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A Novel Four-Dimensional Memristive Hyperchaotic Map Based on a Three-Dimensional Parabolic Chaotic Map with a Discrete Memristor

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Abstract: Recently, the application of memristors in chaotic systems has been extensively studied. Unfortunately, there is limited literature on the introduction of discrete memristors into chaotic maps, especially into non-classical multidimensional maps. For this reason, this paper establishes a new three-dimensional parabolic chaotic map model; in order to improve the complexity and randomness of the map, it is coupled with a square-charge-controlled discrete memristor to design a new four-dimensional memristive hyperchaotic map. Firstly, the stability of the two maps is discussed. And their dynamical properties are compared using Lyapunov exponential spectra and bifurcation diagrams. Then, the phase diagram and iteration sequence of the 4D memristive hyperchaotic map are obtained. Meanwhile, we investigate the hyperchaotic states, the transient chaos, state transfer and attractor coexistence phenomena of the four-dimensional memristive map. In particular, the special state transfer phenomenon of switching from a periodic attractor to a quasi-periodic attractor and the special coexistence phenomenon of a quasi-periodic attractor coexisting with a quasi-periodic attractor around fixed points are found, which have not been observed in other systems. Finally, the phase-track diagrams and iterative sequence diagrams of the four-dimensional memristive map are verified on a digital experimental platform, revealing its potential for practical applications.

Keywords: chaotic map; coupled chaos; discrete memristor; hyperchaos; bifurcation; attractor coexistence



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1. Introduction

Chaos has multiple properties as a nonlinear phenomenon, which include initial condition sensitivity, internal randomness, fractal dimension, and non-periodic fixed normality. Due to these properties, it finds applications in the fields of secure communication [1,2], electronic engineering [3–5], biological engineering [6–8], chemical engineering [9] and so on. Discrete chaotic system can utilize different parameter or initial value settings to generate different random sequences. Therefore, increasing the number of characteristic adjustment parameters of chaotic maps as well as the dimensionality of chaotic maps is an effective way to improve their complexity and randomness.

Memristor [10] is a type of nonlinear element that can exhibit complicated dynamic behavior. It can be effectively implemented in chaotic systems, which is one of the currently trending issues in the field of chaos; however, the majority of these research concentrate on applying memristors to continuous chaotic systems [11–19]. For the past few years, Bao et al. [20] and Peng et al. [21] have successively proposed different methods for converting continuous memristors into discrete memristors. He et al. [22] and Peng et al. [23,24] developed fractional and integer-order discrete HP memristors, respectively, using the

HP continuous memristor model. Liang et al. [25] and Wei et al. [26] developed two-dimensional memristive chaotic maps based on discrete HP memristors to improve the performance of the original maps, respectively. Deng et al. [27] constructed, for the first time, a non-autonomous discrete system based on a discrete memristor (DM) with bursty oscillations. A novel n -dimensional generalized DM model was also created [28]. Yuan et al. [29] proposed a cascade method to generate chaotic and hyperchaotic discrete memory maps. Li et al. [30] discretized a local active continuous memristor using the rectangular pulse discrete sampling method to obtain a DM, and subsequently introduced it into a logistic map. After proposing a new two-dimensional memristive map that has infinite immovable points, Lai et al. coupled a local active DM with a 2D generalized map to produce a 3D memristive map [31,32]. For the purpose of examining the memristive maps without fixed points while adequately accounting for their hidden Naimark–Seik bifurcation, Rong et al. [33] presented a reduced dimensional transformation approach. By utilizing the process of period doubling and the occurrence of crises, Laskaridis et al. [34] identified a pathway into chaos, proposing a model of the Ikeda map based on a memristor. However, multi-dimensional memristive maps based on non-classical maps are yet to be investigated. Therefore, the design of a new multidimensional map and then the introduction of discrete memristors to construct a more complex multidimensional memristive map are worth studying.

The remainder of this paper is arranged as follows. In part 2, a 3D parabolic chaotic map is created. Then, a DM Simulink model is established. By adding this DM to this 3D chaotic map, a novel four-dimensional memristive hyperchaotic map is established. In addition, fixed point analyses are performed for both maps. In part 3, the dynamical behavior of the two maps is analyzed and some phenomena are found by varying different parameters of the maps using Lyapunov exponential spectra, bifurcation diagrams, phase trace diagrams and iterative sequences. Examples include transient chaos, state transfer and attractor coexistence for four-dimensional memristive hyperchaotic map. In particular, a special state transfer phenomenon of switching from a periodic attractor to a quasi-periodic attractor and a special coexistence phenomenon of a quasi-periodic attractor coexisting with a quasi-periodic attractor around an fixed points, which have not been found in other systems, are found. In part 4, the 4D hyperchaotic map is verified on a digital experimental platform. Finally, part 5 summarizes this entire paper.

2. A 4D Parabolic Memristive Hyperchaotic Map Model

In this section, firstly, a 3D parabolic chaotic map is obtained via generalizing and nonlinearly coupling the two-dimensional parabolic map [35] with the tent map [36]. Then, a DM is coupled with this 3D parabolic chaotic map to obtain a 4D parabolic memristor hyperchaotic map.

2.1. A 3D Parabolic Chaotic Map

The equations for the 2D parabolic map and the tent map equations are shown in Equation (1) and Equation (2), respectively.

$$\begin{cases} x_{n+1} = ay_n^2 \\ y_{n+1} = by_n(1 - x_n) \end{cases} \quad (1)$$

$$x_{n+1} = \begin{cases} ax_n, 0 \leq x_n \leq 0.5 \\ a(1 - x_n), 0.5 \leq x_n < 1 \end{cases} \quad (2)$$

By generalizing and linearly coupling the 2D parabolic map with the second equation of the Tent map, a 3D parabolic map is derived; its mathematical model is expressed as

$$\begin{cases} x_{n+1} = ay_n^2 \\ y_{n+1} = by_n(1 - z_n) \\ z_{n+1} = c(1 - x_n) \end{cases} \quad (3)$$

where the parameters a , b and c are all positive.

The fixed point of this 3D chaotic map is $S = (x^*, y^*, z^*)$. The fixed point satisfies

$$\begin{cases} x^* = ay^{*2} \\ y^* = by^*(1 - z^*) \\ z^* = c(1 - x^*) \end{cases} \quad (4)$$

When $1 + bc - b \neq abcy^2$, Equation (4) has only zero solutions; that is, the map (3) has only one origin fixed point $S_0 = (0, 0, 0)$. When $b > 1$, in addition to one zero solution, Equation (4) has two non-zero real solutions; i.e., the map (3) has three fixed points. They are shown below as

$$\begin{cases} S_0 = (0, 0, 0) \\ S_+ = \left(\frac{1+bc-b}{bc}, \sqrt{\frac{1+bc-b}{abc}}, \frac{b-1}{b} \right) \\ S_- = \left(\frac{1+bc-b}{bc}, -\sqrt{\frac{1+bc-b}{abc}}, \frac{b-1}{b} \right) \end{cases} \quad (5)$$

The Jacobi matrix of map at the fixed points S is

$$J_s = \begin{bmatrix} -\lambda & 2ay^* & 0 \\ 0 & b(1 - z^*) - \lambda & -by^* \\ -c & 0 & -\lambda \end{bmatrix} \quad (6)$$

The characteristic equation of map is

$$\lambda^3 - b(1 - z)\lambda^2 - abcy^2 = 0 \quad (7)$$

Based on Equation (5), two cases are discussed.

