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# Exploring a Special Class of Bi-Univalent Functions: $q$-Bernoulli Polynomial, $q$-Convolution, and $q$-Exponential Perspective

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**Abstract:** This research article introduces a novel operator termed $q$-convolution, strategically integrated with foundational principles of $q$-calculus. Leveraging this innovative operator alongside $q$-Bernoulli polynomials, a distinctive class of functions emerges, characterized by both analyticity and bi-univalence. The determination of initial coefficients within the Taylor-Maclaurin series for this function class is accomplished, showcasing precise bounds. Additionally, explicit computation of the second Hankel determinant is provided. These pivotal findings, accompanied by their corollaries and implications, not only enrich but also extend previously published results.

**Keywords:** $q$-Bernoulli polynomials; Hankel determinant; $q$-convolution operator; Fekete–Szegö problem; bi-univalent functions

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## 1. Introduction and Definitions

Fractional calculus, which originated from the work of Liouville in 1832, is a well-established field of mathematical analysis. It is one among several special functions that have been extensively studied. Even today, fractional calculus remains an active and vibrant area of research, as evidenced by recent investigations [1]. Bernoulli polynomials, a specific area of mathematics, are named in honor of Jacob Bernoulli (1654–1705). In recent times, considerable attention and research focus have been directed towards orthogonal polynomials. This interest arises from the significance of orthogonal polynomials in various fields, including engineering, mathematical statistics, and mathematical physics. The Hermite, Laguerre, Jacobi, and Bernoulli polynomials, which are well-known classical orthogonal polynomials, find extensive use in diverse applications. Several recent studies [2–5] highlight the relationship between geometric function theory (GFT) and conventional orthogonal polynomials, further advancing our understanding in this area.

In 1915, Alexander [6] introduced the original integral operator, which was utilized to investigate analytic functions. This field of research, which includes both fractional and ordinary derivative operators, remains an active research topic in the field of complex analysis, specifically within the framework of geometric function theory (GFT). Researchers continue to explore various combinations of these operators [7,8]. Recent works, such as [9], demonstrate the significance of integral fractional and differential operators in the field. Moreover, intriguing results have emerged from recent investigations into quantum (or $q$-) calculus, which provide alternative perspectives on differential and integral operators, and have implications in diverse areas of physics and mathematics. The utilization of

$q$-calculus by researchers in GFT has led to the development and exploration of distinctive subclasses of analytic functions. The official inception of $q$-calculus can be attributed to Jackson's formulation of $q$-derivatives and $q$-integrals in 1909 [10,11]. Jackson also introduced the $q$-difference and $q$-calculus operators ($D_q$). The applications of $q$-calculus extend across various domains, including mechanics, statistics, number theory, combinatorics, relativity, and control theory.

Quantum calculus, often referred to as $q$-calculus, is a field of mathematics that extends classical calculus to non-commutative settings. It plays a crucial role in various areas of quantum science, particularly in the study of $q$-deformed oscillators. These oscillators are quantum mechanical analogs of classical harmonic oscillators, but with a modified algebraic structure. Furthermore, they find practical use in Quantum Field Theory (QFT), a fundamental framework in theoretical physics that describes the behavior of quantum fields. Recently, there has been an incorporation of $q$-calculus in the study of fields that have undergone $q$-deformation, where standard commutation relations are replaced by their $q$-counterparts. This allows for a more adaptable approach to specific quantum field theories. Additionally, they have applications in Quantum Information Theory, a prominent field, especially in the investigation of $q$-deformed quantum channels. These channels describe the evolution of quantum states in non-standard quantum systems and are useful in quantum communication and computation. Further applications extend to Models of $q$-Deformed Quantum Harmonic Oscillators, which have been utilized in various models of quantum mechanics. For example, they have been employed in the analysis of particle behavior in non-standard potentials and within settings with non-trivial geometric attributes. Lastly, they play a role in $q$-Deformed Quantum Optics by utilizing $q$-calculus in crucial studies of versions of quantum optical systems that have undergone $q$-deformation. This encompasses $q$-deformed coherent states and $q$-deformed photon number states, both of which are significant in quantum optics and the processing of quantum information.

Mathematically, the $q$-deformed oscillators are defined by introducing a deformation parameter '$q$' that modifies the standard commutation relations between position and momentum operators. Instead of the canonical commutation relations $[x, p] = i\hbar$, one has $[x, p] = i\hbar q^N$, where 'N' is a number operator associated with the oscillator. This modification leads to a non-trivial deformation of the underlying algebraic structures. The $q$-calculus comes into play when manipulating functions and operators within this deformed algebraic framework. Derivatives, integrals, and other calculus operations are defined in a way consistent with the $q$-deformed algebra, allowing for the development of a $q$-calculus analogous to classical calculus. In summary, $q$-calculus and $q$-deformed oscillators find applications in various domains of quantum science, from quantum field theory to quantum information theory, providing a powerful mathematical framework to describe and analyze systems in non-commutative settings. For more details, see [12–15].

The groundbreaking research conducted by Ismail et al. [16] involved the construction of an extended version of starlike functions using $q$-calculus within the context of GFT. The recently introduced category of functions, which are defined using $q$-derivatives, is referred to as the class of $q$-starlike functions. Anastassiu and Gal [17,18] have made a valuable contribution to this field by further advancing the $q$-generalizations and investigating complex operators.

The progress in this area has been gradual. Aldweby et al. [19,20] developed the $q$-analogues of several operators using methods based on analytic function convolution. They also examined the structure of $q$-operators in relation to analytical functions incorporating $q$-versions of hypergeometric functions. With the recent surge of interest among researchers in $q$-calculus, several papers [21–26] have presented novel findings and insights.

The exploration of symmetric points of convex and starlike functions has received comprehensive attention in [27,28] and related sources. In our current study, we introduce a novel $q$-differential operator as part of the continuous investigation into differential and integral operators. By utilizing this operator, we propose the creation of a new family of analytic functions that possess geometrically bounded turning properties.

For any functions $l$ that are analytic within the region $\mathcal{E}$ and fulfill the specified normalization condition indicated by the function, $l$ has an initial value of 0 and its derivative at 0 is equal to 1, where

$$\mathcal{E} = \{\xi \,:\, \xi \in \mathbb{C},\ |\xi| < 1\}$$

are found in class $\mathcal{A}$, where $\mathcal{A}$ denotes the class of normalized analytic functions and $\mathbb{C}$ is the set of complex numbers. Clearly, the following expressions can be used to represent each function $l \in \mathcal{A}$:

$$l(\xi) = \xi + \sum_{m=2}^{\infty} d_m \xi^m \qquad (\xi \in \mathcal{E}); \tag{1}$$

also, the univalent functions in $\mathcal{E}$ comprise the class $\mathcal{S} \subset \mathcal{A}$.

Now, suppose we have two elements, $l_1$ and $l_2$, belonging to the set $\mathcal{A}$. In the context of $\mathcal{E}$, if $l_1$ is deemed to be subordinate to $l_2$, it is denoted by

$$l_1(z) \prec l_2(\xi) \quad (\xi \in \mathcal{E}). \tag{2}$$

Let us assume there is a Schwarz function denoted as $\mathcal{V}$ belonging to the set $\mathcal{E}$. This function satisfies certain conditions: $\mathcal{V}$ is also an element of the set $\mathcal{A}$, with the property that $|\mathcal{V}(\xi)|$ is less than 1 for all $\xi$, and $\mathcal{V}(0)$ is equal to 0.

$$l_1(\xi) = l_2(\mathcal{V}(\xi)) \quad (\xi \in \mathcal{E}).$$

The Koebe one-quarter theorem (see [29]) states that the image of $\mathcal{E}$ under every $l(\xi) \in \mathcal{S}$ contains a disk of radius one-quarter centered at the origin. So, there is an inverse function $l^{-1} = g$ for any function $l \in \mathcal{S}$ and

$$g(l(\xi)) = \xi \quad (\xi \in \mathcal{E})$$

and

$$l(g(\varpi)) = \varpi \quad (|\varpi| < t_o(l), t_0(l) \geq \frac{1}{4}.$$

The expression provided represents the series expansion of the inverse function $g(\varpi)$.

$$g(\varpi) = l^{-1}(\varpi) = \varpi - d_2\varpi^2 + (2d_2^2 - d_3)\varpi^3 - (5d_2^3 - 5d_2d_3 + d_4)\varpi^4 + \cdots. \tag{3}$$

Let $l$ be the univalent function and, if its inverse $(l^{-1})$ is univalent in $\mathcal{E}$, then the function $l$ is bi-univalent in $\mathcal{E}$. The group of bi-univalent functions within $\mathcal{E}$ is denoted by $\Sigma$. In geometric function theory (GFT), establishing limits for the coefficients in analytic functions has always been crucial. The magnitude of these coefficients can influence various properties, including univalency, growth rate, and distortion. In 2010, Srivastava et al. [30] made groundbreaking contributions, reinvigorating the study of analytical and bi-univalent functions. Since then, many researchers have proposed various new categories of bi-univalent functions and derived initial constraints on the coefficients, as documented in sources such as [3,4,31–35].

The essential terminology and concepts related to the $q$-calculus must now be reviewed in order to comprehend the topics of this article.

**Definition 1** ([10,11]).

1. *The notation $[m]_q!$ denotes the q-factorial, which can be expressed by:*

$$[m]_q! = \begin{cases} [m]_q\ [m-1]_q[m-2]_q[m-3]_q \cdots [3]_q[2]_q[1]_q, & \text{if } m = 1, 2, \cdots, \\ 1, & \text{if } m = 0, \end{cases} \tag{4}$$

*where*

$$\frac{q^m - 1}{q - 1} = [m]_q.$$

2. *A definition has been established for the q-derivative operator when* $0 < q < 1$.

$$\mathcal{D}_q l(\xi) = \begin{cases} \frac{l(q\xi)-l(\xi)}{(q-1)\xi}, & \text{if } \xi \neq 0 \\ l'(0), & \text{if } \xi = 0. \end{cases} \tag{5}$$

Let $e_q$ be the $q$-exponential function; see [36] for details.

$$e_q(\xi) = \sum_{m=0}^{\infty} \frac{\xi^m}{[m]_q!}, \quad (z \in \mathcal{E}).$$

In a further development, the $q$-binomial series is defined (see [37]):

$$(1-\alpha)_q^{\rho} = \sum_{m=0}^{\rho} \binom{\rho}{m}_q (-1)^m \alpha^m, \quad \rho \in \mathbb{N}, \quad m \in \mathbb{N}_0,$$

where

$$\binom{\rho}{m}_q = \frac{[\rho]_q!}{[m]_q![\rho-m]_q!},$$

stands for the $q$-binomial coefficients.

The rationale behind the selection of $q$-numbers, Arik–Coon oscillators, and $q$-deformed oscillators lies in their relevance to quantum physics and their applications in various contexts.

- $q$-Numbers: Rationale and applications: $q$-numbers, an expansion of conventional numbers, emerge within the realm of quantum groups and quantum algebras. They find validation and practical utility in various domains. In particular, in the realm of quantum groups, they play a crucial role. This is significant in disciplines such as quantum field theory, condensed matter physics, and statistical mechanics. Q-numbers are instrumental in representing systems characterized by a non-commutative algebra, a fundamental element in the framework of quantum mechanics.
- Arik–Coon Oscillator: Rationale and applications: The Arik–Coon oscillator is a variant of the harmonic oscillator that incorporates a parameter called "$q$" for deformation. This alteration introduces a broader uncertainty principle, with potential significance in exploring non-commutative geometry, as well as in the realms of quantum optics and quantum information theory. This modification becomes crucial in cases where standard quantum mechanics needs to be expanded, and it can also serve as a tool for simulating systems in the presence of particular background fields.
- $q$-Deformed Oscillator: Rationale and applications: Similar to the Arik–Coon oscillator, $q$-deformed oscillators utilize the deformation parameter '$q$' to modify the characteristics of regular oscillators. They find utility across diverse fields of physics, including quantum field theory, quantum optics, and nuclear physics, providing insights into systems operating within unconventional quantum mechanical contexts. These oscillators play a pivotal role in the exploration of quantum algebras and their corresponding representations.

