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Classification of Data Mining Techniques under the Environment of T-Bipolar Soft Rings

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Abstract: Data mining evaluation is very critical in the sense that it determines how well a classification model performs and how well it can generate accurate predictions on brand-new, unexplored data. It is especially important for classification tasks. There are several methods for evaluating classification models, and the choice of evaluation strategies depends on the particular situation, the available data, and the desired outcomes. The notion of a T-bipolar soft set (TBSS) is a valuable parameterization tool and is closer to the concept of bipolarity. Moreover, algebraic structures like groups, rings, and modules, etc., are basic tools that can be helpful not only in mathematics but also in other scientific areas due to their symmetric properties. In this article, based on the novelty of TBSS and the characteristics of rings, we have generalized these two notions to deliver and introduce the notion of T-bipolar soft rings (TBSRs). Additionally, the concepts of AND product, OR product, extended union, extended intersection, restricted union, and restricted intersection for two TBSRs is introduced, and the related results are conferred. To support these proposed notions, we have delivered examples related to these ideas. For the applicability of the developed approach, an algorithm is defined based on the delivered approach. An illustrative example regarding the classification of data mining techniques is developed to show the applications of the introduced work. We can see that there are four alternatives, and their score values are, respectively, given by -4, 42, 0, and -32. Based on these results, we can evaluate the best data mining technique. So, the defined algorithm makes it easy for us to classify the data mining techniques. Further asymmetric data are frequently employed for selecting the best alternative in decision-making problems because the information regarding alternatives is not necessarily always symmetric. Therefore, asymmetric information can be discussed using these proposed concepts.

Keywords: T-bipolar soft set; T-bipolar soft rings; data mining techniques

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1. Introduction

Numerous real-world issues in the fields of medicine, engineering, and economics, etc., contain vague data. The solution to these issues involves mathematical ideas based on vague and uncertain data. Some of these issues are objective, and others are subjective. So, some developments have been made to tackle the vague and uncertain data. These structures include the fuzzy set (FS) [1], the intuitionistic fuzzy set (IFS) [2], the vague set [3], and the rough set [4]. But all these ideas can be seen to contain inherent limitations, which is a limitation of parameterization tools. Different fuzzy structures are combined with ring theory to introduce new structures, like Emniyet and Sahin [5], to develop the theory of fuzzy normed rings. Moreover, Razaq and Alhamzi [6] used the Pythagorean fuzzy set (PyFS) and developed the theory on PyF ideals of classical rings. Also, Razzaque et al. [7] used the more advanced structure of the q-rung orthopair fuzzy set (q-ROFS) and discussed a detailed study of mathematical rings in the q-ROF framework. Furthermore, Alghazzwi et al. [8] introduced a novel structure of q-ROFS in ring theory. Alolaiyan et al. [9] used the bipolar fuzzy structure and developed a certain structure of bipolar fuzzy subrings.

The idea of a soft set (SS) [10] is an interesting parameterization tool for dealing with vague and uncertain data. We can see that SS is free from all the limitations that all the above theories face in terms of their structure. Ali et al. [11] used the SS idea, proposed some new operations, and proved different laws in this theory. Maji et al. [12] delivered the application of SS in decision-making problems. Moreover, these developments have been used in the medical field, and Yuksel et al. [13] proposed the application of SS for the diagnosis of prostate cancer risk. Also, SS has applications in ring theory, and many new developments have been made in this regard. Tunçay and Sezgin [14] introduced the notion of soft union rings and established their applications in ring theory. Moreover, we can see that Jana et al. [15] produced the structure of (α, β)-soft intersectional rings and ideals with their applications. Sezgin et al. [16] introduced the theory of near-rings with soft union ideals and applications. Moreover, Acar et al. [17] used the structure of SS and developed the theory of soft rings. Also, Celik et al. [18] proposed a new view on soft rings.