Case 1: For the origin fixed point, the eigenvalues are

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = b \end{cases} \quad (8)$$

Case 2: For two non-zero fixed points, the eigenvalues are

$$\begin{cases} \lambda_1 = \gamma + \beta + \frac{1}{3} \\ \lambda_2 = \gamma \times \frac{-1+\sqrt{3}i}{2} - \beta \times \frac{1+\sqrt{3}i}{2} + \frac{1}{3} \\ \lambda_3 = \beta \times \frac{-1+\sqrt{3}i}{2} - \gamma \times \frac{1+\sqrt{3}i}{2} + \frac{1}{3} \end{cases} \quad (9)$$

where

$$\gamma = \left(\frac{28}{27} - b + bc + \sqrt{b^2 - \frac{56}{27}b + \frac{783}{729} - 2b^2c + \frac{56}{27}bc + b^2c^2} \right) \quad (10)$$

$$\beta = \left(\frac{28}{27} - b + bc - \sqrt{b^2 - \frac{56}{27}b + \frac{783}{729} - 2b^2c + \frac{56}{27}bc + b^2c^2} \right) \quad (11)$$

As can be seen above, the magnitude of the eigenvalue is jointly determined by b and c , and has nothing to do with a . This means that changing b and c can change the dynamic characteristics, whereas changing a can only affect the magnitude. Therefore, both b and c are characteristic adjustment parameters and a is a magnitude adjustment characteristic parameter. This 3D parabolic map has an additional control parameter compared to the 2D parabolic map.

Using a structured design approach, the 3D parabolic chaotic map can be designed as the structural map illustrated in Figure 1. It can be seen that the system has three delay modules (green part of the figure), two constant modules (orange part of the figure), five

gain modules (yellow part of the figure), two adders (blue part of the figure), and two multipliers (gray part of the figure).

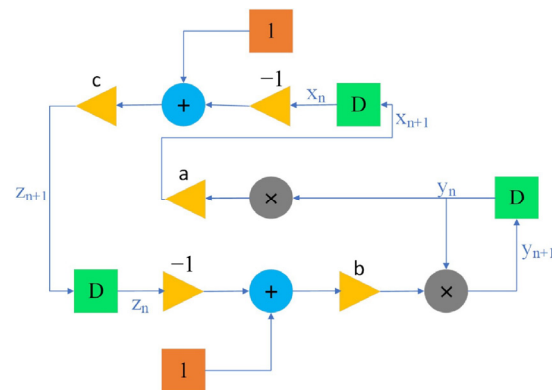


Figure 1. System structure diagram of 3D parabolic chaotic map.

2.2. A Discrete Memristor Model

The memristor utilized in the paper is a square function charge-controlled memristor with a structure similar to that of [37]. In comparison to [37], the memristor employed in this study changed the memristor's parameters ε and η . The mathematical models of continuous memristor and discrete memristor are shown in Equation (12) and Equation (13), respectively.

$$\begin{cases} v(t) = M(q(t))i(t) \\ M(q(t)) = \varepsilon + \eta q^2(t) \\ \frac{dq(t)}{dt} = i(t) \end{cases} \quad (12)$$

$$\begin{cases} v_n = M(q_n)i_n \\ M(q_n) = \varepsilon + \eta q_n^2 \\ q_{n+1} = q_n + ki_n \end{cases} \quad (13)$$

where the memristor parameters $\varepsilon = 1$ and $\eta = -0.5$.

The Simulink model of the DM is illustrated in Figure 2. To check whether this DM still satisfies the memristor characteristics after adjusting its parameters, the volt–ampere curves under the conditions of different signal frequencies and different signal amplitudes are drawn. The volt–ampere curves in Figure 3 are produced by firstly setting the signal's amplitude to 1A and its frequency to 2 rad/s, 3 rad/s, and 4 rad/s, respectively. The voltammetric curves illustrated in Figure 4 are the results of setting the signal's frequency to 2 rad/s and its amplitude to 1 A, 0.8 A, and 0.5 A, respectively.

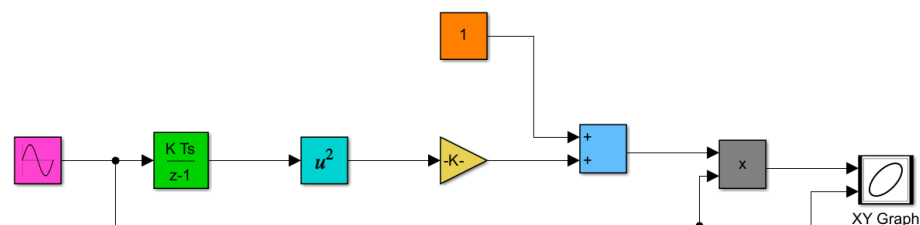


Figure 2. Simulink model of the DM.

As seen in Figure 3, the hysteresis loop gets flatter as the input signal frequency increases for a certain input current amplitude. As seen in Figure 4, the hysteresis loop decreases with decreasing input signal amplitude for a certain input current frequency.

Following the analysis, it is evident that the charge-controlled DM satisfies the definition of memristor, and thus can be used to analyze memristive discrete maps.

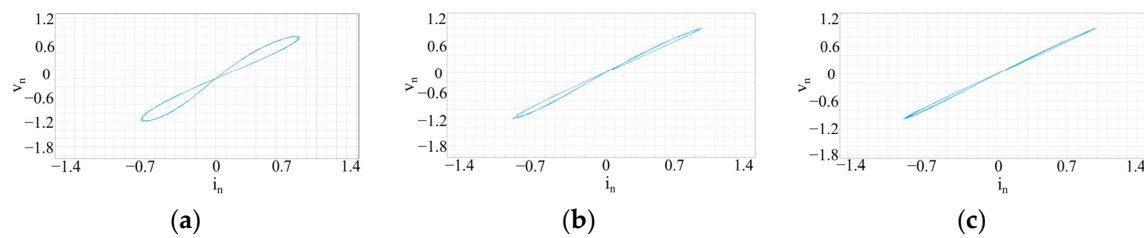


Figure 3. The DM frequency-dependent hysteresis loops for the same amplitude condition. (a) Input signal frequency 2 rad/s. (b) Input signal frequency 3 rad/s. (c) Input signal frequency 4 rad/s.

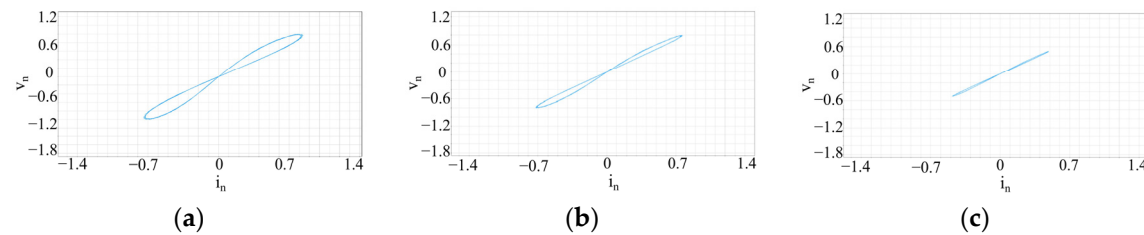


Figure 4. The DM amplitude-dependent hysteresis loops for the same frequency condition. (a) Input signal amplitude of 1A. (b) Input signal amplitude of 0.8A. (c) Input signal amplitude of 0.5A.

2.3. A 4D Parabolic Memristive Hyperchaotic Map

Coupling a 3D chaotic map with DM yields a new 4D memristor hyperchaotic map, as shown in the coupling Equation (14).

$$\begin{cases} x_{n+1} = ay_n^2 \\ y_{n+1} = b(\varepsilon + \eta w_n^2)y_n(1 - z_n) \\ z_{n+1} = c(1 - x_n) \\ w_{n+1} = w_n + y_n \end{cases} \quad (14)$$

Let $S' = (x'^*, y'^*, z'^*, w'^*)$ be the fixed point of the 4D memristive hyperchaotic map, and the fixed point satisfies

$$\begin{cases} x'^* = ay'^{*2} \\ y'^* = b(\varepsilon + \eta w'^{*2})y'^*(1 - z'^*) \\ z'^* = c(1 - x'^*) \\ w'^* = w'^* + y'^* \end{cases} \quad (15)$$

Solving it for its fixed points as

$$\begin{cases} x'^* = y'^* = 0 \\ z'^* = c \\ w'^* = \theta \end{cases} \quad (16)$$

where θ is arbitrary, which means that it can take any value; that is, the map has an infinite number of fixed points.