In essence, these adapted oscillators and mathematical structures empower physicists to explore scenarios that deviate from standard quantum mechanics. This becomes particularly significant in situations involving systems subjected to extreme conditions, like those encountered in high-energy physics, the study of the early universe, or in the vicinity of black holes, where quantum gravity effects could play a substantial role. These mathematical methods provide a framework for probing and elucidating such phenomena; for more details, see [13–15,38].

We introduce our novel $q$-differential operator $\mathfrak{S}^{n,\zeta,v,q}_{\alpha,\rho,\omega,\tau} : \mathcal{A} \longrightarrow \mathcal{A}$ by using these $q$-binomial coefficients. Hence, for $\zeta > 0$, $v \geq 0$, $\alpha > 0$, $0 \leq \tau \leq \omega$, and $z \in \mathcal{E}$. We have

$$\mathfrak{S}^{\zeta,v,q,0}_{\alpha,\rho,\omega,\tau} l(\xi) = l(\xi), \tag{6}$$

$$\mathfrak{S}^{\zeta,\upsilon,q,1}_{\alpha,\rho,\omega,\tau}l(\xi) = \left(1+\zeta(M^{\rho}_q(\alpha)-\upsilon)(\omega-\tau-1)\right)l(\xi) - \zeta\xi\left(M^{\rho}_q(\alpha)-\upsilon\right)(\tau-\omega)$$
$$+\zeta\left(M^{\rho}_q(\alpha)-\upsilon\right)\xi d_q(l(\xi)), \tag{7}$$

$$\vdots \qquad \vdots \quad \vdots \quad \vdots$$

$$\mathfrak{S}^{\zeta,\upsilon,q,n}_{\alpha,\rho,\omega,\tau}l(\xi) = \mathfrak{S}^{\zeta,\upsilon,q,1}_{\alpha,\rho,\omega,\tau}\left(\mathfrak{S}^{\zeta,\upsilon,q,n-1}_{\alpha,\rho,\omega,\tau}l(\xi)\right). \tag{8}$$

Then from the functions (1) and (8), we have

$$\mathfrak{S}^{n,\zeta,\upsilon,q}_{\alpha,\rho,\omega,\tau}l(\xi) = \xi + \sum_{m=2}^{\infty}\left(1+\zeta\left(M^{\rho}_q(\alpha)-\upsilon\right)([m]_q+\omega-\tau-1)\right)^n d_m\xi^m,\ (n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}) \tag{9}$$

where

$$M^{\rho}_q(\alpha) = \sum_{m=1}^{\rho}\binom{\rho}{m}_q(-1)^{m+1}\alpha^m.$$

**Remark 1.** *The aforementioned operators are specific instances of our novel operator $\mathfrak{S}^{n,\zeta,\upsilon,q}_{\alpha,\rho,\omega,\tau}$ that have been established by different authors.*

1. *Setting $\rho=1$, $\omega=\tau$, and $\mathfrak{S}^{n,\zeta,\upsilon,q}_{\alpha,1,\omega,\omega}$ , we obtain the operator defined by Hadi and Darus [39].*
2. *Setting $\omega=\tau$, $\zeta=1$, $\upsilon=0$, and $\mathfrak{S}^{n,1,0,q}_{\alpha,\rho,\omega,\omega}$ , we obtain the operator defined by Hadi et al. [40].*
3. *Setting $\rho=1$, $\zeta=1$, $\upsilon=0$, and $\mathfrak{S}^{n,1,1,q}_{\alpha,1,\omega,\tau}$, we obtain the operator defined by Lasode and Opoola [41].*
4. *Setting $\rho=1$, $\zeta=1$, $\upsilon=0$, $\omega=\tau$, and $\mathfrak{S}^{n,1,1,q}_{\alpha,1,\omega,\omega}$, the q-Al-Oboudi operator, originally introduced by Aouf et al. in their work cited as [42], is available to us.*
5. *Setting $\rho=1$, $\zeta=1$, $\upsilon=0$, $\omega=\tau$, $\alpha=1$, and $\mathfrak{S}^{n,1,1,q}_{1,1,\omega,\omega}$, the q-Salagean operator, introduced by Govindaraj and Sivasubramanian [43], is available to us.*
6. *Setting $q\longrightarrow 1$, $\rho=1$, $\omega=\tau$, and $\mathfrak{S}^{n,\zeta,\upsilon,1}_{\alpha,1,\omega,\omega}$, we acquire the operator that is defined by Darus and Ibrahim in their work [44].*
7. *Setting $q\longrightarrow 1$, $\omega=\tau$, $\zeta=1$, $\upsilon=0$, and $\mathfrak{S}^{n,1,0,1}_{\alpha,\rho,\omega,\omega}$, we obtain the operator defined by Frasin [45].*
8. *Setting $q\longrightarrow 1$, $\rho=1$, $\zeta=1$, $\upsilon=0$, and $\mathfrak{S}^{n,1,1,1}_{\alpha,1,\omega,\tau}$, we obtain the operator defined by Opoola [46].*
9. *Setting $q\longrightarrow 1$, $\rho=1$, $\zeta=1$, $\upsilon=0$, $\omega=\tau$, and $\mathfrak{S}^{n,1,1,1}_{\alpha,1,\omega,\omega}$, we have the Al-Oboudi operator that Al-Oboudi [47] presented.*
10. *Setting $\rho=1$, $q\longrightarrow 1$, $\zeta=1$, $\omega=\tau$, $\upsilon=0$, $\alpha=1$, and $\mathfrak{S}^{n,1,1,1}_{1,1,\omega,\omega}$, we have the Salagean operator that Salagean [48] presented.*

**Definition 2.** *By utilizing the convolution principle, we introduce a novel operator known as the q-convolution, denoted as $T^{\alpha,\rho,\omega,\tau}_{n,\zeta,\upsilon,q}:\mathcal{A}\longrightarrow\mathcal{A}$, where the function $l\in\mathcal{A}$ is involved in the definition, provided as follows:*

$$T^{\alpha,\rho,\omega,\tau}_{n,\zeta,\upsilon,q}l(\xi) = \mathfrak{S}^{n,\zeta,\upsilon,q}_{\alpha,\rho,\omega,\tau}l(\xi) * e_q.$$

The preceding definition implies that

$$T^{\alpha,\rho,\omega,\tau}_{n,\zeta,\upsilon,q}l(\xi) = \xi + \sum_{m=2}^{\infty}\Omega(m)d_m\xi^m, \quad (\xi\in\mathcal{E}), \tag{10}$$

with

$$\Omega(m) = \frac{\left(1+\zeta\left(M_q^{\rho}(\alpha)-v\right)\left([m]_q+\omega-\tau-1\right)\right)^n}{[m]_q!}. \tag{11}$$

In addition, the identity that follows can be easily demonstrated using (9):

$$\begin{aligned}\xi\zeta\left(M_q^{\rho}(\alpha)-v\right)\mathcal{D}_q(\mathfrak{S}_{\alpha,\rho,\omega,\tau}^{n,\zeta,v,q}l(\xi)) = \mathfrak{S}_{\alpha,\rho,\omega,\tau}^{n+1,\zeta,v,q}l(\xi) - \zeta\xi\left(M_q^{\rho}(\alpha)-v\right)(\tau-\omega)\\ -\left(1+\zeta(M_q^{\rho}(\alpha)-v)(\omega-\tau-1)\right)\mathfrak{S}_{\alpha,\rho,\omega,\tau}^{n,\zeta,v,q}l(\xi).\end{aligned}$$

The Fekete–Szegö problem is solved using Loewner's method for the coefficients of $l$ belonging to the set $\mathcal{S}$ in equation [49].

$$\left|d_3-\delta d_2^2\right| \le 1+exp\left(\frac{2\delta}{\delta-1}\right)^2 \quad \text{for} \quad 0\le\delta<1.$$

The well known inequality $|d_3-d_2^2|\le 1$ can be obtained as $\delta \longrightarrow 1^{-1}$. The (GFT) is significantly influenced by the coefficient functional

$$W_\delta(l) = d_3-\delta d_2^2$$

on the normalized analytic functions $l\in\mathcal{E}$. The Fekete–Szegö problem pertains to the maximization of the functional $|W_\delta(l)|$. It was originally proposed in 1933 by Fekete [49]. Scholars have since raised concerns about various types of univalent functions in relation to the Fekete–Szegö problem. These concerns have been addressed in several studies [33,50]. As a result, equivalent inequalities have been established for bi-univalent functions. Recent sources, such as [50,51], provide compelling evidence that this subject matter continues to yield fascinating results.

Let $\mathcal{B}_{q,m}(u)$ be $q$-Bernoulli polynomials defined by making use of the following generating function (see, for instance, [52]):

$$\mathcal{B}_q(u,h) = \frac{h}{e_q(h)-1}e_q(hu) = \sum_{m=0}^{\infty}\mathcal{B}_{q,m}(u)\frac{h^m}{[m]_q!}, \quad |h|<2\pi, \tag{12}$$

where polynomials in $u$ called $\mathcal{B}_m(u)$ exist with respect to every nonnegative integer $m$.

Considering the $q$-Bernoulli Polynomials, the following linear homogeneous recurrence relation remains true:

$$\mathcal{B}_{q,m}(u) = q^m\left(u-\frac{1}{q[2]_q}\right)\mathcal{B}_{q,m-1}(u) - \frac{1}{[m]_q}\sum_{j=0}^{m-2}\begin{bmatrix} m \\ j \end{bmatrix}_q q^{j-1}b_{m-j,q}\mathcal{B}_{j,q}(u);$$

for more details, see [53].

These happen to be the initial few polynomials:

$$\begin{aligned}\mathcal{B}_{q,0}(u) &= 1,\\ \mathcal{B}_{q,1}(u) &= -\left(\frac{q-[2]_q u}{[2]_q}\right),\\ \mathcal{B}_{q,2}(u) &= \frac{q}{[2]_q[3]_q}-u+u^2,\\ \mathcal{B}_{q,3}(u) &= u^3+\frac{1}{q[2]_q}u+\frac{q-1}{[4]_q[2]_q}-\frac{[3]_q}{q[2]_q}u^2.\end{aligned}$$

The relationship between bi-univalent functions and orthogonal polynomials has recently come under the scrutiny of various authors (see, for example, [24,33,54]).

The current investigation introduces a novel $q$-convolution operator, comprising the $q$-exponential function and $q$-binomial series. Alongside this development, the study aims to establish upper bounds for various families of analytic functions that are defined through subordination. Based on our current understanding, no previous research studies have been published on bi-univalent functions concerning the $q$-Bernoulli polynomials associated with the newly proposed $q$-convolution operator. Inspired by the work of Buyankara et al. [54], the main goal of this study is to investigate the characteristics of bi-univalent functions in connection with $q$-Bernoulli polynomials. The subsequent statements serve as definitions for the purposes of this study.

