SPPs and SSPPs have numerous applications in fluid mechanics [6], some models are listed below.

- Formal thin-airfoil expansion problem
- Solution of the thin-airfoil problem
- Non-uniformity for elliptic airfoil problem
- Problem of leading edge drag
- Local solution problem near a round edge
- Problems of matching with solution near round and sharp edges
- Hypersoic flow past thin blunted wedge problem

When real-world phenomena in science and engineering are mathematically modelled, singular singularly perturbed boundary value problems (SSPBVPs) typically appear. According to how setting ϵ approaches zero affects the order of the original differential equation, SPP are classified. Here, the DE's higher derivative is multiplied by a little parameter ϵ. When the differential equation's order is lowered by one, the SPP becomes convection diffusion. The reaction diffusion type is indicated if the order is lowered by two. As a result of the singularity of the derivative term's coefficient, we are now employing the word singular twice. There is extremely little literature on SSPPs compared to SPPs, and these problems are quite difficult to solve.

O'Malley [7] provided singular perturbation theory for the solution of ODEs. PDEs with critical parameters, Kaper and Pieper [8] developed asymptotic and numerical methods. Daba and Duressa [9] worked on artificial viscosity for time dependent singularly perturbed DDEs. Ascher [10] presented some difference schemes for solving SSPBVPs. Zhu [11] contributed to the asymptotic solution using a modified Vasil'eva approach for SSPBVPs of second-order quasilinear systems. It has been shown in a study [12] that the fitted mesh B- spline approach is employed for the second-order SSPBVPs. For the purpose of solving SPBVPs with a delay, see [13] modified reproducing kernel method is implemented. In the order to solve singularly perturbed delay IVPs, a piecewise reproducing kernel method is utilized in [14] by Geng and Qian. Bawa and Natesan [15] has employed a quintic spline to handle self-adjoint SPPs.

A novel QuBS approach for third-order self-adjoint SPBVP was developed by Saini and Mishra in [16]. Lang and Xu [17] proposed a QuBS collocation approach for fifth-order BVPs. For SPP of fourth-order, Gupta and Kumar [18] employed a B-spline based numerical approach. In [19] Deniz and Bildik worked with the adomian decomposition approach to solve SPBVPs of the fourth-order. Development on the fourth-order SPBVPs employing initial value techniques are being done by Mishra and Saini [20]. Wang and Ni [21] talked about the contrast structure for the SSPBVP problem. For the numerical solution of Burger's equation, Jiwari [22] presents a Haar wavelet-based quasi-linearization method. The QBS collocation approach has been extensively used by Lang and Xu [23] for second-order non-linear mixed BVPs. Akram [24] solved the third-order SPBVPs analytically by using QuBS. For the solution of the fourth-order two parameters SPBVP, Mahesh and Phaneendra [25] employed a non-polynomial cubic spline. The use of a novel QBS approximating approach was investigated by Abbas et al. [26] in the numerical analysis of fourth-order SBVPs. The new extended direct algebraic method is used by Nasreen et al. [27] to the solved the coupled nonlinear Schrodinger equations. The conformable ion sound and Langmuir waves dynamical system is solved by Nasreen et al. [28] using new extended direct algebraic method.

The QBS approach has been used in this study to solve a SSPNLBVP of fourth-order. Second-order convergence is made available by this strategy. Think about the subsequent problem type:

$$\epsilon v^{(4)}(\tau) + \frac{\alpha}{\tau} v'''(\tau) + \frac{\beta}{\tau} v''(\tau) + \frac{\gamma}{\tau} v'(\tau) + \frac{\delta}{\tau} v(\tau) = h(\tau, v), \tag{1}$$

where $\tau \in [0,\ 1]$ and

$$v(0) = \rho_0, v''(0) = \rho_1, v(1) = \sigma_0, v''(1) = \sigma_1, \tag{2}$$

where $\rho_0, \rho_1, \sigma_0, \sigma_1 \in \mathbb{R}$, $\epsilon > 0$ is a little number, $h(\tau, v)$ is non-linear term and α, β, γ, and δ are unchanging factors. The reason for considering the above problem mentioned in Equation (1) is that this problem is singular because of the term τ, singulary perturbed because the small parameter multiplied with the highest order derivative term and non non-linear due to the term $h(\tau, v)$. So, by resolving the aforementioned problem Equation (1), the singular singularly-perturbed linear initial/boundary value problem (SSPLIVP/SSPLBVP) addressed in this.

This article has the following structure: In Section 2, a short description of the QBS technique and its derivative is given. In Section 3, the origin of the QBS collocation method for solving the fourth-order SSPNLBVP is explained. Section 4 contains the derivation of uniform convergence. In Section 5, four examples are provided to show how accurate the proposed strategy is. Finish out Section 6 with some closing thoughts

2. Quintic B-Spline Interpolation

In this portion, the interval $[c, d]$ such that $c = \tau_0 < \tau_1 < \ldots < \tau_N = d$ is uniformly divided by $n + 1$ equal-sized knots $\tau_k = \tau_0 + kh$, $k = 0, 1, \ldots, N$, where $N \in \mathbb{Z}^+$ and $h = \frac{d-c}{N}$ being the piecewise uniform width.

Now, the fifth-degree basis spline function $B_{5,k}(\tau)$ at the knot τ is given as:

$$B_{5,k}(\tau) = \frac{1}{h^5} \begin{cases} (\tau - \tau_{k-3})^5, & \tau \in [\tau_{k-3}, \tau_{k-2}] \\ (\tau - \tau_{k-3})^5 - 6(\tau - \tau_{k-2})^5, & \tau \in [\tau_{k-2}, \tau_{k-1}] \\ (\tau - \tau_{k-3})^5 - 6(\tau - \tau_{k-2})^5 + 15(\tau - \tau_{k-1})^5, & \tau \in [\tau_{k-1}, \tau_k] \\ (\tau_{k+3} - \tau)^5 - 6(\tau_{k+2} - \tau)^5 + 15(\tau_{k+1} - \tau)^5, & \tau \in [\tau_k, \tau_{k+1}] \\ (\tau_{k+3} - \tau)^5 - 6(\tau_{k+2} - \tau)^5, & \tau \in [\tau_{k+1}, \tau_{k+2}] \\ (\tau_{k+3} - \tau)^5, & \tau \in [\tau_{k+2}, \tau_{k+3}] \\ 0. & \text{otherwise} \end{cases} \tag{3}$$

The first four derivatives of Equation (3) are given below.

$$B'_{5,k}(\tau) = \frac{1}{h^5} \begin{cases} 5(\tau - \tau_{k-3})^4, & \tau \in [\tau_{k-3}, \tau_{k-2}] \\ 5(\tau - \tau_{k-3})^4 - 30(\tau - \tau_{k-2})^4, & \tau \in [\tau_{k-2}, \tau_{k-1}] \\ 5(\tau - \tau_{k-3})^4 - 30(\tau - \tau_{k-2})^4 + 75(\tau - \tau_{k-1})^4, & \tau \in [\tau_{k-1}, \tau_k] \\ -5(\tau_{k+3} - \tau)^4 + 30(\tau_{k+2} - \tau)^4 - 75(\tau_{k+1} - \tau)^4, & \tau \in [t_k, \tau_{k+1}] \\ -5(\tau_{k+3} - \tau)^4 + 30(\tau_{k+2} - \tau)^4, & \tau \in [\tau_{k+1}, \tau_{k+2}] \\ -5(\tau_{k+3} - \tau)^4, & \tau \in [\tau_{k+2}, \tau_{k+3}] \\ 0. & \text{otherwise} \end{cases} \tag{4}$$

Similarly,

$$B''_{5,k}(\tau) = \frac{1}{h^5}\begin{cases} 20(\tau-\tau_{k-3})^3, & \tau \in [\tau_{k-3}, \tau_{k-2}] \\ 20(\tau-\tau_{k-3})^3 - 120(\tau-\tau_{k-2})^3, & \tau \in [\tau_{k-2}, \tau_{k-1}] \\ 20(\tau-\tau_{k-3})^3 - 120(\tau-\tau_{k-2})^3 + 300(\tau-\tau_{k-1})^3, & \tau \in [\tau_{k-1}, \tau_k] \\ 20(\tau_{k+3}-\tau)^3 - 120(\tau_{k+2}-\tau)^3 + 300(\tau_{k+1}-\tau)^3, & \tau \in [\tau_k, \tau_{k+1}] \\ 20(\tau_{k+3}-\tau)^3 - 120(\tau_{k+2}-\tau)^3, & \tau \in [\tau_{k+1}, \tau_{k+2}] \\ 20(\tau_{k+3}-\tau)^3, & \tau \in [\tau_{k+2}, \tau_{k+3}] \\ 0. & \text{otherwise} \end{cases} \tag{5}$$

$$B'''_{5,k}(\tau) = \frac{1}{h^5}\begin{cases} 60(\tau-\tau_{k-3})^2, & \tau \in [\tau_{k-3}, \tau_{k-2}] \\ 60(\tau-\tau_{k-3})^2 - 360(\tau-\tau_{k-2})^2, & \tau \in [\tau_{k-2}, \tau_{k-1}] \\ 60(\tau-\tau_{k-3})^2 - 360(\tau-\tau_{k-2})^2 + 900(\tau-\tau_{k-1})^2, & \tau \in [\tau_{k-1}, \tau_k] \\ -60(\tau_{k+3}-\tau)^2 + 360(\tau_{k+2}-\tau)^2 - 900(\tau_{k+1}-\tau)^2, & \tau \in [\tau_k, \tau_{k+1}] \\ -60(\tau_{k+3}-\tau)^2 + 360(\tau_{k+2}-\tau)^2, & \tau \in [\tau_{k+1}, \tau_{k+2}] \\ -60(\tau_{k+3}-\tau)^2, & \tau \in [\tau_{k+2}, \tau_{k+3}] \\ 0. & \text{otherwise} \end{cases} \tag{6}$$

$$B^{(4)}_{5,k}(\tau) = \frac{1}{h^5}\begin{cases} 120(\tau-\tau_{k-3}), & \tau \in [\tau_{k-3}, \tau_{k-2}] \\ 120(\tau-\tau_{k-3}) - 720(\tau-\tau_{k-2}), & \tau \in [\tau_{k-2}, \tau_{k-1}] \\ 120(\tau-\tau_{k-3}) - 720(\tau-\tau_{k-2}) + 2700(\tau-\tau_{k-1}), & \tau \in [\tau_{k-1}, \tau_k] \\ 120(\tau_{k+3}-\tau) - 720(\tau_{k+2}-\tau) + 2700(\tau_{k+1}-\tau), & \tau \in [\tau_k, \tau_{k+1}] \\ 120(\tau_{k+3}-\tau) - 720(\tau_{k+2}-\tau), & \tau \in [\tau_{k+1}, \tau_{k+2}] \\ 120(\tau_{k+3}-\tau), & \tau \in [\tau_{k+2}, \tau_{k+3}] \\ 0. & \text{otherwise} \end{cases} \tag{7}$$

Eight additional knot as $\tau_0 > \tau_{-1} > \tau_{-2} > \tau_{-3} > \tau_{-4}$ and $\tau_N > \tau_{N+1} > \tau_{N+2} > \tau_{N+3} > \tau_{N+4}$ are introduced here. It is simple to confirm from Equation (3) that each function $B_{5,k}(\tau)$ is four times continuous and differentiable throughout the whole real line.