Linearization of the system yields its Jacobi matrix as

$$J = \begin{bmatrix} 0 & 2ay & 0 & 0 \\ 0 & b(\varepsilon + \eta w^2)(1 - z) & -b(\varepsilon + \eta w^2)y & 2b\eta wy(1 - z) \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad (17)$$

After substituting the fixed points into the above matrix, we have

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b\theta(1-c) & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad (18)$$

The characteristic equation of map is

$$\lambda^4 - \lambda^3(1 + b\theta(1-c)) + \lambda^2 b\theta(1-c) = 0 \quad (19)$$

The eigenvalues are calculated as

$$\begin{cases} \lambda_1 = \lambda_2 = 0 \\ \lambda_3 = 1 \\ \lambda_4 = b\theta(1-c) \end{cases} \quad (20)$$

According to the judgment rule for the stability of discrete maps, the fixed point is stable if $|\lambda_1| < 1$, $|\lambda_2| < 1$, $|\lambda_3| < 1$, $|\lambda_4| < 1$; otherwise, it is unstable. From Equation (20), it can be seen that λ_1 and λ_2 are always at the center of the unit circle, namely, inside the unit circle, λ_3 is always on the unit circle, and λ_4 depends on the initial state of the parameters b and c as well as the internal state q of the memristor. In other words, the parameters b and c and the memristor-related parameters can change the stability of the four-dimensional memristive map.

Based on this 4D memristive hyperchaotic system, the system structure diagram is shown in Figure 5. Compared to Figure 1, Figure 5 adds the discrete memristor section, which is depicted by the shaded area. This section allows the map to add a delay module, a constant module, a gain module, two adders, and two multipliers.

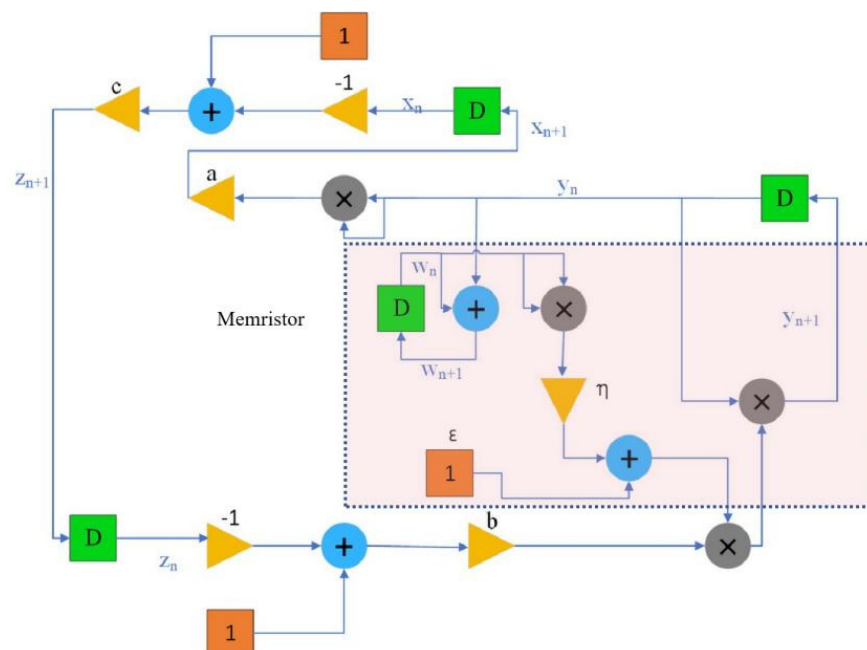


Figure 5. System structure diagram of 4D memristive hyperchaotic map.

3. Dynamical Analyses

This section compares the dynamical behaviors of the 3D parabolic chaotic map with the 4D memristive hyperchaotic map under various parameter settings and identifies several phenomena.

3.1. Lyapunov Exponential Spectrum and Bifurcation

Lyapunov exponential spectrum (LEs) and bifurcation are essential indicators for the evaluation of chaotic dynamical maps. The parameters b and c are known to have an impact on the dynamical behavior of the two maps through fixed point analysis.

To investigate the characteristics of the 3D parabolic chaotic map and 4D memristive hyperchaotic map, Figures 6–11 show the LEs and bifurcations for different parameters.

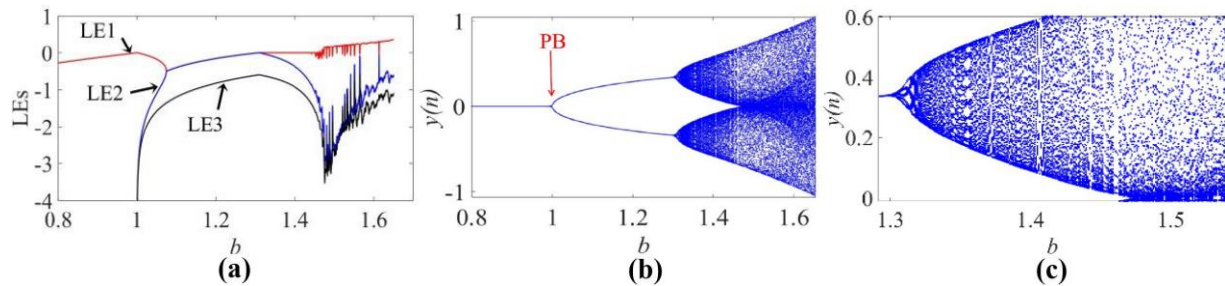


Figure 6. LEs and bifurcations for 3D parabolic chaotic map depending on the parameter b . (a) LEs. (b) Bifurcation diagram. (c) Local view of bifurcation diagram.

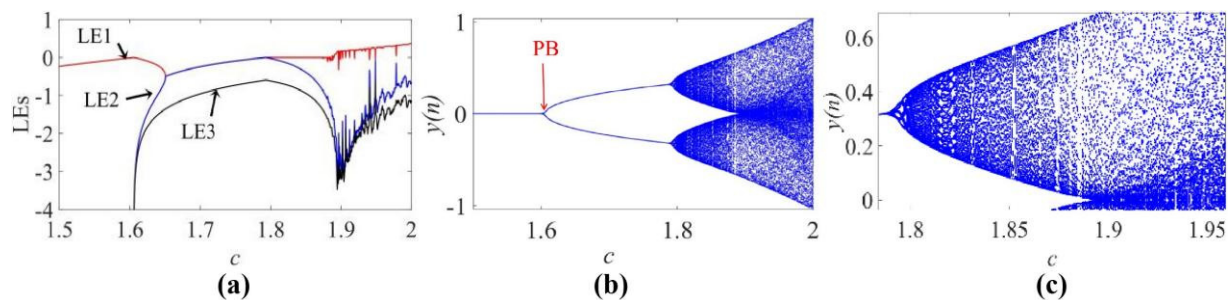


Figure 7. LEs and bifurcations for 3D parabolic chaotic map depending on the parameter c . (a) LEs. (b) Bifurcation. (c) Local view of bifurcation.