**Definition 3.** *Let $l \in \mathcal{RB}^{n,\alpha,\rho}_{\zeta,v,\theta}(\omega,\tau,q;u)$, supposing the following subordinations are valid:*

$$\mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q}l(\xi)) + \frac{e^{i\theta}+1}{2}\xi\mathcal{D}^2_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q}l(\xi)) \prec \mathcal{B}_q(u,\xi) \tag{13}$$

*and*

$$\mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q}g(\varpi)) + \frac{e^{i\theta}+1}{2}\varpi\mathcal{D}^2_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q}g(\varpi)) \prec \mathcal{B}_q(u,\varpi) \tag{14}$$

*where $\prec$ is the subordination principle in (2), $\xi,\varpi \in \mathcal{E}$, $u \in [-\pi,\pi]$, $\mathcal{B}_q(u,\xi)$ is given by (12), and $g(\varpi)$ is given by (3).*

We consider a function $l_t(\xi)$ such that

$$\mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q}l_t(\xi)) + \frac{e^{i\theta}+1}{2}\xi\mathcal{D}^2_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q}l_t(\xi)) = \mathcal{B}_q(u,\xi^t) \tag{15}$$

and

$$\mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q}g_t(\varpi)) + \frac{e^{i\theta}+1}{2}\varpi\mathcal{D}^2_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q}g_t(\varpi)) = \mathcal{B}_q(u,\varpi^t). \tag{16}$$

Thus, we have

$$l_1(\xi) = \xi + \frac{2(u[2]_q-q)}{\Omega(2)[2]^2_q(3+e^{i\theta})}\xi^2 + \frac{2(q-[2]_q[3]_q u(1-u))}{\Omega(3)[2]^2_q[3]^2_q[2+[2]_q(e^{i\theta}+1)]}\xi^3 + \cdots, \tag{17}$$

$$l_2(\xi) = \xi + \frac{2(u[2]_q-q)}{\Omega(3)[2]_q[3]_q[2+[2]_q(e^{i\theta}+1)]}\xi^3 + \frac{2(q-[2]_q[3]_q u(1-u))}{\Omega(4)[2]^2_q[4]_q[3]_q[2+[3]_q(e^{i\theta}+1)]}\xi^4 + \cdots, \tag{18}$$

$$l_3(\xi) = \xi + \frac{2(u[2]_q-q)}{\Omega(4)[2]_q[4]_q[2+[3]_q(e^{i\theta}+1)]}\xi^4 + \frac{2(q-[2]_q[3]_q u(1-u))}{\Omega(5)[2]^2_q[3]_q[5]_q[2+[4]_q(e^{i\theta}+1)]}\xi^5 + \cdots. \tag{19}$$

**Remark 2.** *The class $\mathcal{RB}^{n,\alpha,\rho}_{\zeta,v,\theta}(\omega,\tau,q;u)$ is not empty. At least, the functions defined by (17)–(19) are univalent due to being extremal functions of the class of univalent functions. They all exist in the class $\mathcal{RB}^{n,\alpha,\rho}_{\zeta,v,\theta}(\omega,\tau,q;u)$. To show this, we proceed as follows. When $t = 1$ in (15) and (16), we have*

$$\mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q}l_1(\xi)) + \frac{e^{i\theta}+1}{2}\xi\mathcal{D}^2_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q}l_1(\xi)) = \mathcal{B}_q(u,\xi^1) \tag{20}$$

*and*

$$\mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q}g_1(\varpi)) + \frac{e^{i\theta}+1}{2}\varpi\mathcal{D}^2_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q}g_1(\varpi)) = \mathcal{B}_q(u,\varpi^1). \tag{21}$$

*Firstly, we are going to subtitute $l_1(\xi)$ into (20) and check if it is equal.*

*Now, from the left-hand side (L.H.S) of (20), we have the following:*

$$T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} l_1(\xi) = \xi + \frac{2(u[2]_q - q)}{[2]_q^2(3+e^{i\theta})}\xi^2 + \frac{2(q-[2]_q[3]_q u(1-u))}{[2]_q^2[3]_q^2[2+[2]_q(e^{i\theta}+1)]}\xi^3 + \cdots$$

$$\mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} l_1(\xi)) = 1 + \frac{2(u[2]_q - q)}{[2]_q(3+e^{i\theta})}\xi + \frac{2(q-[2]_q[3]_q u(1-u))}{[2]_q^2[3]_q[2+[2]_q(e^{i\theta}+1)]}\xi^2 + \cdots \tag{22}$$

$$\frac{e^{i\theta}+1}{2}\xi\mathcal{D}_q^2(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} l_1(\xi)) = \frac{(1+e^{i\theta})(u[2]_q - q)}{[2]_q(3+e^{i\theta})}\xi + \frac{(1+e^{i\theta})(q-[2]_q[3]_q u(1-u))}{[2]_q[3]_q[2+[2]_q(e^{i\theta}+1)]}\xi^2 + \cdots . \tag{23}$$

*Combining (22) and (23), we have*

$$\mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} l_1(\xi)) + \frac{e^{i\theta}+1}{2}\xi\mathcal{D}_q^2(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} l_1(\xi)) = 1 + \frac{u[2]_q - q}{[2]_q}\xi + \frac{q-[2]_q[3]_q u(1-u)}{[2]_q^2[3]_q}\xi^2 + \cdots . \tag{24}$$

*Now for the right-hand side (R.H.S) of (20), we have*

$$\mathcal{B}_q(u,\xi^1) = 1 + \frac{u[2]_q - q}{[2]_q}\xi + \frac{q-[2]_q[3]_q u(1-u)}{[2]_q^2[3]_q}\xi^2 + \cdots . \tag{25}$$

*Comparing (24) and (25), we can clearly see that they are equal, so we can say that $l_1(\xi)$ satisfies the first part of Definition 3.*

*Now, we check if $l_1(\xi)$ satisfies the second part of the definition. Hence, we are going to find the inverse of (24) to obtain the L.H.S of the second part of Definition 3 which is $g(\varpi)$*

*Let*

$$w = \mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} l_1(\xi)) + \frac{e^{i\theta}+1}{2}\xi\mathcal{D}_q^2(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} l_1(\xi)) \tag{26}$$

*imply*

$$\xi = \mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} g_1(\varpi)) + \frac{e^{i\theta}+1}{2}\varpi\mathcal{D}_q^2(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} g_1(\varpi)) = \varpi + A_2\varpi^2 + A_3\varpi^3 + A_4\varpi^4 + \cdots . \tag{27}$$

*Now, substituting (27) into (26) gives*

$$\begin{aligned}\varpi =& 1 + \frac{u[2]_q - q}{[2]_q}\left(\varpi + A_2\varpi^2 + A_3\varpi^3 + \cdots\right)\\ &+ \frac{q-[2]_q[3]_q u(1-u)}{[2]_q^2[3]_q}\left(\varpi + A_2\varpi^2 + A_3\varpi^3 + \cdots\right)^2 + \cdots .\end{aligned}$$

*Observe that:*

$$[\varpi + A_2\varpi^2 + A_3\varpi^3 + A_4\varpi^4 + \cdots]^2 = \varpi^2 + 2A_2\varpi^3 + (A_2^2 + 2A_3)\varpi^4 + \cdots , \tag{28}$$

$$[\varpi + A_2\varpi^2 + A_3\varpi^3 + A_4\varpi^4 + \cdots]^3 = \varpi^3 + 3A_2\varpi^4 + \cdots , \tag{29}$$

$$[\varpi + A_2\varpi^2 + A_3\varpi^3 + A_4\varpi^4 + \cdots]^4 = \varpi^4 + 4A_2\varpi^5 + \cdots . \tag{30}$$

*Making use of (28), we have*

$$\varpi = 1 + \frac{u[2]_q - q}{[2]_q}\varpi + \left[\frac{A_2(u[2]_q - q)}{[2]_q} + \frac{q-[2]_q[3]_q u(1-u)}{[2]_q^2[3]_q}\right]\varpi^2 + \cdots$$

$$A_2 = \frac{[2]_q[3]_q u(1-u) - q}{[2]_q[3]_q(u[2]_q - q)}.$$

*Sutituting $A_2$ into (27), we have*

$$\xi = \mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} g_1(\varpi)) + \frac{e^{i\theta}+1}{2}\varpi\mathcal{D}_q^2(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} g_1(\varpi)) = \varpi + \frac{[2]_q[3]_q u(1-u) - q}{[2]_q[3]_q(u[2]_q - q)}\varpi^2 + \cdots. \tag{31}$$

*For the right-hand side (R.H.S) of (21), since the left-hand side (L.H.S) and R.H.S of (20) are equal and we have solved for the inverse of the L.H.S of (21), it is easy to see that the R.H.S will be equal to the L.H.S of (21). That is,*

$$\xi = \left(\mathcal{B}_q(u,\xi^1)\right)^{-1} = \varpi + \frac{[2]_q[3]_q u(1-u) - q}{[2]_q[3]_q(u[2]_q - q)}\varpi^2 + \cdots. \tag{32}$$

*Comparing (31) and (32), we deduce that both sides are equal. Therefore, by applying the same process to $l_2(\xi)$ and $l_3(\xi)$, which gives more degrees, we conclude that they also satisfy both equations in Definition 3.*

*Now, we can conclude that the extremal functions given in (17)–(19) show that our defined class of analytic and bi-univalent functions is not empty and also satisfies both the first and second part of our Definition 3 related to $l(\xi)$ and $g(\varpi)$.*

**Example 1.** *We have the following remarks:*

1. *When $\theta = 0$, we obtain the subfamily*

$$\mathcal{RB}^{n,\alpha,\rho}_{\zeta,v}(\omega,\tau,q;u)$$

*described by*

$$\mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} l(\xi)) + \xi\mathcal{D}_q^2(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} l(\xi)) \prec \mathcal{B}_q(u,\xi) \tag{33}$$

*and*

$$\mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} g(\varpi)) + \varpi\mathcal{D}_q^2(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} g(\varpi)) \prec \mathcal{B}_q(u,\varpi), \tag{34}$$

*where $\xi,\varpi \in \mathcal{E}$, $\mathcal{B}_q(u,\xi)$ is given by (12), and $g(\varpi)$ is given by (3).*

2. *When $\theta = \pi$, we obtain the subfamily*

$$\mathcal{RB}^{n,\alpha,\rho}_{\zeta,v}(\omega,\tau,q;u)$$

*described by*

$$\mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} l(\xi)) \prec \mathcal{B}_q(u,\xi) \tag{35}$$

*and*

$$\mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} g(\varpi)) \prec \mathcal{B}_q(u,\varpi), \tag{36}$$

*where $\xi,\varpi \in \mathcal{E}$, $\mathcal{B}_q(u,\xi)$ is given by (12), and $g(\varpi)$ is given by (3).*

3. *When $\theta = \pi$, $n = 0$, and $q \longrightarrow 1$, we obtain the subfamily*

$$\mathcal{RB}^{0,\alpha,\rho}_{\zeta,v,\pi}(\omega,\tau,q;u) = M_\Sigma(\mathcal{B})$$

*described by*

$$l(\xi) \prec \mathcal{B}(u,\xi) \tag{37}$$

*and*

$$g(\varpi) \prec \mathcal{B}(u,\varpi), \tag{38}$$

*which was studied by Buyankara and Caglar [55].*

To establish our main results, it is necessary to demonstrate the following lemmas.

**Lemma 1.** *Let us assume that $\mathcal{P}$ represents a collection of analytic functions denoted by $s$, which can be expressed in the following manner [29,56]:*

$$s(\xi) = 1 + s_1\xi + s_2\xi^2 + s_3\xi^3 + \cdots = 1 + \sum_{m=1}^{\infty} s_m \xi^m \tag{39}$$

*satisfying $\Re(s(\xi)) > 0$, $\xi \in \mathcal{E}$ and $s(0) = 1$. Then,*

$$|s_m| \leq 2, \; m \in \mathbb{N}.$$

*This inequality is sharp for every natural number $m \in \mathbb{N}$.*

**Lemma 2.** *Let us assume that $\mathcal{P}$ is as (39), satisfying $\Re(s(\xi)) > 0$, $z \in \mathcal{E}$, and $s(0) = 1$. Then,*

$$2s_2 = s_1^2 + (4 - s_1^2)e$$

$$4s_3 = s_1^3 + 2(4 - s_1^2)s_1 e - (4 - s_1^2)s_1 e^2 + 2(4 - s_1^2)(1 - |e|^2)\xi.$$

*There exist values of $e$ and $\xi$ such that their absolute values are less than or equal to 1.*

Throughout this research, we assume that our $\theta$ is 0 and $\pi$, $\zeta > 0$, $v \geq 0$, $\alpha > 0$, $0 \leq \tau \leq \omega$, $u \in [-\pi, \pi]$ throughout our results.