Now, evaluate the QBS function $B_{5,k}(\tau)$ at particular knot $\tau = \tau_m$ as:

$$B_{5,k}(\tau_m) = \begin{cases} 66, & \text{if } k = m \\ 26, & \text{if } k-m = \pm 1 \\ 1, & \text{if } k-m = \pm 2 \\ 0, & \text{if } k-m = \pm 3 \\ 0. & \text{if } k-m = \pm 4 \end{cases} \tag{8}$$

For $\tau < \tau_{k-4}$ and $\tau > \tau_{k+4}$ the QBS function $B_{5,k}(\tau) = 0$.

Similarly,

$$B'_{5,k}(\tau_m) = \begin{cases} 0, & \text{if } k = m \\ \pm\frac{50}{h}, & \text{if } k-m = \pm 1 \\ \pm\frac{5}{h}, & \text{if } k-m = \pm 2 \\ 0, & \text{if } k-m = \pm 3 \\ 0. & \text{if } k-m = \pm 4 \end{cases} \tag{9}$$

For $\tau < \tau_{k-4}$ and $\tau > \tau_{k+4}$ the QBS fucntion $B'_{5,k}(\tau) = 0$.

$$B''_{5,k}(\tau_m) = \begin{cases} -\frac{120}{h^2}, & \text{if } k = m \\ \frac{40}{h^2}, & \text{if } k-m = \pm 1 \\ \frac{20}{h^2}, & \text{if } k-m = \pm 2 \\ 0, & \text{if } k-m = \pm 3 \\ 0. & \text{if } k-m = \pm 4 \end{cases} \tag{10}$$

For $\tau < \tau_{k-4}$ and $\tau > \tau_{k+4}$ the QBS function $B''_{5,k}(\tau) = 0$.

$$B'''_{5,k}(\tau_m) = \begin{cases} 0, & \text{if } k = m \\ \mp\frac{120}{h^3}, & \text{if } k - m = \pm 1 \\ \pm\frac{5}{h^3}, & \text{if } k - m = \pm 2 \\ 0, & \text{if } k - m = \pm 3 \\ 0. & \text{if } k - m = \pm 4 \end{cases} \quad (11)$$

For $\tau < \tau_{k-4}$ and $\tau > \tau_{k+4}$ the QBS function $B'''_{5,k}(\tau) = 0$. And

$$B^{(4)}_{5,k}(\tau_m) = \begin{cases} \frac{720}{h^4}, & \text{if } k = m \\ -\frac{480}{h^4}, & \text{if } k - m = \pm 1 \\ \frac{120}{h^4}, & \text{if } k - m = \pm 2 \\ 0, & \text{if } k - m = \pm 3 \\ 0. & \text{if } k - m = \pm 4 \end{cases} \quad (12)$$

For $\tau < \tau_{k-4}$ and $\tau > \tau_{k+4}$ the QBS function $B^{(4)}_{5,k}(\tau) = 0$.

Table 1 shows the tabular view of QBS function $B_{5,k}(\tau)$ values at a particular knot $\tau = \tau_m$ and its derivatives.

Table 1. The QBS function $B_{5,k}(\tau)$ and its derivatives values at $\tau = \tau_m$.

τ	τ_{i-4}	τ_{i-3}	τ_{i-2}	τ_{i-1}	τ_i	τ_{i+1}	τ_{i+2}	τ_{i+3}	τ_{i+4}
$B_{5,k}(\tau)$	0	0	1	26	66	26	1	0	0
$B'_{5,k}(\tau)$	0	0	$\frac{-5}{h}$	$\frac{-50}{h}$	0	$\frac{50}{h}$	$\frac{5}{h}$	0	0
$B''_{5,k}(\tau)$	0	0	$\frac{20}{h^2}$	$\frac{40}{h^2}$	$\frac{-120}{h^2}$	$\frac{40}{h^2}$	$\frac{20}{h^2}$	0	0
$B'''_{5,k}(\tau)$	0	0	$\frac{-60}{h^3}$	$\frac{120}{h^3}$	0	$\frac{-120}{h^3}$	$\frac{60}{h^3}$	0	0
$B^{(4)}_{5,k}(\tau)$	0	0	$\frac{120}{h^4}$	$\frac{-480}{h^4}$	$\frac{720}{h^4}$	$\frac{-480}{h^4}$	$\frac{120}{h^4}$	0	0

Let $W(\tau)$ represent the QBS interpolation of the knot points for the function $v(\tau)$, so that

$$W(\tau) = \sum_{k=-2}^{N+2} d_k B_{5,k}(\tau), \quad (13)$$

where d_k's are constants (unknown coefficients) and $B_{5,k}(\tau)$'s are QBS basis functions given in Equation (3). Assume that U_k, L_k, M_k, N_k and O_k stand for the corresponding QBS approximations of $v(\tau)$ and its first four derivatives at the kth knot. For the solving fourth-order BVP given in Equation (1), the B-spline approximating function given in Equation (13) evaluated at the particular knot point $\tau = \tau_k$ is necessary. Using Equations (8)–(12) in Equation (13) gives the following relations:

$$U_k = W(\tau_k) = \sum_{k=-2}^{N+2} d_k B_{5,k}(\tau_k) = d_{k-2} + 26d_{k-1} + 66d_k + 26d_{k+1} + d_{k+2}. \quad (14)$$

$$L_k = W'(\tau_k) = \sum_{k=-2}^{N+2} d_k B'_{5,k}(\tau_k) = \frac{5}{h}(-d_{k-2} - 10d_{k-1} + 0d_k + 10d_{k+1} + d_{k+2}). \quad (15)$$

$$M_k = W''(\tau_k) = \sum_{k=-2}^{N+2} d_k B''_{5,k}(\tau_k) = \frac{20}{h^2}(d_{k-2} + 2d_{k-1} - 6d_k + 2d_{k+1} + d_{k+2}). \quad (16)$$

$$N_k = W'''(\tau_k) = \sum_{k=-2}^{N+2} d_k B'''_{5,k}(\tau_k) = \frac{60}{h^3}(-d_{k-2} + 2d_{k-1} + 0d_k - 2d_{k+1} + d_{k+2}). \tag{17}$$

$$O_k = W^{(4)}(\tau_k) = \sum_{k=-2}^{N+2} d_k B^{(4)}_{5,k}(\tau_k) = \frac{120}{h^4}(d_{k-2} - 4d_{k-1} + 6d_k - 4d_{k+1} + d_{k+2}). \tag{18}$$

3. Derivation of the QBS Collocation Technique for SSPNLBVP

For the numerical solution of the fourth-order SSPNLBVP given in Equation (1) with the BCs mentioned in Equation (2), the aforementioned methodology based on the QBS is taken into account in this section. It can be observed that Equation (1) contains the non-linear factor $h(\tau, v)$. Before solving the SSPNLBVP provided in Equation (1), it is necessary to break down the non-linear term $h(\tau, v)$ of the problem into a sequence of linear problems by using the quasi linearization technique. In this technique, $v^{(0)}(\tau)$ stands in for $h(\tau, v)$. Then, $h(\tau, v)$ is spent in terms of the function $v^{(0)}(\tau)$ as:

$$h(\tau, v^{(1)}(\tau)) = h(\tau, v^{(0)}(\tau)) + (v^{(1)}(\tau) - v^{(0)}(\tau)) \times \frac{\partial h}{\partial v(\tau, v^{(0)}(\tau))} + \cdots. \tag{19}$$

Generally, the above Equation (19) can be expressed as:

$$h(\tau, v^{(s+1)}(\tau)) = h(\tau, v^{(s)}(\tau)) + (v^{(s+1)}(\tau) - v^{(s)}(\tau)) \times \frac{\partial h}{\partial v(\tau, v^{(s)}(\tau))} + \cdots, \tag{20}$$

where s is called the iteration index and $s = 0, 1, 2, \ldots$. Then Equation (1) can be approximated as:

$$\begin{aligned}\epsilon v^{(s+1)}_{\tau\tau\tau\tau}(\tau) + \frac{\alpha}{\tau} v^{(s+1)}_{\tau\tau\tau}(\tau) + \frac{\beta}{\tau} v^{(s+1)}_{\tau\tau}(\tau) + \frac{\gamma}{\tau} v^{(s+1)}_{\tau}(\tau) + \frac{\delta}{\tau} v^{(s+1)}(\tau) &= h(\tau, v^{(s)}(\tau)) \\ &+ (v^{(s+1)}(\tau) - v^{(s)}(\tau)) \times \frac{\partial h}{\partial v(\tau, v^{(s)}(\tau))}.\end{aligned} \tag{21}$$

After some simplification,

$$\begin{aligned}\epsilon v^{(s+1)}_{\tau\tau\tau\tau}(\tau) + \frac{\alpha}{\tau} v^{(s+1)}_{\tau\tau\tau}(\tau) + \frac{\beta}{\tau} v^{(s+1)}_{\tau\tau}(\tau) + \frac{\gamma}{\tau} v^{(s+1)}_{\tau}(\tau) + \frac{\delta}{\tau} v^{(s+1)}(\tau) - v^{(s+1)}(\tau) \times \frac{\partial h}{\partial v(\tau, v^{(s)}(\tau))} \\ = h(\tau, v^{(s)}(\tau)) - v^{(s)}(\tau) \times \frac{\partial h}{\partial v(\tau, v^{(s)}(\tau))}.\end{aligned} \tag{22}$$

The Equation (22) is now fourth-order SSPLBVP. It is evident that the Equation (22) contains a singularity at $\tau = 0$ that may be eliminated by applying L. Hospital's method. Accordingly,

$$\begin{cases} (\epsilon + \alpha) v^{(s+1)}_{\tau\tau\tau\tau}(\tau) + \beta v^{(s+1)}_{\tau\tau\tau}(\tau) + \gamma v^{(s+1)}_{\tau\tau}(\tau) + \delta v^{(s+1)}_{\tau}(\tau) & \\ +\xi^s(\tau) v^{(s+1)}(\tau) = g^s(\tau), & \text{for } \tau = 0, \\ \epsilon v^{(s+1)}_{\tau\tau\tau\tau}(\tau) + \frac{\alpha}{\tau} v^{(s+1)}_{\tau\tau\tau}(\tau) + \frac{\beta}{\tau} v^{(s+1)}_{\tau\tau}(\tau) + \frac{\gamma}{\tau} v^{(s+1)}_{\tau}(\tau) & \\ +\frac{\delta}{\tau} v^{(s+1)}(\tau) + \xi^s(\tau) v^{(s+1)}(\tau) = g^s(\tau), & \text{for } \tau \neq 0, \end{cases} \tag{23}$$

where $\xi^s(\tau) = -\frac{\partial h}{\partial v(\tau, u^{(s)}(\tau))}$ and $g^s(\tau) = h(\tau, v^{(s)}(\tau)) - v^{(s)}(\tau) \times \frac{\partial h}{\partial v(\tau, v^{(s)}(\tau))}$. Above Equation (23) can be evaluated at a particular point $\tau = \tau_k$ which gives the following equations:

$$\begin{cases} (\epsilon+\alpha)v^{(s+1)}_{\tau\tau\tau\tau}(\tau_k) + \beta v^{(s+1)}_{\tau\tau\tau}(\tau_k) + \gamma v^{(s+1)}_{\tau\tau}(\tau_k) & \\ +\delta v^{(s+1)}_{\tau}(\tau_k) + \xi^s(\tau_k)v^{(s+1)}(\tau_k) = g^s(\tau_k), & \text{for } k = 0, \\ \epsilon v^{(s+1)}_{\tau\tau\tau\tau}(\tau_k) + \frac{\alpha}{\tau_k}v^{(s+1)}_{\tau\tau\tau}(\tau_k) + \frac{\beta}{\tau_k}v^{(s+1)}_{\tau\tau}(\tau_k) & \\ +\frac{\gamma}{\tau_k}v^{(s+1)}_{\tau}(\tau_k) + \frac{\delta}{\tau_k}u^{(s+1)}(\tau_k) + \xi^s(\tau_k)v^{(s+1)}(\tau_k) = g^s(\tau_k), & \text{for } k = 0, 1, \ldots, N. \end{cases} \tag{24}$$