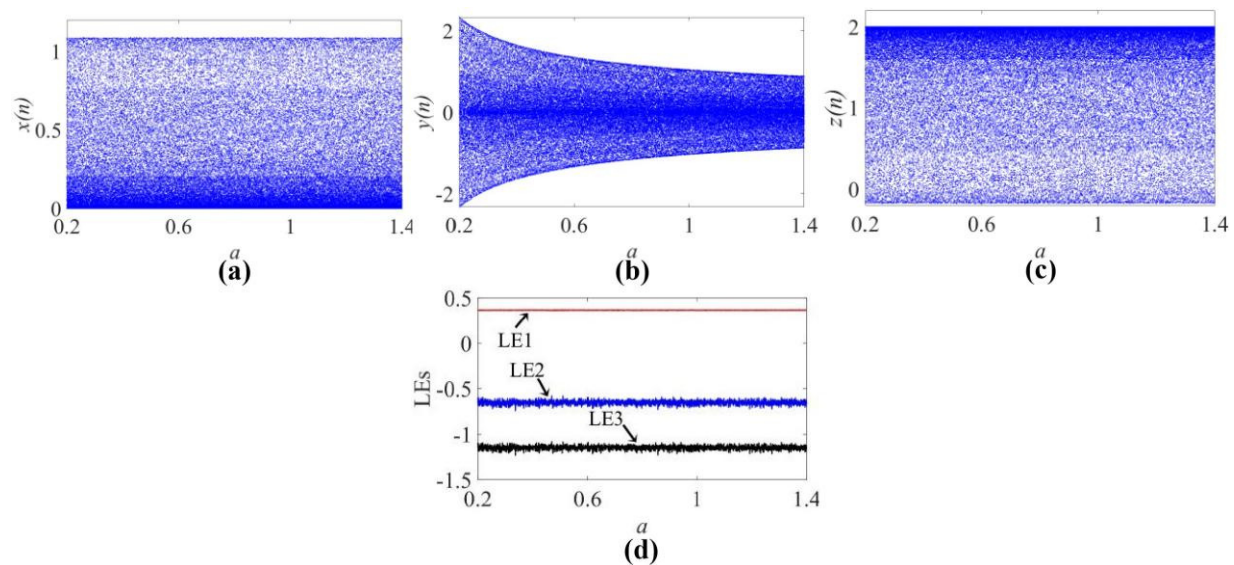


Figure 8. LEs and bifurcations for 3D parabolic chaotic map depending on the parameter a . (a) Bifurcation of states x . (b) Bifurcation of states y . (c) Bifurcation of states z (d) LEs.

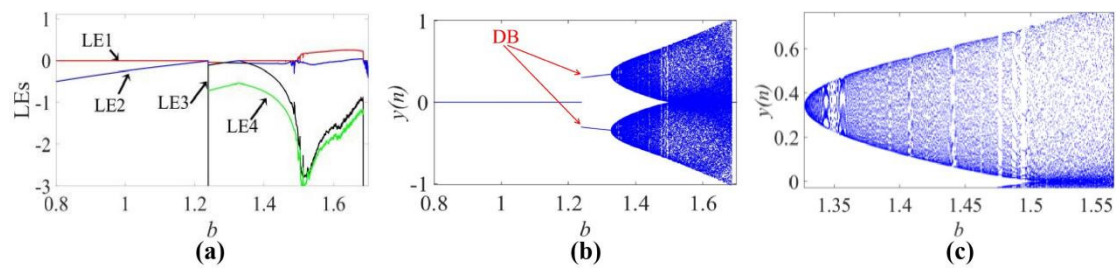


Figure 9. LEs and bifurcations for 4D memristive hyperchaotic map depending on the parameter b . (a) LEs. (b) Bifurcation. (c) Local view of the bifurcation.

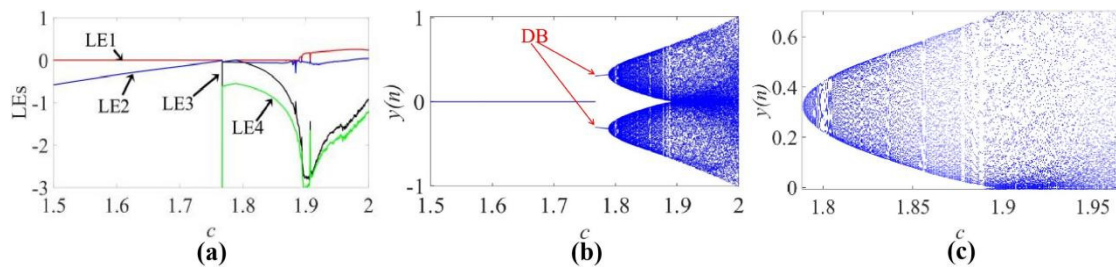


Figure 10. LEs and bifurcations for 4D memristive hyperchaotic map depending on the parameter c . (a) LEs. (b) Bifurcation. (c) Local view of the bifurcation.

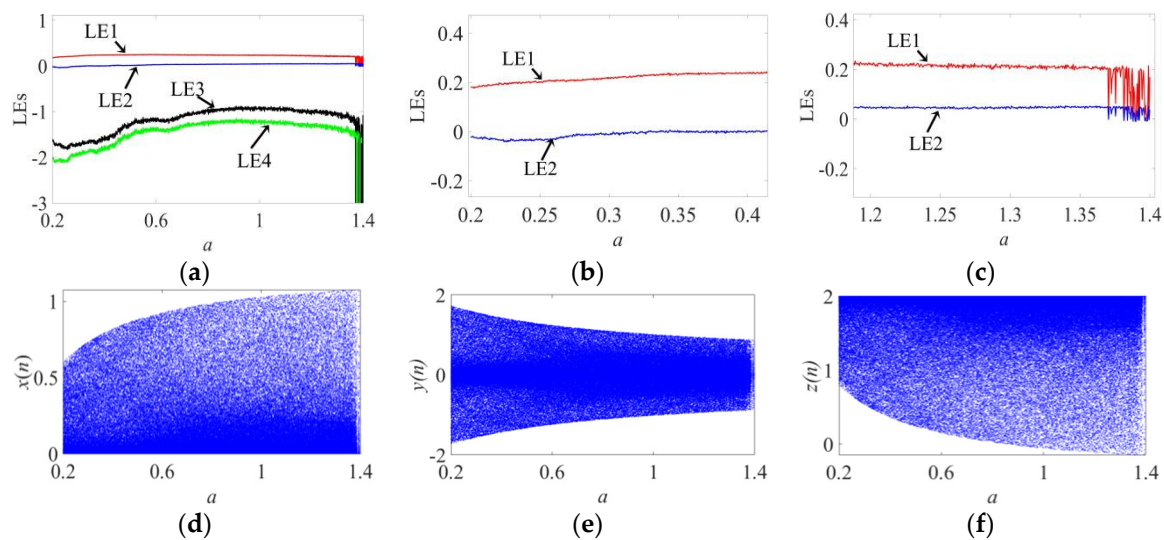


Figure 11. LEs and bifurcations for 4D memristive hyperchaotic map depending on the parameter a . (a) LEs. (b) Local view LEs between the intervals $[0.2, 0.4]$. (c) Local view LEs between the intervals $[1.2, 1.4]$. (d) Bifurcation of states x . (e) Bifurcation of states y . (f) Bifurcation of states z .

Comparing the LEs of the two maps, we can see that a more complex hyperchaotic state occurs for the 4D map compared to the 3D map. Moreover, the LEs in the 3D map varying with the parameter a have a constant Lyapunov exponential behavior and exhibit a continuous chaotic state [38]. In contrast, the Lyapunov exponential of the LEs in the 4D memristive map varying with the parameter a is not constant, and it exhibits a short period of time in a continuous chaotic state and a continuous hyperchaotic state, and finally a discontinuous chaotic state, respectively.

Comparing the bifurcations of the two maps, we can see that the 3D map initially experiences pitchfork bifurcation (PB) [39] as b and c increase, whereas pitchfork bifurcation disappears and defect bifurcation (DB) [40] occurs in the 4D memristive map. Then, the two maps have a large number of periodic windows in the passage to the singular attractor after the appearance of the Hopf bifurcation (HB) [40].

3.2. Phase Diagram and Iterative Sequence

The most straightforward way to monitor the behavior of the dynamics in the map is to look at the phase diagram, which provides the record of the map's trajectory and provides a reflection of the changes in the state of the system. From Figure 12, the phase diagram of the 4D memristive hyperchaotic map can be observed. It shows the evolution of the attractor as the map varies with parameter b (the process of evolution with c is similar).