## 2. Bounds on the Initial Coefficients for Several Families Related to the $q$-Bernoulli Polynomials

In this section, we present a theorem that establishes upper-bound estimates for the first three coefficients of functions within the class $\mathcal{RB}_{\zeta,v,\theta}^{n,\alpha,\rho}(\omega,\tau,q;u)$.

**Theorem 1.** *Let $l \in \mathcal{RB}_{\zeta,v,\theta}^{n,\alpha,\rho}(\omega,\tau,q;u)$. Then,*

$$|d_2| \leq \frac{2|u[2]_q - q|}{\Omega(2)[2]_q^2|3 + e^{i\theta}|},$$

$$|d_3| \leq \frac{2|u[2]_q - q|}{\Omega(3)[2]_q[3]_q|2 + [2]_q(e^{i\theta} + 1)|},$$

$$|d_4| \leq \frac{2|u[2]_q - q|}{\Omega(4)[2]_q[4]_q|2 + [3]_q(e^{i\theta} + 1)|}.$$

*All bounds of Theorem 1 are sharp for the functions given in (57)–(59).*

**Proof.** Let $l \in \mathcal{RB}_{\zeta,v,\theta}^{n,\alpha,\rho}(\omega,\tau,q;u)$. Next, we have the concept of analytic functions $w : \mathcal{E} \longrightarrow \mathcal{E}$, $v : \mathcal{E} \longrightarrow \mathcal{E}$ with $w(0) = 0 = v(0)$, $|w(\xi)| \leq 1$, $|v(\varpi)| \leq 1$ fulfilling the following conditions:

$$\mathcal{D}_q(T_{n,\zeta,v,q}^{\alpha,\rho,\omega,\tau} l(\xi)) + \frac{e^{i\theta} + 1}{2} \xi \mathcal{D}_q^2(T_{n,\zeta,v,q}^{\alpha,\rho,\omega,\tau} l(\xi)) = \mathcal{B}_q(u, w(\xi)) \tag{40}$$

and

$$\mathcal{D}_q(T_{n,\zeta,v,q}^{\alpha,\rho,\omega,\tau} g(\varpi)) + \frac{e^{i\theta} + 1}{2} \varpi \mathcal{D}_q^2(T_{n,\zeta,v,q}^{\alpha,\rho,\omega,\tau} g(\varpi)) = \mathcal{B}_q(u, v(\varpi)). \tag{41}$$

The function $s, r \in \mathcal{P}$ are defined as follows:

$$s(\xi) = \frac{1 + w(\xi)}{1 - w(\xi)} = 1 + \sum_{m=1}^{\infty} s_m \xi^m, \;\; \xi \in \mathcal{E}$$

and

$$r(\varpi) = \frac{1+v(\varpi)}{1-v(\varpi)} = 1 + \sum_{m=1}^{\infty} r_m \varpi^m, \quad \varpi \in \mathcal{E}.$$

As a result,

$$w(\xi) = \frac{s(\xi)-1}{s(\xi)+1} = \frac{1}{2}\left[s_1\xi + \left(s_2 - \frac{s_1^2}{2}\right)\xi^2 + \left(s_3 - s_1 s_2 + \frac{s_1^3}{4}\right)\xi^3 + \cdots\right], \ \xi \in \mathcal{E} \tag{42}$$

and

$$v(\varpi) = \frac{r(\varpi)-1}{r(\varpi)+1} = \frac{1}{2}\left[r_1\varpi + \left(r_2 - \frac{r_1^2}{2}\right)\varpi^2 + \left(r_3 - r_1 r_2 + \frac{r_1^3}{4}\right)\varpi^3 + \cdots\right], \ \varpi \in \mathcal{E}. \tag{43}$$

Upon replacing the expressions of the functions $w(\xi)$ and $v(\varpi)$ in Equations (40) and (41) with the ones given in Equations (42) and (43), we obtain the following result:

$$\begin{aligned}
\mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} l(\xi)) + \frac{e^{i\theta}+1}{2}\xi\mathcal{D}_q^2(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} l(\xi)) &= 1 + \frac{\mathcal{B}_{q,1}(u)}{2}s_1\xi + \left(\frac{\mathcal{B}_{q,1}(u)}{2}\left(s_2 - \frac{s_1^2}{2}\right)\right.\\
&+ \left.\frac{\mathcal{B}_{q,2}(u)}{4[2]_q!}s_1^2\right)\xi^2 + \left(\frac{\mathcal{B}_{q,1}(u)}{2}\left(s_3 - s_1 s_2 + \frac{s_1^3}{4}\right) + \frac{\mathcal{B}_{q,2}(u)}{2[2]_q!}s_1\left(s_2 - \frac{s_1^2}{2}\right)\right.\\
&+ \left.\frac{\mathcal{B}_{q,3}(u)}{8[3]_q!}s_1^3\right)\xi^3 + \cdots,
\end{aligned} \tag{44}$$

$$\begin{aligned}
\mathcal{D}_q(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} g(\varpi)) + \frac{e^{i\theta}+1}{2}\varpi\mathcal{D}_q^2(T^{\alpha,\rho,\omega,\tau}_{n,\zeta,v,q} g(\varpi)) &= 1 + \frac{\mathcal{B}_{q,1}(u)}{2}r_1\varpi + \left(\frac{\mathcal{B}_{q,1}(u)}{2}\left(r_2 - \frac{r_1^2}{2}\right)\right.\\
&+ \left.\frac{\mathcal{B}_{q,2}(u)}{4[2]_q!}r_1^2\right)\varpi^2 + \left(\frac{\mathcal{B}_{q,1}(u)}{2}\left(r_3 - r_1 r_2 + \frac{r_1^3}{4}\right) + \frac{\mathcal{B}_{q,2}(u)}{2[2]_q!}r_1\left(r_2 - \frac{r_1^2}{2}\right)\right.\\
&+ \left.\frac{\mathcal{B}_{q,3}(u)}{8[3]_q!}r_1^3\right)\varpi^3 + \cdots.
\end{aligned} \tag{45}$$

Equations (44) and (45) yield expressions for terms of equal degree, specifically $d_2$, $d_3$, and $d_4$, once the operations and simplifications on the left side are performed.

$$\frac{\Omega(2)[2]_q(3+e^{i\theta})}{2}d_2 = \frac{\mathcal{B}_{q,1}(u)}{2}s_1, \tag{46}$$

$$\frac{[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}{2}d_3 = \frac{\mathcal{B}_{q,1}(u)}{2}\left(s_2 - \frac{s_1^2}{2}\right) + \frac{\mathcal{B}_{q,2}(u)}{4[2]_q!}s_1^2, \tag{47}$$

$$\begin{aligned}
\frac{[2+[3]_q(e^{i\theta}+1)]\Omega(4)[4]_q}{2}d_4 &= \frac{\mathcal{B}_{q,1}(u)}{2}\left(s_3 - s_1 s_2 + \frac{s_1^3}{4}\right) + \frac{\mathcal{B}_{q,2}(u)}{2[2]_q!}s_1\left(s_2 - \frac{s_1^2}{2}\right)\\
&+ \frac{\mathcal{B}_{q,3}(u)}{8[3]_q!}s_1^3
\end{aligned} \tag{48}$$

and

$$-\frac{\Omega(2)[2]_q(3+e^{i\theta})}{2}d_2=\frac{\mathcal{B}_{q,1}(u)}{2}r_1, \tag{49}$$

$$[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q d_2^2-\frac{[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}{2}d_3=\frac{\mathcal{B}_{q,2}(u)}{4[2]_q!}r_1^2$$
$$+\frac{\mathcal{B}_{q,1}(u)}{2}\left(r_2-\frac{r_1^2}{2}\right), \tag{50}$$

$$-\frac{5[2+[3]_q(e^{i\theta}+1)]\Omega(4)[4]_q}{2}d_2^3+\frac{5[2+[3]_q(e^{i\theta}+1)]\Omega(4)[4]_q}{2}d_2d_3$$
$$-\frac{[2+[3]_q(e^{i\theta}+1)]\Omega(4)[4]_q}{2}d_4=\frac{\mathcal{B}_{q,1}(u)}{2}\left(r_3-r_1r_2+\frac{r_1^3}{4}\right)+\frac{\mathcal{B}_{q,2}(u)}{2[2]_q!}r_1\left(r_2-\frac{r_1^2}{2}\right)$$
$$+\frac{\mathcal{B}_{q,3}(u)}{8[3]_q!}r_1^3. \tag{51}$$

Using Equations (46) and (49), we write:

$$s_1=-r_1. \tag{52}$$

The initial result of the theorem can be easily deduced from this fact along with Lemma 1.

By taking into account the equality $s_1=-r_1$ and subtracting Equation (50) from Equation (47), we can derive the following expression:

$$d_3=d_2^2+\frac{\mathcal{B}_{q,1}(u)(s_2-r_2)}{2[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}.$$

In addition,

$$d_3=\frac{s_1^2\mathcal{B}_{q,1}^2(u)}{\Omega^2(2)[2]_q^2(3+e^{i\theta})^2}+\frac{\mathcal{B}_{q,1}(u)(s_2-r_2)}{2[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}. \tag{53}$$

Furthermore, through the subtraction of Equation (51) from Equation (48), while considering the relationships presented in Equations (52) and (53), we reach the conclusion that

$$d_4=\frac{5\mathcal{B}_{q,1}^2(u)s_1(s_2-r_2)}{4\Omega(2)\Omega(3)[2]_q[3]_q(3+e^{i\theta})[2+[2]_q(e^{i\theta}+1)]}+\frac{\mathcal{B}_{q,1}(u)(s_3-r_3)}{2[2+[3]_q(e^{i\theta}+1)]\Omega(4)[4]_q}$$
$$+\frac{[\mathcal{B}_{q,2}(u)-[2]_q!\mathcal{B}_{q,1}(u)]s_1(s_2+r_2)}{2[2]_q![2+[3]_q(e^{i\theta}+1)]\Omega(4)[4]_q}+\frac{[2]_q![3]_q!\mathcal{B}_{q,1}(u)-2[3]_q!\mathcal{B}_{q,2}(u)+[2]_q!\mathcal{B}_{q,3}(u)}{4[2]_q![3]_q![2+[3]_q(e^{i\theta}+1)]\Omega(4)[4]_q}. \tag{54}$$

Moreover, Lemma 2 states that due to the fact that $s_1$ is equal to $-r_1$, we have the ability to express:

$$s_2-r_2=\frac{4-s_1^2}{2}(e-\mu),\quad s_2+r_2=s_1^2+\frac{4-s_1^2}{2}(e+\mu) \tag{55}$$

and

$$s_3-r_3=\frac{s_1^3}{2}+\frac{(4-s_1^2)(e+\mu)}{2}s_1-\frac{(4-s_1^2)(e^2+\mu^2)}{4}s_1$$
$$+\frac{4-s_1^2}{2}\left[\left(1-|e|^2\right)\xi-\left(1-|\mu|^2\right)\varpi\right] \tag{56}$$

for some $e, \varpi, \xi, \mu$ with $|e| \leq 1$, $|\varpi| \leq 1$, $|\xi| \leq 1$, $|\mu| \leq 1$.