Assume that the precise solution $v(\tau)$ of Equation (1) has an approximate solution in the form of QBS interpolation $W(\tau)$ as:

$$W^{(s+1)}(\tau) = \sum_{k=-2}^{N+2} d_k^{(s+1)} B_{5,k}(\tau). \tag{25}$$

Thus, the above Equation (25) satisfy Equation (24)

$$\begin{cases} (\epsilon+\alpha)W^{(s+1)}_{\tau\tau\tau\tau}(\tau_k) + \beta W^{(s+1)}_{\tau\tau\tau}(\tau_k) + \gamma W^{(s+1)}_{\tau\tau}(\tau_k) & \\ +\delta W^{(s+1)}_{\tau}(\tau_k) + \xi^s(\tau_k)W^{(s+1)}(\tau_k) = g^s(\tau_k), & \text{for } k = 0, \\ \epsilon W^{(s+1)}_{\tau\tau\tau\tau}(\tau_k) + \frac{\alpha}{\tau_k}W^{(s+1)}_{\tau\tau\tau}(\tau_k) + \frac{\beta}{\tau_k}W^{(s+1)}_{\tau\tau}(\tau_k) & \\ +\frac{\gamma}{\tau_k}W^{(s+1)}_{\tau}(\tau_k) + \delta'^s(\tau_k)W^{(s+1)}(\tau_k) = g^s(\tau_k), & \text{for } k = 0, 1, \ldots, N, \end{cases} \tag{26}$$

with the B.Cs

$$\begin{cases} W(\tau_k) = \rho_0, & \text{for } k = 0 \\ W''(\tau_k) = \rho_1, & \text{for } k = 0 \\ W(\tau_k) = \sigma_0, & \text{for } k = N \\ W''(\tau_k) = \sigma_1. & \text{for } k = N \end{cases} \tag{27}$$

Apply Table 1 to the above Equation (26), and after some simplification, the first equation in (26) yields

$$\begin{aligned} &[120(\epsilon+\alpha) - 60\beta h + 20\gamma h^2 - 5\delta h^3 + \xi^s(\tau_k)h^4]d_{-2}^{(s+1)} + [-480(\epsilon+\alpha) + 120\beta h \\ &+ 40\gamma h^2 - 50\delta h^3 + 26\xi^s(\tau_k)h^4]d_{-1}^{(s+1)} + [720(\epsilon+\alpha) - 120\gamma h^2 + 66\xi^s(\tau_k)h^4]d_0^{(s+1)} + \\ &[-480(\epsilon+\alpha) - 120\beta h + 40\gamma h^2 + 50\delta h^3 + 26\xi^s(\tau_k)h^4]d_1^{(s+1)} + [120(\epsilon+\alpha) + 60\beta h \\ &+ 20\gamma h^2 + 5\delta h^3 + \xi^s(\tau_k)h^4]d_2^{(s+1)} = g^s(\tau_0). \end{aligned}$$

The above equation can be written as:

$$\eta_0 d_{-2}^{(s+1)} + \zeta_0 d_{-1}^{(s+1)} + \theta_0 d_0^{(s+1)} + \lambda_0 d_1^{(s+1)} + \mu_0 d_2^{(s+1)} = g^s(\tau_0), \tag{28}$$

where

$\eta_0 = 120(\epsilon+\alpha) - 60\beta h + 20\gamma h^2 - 5\delta h^3 + \xi^s(\tau_k)h^4$.
$\zeta_0 = -480(\epsilon+\alpha) + 120\beta h + 40\gamma h^2 - 50\delta h^3 + 26\xi^s(\tau_k)h^4$.
$\theta_0 = 720(\epsilon+\alpha) - 120\gamma h^2 + 66\xi^s(\tau_k)h^4$.
$\lambda_0 = -480(\epsilon+\alpha) - 120\beta h + 40\gamma h^2 + 50\delta h^3 + 26\xi^s(\tau_k)h^4$.
$\mu_0 = 120(\epsilon+\alpha) + 60\beta h + 20\gamma h^2 + 5\delta h^3 + \xi^s(\tau_k)h^4$.

Use Table 1 on Equation (26), and after some simplification, the second part of Equation (26) gives

$$(\frac{120\epsilon}{h^4}-\frac{60\alpha}{t_kh^3}+\frac{20\beta}{t_kh^2}-\frac{5\gamma}{t_kh}+\frac{\delta'^s(\tau_k)}{t_k})d_{k-2}^{(s+1)}+(-\frac{480\epsilon}{h^4}+\frac{120\alpha}{t_kh^3}+\frac{40\beta}{t_kh^2}-\frac{50\gamma}{t_kh}$$
$$+\frac{26\delta'^s(\tau_k)}{t_k})d_{k-1}^{(s+1)}+(\frac{720\epsilon}{h^4}-\frac{120\beta}{t_kh^2}+\frac{66\delta'^s(\tau_k)}{t_k})d_{k}^{(s+1)}+(-\frac{480\epsilon}{h^4}-\frac{120\alpha}{t_kh^3}+\frac{40\beta}{t_kh^2}+\frac{50\gamma}{t_kh}$$
$$+\frac{26\delta'^s(\tau_k)}{t_k})d_{k+1}^{(s+1)}+(\frac{120\epsilon}{h^4}+\frac{60\alpha}{t_kh^3}+\frac{20\beta}{t_kh^2}+\frac{5\gamma}{t_kh}+\frac{\delta'^s(\tau_k)}{t_k})d_{k+2}^{(s+1)}=g^s(\tau_k),$$

where $k=1,2,\dots,N$. The above equation can be written as:

$$\eta_k d_{k-2}^{(s+1)}+\zeta_k d_{k-1}^{(s+1)}+\theta_k d_{k}^{(s+1)}+\lambda_k d_{k+1}^{(s+1)}+\mu_k d_{k+2}^{(s+1)}=g^s(\tau_k), \tag{29}$$

where
$\eta_k=\frac{120\epsilon}{h^4}-\frac{60\alpha}{t_kh^3}+\frac{20\beta}{t_kh^2}-\frac{5\gamma}{t_kh}+\frac{\delta'^s(\tau_k)(\tau_k)}{t_k}$.
$\zeta_k=-\frac{480\epsilon}{h^4}+\frac{120\alpha}{t_kh^3}+\frac{40\beta}{t_kh^2}-\frac{50\gamma}{t_kh}+\frac{26\delta'^s(\tau_k)(\tau_k)}{t_k}$.
$\theta_k=\frac{720\epsilon}{h^4}-\frac{120\beta}{t_kh^2}+\frac{66\delta'^s(\tau_k)(\tau_k)}{t_k}$.
$\lambda_k=-\frac{480\epsilon}{h^4}-\frac{120\alpha}{t_kh^3}+\frac{40\beta}{t_kh^2}+\frac{50\gamma}{t_kh}+\frac{26\delta'^s(\tau_k)(\tau_k)}{t_k}$.
$\mu_k=\frac{120\epsilon}{h^4}+\frac{60\alpha}{t_kh^3}+\frac{20\beta}{t_kh^2}+\frac{5\gamma}{t_kh}+\frac{\delta'^s(\tau_k)(\tau_k)}{t_k}$.
For $k=1,2,\dots,N$ in Equation (29)

$$\begin{cases}\eta_1 d_{-1}^{(s+1)}+\zeta_1 d_{0}^{(s+1)}+\theta_1 d_{1}^{(s+1)}+\lambda_1 d_{2}^{(s+1)}+\mu_1 d_{3}^{(s+1)}=g^s(\tau_1), & \text{for } k=1\\ \eta_2 d_{0}^{(s+1)}+\zeta_2 d_{1}^{(s+1)}+\theta_2 d_{2}^{(s+1)}+\lambda_2 d_{3}^{(s+1)}+\mu_2 d_{4}^{(s+1)}=g^s(\tau_2), & \text{for } k=2\\ \vdots & \vdots\\ \eta_{N-1} d_{N-3}^{(s+1)}+\zeta_{N-1} d_{N-2}^{(s+1)}+\theta_{N-1} d_{N-1}^{(s+1)}+\lambda_{N-1} d_{N}^{(s+1)}+\mu_{N-1} d_{N+1}^{(s+1)}=g^s(\tau_{N-1}), & \text{for } k=N-1\\ \eta_N d_{N-2}^{(s+1)}+\zeta_N d_{N-1}^{(s+1)}+\theta_N d_{N}^{(s+1)}+\lambda_N d_{N+1}^{(s+1)}+\mu_N d_{N+2}^{(s+1)}=g^s(\tau_N). & \text{for } k=N\end{cases} \tag{30}$$

From Equations (28) and (29), the following matrix form develop:

$$\begin{pmatrix}\eta_0 & \zeta_0 & \theta_0 & \lambda_0 & \mu_0 & 0 & 0 & \cdots & 0 & 0\\ 0 & \eta_1 & \zeta_1 & \theta_1 & \lambda_1 & \mu_1 & 0 & \cdots & 0 & 0\\ 0 & 0 & \eta_2 & \zeta_2 & \theta_2 & \lambda_2 & \mu_2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & \eta_{N-1} & \zeta_{N-1} & \theta_{N-1} & \lambda_{N-1} & \mu_{N-1} & 0\\ 0 & 0 & \cdots & 0 & 0 & 0\eta_N & \zeta_N & \theta_N & \lambda_N & \mu_N\end{pmatrix}\begin{pmatrix}d_{-2}^{(s+1)}\\ d_{-1}^{(s+1)}\\ d_{0}^{(s+1)}\\ \vdots\\ d_{N+1}^{(s+1)}\\ d_{N+2}^{(s+1)}\end{pmatrix}=\begin{pmatrix}g^s(\tau_0)\\ g^s(\tau_1)\\ g^s(\tau_2)\\ \vdots\\ g^s(\tau_{N-1})\\ g^s(\tau_N)\end{pmatrix} \tag{31}$$

Above Equation (31) can also be written as:

$$AD=F,$$

where

$$A=\begin{pmatrix}\eta_0 & \zeta_0 & \theta_0 & \lambda_0 & \mu_0 & 0 & 0 & \cdots & 0 & 0\\ 0 & \eta_1 & \zeta_1 & \theta_1 & \lambda_1 & \mu_1 & 0 & \cdots & 0 & 0\\ 0 & 0 & \eta_2 & \zeta_2 & \theta_2 & \lambda_2 & \mu_2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & \eta_{N-1} & \zeta_{N-1} & \theta_{N-1} & \lambda_{N-1} & \mu_{N-1} & 0\\ 0 & 0 & \cdots & 0 & 0 & 0\eta_N & \zeta_N & \theta_N & \lambda_N & \mu_N\end{pmatrix}, D=\begin{pmatrix}d_{-2}^{(s+1)}\\ d_{-1}^{(s+1)}\\ d_{0}^{(s+1)}\\ \vdots\\ d_{N+1}^{(s+1)}\\ d_{N+2}^{(s+1)}\end{pmatrix}, F=\begin{pmatrix}g^s(\tau_0)\\ g^s(\tau_1)\\ g^s(\tau_2)\\ \vdots\\ g^s(\tau_{N-1})\\ g^s(\tau_N)\end{pmatrix},$$

where D and F are column matrices and A is a non-singular square matrix of order $(N + 1) \times (N + 5)$. Four more equations, which may be stated using B.Cs given in Equation (27), are required for a unique solution.