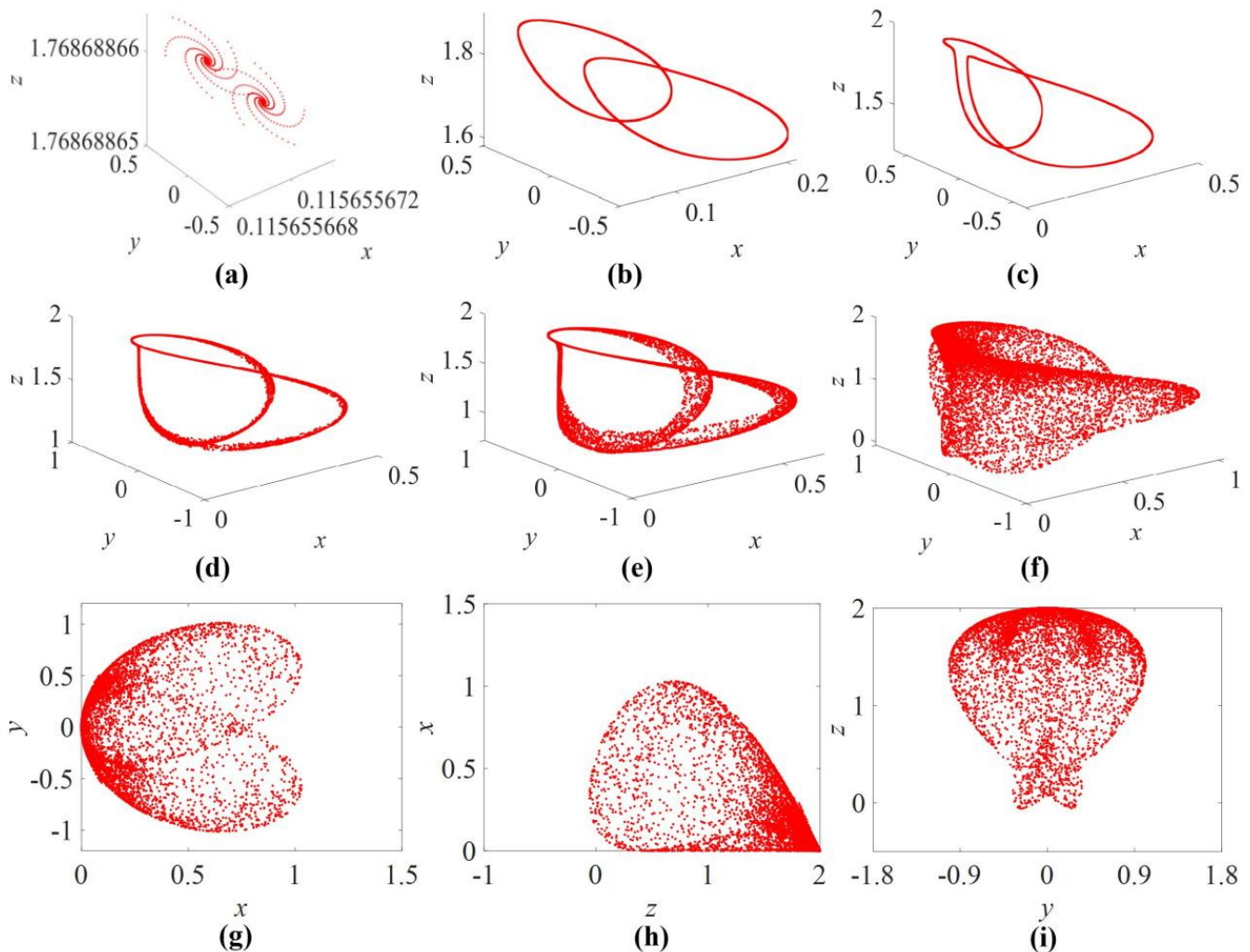


Figure 12. Phase diagrams for 4D memristive hyperchaotic map. (a) $b = 1.31$, (b) $b = 1.34$, (c) $b = 1.45$, (d) $b = 1.48$, (e) $b = 1.56$, (f) $b = 1.68$, and (g) x - y plane at $b = 1.68$, (h) z - x plane at $b = 1.68$, and (i) y - z plane at $b = 1.68$.

The phase diagram makes it clear that the attractor's evolution, and the change in LEs go hand in hand. At $b = 1.31$, the map is already in the fixed points state. Then, at $b = 1.34$, the map evolves from the fixed point to the limit cycle. At $b = 1.45$, the limit cycle experiences an increase in deformation. At $b = 1.48$ and $b = 1.56$, this map appears to be chaos. At $b = 1.68$, the map reaches the most complex hyperchaotic state. In order to be able to observe the hyperchaotic state of this four-dimensional memristive map more clearly, Figure 12g–i show the x - y , y - z , and z - x planes of Figure 12f, which are similar in shape to a peach heart, a mouse, and a vase, respectively.

Figure 13 shows the sequence of iterations for different states. It can be noticed that the iteration sequence becomes more and more chaotic as the state changes.

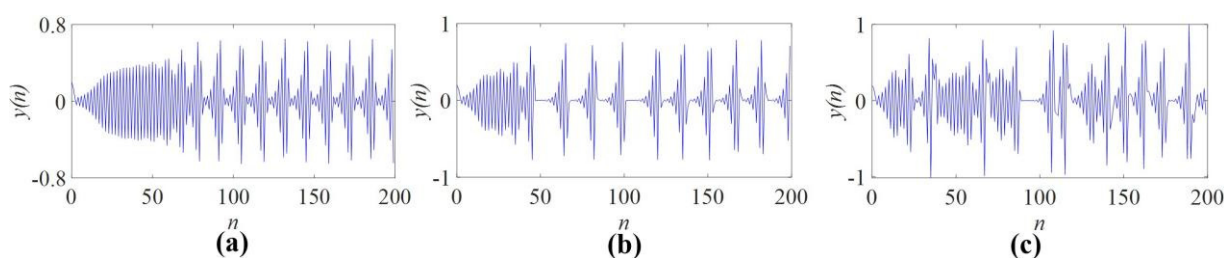


Figure 13. Iterative sequences for 4D memristive hyperchaotic map. (a) Iterative sequence of the limit cycle state. (b) Iterative sequence of the chaotic state. (c) Iterative sequence of the hyperchaotic state.

3.3. Transient Chaos and State Transfer

Transient chaos refers to the phenomenon when a map under the fixed parameters and initial conditions suddenly enters into a chaotic state during a short iteration and eventually enters into a normal motion state.

When $b = 1.492$, transient chaos appears in the 4D memristive hyperchaotic map. From Figure 14a, it can be seen that the transient chaotic state returns to the periodic state when the number of iterations $n = 80500$. The iteration sequence and phase diagram of the transient chaotic state are shown in Figure 14b and Figure 14d, respectively. The iteration sequence and phase diagram of the periodic state are shown in Figure 14c and Figure 14e, respectively.

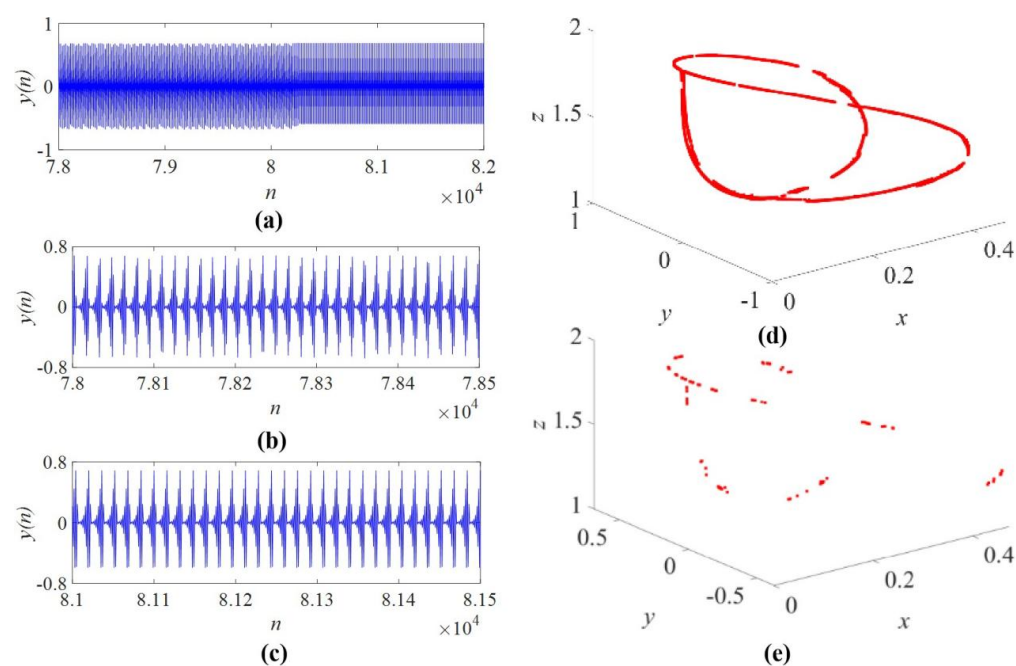


Figure 14. Iterative sequence and attractor diagram of transient chaotic phenomena with parameter $b = 1.492$. (a) Iterative sequence of y at transient chaos. (b) Chaotic sequence. (c) Periodic sequence. (d) Chaotic attractor. (e) Periodic attractor.