The SS has been combined into different structures, and many new developments have been introduced, like fuzzy SS (FSS) [19], intuitionistic fuzzy soft set (IFSS) [20], and picture fuzzy soft set (PFSS) [21]. All these notions use the parameterization tool, and these ideas have been used in different fields. Celik et al. [22] used the FSS in ring theory. The application of FSS in medical diagnosis using the notion of fuzzy soft complement is given in [23]. Zulqarnain et al. [24] used IFS matrices for disease diagnosis. Muthukumar and Krishnan [25] delivered the notion of similarity measures based on IFSS and established the utilization of these notions in medical diagnosis. Zhang [26] utilized the notion of IFSS in algebraic structures and proposed the notion of IFS rings.

To discuss bipolarity, two kinds of attempts have been made, first by Shabir and Naz [27] and the other by Karaaslan [28], called bipolar soft set (BSS). But both of these notions have some shortcomings. (1) If we discuss the FS and SS, then we can note that both are characterized by a single function; both of these notions use the single set as a domain set, and their codomain set is lattice in either case. (2) Also, note that in the case of IFS and double-framed SS [29], both of these structures use two functions; both of these ideas use a single set as a domain set for both functions, and their codomain set is lattice in either case for both functions. However, we can notice that this is not the case for bipolar-valued fuzzy sets and BSS. We can note that all the attempts used for BSS in [27,28] do not fill the space. To fill up this space, the idea of a T-bipolar soft set (TBSS) has been delivered by Mahmood [30]. The idea of TBSS is a remarkable achievement and an interesting tool that has gained attention, and many new developments have been made in this regard. Mahmood et al. [31] established the TOPSIS method based on lattice-ordered TBSSs and proposed its applications.

The T-bipolar soft set is a parameterization tool that can handle uncertainty and ambiguous data. Many decision-making situations are based on two-sided aspects, like the effects and side effects of medicine. So the T-bipolar soft set can handle such situations. A subfield of abstract algebra called ring theory analyses rings as algebraic structures. Rings are key tools in many branches of mathematics and their applications because they combine the attributes of addition and multiplication. Keeping in view the advantages of TBSSs that we have discussed in the aforementioned discussion and the algebraic structure of rings, our main goal of this study is to give space to all such kinds of situations that are based on two-sided aspects. To cover these issues, we have combined the notions of T-bipolar soft sets with ring theory to develop the notion of T-bipolar soft rings. Moreover, the notion of a TBSR subset is proposed. Additionally, the concepts of AND product for two TBSRs and OR product for two TBSRs have been introduced. Moreover, we have delivered the ideas of an extended union and an extended intersection for two TBSRs. Furthermore, we have elaborated on the notion of a restricted union and restricted intersection for two TBSRs.

Moreover, considering the application perspectives of these ideas, our main objective is to develop an algorithm that further allows us to utilize these notions in decision-making scenarios. The developed ideas allow us to solve all those situations when data are given in the form of T-bipolar soft rings and/or to discuss those decision-making situations where two-sided aspects are given.

The remaining article is arranged as follows: Section 2 discusses the notion of the SS, BSS, TBSS, and TBS subsets. Section 3 discusses the notion of TBSRs, TBSR subset, AND product, OR product, extended union, extended intersection, restricted union, and restricted intersection. We have also elaborated on the examples and theorems to discuss the authenticity of the developed approach. Section 4 is about the application of the developed approach using the defied algorithm in the environment of delivered work. Section 5 is based on the conclusion remarks.

2. Preliminaries

Definition 1 [10]. *Let* U *be a universal set and* E *be the set of parameters. Now for any* $\tilde{\mathrm{A}} \subseteq \mathrm{E}$, *the soft set is a pair* $(\mathrm{F}, \tilde{\mathrm{A}})$ *where* $\mathrm{F} : \tilde{\mathrm{A}} \to \mathrm{P}(\mathrm{U})$ *is set-valued mapping.*

Two kinds of attempts have been made to define the notion of bipolar soft sets. But the idea of a T-bipolar soft set is much closer to bipolarity. These notions are given as follows:

Definition 2 [27]. Assume that U is the universal set and $\tilde{\mathrm{A}} \subseteq \mathrm{E}$. Also, $\neg\tilde{\mathrm{A}} = \{\neg\mathfrak{z}, \mathfrak{z} \in \tilde{\mathrm{A}}\}$ represent the NOT set of $\tilde{\mathrm{A}}$. Then, triplet $\left(\mathrm{F}_{\mathrm{bp}}, \mathrm{G}_{\mathrm{bp}}, \tilde{\mathrm{A}}\right)$ is called BSS where $\mathrm{F}_{\mathrm{bp}} : \tilde{\mathrm{A}} \to \mathrm{P}(\mathrm{U})$ and $\mathrm{G}_{\mathrm{bp}} : \neg\tilde{\mathrm{A}} \to \mathrm{P}(\mathrm{U})$ and $\mathrm{F}_{\mathrm{bp}}(\mathfrak{z}) \bigcap \mathrm{G}_{\mathrm{bp}}(\neg\mathfrak{z}) = \phi(\text{null set})$.

Definition 3 [28]. *Assume that* $\tilde{\mathrm{A}}$ *represent the set of parameters and* $\tilde{\mathrm{A}}_1 \subseteq \tilde{\mathrm{A}}$, $\tilde{\mathrm{A}}_2 \subseteq \tilde{\mathrm{A}}$ *such that* $\tilde{\mathrm{A}}_1 \cup \tilde{\mathrm{A}}_2 = \tilde{\mathrm{A}}$ *and* $\tilde{\mathrm{A}}_1 \cap \tilde{\mathrm{A}}_2 = \phi(\text{null set})$. *Then, triplet* $\left(\mathrm{F}_{\mathrm{bp}}, \mathrm{G}_{\mathrm{bp}}, \tilde{\mathrm{A}}\right)$ *is called BSS where* $\mathrm{F}_{\mathrm{bp}} : \tilde{\mathrm{A}}_1 \to \mathrm{P}(\mathrm{U})$ *and* $\mathrm{G}_{\mathrm{bp}} : \tilde{\mathrm{A}}_2 \to \mathrm{P}(\mathrm{U})$ *with* $\mathrm{F}_{\mathrm{bp}}(\mathfrak{z}) \cap \mathrm{G}_{\mathrm{bp}}(\mathrm{g}(\mathfrak{z})) = \phi$ *where* $\mathrm{g} : \tilde{\mathrm{A}}_1 \to \tilde{\mathrm{A}}_2$ *is bijective mapping.*

Definition 4 [30]. *Let* U *denote the universal set, and* E *be the set of parameters and* $\tilde{\mathrm{A}} \subseteq \mathrm{E}$. *Also, let* $\mathrm{X} \subseteq \mathrm{U}$ *and* $\mathrm{Y} = \mathrm{U} - \mathrm{X}$. *then the triplet* $(\mathrm{F}_{\mathrm{tb}}, \mathrm{G}_{\mathrm{tb}}, \tilde{\mathrm{A}})$ *is called TBSS over* U, *where* $\mathrm{F}_{\mathrm{tb}} : \tilde{\mathrm{A}} \to \mathrm{P}(\mathrm{X})$ *and* $\mathrm{G}_{\mathrm{tb}} : \tilde{\mathrm{A}} \to \mathrm{P}(\mathrm{Y})$. *So* $\mathrm{T} - \mathrm{B}_{\mathrm{p}}\mathrm{S}_{\mathrm{ft}}\mathrm{S}$ *is given by simply* $(\mathrm{F}_{\mathrm{tb}}, \mathrm{G}_{\mathrm{tb}}, \tilde{\mathrm{A}}) = \{\mathfrak{z}, \mathrm{F}_{\mathrm{tb}}(\mathfrak{z}), \mathrm{G}_{\mathrm{tb}}(\mathfrak{z}) : \mathrm{F}_{\mathrm{tb}}(\mathfrak{z}) \in \mathrm{P}(\mathrm{X}) and \mathrm{G}_{\mathrm{tb}}(\mathfrak{z}) \in \mathrm{P}(\mathrm{Y})\}$.