$$\begin{cases} d_{-2}^{(s+1)} + 26d_{-1}^{(s+1)} + 66d_{0}^{(s+1)} + 26d_{1}^{(s+1)} + d_{2}^{(s+1)} = \rho_0, \\ \frac{20}{h^2}(d_{-2}^{(s+1)} + 2d_{-1}^{(s+1)} - 6d_{0}^{(s+1)} + 2d_{1}^{(s+1)} + d_{2}^{(s+1)}) = \rho_1, \\ d_{N-2}^{(s+1)} + 26d_{N-1}^{(s+1)} + 66d_{N}^{(s+1)} + 26d_{N+1}^{(s+1)} + d_{N+2}^{(s+1)} = \sigma_0, \\ \frac{20}{h^2}(d_{N-2}^{(s+1)} + 2d_{N-1}^{(s+1)} - 6d_{N}^{(s+1)} + 2d_{N+1}^{(s+1)} + d_{N+2}^{(s+1)}) = \sigma_1. \end{cases} \tag{32}$$

Substituting Equation (32) into Equation (31)

$$\begin{pmatrix} \nu_{1,1} & \nu_{1,2} & \nu_{1,3} & \nu_{1,4} & \nu_{1,5} & 0 & 0 & \cdots & 0 & 0 \\ \nu_{2,1} & \nu_{2,2} & \nu_{2,3} & \nu_{2,4} & \nu_{2,5} & 0 & 0 & \cdots & 0 & 0 \\ \nu_{3,1} & \nu_{3,2} & \nu_{3,3} & \nu_{3,4} & \nu_{3,5} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \nu_{4,1} & \nu_{4,2} & \nu_{4,3} & \nu_{4,4} & \nu_{4,5} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \nu_{5,1} & \nu_{5,2} & \nu_{5,3} & \nu_{5,4} & \nu_{5,5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \nu_{N+2,N} & \nu_{N+2,N+1} & \nu_{N+2,N+2} & \nu_{N+2,N+3} & \nu_{N+2,N+4} & 0 \\ 0 & 0 & \cdots & 0 & 0 & \nu_{N+3,N+1} & \nu_{N+3,N+2} & \nu_{N+3,N+3} & \nu_{N+3,N+4} & \nu_{N+3,N+5} \\ 0 & 0 & \cdots & 0 & 0 & \nu_{N+4,N+1} & \nu_{N+4,N+2} & \nu_{N+4,N+3} & \nu_{N+4,N+4} & \nu_{N+4,N+5} \\ 0 & 0 & \cdots & 0 & 0 & \nu_{N+5,N+1} & \nu_{N+5,N+2} & \nu_{N+5,N+3} & \nu_{N+5,N+4} & \nu_{N+5,N+5} \end{pmatrix} \begin{pmatrix} d_{-2}^{(s+1)} \\ d_{-1}^{(s+1)} \\ d_{0}^{(s+1)} \\ d_{1}^{(s+1)} \\ d_{2}^{(s+1)} \\ \vdots \\ d_{N-1}^{(s+1)} \\ d_{N}^{(s+1)} \\ d_{N+1}^{(s+1)} \\ d_{N+2}^{(s+1)} \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \sigma_0 \\ g^s(\tau_0) \\ g^s(\tau_1) \\ g^s(\tau_2) \\ \vdots \\ g^s(\tau_{N-1}) \\ g^s(\tau_N) \\ \sigma_1 \\ \rho_1 \end{pmatrix}.$$

Above equation can also be written as:

$$AD = F, \tag{33}$$

where

$$A = \begin{pmatrix} \nu_{1,1} & \nu_{1,2} & \nu_{1,3} & \nu_{1,4} & \nu_{1,5} & 0 & 0 & \cdots & 0 & 0 \\ \nu_{2,1} & \nu_{2,2} & \nu_{2,3} & \nu_{2,4} & \nu_{2,5} & 0 & 0 & \cdots & 0 & 0 \\ \nu_{3,1} & \nu_{3,2} & \nu_{3,3} & \nu_{3,4} & \nu_{3,5} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \nu_{4,1} & \nu_{4,2} & \nu_{4,3} & \nu_{4,4} & \nu_{4,5} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \nu_{5,1} & \nu_{5,2} & \nu_{5,3} & \nu_{5,4} & \nu_{5,5} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \nu_{N+2,N} & \nu_{N+2,N+1} & \nu_{N+2,N+2} & \nu_{N+2,N+3} & \nu_{N+2,N+4} & 0 \\ 0 & 0 & \cdots & 0 & 0 & \nu_{N+3,N+1} & \nu_{N+3,N+2} & \nu_{N+3,N+3} & \nu_{N+3,N+4} & \nu_{N+3,N+5} \\ 0 & 0 & \cdots & 0 & 0 & \nu_{N+4,N+1} & \nu_{N+4,N+2} & \nu_{N+4,N+3} & \nu_{N+4,N+4} & \nu_{N+4,N+5} \\ 0 & 0 & \cdots & 0 & 0 & \nu_{N+5,N+1} & \nu_{N+5,N+2} & \nu_{N+5,N+3} & \nu_{N+5,N+4} & \nu_{N+5,N+5} \end{pmatrix},$$

$$D = \begin{pmatrix} d_{-2}^{(s+1)} \\ d_{-1}^{(s+1)} \\ d_{0}^{(s+1)} \\ d_{1}^{(s+1)} \\ d_{2}^{(s+1)} \\ \vdots \\ d_{N-1}^{(s+1)} \\ d_{N}^{(s+1)} \\ d_{N+1}^{(s+1)} \\ d_{N+2}^{(s+1)} \end{pmatrix} \text{ and } F = \begin{pmatrix} \rho_0 \\ \sigma_0 \\ g^s(\tau_0) \\ g^s(\tau_1) \\ g^s(\tau_2) \\ \vdots \\ g^s(\tau_{N-1}) \\ g^s(\tau_N) \\ \sigma_1 \\ \rho_1 \end{pmatrix}.$$

If matrix A is a non-singular square matrix, then Equation (33) gives us a solution; otherwise, SSPNLBVP provided in Equation (1) has no solution. The order of A is $(N + 5) \times (N + 5)$. The order of D is $(N + 5) \times (1)$ and the order of F is the same as that of D. After calculating values of $d_{-2}^{(s+1)}, d_{-1}^{(s+1)}, d_{0}^{(s+1)}, \ldots, d_{N}^{(s+1)}, d_{N+1}^{(s+1)}, d_{N+2}^{(s+1)}$ from

Equation (33) and then substitute these values into Equation (33). Then Equation (33) can provide us with a very accurate approximate solution that guarantees to be matchable with the given exact solution, if any exist.

Where the matrix A has following entries:

$\nu_{1,1} = 1, \nu_{1,2} = 26, \nu_{1,3} = 66, \nu_{1,4} = 26, \nu_{1,5} = 1.$
$\nu_{2,1} = 20, \nu_{2,2} = 40, \nu_{2,3} = -120, \nu_{2,4} = 40, \nu_{5,5} = 20.$
$\nu_{3,1} = \eta_0 = 120(\epsilon + \alpha) - 60\beta h + 20\gamma h^2 - 5\delta h^3 + \xi^s h^4,$
$\nu_{3,2} = \zeta_0 = -480(\epsilon + \alpha) + 20\beta h + 40\gamma h^2 - 50\delta h^3 + 26\xi^s h^4,$
$\nu_{3,3} = \theta_0 = 720(\epsilon + \alpha) - 120\gamma h^2 + 66\xi^s h^4,$
$\nu_{3,4} = \lambda_0 = -480(\epsilon + \alpha) - 120\beta h + 40\gamma h^2 + 50\delta h^3 + 26\xi^s h^4,$
$\nu_{3,5} = \mu_0 = 120(\epsilon + \alpha) + 60\beta h + 20\gamma h^2 + 5\delta h^3 + \xi^s h^4.$
$\nu_{4,1} = \eta_1 = \frac{120\epsilon}{h^4} - \frac{60\alpha}{t_1 h^3} + \frac{20\beta}{t_1 h^2} - \frac{5\gamma}{t_1 h} + \frac{\delta'^s}{t_1},$
$\nu_{4,2} = \zeta_1 = -\frac{480\epsilon}{h^4} + \frac{120\alpha}{t_1 h^3} + \frac{40\beta}{t_1 h^2} - \frac{50\gamma}{t_1 h} + \frac{26\delta'^s}{t_1},$
$\nu_{4,3} = \theta_1 = \frac{720\epsilon}{h^4} - \frac{120\beta}{t_1 h^2} + \frac{66\delta'^s}{t_1},$
$\nu_{4,4} = \lambda_1 = -\frac{480\epsilon}{h^4} - \frac{120\alpha}{t_1 h^3} + \frac{40\beta}{t_1 h^2} + \frac{50\delta'^s}{t_1 h} + \frac{26\delta}{t_1},$
$\nu_{4,5} = \mu_1 = \frac{120\epsilon}{h^4} + \frac{60\alpha}{t_1 h^3} + \frac{20\beta}{t_1 h^2} + \frac{5\delta'^s}{t_1 h} + \frac{\delta'^s}{t_1}.$
$\nu_{5,1} = \eta_2 = \frac{120\epsilon}{h^4} - \frac{60\alpha}{t_2 h^3} + \frac{20\beta}{t_2 h^2} - \frac{5\gamma}{t_2 h} + \frac{\delta'^s}{t_2},$
$\nu_{5,2} = \zeta_2 = -\frac{480\epsilon}{h^4} + \frac{120\alpha}{t_2 h^3} + \frac{40\beta}{t_2 h^2} - \frac{50\gamma}{t_2 h} + \frac{26\delta'^s}{t_2},$
$\nu_{5,3} = \theta_2 = \frac{720\epsilon}{h^4} - \frac{120\beta}{t_2 h^2} + \frac{66\delta'^s}{t_2},$
$\nu_{5,4} = \lambda_2 = -\frac{480\epsilon}{h^4} - \frac{120\alpha}{t_2 h^3} + \frac{40\beta}{t_2 h^2} + \frac{50\gamma}{t_2 h} + \frac{26\delta'^s}{t_2},$
$\nu_{5,5} = \mu_2 = \frac{120\epsilon}{h^4} + \frac{60\alpha}{t_2 h^3} + \frac{20\beta}{t_2 h^2} + \frac{5\gamma}{t_2 h} + \frac{\delta'^s}{t_2}.$