The 4D memristive hyperchaotic map undergoes the phenomenon of state transfer, as shown in Figures 15–17. When $b = 1.34$, the map switches from the periodic attractor to the quasi-periodic attractor. In the quasi-periodic part, there is a fast and slow period oscillation effect, which is caused by the fact that $LE1$ is zero in this parameter interval, and the limit cycle formed by the trajectory is critically stable at this time. This state transfer phenomenon is not yet seen in other systems. Moreover, the map undergoes a transfer from a chaotic to a periodic state when $b = 1.4985$ and $c = 1.907$.

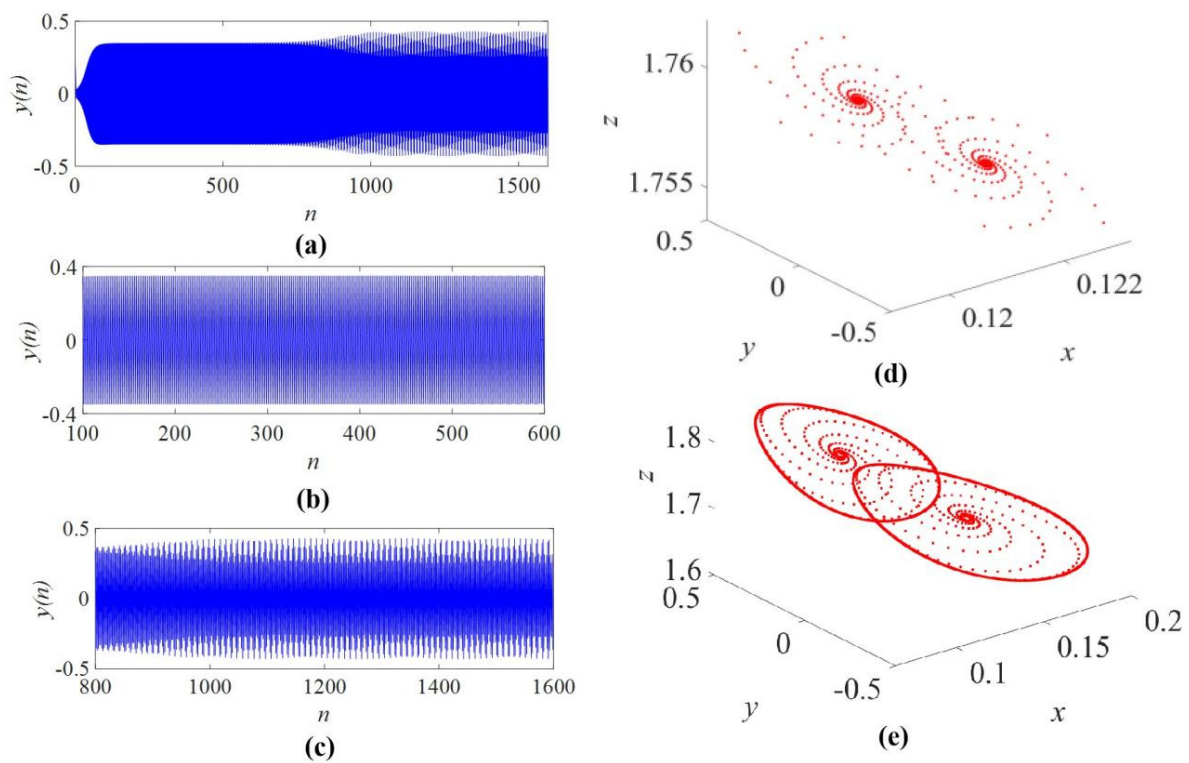


Figure 15. Iterative sequences and phase diagrams of the state transition phenomenon with parameter $b = 1.34$. (a) Iterative sequence of y at state transfer. (b) Periodic sequence. (c) Quasi-periodic sequence. (d) Periodic chaotic attractor. (e) Quasi-periodic attractor.

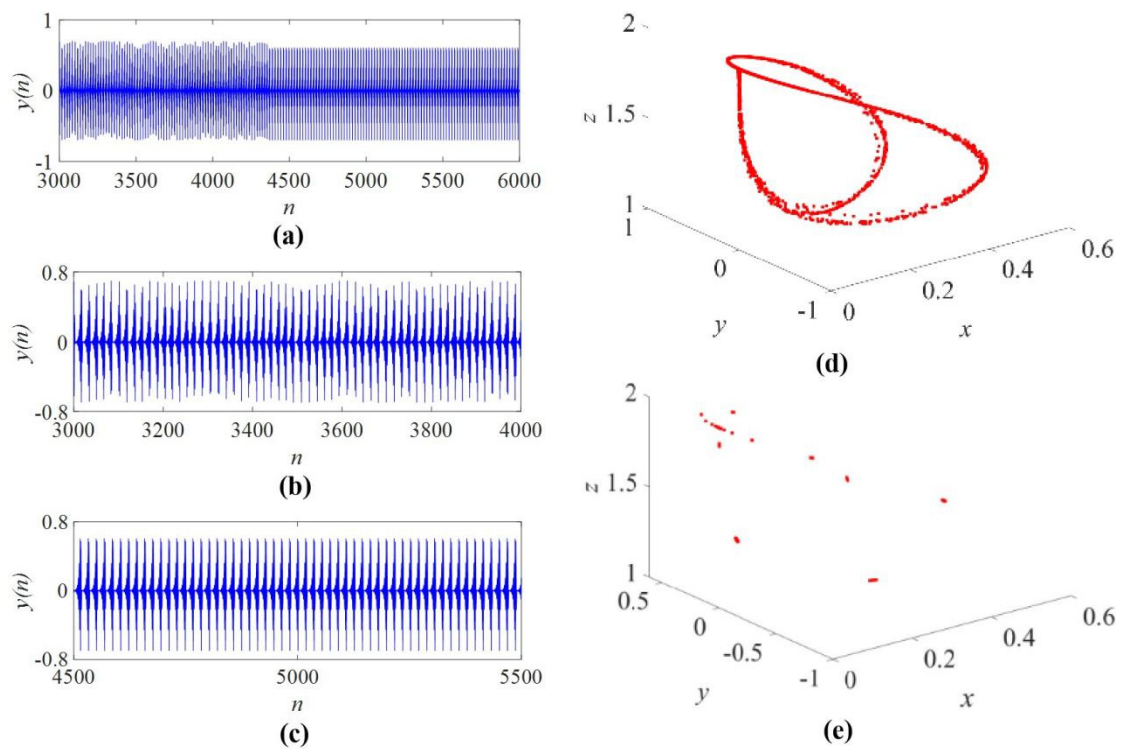


Figure 16. Iterative sequences and phase diagrams of the state transition phenomenon with parameter $b = 1.4985$. (a) Iterative sequence of y at state transfer. (b) Chaotic sequence. (c) Periodic sequence. (d) Chaotic attractor. (e) Periodic attractor.