Definition 5 [30]. *Let* $(\mathrm{F}_{\mathrm{tb}-1}, \mathrm{G}_{\mathrm{tb}-1}, \tilde{\mathrm{A}})$ *and* $(\mathrm{F}_{\mathrm{tb}}, \mathrm{G}_{\mathrm{tb}}, \mathfrak{B})$ *be two TBSSs. Then,* $(\mathrm{F}_{\mathrm{tb}-1}, \mathrm{G}_{\mathrm{tb}-1}, \tilde{\mathrm{A}})$ *is called the TBS subset of* $(\mathrm{F}_{\mathrm{tb}-2}, \mathrm{G}_{\mathrm{tb}-2}, \mathfrak{B})$ if

1. $\tilde{\mathrm{A}} \subseteq \mathfrak{B}$
2. $\mathrm{F}_{\mathrm{tb}-1}(\mathfrak{z}) \subseteq \mathrm{F}_{\mathrm{tb}-2}(\mathfrak{z})$ *and* $\mathrm{G}_{\mathrm{tb}-1}(\mathfrak{z}) \supseteq \mathrm{G}_{\mathrm{tb}-2}(\mathfrak{z})$ *for all* $\mathfrak{z} \in \tilde{\mathrm{A}}$.

3. T-Bipolar Soft Rings (TBSRs)

This section of the article is devoted to defining the notion of TBSR. We have also proposed the definition of basic operational rules for TBSRs. We have also introduced the notion of T-bipolar soft subring. The notions of AND product, OR product, extended union, extended intersection, restricted union, and restricted intersection have been delivered. To illustrate all these developed notions, we have established an example to support the initiated notions.

Definition 6. *Let* X *and* $\mathrm{\gamma}$ *denote the two distinct commutative rings such that* $\mathrm{U} = \mathrm{X} \cup \mathrm{\gamma}$, *then for any set* $\tilde{\mathrm{A}}$, *a T-bipolar soft set* $(\mathrm{F}_{\mathrm{tb}}, \mathrm{G}_{\mathrm{tb}}, \tilde{\mathrm{A}})$ *is called a TBSR if and only if* $\mathrm{F}_{\mathrm{tb}.}(\mathfrak{z})$ *is a subring of* X *and* $\mathrm{G}_{\mathrm{tb}.}(\mathfrak{z})$ *is a subring of* $\mathrm{\gamma}$ *for all* $\mathfrak{z} \in \tilde{\mathrm{A}}$ *where* $\mathrm{F}_{\mathrm{tb}} : \tilde{\mathrm{A}} \to \mathrm{P}(X)$ *and* $\mathrm{G}_{\mathrm{tb}} : \tilde{\mathrm{A}} \to \mathrm{P}(\mathrm{\gamma})$.

Example 1. *Let* $\mathrm{X} = \mathrm{Z}_6$ *and* $\gamma = \mathrm{Z}_8$ *be two rings and* $\mathrm{U} = \mathrm{X} \cup \gamma$. *Also, assume that* $\tilde{\mathrm{A}} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3, \mathfrak{z}_4\}$.

Now assume that a mapping $\mathrm{F}_{\mathrm{tb.}} : \tilde{\mathrm{A}} \to \mathrm{P}(\mathrm{X})$ is *defined by*

$$\mathrm{F}_{\mathrm{tb}}(\mathfrak{z}_1) = \left\{\overline{0}\right\}, \mathrm{F}_{\mathrm{tb}}(\mathfrak{z}_2) = \left\{\overline{0}, \overline{3}\right\}, \mathrm{F}_{\mathrm{tb}}(\mathfrak{z}_3) = \left\{\overline{0}, \overline{2}, \overline{4}\right\} \text{and} \mathrm{F}_{\mathrm{tb}}(\mathfrak{z}_4) = \mathrm{Z}_6.$$