$$\vdots$$

$\nu_{N+2,N} = \eta_{N-1} = \frac{120\epsilon}{h^4} - \frac{60\alpha}{t_{N-1} h^3} + \frac{20\beta}{t_{N-1} h^2} - \frac{5\gamma}{t_{N-1} h} + \frac{\delta'^s}{t_{N-1}},$
$\nu_{N+2,N+1} = \zeta_{N-1} = -\frac{480\epsilon}{h^4} + \frac{120\alpha}{t_{N-1} h^3} + \frac{40\beta}{t_{N-1} h^2} - \frac{50\gamma}{t_{N-1} h} + \frac{26\delta'^s}{t_{N-1}},$
$\nu_{N+2,N+2} = \theta_{N-1} = \frac{720\epsilon}{h^4} - \frac{120\beta}{t_{N-1} h^2} + \frac{66\delta'^s}{t_{N-1}},$
$\nu_{N+2,N+3} = \lambda_{N-1} = -\frac{480\epsilon}{h^4} - \frac{120\alpha}{t_{N-1} h^3} + \frac{40\beta}{t_{N-1} h^2} + \frac{50\gamma}{t_{N-1} h} + \frac{26\delta'^s}{t_{N-1}},$
$\nu_{N+2,N+4} = \mu_{N-1} = \frac{120\epsilon}{h^4} + \frac{60\alpha}{t_{N-1} h^3} + \frac{20\beta}{t_{N-1} h^2} + \frac{5\gamma}{t_{N-1} h} + \frac{\delta'^s}{t_{N-1}}.$
$\nu_{N+3,N+1} = \eta_N = \frac{120\epsilon}{h^4} - \frac{60\alpha}{t_N h^3} + \frac{20\beta}{t_N h^2} - \frac{5\gamma}{t_N h} + \frac{\delta'^s}{t_N},$
$\nu_{N+3,N+2} = \zeta_N = -\frac{480\epsilon}{h^4} + \frac{120\alpha}{t_N h^3} + \frac{40\beta}{t_N h^2} - \frac{50\gamma}{t_N h} + \frac{26\delta'^s}{t_N},$
$\nu_{N+3,N+3} = \theta_N = \frac{720\epsilon}{h^4} - \frac{120\beta}{t_N h^2} + \frac{66\delta'^s}{t_N},$
$\nu_{N+3,N+4} = \lambda_N = -\frac{480\epsilon}{h^4} - \frac{120\alpha}{t_N h^3} + \frac{40\beta}{t_N h^2} + \frac{50\gamma}{t_N h} + \frac{26\delta'^s}{t_N},$
$\nu_{N+3,N+5} = \mu_N = \frac{120\epsilon}{h^4} + \frac{60\alpha}{t_N h^3} + \frac{20\beta}{t_N h^2} + \frac{5\gamma}{t_N h} + \frac{\delta'^s}{t_N}.$
$\nu_{N+4,N+1} = 20, \nu_{N+4,N+2} = 40, \nu_{N+4,N+3} = -120, \nu_{N+4,N+4} = 40, \nu_{N+4,N+5} = 20.$
$\nu_{N+5,N+1} = 1, \nu_{N+5,N+2} = 26, \nu_{N+5,N+3} = 66, \nu_{N+5,N+4} = 26, \nu_{N+5,N+5} = 1.$

4. Error Analysis

This section designates a method for computing TE for the QBS technique across the range $0 \leq \tau \leq 1$. In [17,26], the QBS approximations are used to provide the following relations, which may be shown using Equations (14)–(18).

Equations (14) and (15) give the following relation:

$$h[W'(\tau_{k-2}) + 26W'(\tau_{k-1}) + 66W'(\tau_k) + 26W'(\tau_{k+1}) + W'(\tau_{k+2})] \\ = 5[-v(\tau_{k-2}) - 10v(\tau_{k-1}) + 10v(\tau_{k+1}) + v(\tau_{k+2})]. \quad (34)$$

Equations (14) and (16) give the following relation:

$$h^2[W''(\tau_{k-2})+26W''(\tau_{k-1})+66W''(\tau_k)+26W''(\tau_{k+1})+W''(\tau_{k+2})]$$
$$=20[v(\tau_{k-2})+2v(\tau_{k-1})-6v(\tau_k)+2v(\tau_{k+1})+v(\tau_{k+2})]. \quad (35)$$

Equations (14) and (17) give the following relation:

$$h^3[W'''(\tau_{k-2})+26W'''(\tau_{k-1})+66W'''(\tau_k)+26W'''(\tau_{k+1})+W'''(\tau_{k+2})]$$
$$=60[-v(\tau_{k-2})+2v(\tau_{k-1})-2v(\tau_{k+1})+v(\tau_{k+2})]. \quad (36)$$

Equations (14) and (18) give the following relation:

$$h^4[W^{(4)}(\tau_{k-2})+26W^{(4)}(\tau_{k-1})+66W^{(4)}(\tau_k)+26W^{(4)}(\tau_{k+1})+W^{(4)}(\tau_{k+2})]$$
$$=120[v(\tau_{k-2})-4v(\tau_{k-1})+6v(\tau_k)-4v(\tau_{k+1})+v(\tau_{k+2})]. \quad (37)$$

By means of the operator notation $E^m(W^{(n)}(\tau_j)) = W^{(n)}_{j+m}$, m $\in \mathbb{Z}$ and n is any order of derivative, Equations (34)–(37) can be expressed as:

$$h[E^{-2}+26E^{-1}+66+26E^1+E^2]W'(\tau_k)=5[-E^{-2}-10E^{-1}+10E^1+E^2]v(\tau_k), \quad (38)$$

$$h^2[E^{-2}+26E^{-1}+66+26E^1+E^2]W''(\tau_k)=20[E^{-2}+2E^{-1}-6+2E^1+E^2]v(\tau_k), \quad (39)$$

$$h^3[E^{-2}+26E^{-1}+66+26E^1+E^2]W'''(\tau_k)=60[-E^{-2}+2E^{-1}-2E^1+E^2]v(\tau_k), \quad (40)$$

$$h^4[E^{-2}+26E^{-1}+66+26E^1+E^2]W^{(4)}(\tau_k)=120[E^{-2}-4E^{-1}+6-4E^1+E^2]v(\tau_k). \quad (41)$$

After some simplification

$$W'(\tau_k)=\frac{5(-E^{-2}-10E^{-1}+10E^1+E^2)}{h(E^{-2}+26E^{-1}+66+26E^1+E^2)}v(\tau_k), \quad (42)$$

$$W''(\tau_k)=\frac{20(E^{-2}+2E^{-1}-6+2E^1+E^2)}{h^2(E^{-2}+26E^{-1}+66+26E^1+E^2)}v(\tau_k), \quad (43)$$

$$W'''(\tau_k)=\frac{60(-E^{-2}+2E^{-1}-2E^1+E^2)}{h^3(E^{-2}+26E^{-1}+66+26E^1+E^2)}v(\tau_k), \quad (44)$$

$$W^{(4)}(\tau_k)=\frac{120(E^{-2}-4E^{-1}+6-4E^1+E^2)}{h^4(E^{-2}+26E^{-1}+66+26E^1+E^2)}v(\tau_k). \quad (45)$$

Employing $E^m = e^{mhD}$ where $m \in \mathbb{Z}$ and $Dv(\tau_k) = v'(\tau_k)$, $D^2v(\tau_k) = v''(\tau_k), \ldots,$ $D^{(n)}v(\tau_k) = v^{(n)}(\tau_k)$. Equations (42)–(45) can be written as:

$$W'(\tau_k)=\frac{5(-e^{-2hD}-10e^{-hD}+10e^{hD}+e^{2hD})}{h(e^{-2hD}+26e^{-hD}+66+26e^{hD}+e^{2hD})}v(\tau_k), \quad (46)$$

$$W''(\tau_k)=\frac{20(e^{-2hD}+2e^{-hD}-6+2e^{hD}+e^{2hD})}{h^2(e^{-2hD}+26e^{-hD}+66+26e^{hD}+e^{2hD})}v(\tau_k), \quad (47)$$

$$W'''(\tau_k)=\frac{60(-e^{-2hD}+2e^{-hD}-2e^{hD}+e^{2hD})}{h^3(e^{-2hD}+26e^{-hD}+66+26e^{hD}+e^{2hD})}v(\tau_k), \quad (48)$$

$$W^{(4)}(\tau_k) = \frac{120(e^{-2hD} - 4e^{-hD} + 6 - 4e^{hD} + e^{2hD})}{h^4(e^{-2hD} + 26e^{-hD} + 66 + 26e^{hD} + e^{2hD})} v(\tau_k). \tag{49}$$

Expand the series expansions of exponential functions in powers of hD.

$$e^{hD} = 1 + Dh + \frac{D^2h^2}{2} + \frac{D^3h^3}{6} + \frac{D^4h^4}{24} + \frac{D^5h^5}{120}. \tag{50}$$

$$e^{-hD} = 1 - Dh + \frac{D^2h^2}{2} - \frac{D^3h^3}{6} + \frac{D^4h^4}{24} - \frac{D^5h^5}{120}. \tag{51}$$

$$e^{2hD} = 1 + 2Dh + 2D^2h^2 + \frac{4D^3h^3}{3} + \frac{2D^4h^4}{3} + \frac{4D^5h^5}{15}. \tag{52}$$

$$e^{-2hD} = 1 - 2Dh + 2D^2h^2 - \frac{4D^3h^3}{3} + \frac{2D^4h^4}{3} - \frac{4D^5h^5}{15}. \tag{53}$$

Substituting Equations (50)–(53) into Equations (46)–(49) then simplifying gives the following equations:

$$\begin{aligned} W'(\tau_k) = v'(\tau_k) &+ \frac{1}{5040}h^6 v^{(7)}(\tau_k) - \frac{1}{21600}h^8 v^{(9)}(\tau_k) + \frac{1}{190080}h^{10} v^{(11)}(\tau_k) \\ &- \frac{583}{1415232000}h^{12} v^{(13)}(\tau_k) + \frac{19}{870912000}h^{14} v^{(15)}(\tau_k) + O(h^{15}), \end{aligned} \tag{54}$$

$$\begin{aligned} W''(\tau_k) = v''(\tau_k) &+ \frac{1}{720}h^4 v^{(6)}(\tau_k) - \frac{1}{3360}h^6 v^{(8)}(\tau_k) + \frac{1}{86400}h^8 v^{(10)}(\tau_k) \\ &+ \frac{221}{239500800}h^{10} v^{(12)}(\tau_k) - \frac{1681}{8805888000}h^{12} v^{(14)}(\tau_k) + \frac{433}{19160064000}h^{14} v^{(16)}(\tau_k) + O(h^{15}), \end{aligned} \tag{55}$$

$$\begin{aligned} W'''(\tau_k) = v'''(\tau_k) &- \frac{1}{240}h^4 v^{(7)}(\tau_k) + \frac{11}{30240}h^6 v^{(9)}(\tau_k) - \frac{1}{28800}h^8 v^{(11)}(\tau_k) \\ &+ \frac{37}{11404800}h^{10} v^{(13)}(\tau_k) - \frac{2993}{15850598400}h^{12} v^{(15)}(\tau_k) - \frac{1}{6386688000}h^{14} v^{(17)}(\tau_k) + O(h^{15}), \end{aligned} \tag{56}$$

$$\begin{aligned} W^{(4)}(\tau_k) = v^{(4)}(\tau_k) &- \frac{1}{12}h^2 v^{(6)}(\tau_k) + \frac{1}{240}h^4 v^{(8)}(\tau_k) - \frac{1}{7560}h^6 v^{(10)}(\tau_k) - \frac{13}{907200}h^8 v^{(12)}(\tau_k) \\ &+ \frac{643}{159667200}h^{10} v^{(14)}(\tau_k) - \frac{465737}{871782912000}h^{12} v^{(16)}(\tau_k) + \frac{196843}{3923023104000}h^{14} v^{(18)}(\tau_k) + O(h^{15}). \end{aligned} \tag{57}$$