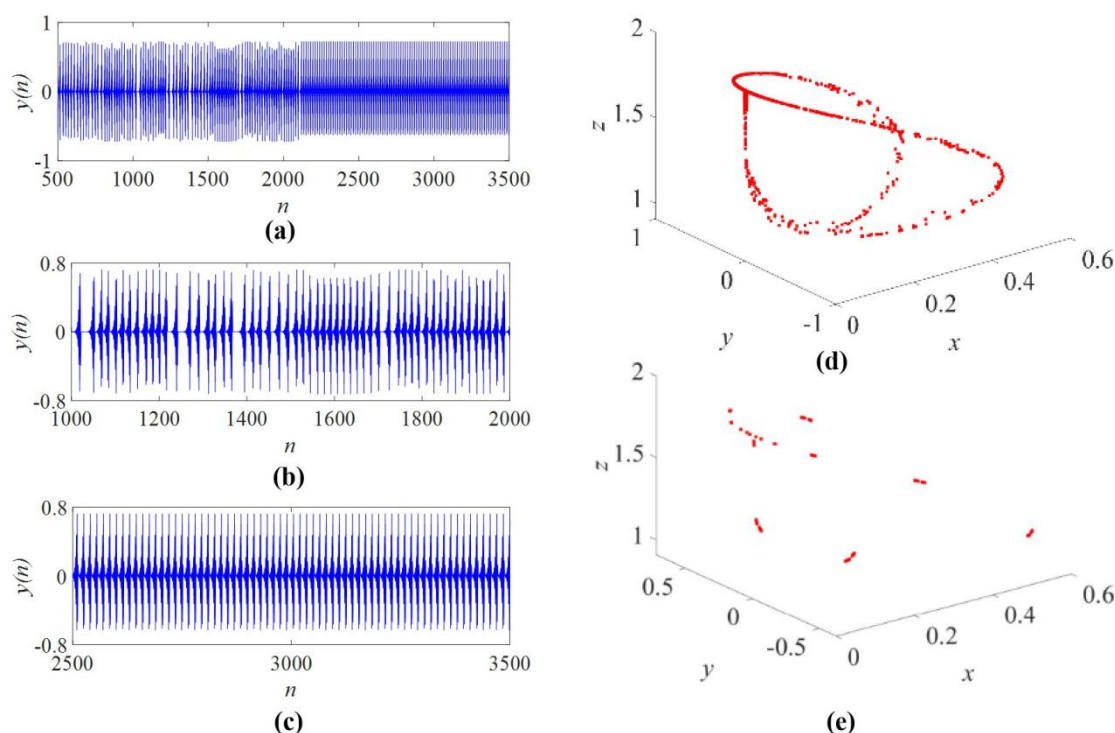


Figure 17. Iterative sequences and phase diagrams of the state transition phenomenon with parameter $c = 1.907$. (a) Iterative sequence of y at state transfer. (b) Chaotic sequence. (c) Periodic sequence. (d) Chaotic attractor. (e) Periodic attractor.

3.4. Multi Stability Analysis

Attractor coexistence is the phenomenon of obtaining different attractor tracks by varying the system's initial values under an identical parameter condition. If there are coexisting attractors in a system, then it indicates that this system has multi stability.

From the previous analysis of 4D map fixed points, it is clear that they have numerous fixed points, suggesting that there is a possibility of several coexistence solutions, and thus, it is highly likely that the map has multi stability.

In general, we use bifurcation diagrams and LEs to find the attractors coexisting in chaotic systems. Fixing the parameters and setting different initial values, the bifurcation maps and LEs vary with b and c , as shown in Figure 18 and Figure 19, respectively.

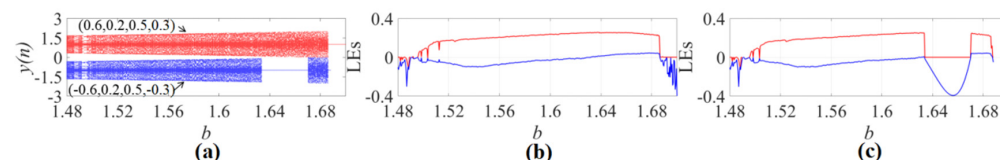


Figure 18. Dynamic behavior of parameters b with different initial values, (a) LEs with initial values $(0.6, 0.2, 0.5, 0.3)$, (b) bifurcation diagram with initial values $(0.6, 0.2, 0.5, 0.3)$, (c) LEs with initial values $(-0.6, 0.2, 0.5, -0.3)$.

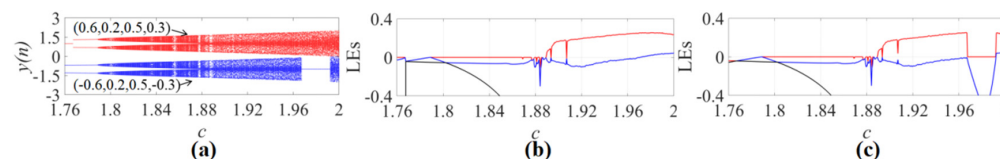


Figure 19. Dynamic behavior of parameters c with different initial values. (a) LEs with initial values $(0.6, 0.2, 0.5, 0.3)$, (b) bifurcation diagram with initial values $(0.6, 0.2, 0.5, 0.3)$, (c) LEs with initial values $(-0.6, 0.2, 0.5, -0.3)$.

Comparing the bifurcation diagrams and LEs at different initial values, respectively, it can be found that when fixing a and c , the parameter b has attractor coexistence at 1.492, 1.6875, and (1.6335, 1.67) intervals; the coexisting states are the coexistence of periodic $(-0.6, 0.2, 0.5, -0.3)$ and chaotic $(0.6, 0.2, 0.5, 0.3)$ states, the coexistence of periodic $(-0.6, 0.2, 0.5, -0.3)$ and hyperchaotic $(0.6, 0.2, 0.5, 0.3)$ states, and the coexistence of hyperchaotic $(-0.6, 0.2, 0.5, -0.3)$ and periodic $(0.6, 0.2, 0.5, 0.3)$ states, respectively. When fixing a and b , the parameter c has attractor coexistence at 1.7675, 1.907, and (1.967, 1.992) intervals; the coexisting states are the coexistence of quasi-periodic $(-0.6, 0.2, 0.5, -0.3)$ and quasi-periodic (around the fixed points) $(0.6, 0.2, 0.5, 0.3)$ states, the coexistence of chaotic $(-0.6, 0.2, 0.5, -0.3)$ and chaotic $(0.6, 0.2, 0.5, 0.3)$ states, and the coexistence of periodic $(-0.6, 0.2, 0.5, -0.3)$ and hyperchaotic $(0.6, 0.2, 0.5, 0.3)$ states. In particular, the coexistence of a quasi-periodic state with a quasi-periodic state around the fixed points is not seen in other systems. The coexistence phase diagram of these coexisting attractors is shown in Figure 20. The red traces indicate phase diagrams with initial values of $(0.6, 0.2, 0.5, 0.3)$, and the blue traces denote phase diagrams with initial values of $(-0.6, 0.2, 0.5, -0.3)$.