To derive the expression for the coefficient $d_3$, we put the first Equation (55) into Equation (53), resulting in the following formulation:

$$d_3 = \frac{s_1^2 \mathcal{B}_{q,1}^2(u)}{\Omega^2(2)[2]_q^2(3+e^{i\theta})^2} + \frac{\mathcal{B}_{q,1}(u)(4-s_1^2)(e-\mu)}{4[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}.$$

Let $s_1 = c$. Essentially, we can consider the value of $c$ to be in the range of [0, 2]. Under this condition, the inequality for $|d_3|$ can be expressed as follows:

$$|d_3| \leq \frac{c^2|\mathcal{B}_{q,1}^2(u)|}{\Omega^2(2)[2]_q^2|3+e^{i\theta}|^2} + \frac{|\mathcal{B}_{q,1}(u)|(4-c^2)}{4|2+[2]_q(e^{i\theta}+1)|\Omega(3)[3]_q}(\lambda+\sigma) = M(\lambda,\sigma), \;\; (\lambda,\sigma) \in [0,1]\times[0,1]$$

using the triangle inequality and assigning the values of $|e|$ as $\lambda$ and $|\mu|$ as $\sigma$.

The function $M : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is now defined in the following manner:

$$M(\lambda,\sigma) = \frac{c^2\mathcal{B}_{q,1}^2(u)}{\Omega^2(2)[2]_q^2(3+e^{i\theta})^2} + \frac{\mathcal{B}_{q,1}(u)(4-c^2)}{4[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}(\lambda+\sigma), \;\; (\lambda,\sigma) \in [0,1]^2.$$

To achieve the desired outcome, it is essential to optimize the function $M$ within the confines of a square $X$, which is closed and defined as $X = [(\lambda,\sigma) : (\lambda,\sigma) \in [0,1]^2]$.

The location of the maximum value of the function $M$ within the closed square $X$ is clearly evident. By employing the parameter $\lambda$ to distinguish the function $M(\lambda,\sigma)$, we can derive the following:

$$M_\lambda(\lambda,\sigma) = \frac{\mathcal{B}_{q,1}(u)(4-c^2)}{4[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}.$$

Since $M_\lambda(\lambda,\sigma) \geq 0$, the function $M(\lambda,\sigma)$ exhibits an increasing trend as $\lambda$ increases and attains its maximum value when $\lambda$ equals 1. Therefore,

$$\max\{M(\lambda,\sigma) : \lambda \in [0,1]\} = M(1,\sigma) = \frac{c^2\mathcal{B}_{q,1}^2(u)}{\Omega^2(2)[2]_q^2(3+e^{i\theta})^2} + \frac{\mathcal{B}_{q,1}(u)(4-c^2)(1+\sigma)}{4[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}$$

for every $\sigma$ belonging to the interval [0, 1] and every $c$ belonging to the interval [0, 2].

After taking the derivative of the function $M(1,\sigma)$, our result is now:

$$M'(1,\sigma) = \frac{\mathcal{B}_{q,1}(u)(4-c^2)}{4[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}.$$

Because $M'(1,\sigma) \geq 0$, it can be concluded that the function $M(1,\sigma)$ exhibits a consistent upward trend, meaning it increases as $\sigma$ increases. The maximum value of the function is attained when $\sigma = 1$. Consequently,

$$\max\{M(1,1) : \lambda \in [0,1]\} = M(1,1) = \frac{c^2\mathcal{B}_{q,1}^2(u)}{\Omega^2(2)[2]_q^2(3+e^{i\theta})^2} + \frac{\mathcal{B}_{q,1}(u)(4-c^2)}{2[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q},$$

where $c \in [0,2]$.

As a result, we have

$$M(\lambda,\sigma) \leq \max\{M(\lambda,\sigma) : (\lambda,\sigma) \in X\} = M(1,1) = \frac{c^2\mathcal{B}_{q,1}^2(u)}{\Omega^2(2)[2]_q^2|3+e^{i\theta}|^2} + \frac{\mathcal{B}_{q,1}(u)(4-c^2)}{2|2+[2]_q(e^{i\theta}+1)|\Omega(3)[3]_q}.$$

Given that the absolute value of $a_3$ is less than or equal to the function $M(\lambda, \sigma)$, it can be concluded that:

$$|d_3| \leq \Psi(q, \theta) \times c^2 + \frac{2|\mathcal{B}_{q,1}(u)|}{|2 + [2]_q(e^{i\theta} + 1)|\Omega(3)[3]_q}, \quad c \in [0, 2],$$

where

$$\Psi(q, \theta) = |\mathcal{B}_{q,1}(u)| \left[ \frac{|\mathcal{B}_{q,1}(u)|}{\Omega^2(2)[2]_q^2|3 + e^{i\theta}|^2} - \frac{1}{2|2 + [2]_q(e^{i\theta} + 1)|\Omega(3)[3]_q} \right].$$

Now, let us find the highest value of the function $H : \mathbb{R} \longrightarrow \mathbb{R}$, which is defined in the following way:

$$H(c) = \Psi(q, \theta) \times c^2 + \frac{2\mathcal{B}_{q,1}(u)}{[2 + [2]_q(e^{i\theta} + 1)]\Omega(3)[3]_q}$$

in the range of $c \in [0, 2]$ and $u \in [-\pi, \pi]$.

Moreover, if we take the derivative of the function $H(c)$, denoted as $H'(c)$, we obtain the expression $H'(c) = 2\Psi(q, \theta)c$, where $c$ belongs to the interval [0, 2]. It is known that when $\Psi(q, \theta) \leq 0$ ($u = 0$, $q = \frac{1}{2}$, and $\theta = \pi$ and 0), the value of $H'(c)$ is less than or equal to zero. This implies that the function $H(c)$ is a decreasing function, and its maximum value occurs at $c = 0$. Therefore, we define $H : \mathbb{R} \longrightarrow \mathbb{R}$ as a function with a maximum value, which can be described as follows.

$$\max\{H(c) : c \in [0, 2]\} = H(0) = \frac{2\mathcal{B}_{q,1}(u)}{[2 + [2]_q(e^{i\theta} + 1)]\Omega(3)[3]_q};$$

furthermore, if $\Psi(q, \theta, q) \geq 0$ ($u = \pi$, $q = \frac{1}{2}$, and $\theta = \pi$ and 0), the condition $H'(c) \geq 0$ holds. The function $H(c)$ is monotonically increasing, and its maximum value is attained when $c = 2$:

$$\max\{H(c) : c \in [0, 2]\} = H(2) = \frac{4\mathcal{B}_{q,1}^2(u)}{\Omega^2(2)[2]_q^2(3 + e^{i\theta})^2}.$$

Therefore, we derive the maximum limit approximation for the absolute value of $d_3$, as provided subsequently:

$$|d_3| \leq \frac{2|u[2]_q - q|}{\Omega(3)[2]_q[3]_q|2 + [2]_q(e^{i\theta} + 1)|}.$$

Using Equations (55) and (56), and the triangle inequality, we obtain the subsequent inequality for the absolute value of $d_4$ based on (54):

$$|d_4| \leq y_1(c) + y_2(c)(\lambda + \sigma) + y_3(c)(\lambda^2 + \sigma^2) := L(\lambda, \sigma)$$

where

$$y_1(c) = \frac{|\mathcal{B}_{q,3}(u)|}{4[3]_q!|2 + [3]_q(e^{i\theta} + 1)|\Omega(4)[4]_q} + \frac{|\mathcal{B}_{q,1}(u)|(4 - c^2)}{2|2 + [3]_q(e^{i\theta} + 1)|\Omega(4)[4]_q}$$

$$y_2(c) = \frac{|\mathcal{B}_{q,2}(u)|(4 - c^2)}{4[2]_q!|2 + [3]_q(e^{i\theta} + 1)|\Omega(4)[4]_q}c + \frac{5|\mathcal{B}_{q,1}^2(u)|(4 - c^2)}{8\Omega(2)\Omega(3)[2]_q[3]_q|3 + e^{i\theta}||2 + [2]_q(e^{i\theta} + 1)|}$$

$$y_3(c) = \frac{|\mathcal{B}_{q,1}(u)|(4 - c^2)(c - 2)}{8|2 + [3]_q(e^{i\theta} + 1)|\Omega(4)[4]_q}.$$

We are required to find the maximum value of the function $L(\lambda, \sigma)$ over the set $X$ for all values of $c$ between 0 and 2.

Because the coefficients $y_1(c)$, $y_2(c)$, and $y_3(c)$ of the function $H(\lambda,\sigma)$ depend on the parameter $c$, we need to analyze the highest value of the function $H(\lambda,\sigma)$ for different values of $c$.

Given that $y_2(0)=0$, we can set $c=0$ :

$$y_1(0)=\frac{2\mathcal{B}_{q,1}(u)}{[2+[3]_q(e^{i\theta}+1)]\Omega(4)[4]_q}$$
$$\text{and}\quad y_3(0)=-\frac{\mathcal{B}_{q,1}(u)}{[2+[3]_q(e^{i\theta}+1)]\Omega(4)[4]_q}.$$

Moreover, we obtain

$$L(\lambda,\sigma)=\frac{2\mathcal{B}_{q,1}(u)}{[2+[3]_q(e^{i\theta}+1)]\Omega(4)[4]_q}-\frac{\mathcal{B}_{q,1}(u)}{[2+[3]_q(e^{i\theta}+1)]\Omega(4)[4]_q}(\lambda^2+\sigma^2),\ (\lambda,\sigma)\in[0,1]\times[0,1].$$

Therefore, we have the following:

$$L(\lambda,\sigma)\le\max\{L(\lambda,\sigma):(\lambda,\sigma)\in X\}=L(0,0)=\frac{2|\mathcal{B}_{q,1}(u)|}{|2+[3]_q(e^{i\theta}+1)|\Omega(4)[4]_q}.$$

Assuming $c=2$, we have $y_2(2)=y_3(2)=0$. Consequently,

$$y_1(2)=\frac{2\mathcal{B}_{q,3}(u)}{[3]_q![2+[3]_q(e^{i\theta}+1)]\Omega(4)[4]_q}.$$

The function $L(\lambda,\sigma)$ remains unchanged:

$$L(\lambda,\sigma)=y_1(2)=\frac{2\mathcal{B}_{q,3}(u)}{[3]_q![2+[3]_q(e^{i\theta}+1)]\Omega(4)[4]_q}.$$

Consequently, we have the ability to prove that the function $L(\lambda,\sigma)$ does not have a maximum on the set $X$ when the value of $c$ falls within the interval (0, 2) as follows:

$$|d_4|\le\frac{2|u[2]_q-q|}{\Omega(4)[4]_q[2]_q|2+[3]_q(e^{i\theta}+1)|}.$$

The results obtained in the theorem are sharp. Really, the obtained results hold with equalities for the following function

$$l_1(\xi)=\xi+\frac{2(u[2]_q-q)}{\Omega(2)[2]_q^2(3+e^{i\theta})}\xi^2+\frac{2(q-[2]_q[3]_qu(1-u))}{\Omega(3)[2]_q^2[3]_q^2[2+[2]_q(e^{i\theta}+1)]}\xi^3+\cdots, \tag{57}$$

$$l_2(\xi)=\xi+\frac{2(u[2]_q-q)}{\Omega(3)[2]_q[3]_q[2+[2]_q(e^{i\theta}+1)]}\xi^3+\frac{2(q-[2]_q[3]_qu(1-u))}{\Omega(4)[2]_q^2[4]_q[3]_q[2+[3]_q(e^{i\theta}+1)]}\xi^4+\cdots, \tag{58}$$

$$\text{and}\quad l_3(\xi)=\xi+\frac{2(u[2]_q-q)}{\Omega(4)[2]_q[4]_q[2+[3]_q(e^{i\theta}+1)]}\xi^4+\frac{2(q-[2]_q[3]_qu(1-u))}{\Omega(5)[2]_q^2[3]_q[5]_q[2+[4]_q(e^{i\theta}+1)]}\xi^5+\cdots. \tag{59}$$

□

Based on the results derived from Theorem 1, specific parameter values yield the following conclusions.