Currently, the error term is defined at the kth knot as $e(\tau_k) = W(\tau_k) - v(\tau_k)$. Substituting Equations (54)–(57) into error term,

$$\begin{aligned} e'(\tau_k) = W'(\tau_k) - v'(\tau_k) &= \frac{1}{5040}h^6 v^{(7)}(\tau_k) - \frac{1}{21600}h^8 v^{(9)}(\tau_k) + \frac{1}{190080}h^{10} v^{(11)}(\tau_k) \\ &- \frac{583}{1415232000}h^{12} v^{(13)}(\tau_k) + \frac{19}{870912000}h^{14} v^{(15)}(\tau_k) + O(h^{15}), \end{aligned} \tag{58}$$

$$\begin{aligned} e''(\tau_k) = W''(\tau_k) - v''(\tau_k) &= \frac{1}{720}h^4 v^{(6)}(\tau_k) - \frac{1}{3360}h^6 v^{(8)}(\tau_k) + \frac{1}{86400}h^8 v^{(10)}(\tau_k) \\ &+ \frac{221}{239500800}h^{10} v^{(12)}(\tau_k) - \frac{1681}{8805888000}h^{12} v^{(14)}(\tau_k) + \frac{433}{19160064000}h^{14} v^{(16)}(\tau_k) + O(h^{15}), \end{aligned} \tag{59}$$

$$e'''(\tau_k) = W'''(\tau_k) - v'''(\tau_k) = -\frac{1}{240}h^4 v^{(7)}(\tau_k) + \frac{11}{30240}h^6 v^{(9)}(\tau_k) - \frac{1}{28800}h^8 v^{(11)}(\tau_k) + \frac{37}{11404800}h^{10} v^{(13)}(\tau_k) - \frac{2993}{15850598400}h^{12} v^{(15)}(\tau_k) - \frac{1}{6386688000}h^{14} v^{(17)}(\tau_k) + O(h^{15}), \quad (60)$$

$$e^{(4)}(\tau_k) = W^{(4)}(\tau_k) - v^{(4)}(\tau_k) = -\frac{1}{12}h^2 v^{(6)}(\tau_k) + \frac{1}{240}h^4 v^{(8)}(\tau_k) - \frac{1}{7560}h^6 v^{(10)}(\tau_k) - \frac{13}{907200}h^8 v^{(12)}(\tau_k) + \frac{643}{159667200}h^{10} v^{(14)}(\tau_k) - \frac{465737}{871782912000}h^{12} v^{(16)}(\tau_k) + \frac{196843}{3923023104000}h^{14} v^{(18)}(\tau_k) + O(h^{15}). \quad (61)$$

Employing the Equations (58)–(61) in the TS expansion of the error term

$$e(\tau_k + \phi h) = \frac{\phi^2(1-5\phi^2)}{1440}h^6 v^{(6)}(\tau_k) + \frac{\phi(2-7\phi^2)}{10080}h^7 v^{(7)}(\tau_k) + \frac{\phi^2(-6+7\phi^2)}{40320}h^8 v^{(8)}(\tau_k) + \frac{\phi(-42+55\phi^2)}{907200}h^9 v^{(9)}(\tau_k) + \frac{\phi^2(21-20\phi^2)}{3628800}h^{10} v^{(10)}(\tau_k) + \frac{\phi(10-11\phi^2)}{1900800}h^{11} v^{(11)}(\tau_k) - \frac{\phi^2(-17+22\phi^2)}{479001600}h^{12} v^{(12)}(\tau_k) + \frac{\phi(-12826+16835\phi)}{31135104000}h^{13} v^{(13)}(\tau_k) + \frac{\phi^2(-166419+292565\phi^2)}{1743565824000}h^{14} v^{(14)}(\tau_k) + O(h^{15}), \quad (62)$$

where, $0 \leq \phi \leq 1$. Observe Equation (62), where the TE of the improved QBS approximation is clearly $O(h^6)$.

5. Numerical Findings and Consensus

To demonstrate the effectiveness and dependability of the suggested QBS approach for the solution of fourth-order SSPLBVPs and SSPNLBVPs, four problems are taken into consideration in this section. The experimental outcomes of the novel QBS approximation approach are also displayed. The residual error represents the differences between the observed $v(\tau)$ values and the corresponding approximate values $W(\tau)$.

Example 1. *Take into account the subsequent fourth-order SSPLBVP [29]:*

$$\epsilon v^{(4)}(\tau) + \frac{1}{\tau}v''(\tau) + \frac{1}{\tau}v(\tau) = e^{\tau}\left(\epsilon(\tau+4) + 2 + \frac{2}{\tau}\right) - \frac{2}{\tau} + \frac{8}{3} - \frac{7}{2}e - \tau + \left(\frac{1}{3} - \frac{e}{2}\right)\tau^2, \ 0 \leq \tau \leq 1, \quad (63)$$

with the B.Cs

$$v(0) = v(1) = 0, v''(0) = v''(1) = 0. \quad (64)$$

Example 1's exact solution is $v(\tau) = \tau e^{\tau} + (\frac{2}{3} - \frac{1}{2}e)\tau - \tau^2 + (\frac{1}{3} - \frac{1}{2}e)\tau^3$. The results of the numerical calculations used in Example 1 are displayed in Tables 2 and 3. The point-wise AEs are shown in Tables 2 and 3, respectively, for $\epsilon = 0.0625$ and $N = 10$ and $\epsilon = 0.0001$ and $N = 10$, respectively. When $\epsilon = 0.0001$ and $\epsilon = 0.0625$, Figure 1 illustrates how the exact and approximate solutions to Example 1 behave. Table 4 shows the maximum AEs for various tiny values of ϵ for various values of N. When $N = 500$ and $\epsilon = 0.0001$, Table 5 compares point-wise AEs for the solution to Example 1 between the proposed technique and QBS in [29].

Table 2. With $h = \frac{1}{10}$ and $\epsilon = \frac{1}{16}$, point-by-point errors in the solution to Example 1.

τ	Exact Solution	Approximate Solution	Absolute Error
0.0	-0.00000	-0.00000	0.00000×10^{-00}
0.1	-0.07994	-0.07994	9.15934×10^{-16}
0.2	-0.15252	-0.15252	7.21645×10^{-16}
0.3	-0.21126	-0.21126	1.83187×10^{-15}
0.4	-0.25086	-0.25086	4.05231×10^{-15}
0.5	-0.26739	-0.26739	5.55112×10^{-16}
0.6	-0.25857	-0.25857	2.66454×10^{-15}
0.7	-0.22406	-0.22406	3.05311×10^{-15}
0.8	-0.16582	-0.16582	3.33067×10^{-16}
0.9	-0.08852	-0.08852	2.85882×10^{-15}
1.0	-0.00000	-0.00000	0.00000×10^{-00}

Table 3. With $h = \frac{1}{10}$ and $\epsilon = \frac{1}{10{,}000}$, point-by-point errors in the solution to Example 1.

τ	Exact Solution	Approximate Solution	Absolute Error
0.0	-0.00000	-0.00000	0.00000×10^{-00}
0.1	-0.07994	-0.07994	0.00000×10^{-00}
0.2	-0.15252	-0.15252	2.77556×10^{-17}
0.3	-0.21126	-0.21126	0.00000×10^{-00}
0.4	-0.25086	-0.25086	5.55112×10^{-17}
0.5	-0.26739	-0.26739	2.22045×10^{-16}
0.6	-0.25857	-0.25857	5.55112×10^{-17}
0.7	-0.22406	-0.22406	0.00000×10^{-00}
0.8	-0.16582	-0.16582	2.77556×10^{-17}
0.9	-0.08852	-0.08852	1.38778×10^{-16}
1.0	-0.00000	-0.00000	2.77556×10^{-17}

Table 4. Maximum AEs of Example 1 with varying combinations of N and $\epsilon = 10^{-m}$.

$\epsilon = 10^{-m}$	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
10^{-00}	1.05471×10^{-15}	4.48530×10^{-14}	1.36713×10^{-12}	2.63349×10^{-11}	3.84210×10^{-10}
10^{-01}	8.88178×10^{-16}	7.53841×10^{-14}	1.07947×10^{-12}	2.24331×10^{-11}	2.77389×10^{-10}
10^{-02}	9.99201×10^{-16}	4.06342×10^{-14}	5.46119×10^{-13}	1.11223×10^{-11}	1.48288×10^{-10}
10^{-03}	1.11022×10^{-16}	5.55112×10^{-15}	9.25926×10^{-14}	1.50002×10^{-12}	3.02446×10^{-11}
10^{-04}	2.77556×10^{-17}	4.44089×10^{-16}	1.25455×10^{-14}	1.59650×10^{-13}	3.13072×10^{-12}
10^{-05}	2.77556×10^{-17}	5.55112×10^{-17}	1.33227×10^{-15}	2.57572×10^{-14}	2.71450×10^{-13}
10^{-06}	1.38778×10^{-17}	2.77556×10^{-17}	1.11022×10^{-16}	1.99840×10^{-15}	2.69784×10^{-14}
10^{-07}	1.38778×10^{-17}	2.77556×10^{-17}	2.77556×10^{-17}	2.22045×10^{-16}	3.77476×10^{-15}
10^{-08}	1.38778×10^{-17}	1.38778×10^{-17}	2.77556×10^{-17}	2.77556×10^{-17}	3.33067×10^{-16}
10^{-09}	1.38778×10^{-17}	2.77556×10^{-17}	2.77556×10^{-17}	2.77556×10^{-17}	5.55112×10^{-17}
10^{-10}	1.38778×10^{-17}	2.77556×10^{-17}	2.77556×10^{-17}	2.77556×10^{-17}	2.77556×10^{-17}

Table 5. Point-by-point AEs comparison between the proposed method and QBS in [29] for the solution of Example 1, when $N = 500$ and $\epsilon = 0.0001$.

τ	Method in [29]	Proposed Method
0.000	0.00000×10^{-00}	0.00000×10^{-00}
0.002	4.27811×10^{-14}	1.55989×10^{-16}
0.018	3.96132×10^{-13}	3.21670×10^{-16}
0.034	7.40092×10^{-13}	1.32567×10^{-15}
0.124	2.70092×10^{-12}	3.79909×10^{-15}
0.220	4.56300×10^{-12}	6.30401×10^{-15}
0.500	9.10411×10^{-12}	3.34622×10^{-14}
0.720	1.10860×10^{-11}	4.73746×10^{-14}
0.780	1.07286×10^{-11}	1.89879×10^{-14}
0.876	8.47087×10^{-12}	3.25088×10^{-14}
1.982	3.18125×10^{-12}	4.19127×10^{-14}
1.996	2.10516×10^{-12}	2.10516×10^{-14}
1.000	0.00000×10^{-00}	0.00000×10^{-00}

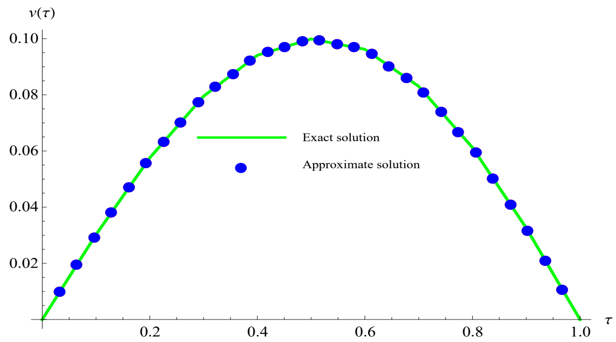

Figure 1. When $h = \frac{1}{10}$, $\epsilon = 0.0625$, and $\epsilon = 0.0001$, the attitude of exact and approximate solution of Example 1.