Then, we can see that $\left\{\overline{0}\right\}, \left\{\overline{0}, \overline{3}\right\}, \left\{\overline{0}, \overline{2}, \overline{4}\right\}$ *and* Z_6 *all are the subrings of* Z_6.
Also, for the mapping $\mathrm{G}_{\mathrm{tb.}} : \tilde{\mathrm{A}} \to \mathrm{P}(\gamma)$ *defined by*

$$\mathrm{G}_{\mathrm{tb}}(\mathfrak{z}_1) = \left\{\overline{0}, \overline{4}\right\}, \mathrm{G}_{\mathrm{tb}}(\mathfrak{z}_2) = \left\{\overline{0}\right\}, \mathrm{G}_{\mathrm{tb}}(\mathfrak{z}_3) = \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}\right\} \text{ and } \mathrm{G}_{\mathrm{tb}}(\mathfrak{z}_4) = \mathrm{Z}_8.$$

Then, we can note that for all $\mathfrak{z} \in$, $\mathrm{F}_{\mathrm{tb}}(\mathfrak{z})$ *is a subring of* Z_6 *and* $\mathrm{G}_{\mathrm{tb}}(\mathfrak{z})$ *is a subring of* Z_8. *Hence,* $(\mathrm{F}_{\mathrm{tb}}, \mathrm{G}_{\mathrm{tb}}, \tilde{\mathrm{A}})$ *is a TBSR. Hence, we can write*

$$(\mathrm{F}_{\mathrm{tb}}, \mathrm{G}_{\mathrm{tb}}, \tilde{\mathrm{A}}) = \left\{\left(\mathfrak{z}_1, \left\{\overline{0}\right\}, \left\{\overline{0}, \overline{4}\right\}\right), \left(\mathfrak{z}_2, \left\{\overline{0}, \overline{3}\right\}, \left\{\overline{0}\right\}\right), \left(\mathfrak{z}_3, \left\{\overline{0}, \overline{2}, \overline{4}\right\}, \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}\right\}\right), (\mathfrak{z}_4, \mathrm{Z}_6, \mathrm{Z}_8)\right\}$$

Now to make the numerical expression of the above given TBSR, we have the following definition

Definition 7. *Let* $\tilde{\mathrm{A}} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3, \ldots, \mathfrak{z}_{\mathrm{m}}\} \subseteq \mathrm{E}$ *be the set of* m *alternatives and assume that* X *and* γ *are two distinct rings such that* $\mathrm{U} = \mathrm{X} \bigcup \gamma$ *and* $(\mathrm{F}_{\mathrm{tb}}, \mathrm{G}_{\mathrm{tb}}, \tilde{\mathrm{A}})$ *is a TBSR where* $\mathrm{F}_{\mathrm{tb}} : \tilde{\mathrm{A}} \to \mathrm{P}(\mathrm{X})$ *and* $\mathrm{G}_{\mathrm{tb}} : \tilde{\mathrm{A}} \to \mathrm{P}(\gamma)$ *are the set-valued maps. Then, we can represent* $(\mathrm{F}_{\mathrm{tb}}, \mathrm{G}_{\mathrm{tb}}, \tilde{\mathrm{A}})$ *as*

$$\mathfrak{P}_{\mathrm{ijk}} = (\mathcal{Q}_{\mathrm{j}}, \mathcal{S}_{\mathrm{k}}) = \begin{cases} (0,0) \text{ if } \rho_{\mathrm{j}} \notin \mathrm{F}_{\mathrm{tb}}(\mathfrak{z}_{\mathrm{i}}) \text{ and } \sigma_{\mathrm{k}} \notin \mathrm{G}_{\mathrm{tb}}(\mathfrak{z}_{\mathrm{i}}) \\ (1,0) \text{ if } \rho_{\mathrm{j}} \in \mathrm{F}_{\mathrm{tb}}(\mathfrak{z}_{\mathrm{i}}) \text{ and } \sigma_{\mathrm{k}} \notin \mathrm{G}_{\mathrm{tb}}(\mathfrak{z}_{\mathrm{i}}) \\ (0,1) \text{ if } \rho_{\mathrm{j}} \notin \mathrm{F}_{\mathrm{tb}}(\mathfrak{z}_{\mathrm{i}}) \text{ and } \sigma_{\mathrm{k}} \in \mathrm{G}_{\mathrm{tb}}(\mathfrak{z}_{\mathrm{i}}) \\ (1,1) \text{ if } \rho_{\mathrm{j}} \in \mathrm{F}_{\mathrm{tb}}(\mathfrak{z}_{\mathrm{i}}) \text{ and } \sigma_{\mathrm{k}} \in \mathrm{G}_{\mathrm{tb}}(\mathfrak{z}_{\mathrm{i}}) \end{cases}$$
$$\mathfrak{P}^{*}_{\mathrm{ijk}} = \mathcal{Q}_{\mathrm{j}} \text{ and } \mathfrak{P}^{\boxdot}_{\mathrm{ijk}} = \mathcal{S}_{\mathrm{k}}$$