**Corollary 1.** *Let* $l \in \mathcal{RB}_{\zeta,v}^{n,\alpha,\rho}(\omega,\tau,q;u)$. *Then,*

$$|d_2| \leq \frac{|u[2]_q - q|}{2\Omega(2)[2]_q^2},$$
$$|d_3| \leq \frac{|u[2]_q - q|}{\Omega(3)[2]_q[3]_q[1+[2]_q]},$$
$$\text{and} \quad |d_4| \leq \frac{|u[2]_q - q|}{\Omega(4)[2]_q[4]_q[1+[3]_q]}.$$

*The outcomes achieved in this study are precise for the following functions:*

$$l_1(\xi) = \xi + \frac{u[2]_q - q}{\Omega(2)[2]_q^2}\xi^2 + \frac{q - [2]_q[3]_q u(1-u)}{\Omega(3)[2]_q^2[3]_q^2[1+[2]_q]}\xi^3 + \cdots,$$
$$l_2(\xi) = \xi + \frac{u[2]_q - q}{\Omega(3)[2]_q[3]_q[1+[2]_q]}\xi^3 + \frac{q - [2]_q[3]_q u(1-u)}{\Omega(4)[2]_q^2[4]_q[3]_q[1+[3]_q]}\xi^4 + \cdots,$$
$$\text{and} \quad l_3(\xi) = \xi + \frac{u[2]_q - q}{\Omega(4)[2]_q[4]_q[1+[3]_q]}\xi^4 + \frac{q - [2]_q[3]_q u(1-u)}{\Omega(5)[2]_q^2[3]_q[5]_q[1+[4]_q]}\xi^5 + \cdots.$$

**Corollary 2.** *Let* $l \in \mathcal{RB}_{\zeta,v}^{n,\alpha,\rho}(\omega,\tau,q;u)$. *Then,*

$$|d_2| \leq \frac{|u[2]_q - q|}{\Omega(2)[2]_q^2},$$
$$|d_3| \leq \frac{|u[2]_q - q|}{\Omega(3)[2]_q[3]_q},$$
$$\text{and} \quad |d_4| \leq \frac{|u[2]_q - q|}{\Omega(4)[2]_q[4]_q}.$$

*The outcomes achieved in this study are precise for the following function:*

$$l_1(\xi) = \xi + \frac{u[2]_q - q}{\Omega(2)[2]_q^2}\xi^2 + \frac{q - [2]_q[3]_q u(1-u)}{\Omega(3)[2]_q^2[3]_q^2}\xi^3 + \cdots,$$
$$l_2(\xi) = \xi + \frac{u[2]_q - q}{\Omega(3)[2]_q[3]_q}\xi^3 + \frac{q - [2]_q[3]_q u(1-u)}{\Omega(4)[2]_q^2[4]_q[3]_q}\xi^4 + \cdots,$$
$$\text{and} \quad l_3(\xi) = \xi + \frac{u[2]_q - q}{\Omega(4)[2]_q[4]_q}\xi^4 + \frac{q - [2]_q[3]_q u(1-u)}{\Omega(5)[2]_q^2[3]_q[5]_q}\xi^5 + \cdots.$$

**Corollary 3.** *Let* $l \in M_\Sigma(\mathcal{B})$. *Then,*

$$|d_2| \leq \frac{|2u-1|}{4},$$
$$|d_3| \leq \frac{|2u-1|}{6},$$
$$\text{and} \quad |d_4| \leq \frac{|2u-1|}{8}.$$

*The outcomes achieved in this study are precise for the following function:*

$$l_1(\xi) = \xi + \frac{2u-1}{4}\xi^2 + \cdots,$$
$$l_2(\xi) = \xi + \frac{2u-1}{6}\xi^3 + \cdots,$$
$$\text{and} \quad l_3(\xi) = \xi + \frac{2u-1}{8}\xi^4 + \cdots.$$

*The outcomes achieved in this study are precise, leading to an enhancement of the findings presented by Buyankara and Caglar [55] in Theorem* 2.

## 3. The Second Hankel Determinant and Fekete–Szegö Inequality for Several Families Related to the $q$-Bernoulli Polynomials

In this section, we provide an estimate of the upper bound for the second Hankel determinant and the Fekete–Szegö inequality. These estimates are applicable to a function that belongs to a specific class, denoted as $\mathcal{RB}^{n,\alpha,\rho}_{\zeta,v,\theta}(\omega,\tau,q;u)$, as defined in Definition 3. We now introduce a theorem that offers an estimate for the upper bound of the second Hankel determinant.

**Theorem 2.** *Let* $l \in \mathcal{RB}^{n,\alpha,\rho}_{\zeta,v,\theta}(\omega,\tau,q;u)$. *Then,*

$$|d_2d_4 - d_3^2| \leq \left(\frac{2|u[2]_q - q|}{|2+[2]_q(e^{i\theta}+1)|\Omega(3)[3]_q[2]_q}\right)^2.$$

*The results obtained here are sharp for the function:*

$$l_2(\xi) = \xi + \frac{2(u[2]_q - q)}{\Omega(3)[2]_q[3]_q[2+[2]_q(e^{i\theta}+1)]}\xi^3 + \cdots.$$

**Proof.** Suppose $l$ belongs to the set $\mathcal{RB}^{n,\alpha,\rho}_{\zeta,v,\theta}(\omega,\tau,q;u)$. In that case, the expression $d_2d_4 - d_3^2$ can be expressed in the following manner by utilizing Equations (52)–(54):

$$\begin{aligned}
d_2d_4 - d_3^2 &= \frac{[2]_q![3]_q!\mathcal{B}^2_{q,1}(u) - 2[3]_q!\mathcal{B}_{q,1}(u)\mathcal{B}_{q,2}(u) + [2]_q!\mathcal{B}_{q,1}(u)\mathcal{B}_{q,3}(u)}{4\Omega(2)\Omega(4)[2]_q[4]_q[2]_q![3]_q!(3+e^{i\theta})[2+[3]_q(e^{i\theta}+1)]}s_1^4 \\
&- \frac{B^4_{q,1}(u)}{\Omega^4(2)[2]_q^4(3+e^{i\theta})^4}s_1^4 + \frac{5\mathcal{B}^3_{q,1}(u)(s_2 - r_2)}{4\Omega^2(2)\Omega(3)[2]_q^2[3]_q(3+e^{i\theta})^2[2+[2]_q(e^{i\theta}+1)]}s_1^2 \\
&+ \frac{\mathcal{B}^2_{q,1}(u)(s_3 - r_3)}{2\Omega(2)\Omega(4)[2]_q[4]_q(3+e^{i\theta})[2+[3]_q(e^{i\theta}+1)]}s_1 - \frac{\mathcal{B}^2_{q,1}(u)(s_2 - r_2)^2}{4\Omega^2(3)[3]_q^2[2+[2]_q(e^{i\theta}+1)]^2} \\
&+ \frac{\mathcal{B}_{q,1}(u)[\mathcal{B}_{q,2}(u) - [2]_q!\mathcal{B}_{q,1}(u)](s_2 + r_2)}{2\Omega(2)\Omega(4)[2]_q[4]_q[2]_q!(3+e^{i\theta})[2+[3]_q(e^{i\theta}+1)]}s_1^2 \\
&- \frac{\mathcal{B}^3_{q,1}(u)(s_2 - r_2)}{\Omega^2(2)[2]_q^2(3+e^{i\theta})^2[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}s_1^2.
\end{aligned}$$

By utilizing Equations (55) and (56), the triangle inequality, and assuming that $|s_1| = c$, $|e|$ is $\lambda$, and $|\mu|$ is $\sigma$, we can make an approximation for:

$$|d_2d_4 - d_3^2| \leq \Upsilon_1(c) + \Upsilon_2(c)(\lambda+\sigma) + \Upsilon_3(c)(\lambda^2+\sigma^2) + \Upsilon_4(c)(\lambda+\sigma)^2, \tag{60}$$

where

$$\Upsilon_1(c) = \frac{[2]_q!|\mathcal{B}_{q,1}(u)||\mathcal{B}_{q,3}(u)|}{4\Omega(2)\Omega(4)[2]_q[4]_q[2]_q![3]_q!|3+e^{i\theta}||2+[3]_q(e^{i\theta}+1)|}c^4 + \frac{|\mathcal{B}^4_{q,1}(u)|}{\Omega^4(2)[2]^4_q|3+e^{i\theta}|^4}c^4$$
$$+ \frac{|\mathcal{B}^2_{q,1}(u)|(4-c^2)}{2\Omega(2)\Omega(4)[2]_q[4]_q|3+e^{i\theta}||2+[3]_q(e^{i\theta}+1)|}c \geq 0,$$
$$\Upsilon_2(c) = \frac{9|\mathcal{B}^3_{q,1}(u)|(4-c^2)}{8\Omega^2(2)\Omega(3)[2]^2_q[3]_q|3+e^{i\theta}|^2|2+[2]_q(e^{i\theta}+1)|}c^2$$
$$+ \frac{|\mathcal{B}_{q,1}(u)||\mathcal{B}_{q,2}(u)|(4-c^2)}{4\Omega(2)\Omega(4)[2]_q[2]_q![4]_q|3+e^{i\theta}||2+[3]_q(e^{i\theta}+1)|}c^2 \geq 0$$
$$\Upsilon_3(c) = \frac{|\mathcal{B}^2_{q,1}(u)|(4-c^2)c(c-2)}{8\Omega(2)\Omega(4)[2]_q[4]_q|3+e^{i\theta}||2+[3]_q(e^{i\theta}+1)|} \leq 0$$
$$\Upsilon_4(c) = \frac{|\mathcal{B}^2_{q,1}(u)|(4-c^2)^2}{16\Omega^2(3)[3]^2_q|2+[2]_q(e^{i\theta}+1)|^2} \geq 0.$$

The function $D : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is now defined in the following manner:

$$D(\lambda,\sigma) = \Upsilon_1(c) + \Upsilon_2(c)(\lambda+\sigma) + \Upsilon_3(c)(\lambda^2+\sigma^2) + \Upsilon_4(c)(\lambda+\sigma)^2, \quad (\lambda,\sigma) \in [0,1]^2$$

for each $c \in [0,2]$.

Now, we need to maximize the function $D$ on closed square $X$ for each $c \in [0,2]$.

Since the coefficients of the function $D$ depend on the parameter $c$, we must investigate the maximum for different values of the parameter $c$.

1. (**Case 1:** When $c = 0$ (extreme point)). Assuming $c$ equals 0, because $\Upsilon_1(0)$, $\Upsilon_2(0)$, and $\Upsilon_3(0)$ are all equal to zero, and

$$\Upsilon_4(0) = \frac{\mathcal{B}^2_{q,1}(u)}{[2+[2]_q(e^{i\theta}+1)]^2\Omega^2(3)[3]^2_q}.$$

The function $D(\lambda,\sigma)$ can be expressed in the following manner.

$$D(\lambda,\sigma) = \frac{\mathcal{B}^2_{q,1}(u)}{[2+[2]_q(e^{i\theta}+1)]^2\Omega^2(3)[3]^2_q}(\lambda+\sigma)^2, \quad (\lambda,\sigma) \in X.$$

The maximum value of the function $D(\lambda,\sigma)$ can be observed at the edges of the enclosed square $X$.