Example 2. *Think about the subsequent fourth-order SSPLBVP [29]:*

$$-\epsilon v^{(4)}(\tau) - \frac{1}{\tau}v(\tau) = e^{\tau}\Big(\epsilon(8 + 7\tau + \tau^2) - (1-\tau)\Big) + \frac{2}{3}e(1-\tau^2), 0 \le \tau \le 1, \tag{65}$$

with the B.Cs

$$v(0) = v(1) = 0, v''(0) = v''(1) = 0. \tag{66}$$

Example 2's exact solution is $v(\tau) = \tau(1-\tau)e^{\tau} - \frac{2}{3}e\tau(1-\tau^2)$. The results of the numerical calculations used in Example 2 are displayed in Tables 6 and 7. The point-wise AEs are shown in Tables 6 and 7, respectively, for $\epsilon = 0.0625$ and $N = 10$ and $\epsilon = 0.0001$ and $N = 10$, respectively. When $\epsilon = 0.0001$ and $\epsilon = 0.0625$, Figure 2 illustrates how the exact and approximate solutions to Example 2 behave. Table 8 portrays the maximum AEs for various tiny values of ϵ for various values of N. When $N = 500$ and $\epsilon = 0.0001$, Table 9 compares point-wise AEs for the solution to Example 2 between the proposed method and QBS in [29].

Table 6. With $h = \frac{1}{10}$ and $\epsilon = \frac{1}{16}$, point-by-point errors in the solution to Example 2.

τ	Exact Solution	Approximate Solution	Absolute Error
0.0	0.00000	0.00000	0.00000×10^{-00}
0.1	0.03024	0.03024	3.33067×10^{-16}
0.2	0.05758	0.05758	9.15934×10^{-16}
0.3	0.07952	0.07952	4.44089×10^{-16}
0.4	0.09409	0.09409	6.66134×10^{-16}
0.5	0.09990	0.09990	7.77156×10^{-16}
0.6	0.09621	0.09621	1.33227×10^{-15}
0.7	0.08304	0.08304	2.22045×10^{-16}
0.8	0.06124	0.06124	5.73847×10^{-15}
0.9	0.03260	0.03260	4.44089×10^{-16}
1.0	0.00000	0.00000	0.00000×10^{-00}

Table 7. With $h = \frac{1}{10}$ and $\epsilon = \frac{1}{10{,}000}$, point-by-point errors in the solution to Example 2.

τ	Exact Solution	Approximate Solution	Absolute Error
0.0	0.00000	0.00000	0.00000×10^{-00}
0.1	0.03024	0.03024	3.46945×10^{-18}
0.2	0.05758	0.05758	6.93889×10^{-18}
0.3	0.07952	0.07952	4.16334×10^{-17}
0.4	0.09409	0.09409	2.77556×10^{-17}
0.5	0.09990	0.09990	4.16334×10^{-17}
0.6	0.09621	0.09621	0.00000×10^{-00}
0.7	0.08304	0.08304	0.00000×10^{-00}
0.8	0.06124	0.06124	2.77556×10^{-17}
0.9	0.03260	0.03260	6.93889×10^{-18}
1.0	0.00000	0.00000	0.00000×10^{-00}

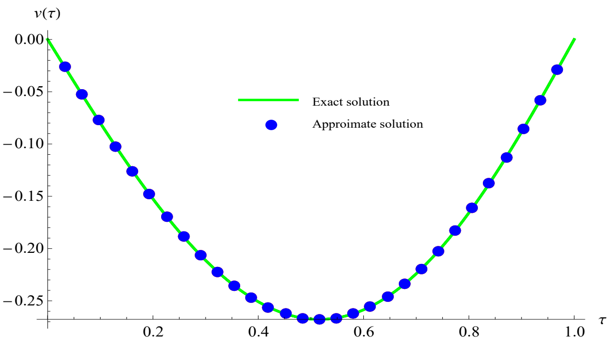

Figure 2. When $h = \frac{1}{10}$, $\epsilon = 0.0625$, and $\epsilon = 0.0001$, the attitude of the exact and approximate solution of Example 2.

Table 8. Maximum AEs of Example 2 with varying combinations of N and $\epsilon = 10^{-m}$.

$\epsilon = 10^{-m}$	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
10^{-00}	2.33147×10^{-15}	1.71085×10^{-13}	3.59152×10^{-12}	7.89434×10^{-11}	1.10196×10^{-09}
10^{-01}	2.48412×10^{-15}	1.82632×10^{-13}	3.68444×10^{-12}	5.45033×10^{-11}	6.64096×10^{-10}
10^{-02}	2.30371×10^{-15}	1.11994×10^{-13}	1.03767×10^{-12}	2.12168×10^{-11}	3.45862×10^{-10}
10^{-03}	5.55112×10^{-16}	2.54241×10^{-14}	4.44700×10^{-13}	4.19526×10^{-12}	8.91627×10^{-11}
10^{-04}	1.11022×10^{-16}	1.27676×10^{-15}	2.78666×10^{-14}	5.57276×10^{-13}	9.21263×10^{-12}
10^{-05}	5.55112×10^{-17}	1.66533×10^{-16}	2.88658×10^{-15}	4.99600×10^{-14}	8.18012×10^{-13}
10^{-06}	5.55112×10^{-17}	5.55112×10^{-17}	1.11022×10^{-16}	6.10623×10^{-16}	9.27036×10^{-15}
10^{-07}	2.77556×10^{-17}	2.77556×10^{-17}	2.77556×10^{-17}	2.22045×10^{-16}	3.77476×10^{-15}
10^{-08}	2.77556×10^{-17}	5.55112×10^{-17}	5.55112×10^{-17}	1.11022×10^{-16}	1.11022×10^{-15}
10^{-09}	2.77556×10^{-17}	2.77556×10^{-17}	5.55112×10^{-17}	5.55112×10^{-17}	1.11022×10^{-16}
10^{-10}	5.55112×10^{-17}	5.55112×10^{-17}	5.55112×10^{-17}	5.55112×10^{-17}	5.55112×10^{-17}

Table 9. Point-by-point AEs comparison between the proposed method and QBS in [29] for the solution of Example 2, when $N = 500$ and $\epsilon = 0.0001$.

τ	Method in [29]	Proposed Method
0.000	0.00000×10^{-00}	0.00000×10^{-00}
0.002	1.42286×10^{-12}	1.27626×10^{-14}
0.018	1.27219×10^{-11}	2.56264×10^{-14}
0.034	2.37708×10^{-11}	1.11115×10^{-13}
0.124	7.88766×10^{-11}	1.33951×10^{-13}
0.220	1.24021×10^{-10}	2.47738×10^{-12}
0.500	3.94267×10^{-10}	3.87702×10^{-12}
0.720	1.30052×10^{-09}	4.12510×10^{-11}
0.780	1.50042×10^{-09}	7.70917×10^{-11}
0.876	1.36220×10^{-09}	4.43062×10^{-11}
1.982	4.83877×10^{-10}	3.30951×10^{-12}
1.996	2.61038×10^{-10}	2.51630×10^{-12}
1.000	0.00000×10^{-00}	0.00000×10^{-00}

Example 3. *Consider the following non-linear Emden-Fowler type initial value problem [26,30]:*

$$v^{(4)}(\tau) + \frac{3}{\tau} v^{(3)}(\tau) = f(\tau, v), 0 \leq \tau \leq 1, \tag{67}$$

with the I.Cs

$$v(0) = v'(0) = v''(0) = v'''(0) = 0. \tag{68}$$

Example 3's exact solution is $v(\tau) = \log(1 + \tau^4)$ and $f(\tau, v) = 96(1 - 10\tau^4 + 5\tau^8)e^{-4v(\tau)}$. The point-wise AEs are shown in Table 10, for $h = \frac{1}{10}$. Figure 3 illustrates how the exact and approximative solutions to Example 3 behave when $h = \frac{1}{10}$. Table 11 gives a point-wise absolute error comparison between the proposed method and the method in [26] for the solution of Example 3 when $N = 100$.

Table 10. With $h = \frac{1}{10}$, point-by-point errors in the solution to Example 3.

τ	Exact Solution	Approximate Solution	Absolute Error
0.0	0.00000×10^{-0}	0.00000×10^{-0}	0.00000×10^{-0}
0.1	9.99950×10^{-5}	5.19552×10^{-5}	4.80398×10^{-6}
0.2	1.59872×10^{-3}	1.50168×10^{-3}	9.70440×10^{-6}
0.3	8.06737×10^{-3}	7.91339×10^{-3}	1.53985×10^{-5}
0.4	2.52778×10^{-2}	2.50354×10^{-2}	2.42396×10^{-5}
0.5	6.06246×10^{-2}	6.02134×10^{-2}	4.11259×10^{-5}
0.6	1.21860×10^{-1}	1.21145×10^{-1}	7.24394×10^{-5}
0.7	2.15190×10^{-1}	2.13984×10^{-1}	1.21564×10^{-4}
0.8	3.43310×10^{-1}	3.41478×10^{-1}	1.83187×10^{-4}
0.9	5.04470×10^{-1}	5.02046×10^{-1}	2.42343×10^{-4}
1.0	6.93140×10^{-1}	6.90341×10^{-1}	2.80740×10^{-4}

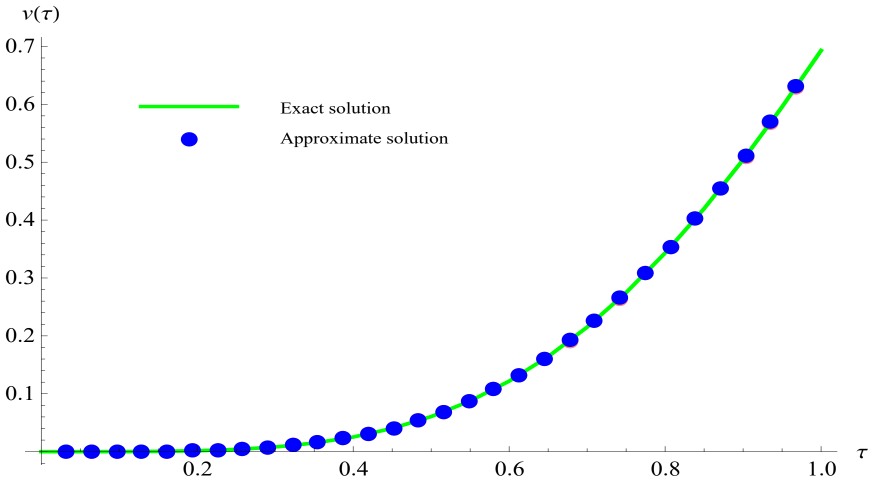

Figure 3. When $h = \frac{1}{10}$, the attitude of the exact and approximate solution of Example 3.

Table 11. Point-by-point AEs comparison between the proposed method and QBS in [26] for the solution of Example 3 when $N = 20$.