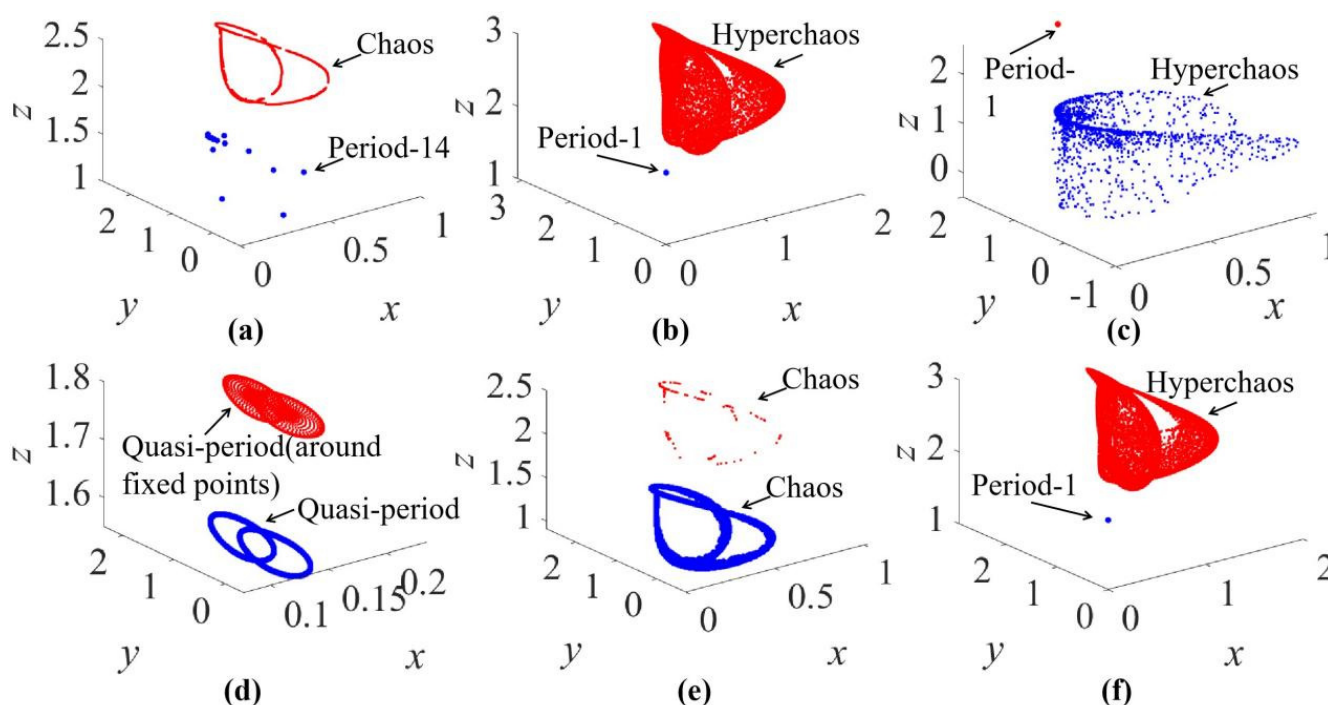


Figure 20. Typical coexistence phase diagrams for the memristive hyperchaotic map having various parameters b and c . (a) $b = 1.492$, (b) $b = 1.6355$, (c) $b = 1.6875$, (d) $c = 1.7675$, (e) $c = 1.907$, (f) $c = 1.9672$.

4. Hardware Experiment

In this section, we first write the 4D memristive hyperchaotic map program using C language based on the map mathematical model from Equation (14), in which its initial values and parameters are set according to the results of the numerical simulations in Figures 12 and 13. It is downloaded into a high-performance STM32F407ZGT6 microcontroller. The digital signals are then transferred into the low power, dual channel, 16/12-bit buffered voltage output analogue-to-digital converter (AD5689). The transformed data are transmitted into an oscilloscope that displays the 4D memristive map hyperchaotic state of each 2D planar phase-track diagram and its iterative sequence in different states. The platform for the experiment is displayed in Figure 21.

Final results obtained with the oscilloscope are shown in Figure 22 and Figure 23, respectively. Comparison of this result with Figures 12 and 13 shows that the phase traces and iteration sequences captured via the oscilloscope in Figure 21 correspond to the

results of the numerical simulation. These experimental results demonstrate the hardware realizability of this 4D memristive map.

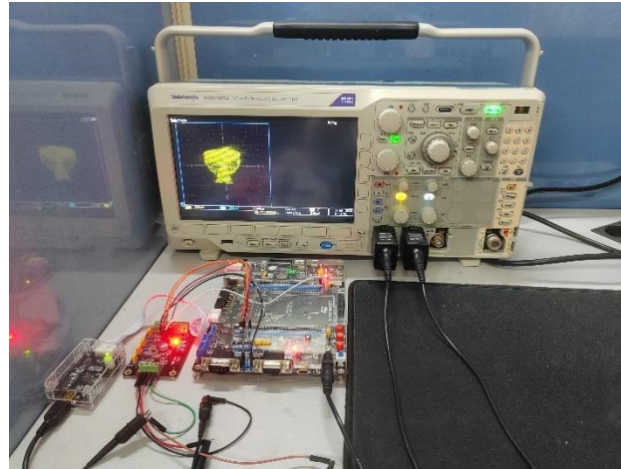


Figure 21. Microcontroller-based hardware experimental platform.

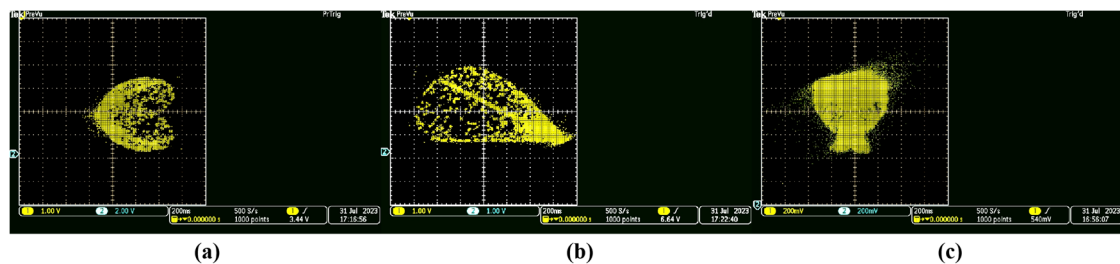


Figure 22. Oscilloscope-captured 2D planar phase diagrams of a 4D memristive map. (a) x–y, (b) z–x, (c) y–z.

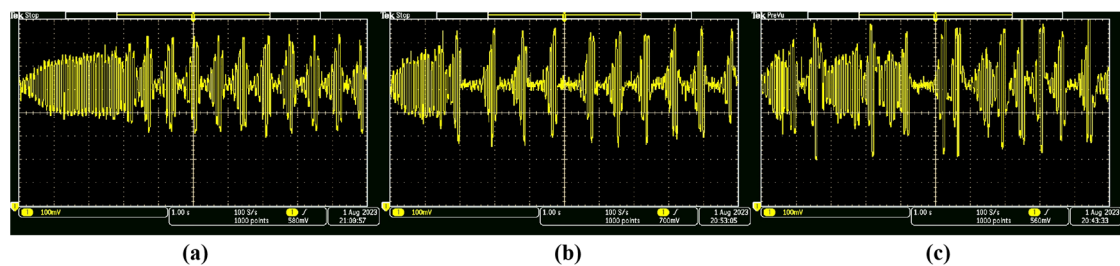


Figure 23. Oscilloscope-captured iterative sequences of different states of a 4D memristive map. (a) limit cycle state, (b) chaotic state, (c) hyperchaotic state.

5. Conclusions

The present paper firstly establishes a 3D parabolic map based on the Tent map and the 2D parabolic map, and its fixed points are analyzed along with the system structure diagram of this map. Then, a Simulink model of the DM is established on the basis of its mathematical model, and it is verified that this DM model meets the memristor's definition under this parameter condition. Moreover, coupling this DM with the 3D parabolic map, a novel 4D memristor hyperchaotic map is obtained, and its fixed points are analyzed; also, the system structure diagram and Simulink model of this map are established. By comparing the fixed points, Lyapunov exponential spectra and bifurcation diagrams of the two maps, it can be found that the new 4D memristive hyperchaotic map possesses more enriched dynamical behaviors than the 3D parabolic chaotic maps without a memristor. Then, the evolution process of attractors and the iterative sequences of the 4D memristive map as the characteristic tuning parameter, which is varied, are given. In addition, transient

chaos, state transfer and attractor coexistence phenomena are found. In particular, state transfer from periodic to quasi-periodic states, which has not appeared in other systems, is found. Finally, the 4D memristive hyperchaotic map is verified on a digital experimental platform. The research mentioned in this paper provides a reference for the study of discrete memristors and discrete maps. Our future work will focus on the implementation of this DM model and discrete memristive maps using neural networks.

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