After applying differentiation techniques to the function $D(\lambda,\sigma)$ with respect to $\lambda$, we obtain the following result.

$$D_\lambda(\lambda,\sigma) = \frac{2\mathcal{B}^2_{q,1}(u)}{[2+[2]_q(e^{i\theta}+1)]^2\Omega^2(3)[3]^2_q}(\lambda+\sigma), \quad \sigma \in [0,1].$$

The function $D(\lambda,\sigma)$ increases as $\lambda$ increases and reaches its maximum when $\lambda$ equals 1, as indicated by the condition $D_\lambda(\lambda,\sigma) \geq 0$. Therefore, we can conclude the following:

$$\max\{D(\lambda,\sigma) : \sigma \in [0,1]\} = D(1,\sigma) = \frac{\mathcal{B}^2_{q,1}(u)(1+\sigma)^2}{[2+[2]_q(e^{i\theta}+1)]^2\Omega^2(3)[3]^2_q}, \ \sigma \in [0,1].$$

By utilizing differentiation techniques on the function $D(1,\sigma)$, we find that if $D'(1,\sigma)$ is greater than zero, then $D(1,\sigma)$ is an ascending function and reaches its maximum value when $\sigma$ is equal to 1. Consequently,

$$\max\{D(1,\sigma) : \sigma \in [0,1]\} = D(1,1) = \frac{4\mathcal{B}_{q,1}^2(u)}{[2+[2]_q(e^{i\theta}+1)]^2\Omega^2(3)[3]_q^2}.$$

Therefore, when $c$ is equal to 0, we obtain the following.

$$D(\lambda,\sigma) \leq \max\{D(\lambda,\sigma) : (\lambda,\sigma) \in [0,1]^2\} = D(1,1) = \left(\frac{2|\mathcal{B}_{q,1}(u)|}{|2+[2]_q(e^{i\theta}+1)|\Omega(3)[3]_q}\right)^2.$$

Since $|d_2d_4 - d_3^2| \leq D(\lambda,\sigma)$, we have

$$|d_2d_4 - d_3^2| \leq \left(\frac{2|\mathcal{B}_{q,1}(u)|}{|2+[2]_q(e^{i\theta}+1)|\Omega(3)[3]_q}\right)^2.$$

2. (**Case 2:** when $c = 2$). Now, when we set $c$ equal to 2, we have $Y_2(2) = Y_3(2) = Y_4(2) = 0$ and

$$Y_1(2) = \frac{4[2]_q!\mathcal{B}_{q,1}(u)\mathcal{B}_{q,3}(u)}{\Omega(2)\Omega(4)[2]_q[4]_q[2]_q![3]_q!(3+e^{i\theta})[2+[3]_q(e^{i\theta}+1)]} + \frac{16B_{q,1}^4(u)}{\Omega^4(2)[2]_q^4(3+e^{i\theta})^4}.$$

the function $D(\lambda,\sigma)$ is a function as follows

$$D(\lambda,\sigma) = \frac{2\Omega^3(2)[2]_q^3(1+e^{i\theta})^3[2]_q!\mathcal{B}_{q,1}(u)\mathcal{B}_{q,3}(u) + 16\Omega(4)[4]_q[2]_q![3]_q![2+[3]_q(e^{i\theta}+1)]}{\Omega^4(2)\Omega(4)[2]_q^4[4]_q[2]_q![3]_q!(3+e^{i\theta})^4[2+[3]_q(e^{i\theta}+1)]}$$

Hence, we have

$$|d_2d_4 - d_3^2| \leq \frac{V}{\Omega^4(2)\Omega(2)\Omega(4)[2]_q^4[4]_q[2]_q![3]_q!|e^{i\theta}+3|^4||2+[3]_q(e^{i\theta}+1)|},$$

where

$$\begin{aligned} V =& 2\Omega^3(2)[2]_q^3|3+e^{i\theta}|^3[2]_q!|\mathcal{B}_{q,1}(u)||\mathcal{B}_{q,3}(u)| + 16|\mathcal{B}_{q,1}^4(u)|\Omega(4)[4]_q[2]_q![3]_q! \\ & |2+[3]_q(e^{i\theta}+1)| \end{aligned}$$

3. (**Case 3:** When $c$ is between 0 and 2). We are given a range for the variable $c$, which lies between 0 and 2. Our objective is to analyze the maximum value of the function $D(\lambda,\sigma)$ while considering the sign of a certain expression denoted by $\chi(D(\lambda,\sigma))$. This expression is given by the equation

$$\chi(D(\lambda,\sigma)) = D_{\lambda\lambda}(\lambda,\mu)D_{\sigma\sigma}(\lambda,\sigma) - (D_{\lambda\sigma}(\lambda,\sigma))^2.$$

We have observed that the equation

$$\chi(D(\lambda,\sigma)) = 4Y_3(c)[Y_3(c) + 2Y_4(c)]$$

is visible to us. We will now examine two instances where we determine the sign of $\chi(D(\lambda,\sigma))$.

(a) In the given interval of $c$ values, specifically between 0 and 2, it is required that $Y_3(c) + 2Y_4(c)$ remains less than or equal to zero. In this particular case, since both $D_{\lambda\sigma}(\lambda,\sigma)$ and $D_{\sigma\lambda}(\lambda,\sigma)$ are equal to $2Y_4(c)$ and greater than or equal to zero, and $\chi(D(\lambda,\sigma))$ is also greater than or equal to zero, the function

$D(\lambda,\sigma)$ cannot achieve its maximum value within the square $X$ as per basic calculus principles.

(b) Additionally, suppose there exists a value of $c$ within the range of 0 to 2 where $\Upsilon_3(c) + 2\Upsilon_4(c) \geq 0$. In this particular case, if $\chi(D) \leq 0$, it is not possible for the function $D(\lambda,\sigma)$ to have a maximum within the square $X = [(\lambda,\sigma) : (\lambda,\sigma) \in [0,1]^2]$.

Due to these three occurrences, In the first case, we take the maximum, which is the extreme point, and then proceed to express it in writing.

$$|d_2 d_4 - d_3^2| \leq \left( \frac{2|\mathcal{B}_{q,1}(u)|}{|2 + [2]_q(e^{i\theta} + 1)|\Omega(3)[3]_q} \right)^2.$$

Therefore, the proof of Theorem 2 is concluded. □

Based on the results derived from Theorem 2, specific parameter values yield the following conclusions.

**Corollary 4.** *Let $l \in \mathcal{RB}^{n,\alpha,\rho}_{\zeta,v}(\omega,\tau,q;u)$. Then,*

$$|d_2 d_4 - d_3^2| \leq \left( \frac{|u[2]_q - q|}{|1 + [2]_q|\Omega(3)[3]_q[2]_q} \right)^2.$$

*The outcomes achieved in this study are precise for the function:*

$$l_2(\xi) = \xi + \frac{u[2]_q - q}{\Omega(3)[2]_q[3]_q[1 + [2]_q]} \xi^3 + \cdots.$$

**Corollary 5.** *Let $l \in \mathcal{RB}^{n,\alpha,\rho}_{\zeta,v}(\omega,\tau,q;u)$. Then,*

$$|d_2 d_4 - d_3^2| \leq \left( \frac{|u[2]_q - q|}{\Omega(3)[3]_q[2]_q} \right)^2.$$

*The outcomes achieved in this study are precise for the function:*

$$l_2(\xi) = \xi + \frac{u[2]_q - q}{\Omega(3)[2]_q[3]_q} \xi^3 + \cdots.$$

**Corollary 6.** *Let $l \in M_\Sigma(\mathcal{B})$. Then,*

$$|d_2 d_4 - d_3^2| \leq \left( \frac{2|u - 1|}{6} \right)^2.$$

*The outcomes achieved in this study are precise for the function:*

$$l_2(\xi) = \xi + \frac{2u - 1}{6} \xi^3 + \cdots.$$

Now, we will present the theorem regarding the Fekete–Szegö inequality.

**Theorem 3.** *Let $l \in \mathcal{RB}^{n,\alpha,\rho}_{\zeta,v,\theta}(\omega,\tau,q;u)$, $\vartheta \in \mathcal{C}$. Then,*

$$\left| d_3 - \vartheta d_2^2 \right| \leq \begin{cases} \frac{4|u[2]_q - q|^2 G(\theta,q)}{\Omega^2(2)[2]_q^4|3+e^{i\theta}|^2} & |1 - \vartheta| \leq G(q) \\ \\ \frac{4|u[2]_q - q|^2|1-\vartheta|}{\Omega^2(2)[2]_q^4|3+e^{i\theta}|^2} & |1 - \vartheta| \geq G(q). \end{cases}$$

*where*

$$G(q) = \frac{\Omega^2(2)[2]_q^3|3+e^{i\theta}|^2}{2|u[2]_q - q||2+[2]_q(e^{i\theta}+1)|\Omega(3)[3]_q}.$$

*The results obtained here are sharp for the function:*

$$l_1(\xi) = \xi + \frac{2(u[2]_q - q)}{\Omega(2)[2]_q^2(3+e^{i\theta})}\xi^2 + \frac{2(q-[2]_q[3]_q u(1-u))}{\Omega(3)[2]_q^2[3]_q^2[2+[2]_q(e^{i\theta}+1)]}\xi^3 + \cdots.$$

**Proof.** Let $l \in \mathcal{RB}_{\zeta,\upsilon,\theta}^{n,\alpha,\rho}(\omega,\tau,q;u)$ and $\vartheta \in \mathcal{C}$. Subsequently, utilizing Equations (52), (53), (55), and (56), we determine the value of the expression $d_3 - \vartheta d_2^2$:

$$d_3 - \vartheta d_2^2 = \frac{s_1^2\mathcal{B}_{q,1}^2(u)}{\Omega^2(2)[2]_q^2(3+e^{i\theta})^2}(1-\vartheta) + \frac{\mathcal{B}_{q,1}(u)(4-s_1^2)}{4[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}(e-\mu) \tag{61}$$

for some $e, \mu$ with $|e| \le 1$ and $|\mu| \le 1$.

By utilizing the triangle inequality on Equation (61), we can determine the maximum value of $|d_3 - \vartheta d_2^2|$. This can be achieved by considering the following conditions: $|e|$ is equal to $\lambda$, $|\mu|$ is equal to $\sigma$, and $|s_1|$ is equal to $c$.

$$|d_3 - \vartheta d_2^2| \le \frac{|1-\vartheta||\mathcal{B}_{q,1}^2(u)|}{\Omega^2(2)[2]_q^2|3+e^{i\theta}|^2}c^2 + \frac{|\mathcal{B}_{q,1}(u)|(4-c^2)}{4|2+[2]_q(e^{i\theta}+1)|\Omega(3)[3]_q}(\lambda+\sigma), \tag{62}$$

for each $c \in [0,2]$.

We have the ability to establish the function $\chi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ in the following manner:

$$\chi(\lambda,\sigma) = \frac{|1-\vartheta|\mathcal{B}_{q,1}^2(u)}{\Omega^2(2)[2]_q^2(3+e^{i\theta})^2}c^2 + \frac{\mathcal{B}_{q,1}(u)(4-c^2)}{4[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}(\lambda-\sigma), \quad (\lambda,\sigma) \in X,$$

for each $c \in (0,2)$. We need to verify that the function $\chi(\lambda,\sigma)$ is maximized on the set $X$ for every value of $c$ ranging from 0 to 2.

It is evident that the function $\chi(\lambda,\sigma)$ achieves its highest value at the edges of the enclosed square $X$. Therefore, by performing straightforward differentiation of the function $\chi(\lambda,\sigma)$ with respect to $\lambda$, we obtain

$$\chi_\lambda(\lambda,\sigma) = \frac{\mathcal{B}_{q,1}(u)(4-c^2)}{4[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}, \quad c \in [0,2]. \tag{63}$$

As the value of $\chi_\lambda(\lambda,\sigma)$ is greater than 0, the function $\chi(\lambda,\sigma)$ exhibits an upward trend with respect to $\lambda$, and its maximum value is attained when $\lambda$ is equal to 1.

$$\begin{aligned}\max\{\chi(\lambda,\sigma) : \sigma \in [0,1]\} = \chi(1,\sigma) &= \frac{|1-\vartheta|\mathcal{B}_{q,1}^2(u)}{\Omega^2(2)[2]_q^2(3+e^{i\theta})^2}c^2 \\ &\quad + \frac{\mathcal{B}_{q,1}(u)(4-c^2)}{4[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}(1-\sigma)\end{aligned}$$

for each $\sigma \in [0,1]$ and $c \in [0,2]$.