τ	Method in [26]	Proposed Method
0.0	0.00000×10^{-0}	0.00000×10^{-00}
0.1	9.99841×10^{-5}	4.12793×10^{-10}
0.2	1.59865×10^{-3}	7.74459×10^{-10}
0.3	8.06714×10^{-3}	1.83463×10^{-09}
0.4	2.52772×10^{-2}	5.82965×10^{-09}
0.5	6.06234×10^{-2}	1.75771×10^{-08}
0.6	1.21860×10^{-1}	4.34990×10^{-08}
0.7	2.15180×10^{-1}	8.71255×10^{-08}
0.8	3.43300×10^{-1}	1.43235×10^{-07}
0.9	5.04460×10^{-1}	1.96797×10^{-07}
1.0	6.93140×10^{-1}	2.29445×10^{-07}

Example 4. *Consider the following non-linear Emden-Fowler type initial value problem:*

$$v^{(4)}(\tau) + \frac{4}{\tau} v^{(3)}(\tau) = f(\tau, v), 0 \leq \tau \leq 1, \tag{69}$$

with the I.Cs

$$v(0) = v'(0) = v''(0) = v'''(0) = 0. \tag{70}$$

Example 4's exact solution is $v(\tau) = -4\log(1+\tau^4)$ and $f(\tau, v) = -32(15 - 129\tau^4 + 49\tau^8 + \tau^{12})e^{v(\tau)}$. The point-wise AEs are shown in Table 12, for $h = \frac{1}{10}$. Figure 4 illustrates how the exact and approximative solutions to Example 4 behave when $h = \frac{1}{10}$.

Table 12. With $h = \frac{1}{10}$, point-by-point errors in the solution to Example 4.

τ	Exact Solution	Approximate Solution	Absolute Error
0.0	-0.00000×10^{-0}	-0.00000×10^{-0}	0.00000×10^{-0}
0.1	-3.99980×10^{-4}	-3.99793×10^{-4}	1.86537×10^{-9}
0.2	-6.39489×10^{-3}	-6.39452×10^{-3}	3.60941×10^{-9}
0.3	-3.22695×10^{-2}	-3.22687×10^{-2}	7.68977×10^{-9}
0.4	-1.01111×10^{-1}	-1.01109×10^{-1}	2.17310×10^{-8}
0.5	-2.42498×10^{-1}	-2.42492×10^{-1}	6.19462×10^{-8}
0.6	-4.87454×10^{-1}	-4.87439×10^{-1}	1.49111×10^{-7}
0.7	-8.60768×10^{-1}	-8.60739×10^{-1}	2.92107×10^{-7}
0.8	-1.37322×10^{-0}	-1.37318×10^{-0}	4.67654×10^{-7}
0.9	-2.01786×10^{-0}	-2.01780×10^{-0}	6.18909×10^{-7}
1.0	-2.77259×10^{-0}	-2.77252×10^{-0}	6.81352×10^{-7}

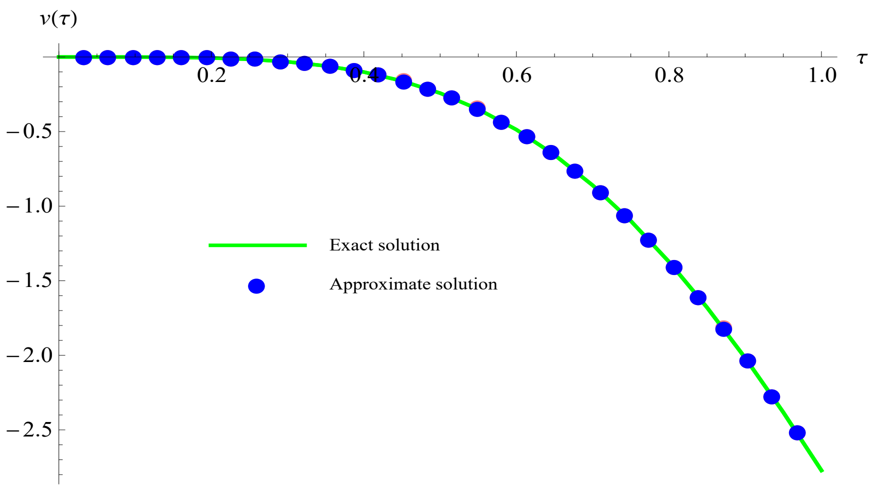

Figure 4. When $h = \frac{1}{10}$, the attitude of the exact and approximate solution of Example 4.

Example 5. *Take into account the subsequent fourth-order SSPLBVP:*

$$\epsilon v^{(4)}(\tau) + \frac{1}{\tau}v(\tau) = e^{\tau}[(1-\tau) - \epsilon(8 + 7\tau + \tau^2)] - \frac{2}{3}e(1-\tau^2), 0 \le \tau \le 1. \quad (71)$$

with the B.Cs

$$v(0) = v(1) = 0, v''(0) = v''(1) = 0. \quad (72)$$

There is no exact solution available in the literature of this example. So, the approximate solution at $\epsilon = 0.001$ and $N = 200$ is as observed exact solution of above SSPLBVP. The results of the numerical calculations used in this example are displayed in Tables 13 and 14. The point-wise residual errors are shown in Table 13 for $\epsilon = 0.01$ and $N = 10$. Figure 5 illustrates the observed exact and approximate solutions when $\epsilon = 0.01$ and $N = 10$. Table 14 shows the maximum residual errors for various tiny values of ϵ for various values of N.

Table 13. With $N = 10$ and $\epsilon = 0.01$, point-by-point residual errors in the solution to Example 5.

τ	Observed Exact Solution	Approximate Solution	Residual Errors
0.0	-0.00000	-0.00000	0.00000×10^{-00}
0.1	-1.69460	-1.69460	1.66755×10^{-15}
0.2	-1.54428	-1.54428	2.82596×10^{-15}
0.3	-1.36562	-1.36562	4.72511×10^{-15}
0.4	-1.16420	-1.16420	1.05449×10^{-14}
0.5	-0.94696	-0.94696	1.23679×10^{-15}
0.6	-0.72249	-0.72249	2.08011×10^{-14}
0.7	-0.50132	-0.50132	6.68576×10^{-14}
0.8	-0.29630	-0.29630	9.85989×10^{-15}
0.9	-0.12295	-0.122951	5.24580×10^{-15}
1.0	-0.00000	-0.00000	0.00000×10^{-00}

Table 14. Maximum residual errors of Example 5 with varying combinations of N and $\epsilon = 10^{-m}$.

$\epsilon = 10^{-m}$	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
10^{-00}	3.59712×10^{-14}	4.14853×10^{-13}	1.01803×10^{-12}	1.21552×10^{-10}	3.71410×10^{-9}
10^{-01}	3.28626×10^{-14}	4.47642×10^{-13}	8.00959×10^{-12}	1.41209×10^{-10}	3.30666×10^{-9}
10^{-02}	1.17684×10^{-14}	2.26708×10^{-13}	3.07954×10^{-12}	5.31172×10^{-11}	9.55786×10^{-10}
10^{-03}	1.33227×10^{-15}	2.97540×10^{-14}	5.61329×10^{-13}	6.96837×10^{-12}	1.24716×10^{-10}
10^{-04}	6.66134×10^{-16}	5.32907×10^{-15}	4.17444×10^{-14}	8.21232×10^{-13}	1.36269×10^{-12}
10^{-05}	1.55431×10^{-15}	8.88178×10^{-16}	4.44089×10^{-16}	9.76996×10^{-15}	2.04503×10^{-13}
10^{-06}	1.15374×10^{-12}	1.96287×10^{-13}	2.66454×10^{-15}	4.44089×10^{-15}	2.22045×10^{-15}
10^{-07}	2.21512×10^{-12}	1.16795×10^{-12}	3.68594×10^{-14}	3.55271×10^{-15}	2.66454×10^{-15}
10^{-08}	1.18527×10^{-12}	1.15896×10^{-12}	1.25694×10^{-14}	2.59674×10^{-15}	5.63214×10^{-15}
10^{-09}	5.25903×10^{-12}	6.59812×10^{-12}	2.56912×10^{-13}	2.77556×10^{-14}	5.55112×10^{-15}
10^{-10}	6.10800×10^{-12}	6.62492×10^{-12}	6.51035×10^{-13}	3.68594×10^{-14}	2.66454×10^{-15}

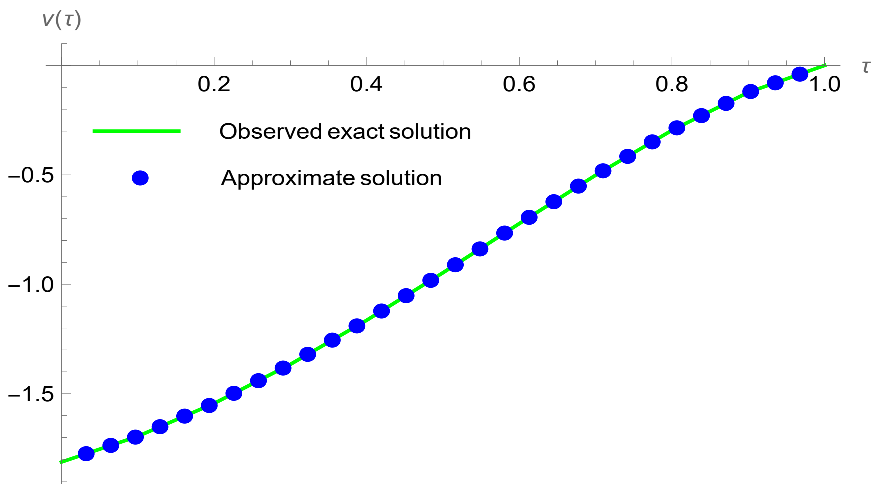

Figure 5. Behaviour of observed exact and approximate solutions of Example 5 when N = 10 and ϵ = 0.01.

6. Concluding Remarks

When different initial and boundary conditions are present, it is frequently exceedingly challenging to derive the analytical solutions to these equations. So, in order to tackle problems, we need to find some trustworthy numerical techniques. The goal of this work is to propose enforceable numerical algorithms for fourth-order SSPNLBVP, SSPNLIVP and SSPBVP using QBS. Additionally, systems involving sparse matrices are produced using B-spline approaches and these systems can be managed by suitable techniques at minimal computational and time complication.

The following are the contributions of this study:

- The previously suggested numerical methodology for the fourth-order SSPPs was based on a effective QBS approximation.

- For SSPNLBVP, SSPNLIVP and SSPBVP the aforementioned method was innovative.
- The approximation solution becomes closer to the precise analytical solution when the step size is decreased, ensuring convergence with the suggested methodology.
- The strategy was created to enhance a QBS for fourth-order problems without lowering lower-order DEs.
- This approach generates a spline function that may be used to find the answer anywhere throughout the range.
- In the whole domain, the scheme is uniformly convergent.

The B-spline approach has several benefits over the standard finite difference formulation because it yields highly precise continuous approximations of the unknown function and its derivatives at each point of the spectrum of integration. Despite its benefits, QBS interpolation has many drawbacks. If no free parameter is involved, the resultant curve cannot be altered. As a result, once the control points have been identified, the curve cannot be altered. Furthermore, it operates globally, therefore any effort to modify the control points will need resolving all associated systems once again. To address SSP linear, nonlinear, initial, and boundary value problems of various orders, we will in the future utilize polynomial, exponential, trigonometric, and hyperbolic trigonometric B-spline functions of various degrees.

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