Moreover, by differentiating $\chi(1,\sigma)$, we obtain the following result.

$$\chi'(1,\sigma) = \frac{\mathcal{B}_{q,1}(u)(4-c^2)}{4[2+[2]_q(e^{i\theta}+1)]\Omega(3)[3]_q}$$

for each $c \in [0,2]$.

If the condition $\chi'(1,\sigma) > 0$ is satisfied, the function $\chi(1,\sigma)$ is monotonically increasing, and its maximum value is achieved when $\sigma$ is equal to 1. Therefore,

$$\chi(\lambda,\mu) \leq \max\{\chi(\lambda,\sigma) : (\lambda,\sigma) \in [0,1] \times [0,1]\} = \chi(1,1) = \frac{|1-\vartheta||\mathcal{B}^2_{q,1}(u)|}{\Omega^2(2)[2]_q^2|3+e^{i\theta}|^2}c^2 + \frac{|\mathcal{B}_{q,1}(u)|(4-c^2)}{2|2+[2]_q(e^{i\theta}+1)|\Omega(3)[3]_q}.$$

Since $|d_3 - \vartheta d_2^2| \leq \chi(\lambda,\sigma)$, we obtain

$$|d_3 - \vartheta d_2^2| \leq |\mathcal{B}^2_{q,1}(u)| \left[ \frac{|1-\vartheta| - G(q)}{\Omega^2(2)[2]_q^2|3+e^{i\theta}|^2} \right] c^2 + \frac{4|\mathcal{B}^2_{q,1}(u)|G(q)}{\Omega^2(2)[2]_q^2|3+e^{i\theta}|^2}$$

where

$$G(q) = \frac{\Omega^2(2)[2]_q^2|3+e^{i\theta}|^2}{2|\mathcal{B}_{q,1}(u)||2+[2]_q(e^{i\theta}+1)|\Omega(3)[3]_q}.$$

Now is the time to determine the maximum value of the function $\Xi : [0,2] \longrightarrow \mathbb{R}$ defined by:

$$\Xi(c) = \mathcal{B}^2_{q,1}(u) \left[ \frac{|1-\vartheta| - G(q)}{\Omega^2(2)[2]_q^2(3+e^{i\theta})^2} \right] c^2 + \frac{4\mathcal{B}^2_{q,1}(u)G(q)}{\Omega^2(2)[2]_q^2(3+e^{i\theta})^2}.$$

When we use the differentiation principle on the function $\Xi(c)$, we obtain the following result.

$$\Xi'(c) = \frac{2\mathcal{B}^2_{q,1}(u)[|1-\vartheta| - G(q)]}{\Omega^2(2)[2]_q^2(3+e^{i\theta})^2}c, \quad c \in [0,2].$$

If the absolute difference between 1 and $\vartheta$ is less than or equal to $G(q)$, and the maximum occurs when $c$ is equal to 0, then the function $\Xi(c)$ is decreasing because its derivative $\Xi'(c)$ is less than or equal to 0.

$$\max\{\Xi(c) : c \in [0,2]\} = \Xi(0) = \frac{4\mathcal{B}^2_{q,1}(u)G(q)}{\Omega^2(2)[2]_q^2(3+e^{i\theta})^2}.$$

If $\Xi'(c) \geq 0$ and $\Xi(c)$ represents an increasing function, then when $|1-\vartheta| \geq G(q)$ and the maximum value is achieved at $c = 2$,

$$\max\{\Xi(c) : c \in [0,2]\} = \Xi(2) = \frac{4\mathcal{B}^2_{q,1}(u)|1-\vartheta|}{\Omega^2(2)[2]_q^2(3+e^{i\theta})^2}.$$

We consequently arrive at

$$\left|d_3 - \vartheta d_2^2\right| \leq \begin{cases} \frac{4|\mathcal{B}^2_{q,1}(u)|G(q)}{\Omega^2(2)[2]_q^2|3+e^{i\theta}|^2} & |1-\vartheta| \leq G(q) \\ \\ \frac{4|\mathcal{B}^2_{q,1}(u)||1-\vartheta|}{\Omega^2(2)[2]_q^2|3+e^{i\theta}|^2} & |1-\vartheta| \geq G(q). \end{cases}$$

The result achieved in this scenario is precise and definitive for $|1-\vartheta| \geq G(q)$. □

Based on the results derived from Theorem 3, specific parameter values yield the following conclusions.

**Corollary 7.** *Let $l \in \mathcal{RB}^{n,\alpha,\rho}_{\zeta,v}(\omega,\tau,q;u)$, $\vartheta \in \mathcal{C}$. Then,*

$$\left|d_3 - \vartheta d_2^2\right| \leq \begin{cases} \frac{|u[2]_q - q|^2 G(q)}{4\Omega^2(2)[2]_q^4} & |1-\vartheta| \leq G(q) \\ \\ \frac{|u[2]_q - q|^2 |1-\vartheta|}{4\Omega^2(2)[2]_q^4} & |1-\vartheta| \geq G(q). \end{cases}$$

*where*

$$G(q) = \frac{4\Omega^2(2)[2]_q^3}{|u[2]_q - q|[1+[2]_q]\Omega(3)[3]_q}.$$

*The outcomes achieved in this study are precise for the function:*

$$l_1(\xi) = \xi + \frac{u[2]_q - q}{\Omega(2)[2]_q^2}\xi^2 + \cdots.$$

**Corollary 8.** *Let $l \in \mathcal{RB}^{n,\alpha,\rho}_{\zeta,v}(\omega,\tau,q;u)$, $\vartheta \in \mathcal{C}$. Then,*

$$\left|d_3 - \vartheta d_2^2\right| \leq \begin{cases} \frac{|u[2]_q - q|^2 G(q)}{\Omega^2(2)[2]_q^4} & |1-\vartheta| \leq G(q) \\ \\ \frac{|u[2]_q - q|^2 |1-\vartheta|}{\Omega^2(2)[2]_q^4} & |1-\vartheta| \geq G(q). \end{cases}$$

*where*

$$G(q) = \frac{\Omega^2(2)[2]_q^3}{|u[2]_q - q|\Omega(3)[3]_q}.$$

*The outcomes achieved in this study are precise for the function:*

$$l_1(\xi) = \xi + \frac{u[2]_q - q}{\Omega(2)[2]_q^2}\xi^2 + \cdots.$$

**Corollary 9.** *Let $l \in M_\Sigma(\mathcal{B})$, $\vartheta \in \mathcal{C}$. Then,*

$$\left|d_3 - \vartheta d_2^2\right| \leq \begin{cases} \frac{|2u-1|}{6} & |1-\vartheta| \leq \frac{8}{3(2u-1)} \\ \\ \frac{|2u-1|^2|1-\vartheta|}{16} & |1-\vartheta| \geq \frac{8}{3(2u-1)}. \end{cases}$$

*The outcomes achieved in this study are precise for the function:*

$$l_1(\xi) = \xi + \frac{2u-1}{4}\xi^2 + \cdots,$$

*leading to an enhancement of the findings presented by Buyankara and Caglar [55] in Theorem* 3.

The theorem referred to as the presentation of Theorem 3 is as follows, assuming that $\vartheta$ belongs to the set of real numbers, denoted as $\mathbb{R}$.

**Theorem 4.** *Let $l \in \mathcal{RB}^{n,\alpha,\rho}_{\zeta,v,\theta}(\omega,\tau,q;u)$, $\vartheta \in \mathbb{R}$. Then:*

$$|d_3 - \vartheta d_2^2| \leq \begin{cases} \frac{4|u[2]_q - q|^2(1-\vartheta)}{\Omega^2(2)[2]_q^4|3+e^{i\theta}|^2} & \text{if } \vartheta \leq 1 - G(q) \\ \\ \frac{4|u[2]_q - q|^2 G(q)}{\Omega^2(2)[2]_q^4|3+e^{i\theta}|^2} & \text{if } 1 - G(q) \leq \vartheta \leq 1 + G(q) \\ \\ \frac{4|u[2]_q - q|^2(\vartheta-1)}{\Omega^2(2)[2]_q^4|3+e^{i\theta}|^2} & \text{if } 1 + G(q) \leq \vartheta, \end{cases} \tag{64}$$

*where*

$$G(q) = \frac{\Omega^2(2)[2]_q^3|3+e^{i\theta}|^2}{2|u[2]_q - q||2+[2]_q 1 + e^{i\theta}|\Omega(3)[3]_q}.$$

**Proof.** Let $l$ belong to a specific class denoted as $\mathcal{RB}^{n,\alpha,\rho}_{\zeta,v,\theta}(\omega,\tau,q;u)$ and let $\vartheta$ be a real number. In the scenario where $\vartheta$ is a real number, the inequalities $|1-\vartheta| \geq G(q)$ and $|1-\vartheta| \leq G(q)$ can be considered as equivalent.

$$\vartheta \leq 1 - G(q) \;\text{ or }\; \vartheta \geq 1 + G(q)$$

and

$$1 - G(q) \leq \vartheta \leq 1 + G(q),$$

respectively. The conclusion of the theorem is obtained based on the findings of Theorem 3. □

Given the value of parameter $\vartheta$ as 1, the subsequent corollary can be stated as follows:

**Corollary 10.** *Let $l \in \mathcal{RB}^{n,\alpha,\rho}_{\zeta,v,\theta}(\omega,\tau,q;u)$, $\vartheta \in \mathbb{R}$. Then:*

$$|d_3 - d_2^2| \leq \frac{2|u[2]_q - q|}{\Omega(3)[2]_q[3]_q|2+[2]_q(e^{i\theta}+1)|}.$$

Given the parameter $\xi = 0$, we can deduce the following corollary:

**Corollary 11.** *Let $l \in \mathcal{RB}^{n,\alpha,\rho}_{\zeta,v,\theta}(\omega,\tau,q;u)$, $\vartheta \in \mathbb{R}$. Then:*

$$|d_3| \leq \begin{cases} \frac{4|u[2]_q - q|^2}{\Omega^2(2)[2]_q^4|3+e^{i\theta}|^2} & \text{if } G(q) \leq 1 \\ \\ \frac{4|u[2]_q - q|^2 G(q)}{\Omega^2(2)[2]_q^4|3+e^{i\theta}|^2} & \text{if } G(q) \geq 1, \end{cases} \tag{65}$$

*where*

$$G(q) = \frac{\Omega^2(2)[2]_q^3|3+e^{i\theta}|^2}{2|u[2]_q - q||2+[2]_q(e^{i\theta}+1)|\Omega(3)[3]_q}.$$

**Remark 3.** *By modifying the parameters in Theorems 1–4, we obtained additional results that are connected to the established operators and are derived using the newly introduced q-convolution operator described in this research paper.*

## 4. Conclusions

This article introduces new subfamilies of $\Sigma$ by utilizing the concepts of the new $q$-convolution operator, bi-univalent functions, and $q$-Bernoulli polynomials. We examined the coefficient bounds, Fekete–Szegö functional, and second Hankel determinant for these newly defined subfamilies, and our results are proven to be precise. Furthermore, our research showcases how the outcomes can be improved and broadened by adjusting the parameters, incorporating some recently discovered findings.

In future research, scientists can choose to investigate alternative extended $q$-operators as substitutes for the $q$-convolution operator. This would enable them to create multiple new subcategories within the bi-univalent function class $\Sigma$. Furthermore, by employing the $q$-Bernoulli polynomial method, researchers interested in this field can analyze coefficient estimates for different recently defined subclasses of bi-univalent functions. Depending on their inspiration and the insights gained from this area, researchers may decide to explore various approaches.

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