Hence, by using Definition 7, the numerical form of data given in Example 1 is given in Table 1.

Table 1. Tabular representation of TBSR $(\mathbf{F_{tb}}, \mathbf{G_{tb}}, \tilde{\mathrm{A}}) = \left\{\left(\mathfrak{z}_1, \left\{\overline{\mathbf{0}}\right\}, \left\{\overline{\mathbf{0}}, \overline{\mathbf{4}}\right\}\right)\right\}, \left\{\left(\mathfrak{z}_2, \mathbf{Z}_6, \left\{\overline{\mathbf{0}}\right\}\right)\right\}, \left\{\left(\mathfrak{z}_3, \left\{\overline{\mathbf{0}}, \overline{\mathbf{2}}, \overline{\mathbf{4}}\right\}, \left\{\overline{\mathbf{0}}, \overline{\mathbf{2}}, \overline{\mathbf{4}}, \overline{\mathbf{6}}\right\}\right), \left(\mathfrak{z}_4, \left\{\overline{\mathbf{0}}, \overline{\mathbf{3}}\right\}, \mathbf{Z}_8\right)\right\}$.

$(\mathbf{F_{tb}}, \mathbf{G_{tb}}, \tilde{\mathrm{A}})$	$(\overline{0}, \overline{0})$	$(\overline{0}, \overline{1})$	$(\overline{0}, \overline{2})$	$(\overline{0}, \overline{3})$	$(\overline{0}, \overline{4})$	$(\overline{0}, \overline{5})$	$(\overline{0}, \overline{6})$	$(\overline{0}, \overline{7})$	$(\overline{1}, \overline{0})$	$(\overline{1}, \overline{1})$
$\mathfrak{z}_1$	(1,1)	(1,0)	(1,0)	(1,0)	(1,1)	(1,0)	(1,0)	(1,0)	(0,1)	(0,0)
$\mathfrak{z}_2$	(1,1)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,1)	(1,0)
$\mathfrak{z}_3$	(1,1)	(1,0)	(1,1)	(1,0)	(1,1)	(1,0)	(1,1)	(1,0)	(0,1)	(0,0)
$\mathfrak{z}_4$	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(0,1)	(0,1)
$(\mathbf{F_{tb}}, \mathbf{G_{tb}}, \tilde{\mathrm{A}})$	$(\overline{1}, \overline{2})$	$(\overline{1}, \overline{3})$	$(\overline{1}, \overline{4})$	$(\overline{1}, \overline{5})$	$(\overline{1}, \overline{6})$	$(\overline{1}, \overline{7})$	$(\overline{2}, \overline{0})$	$(\overline{2}, \overline{1})$	$(\overline{2}, \overline{2})$	$(\overline{2}, \overline{3})$
$\mathfrak{z}_1$	(0,0)	(0,0)	(0,1)	(0,0)	(0,0)	(0,0)	(0,1)	(0,0)	(0,0)	(0,0)
$\mathfrak{z}_2$	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,0)	(1,1)	(1,0)	(1,0)	(1,0)
$\mathfrak{z}_3$	(0,1)	(0,0)	(0,1)	(0,0)	(0,1)	(0,0)	(1,1)	(1,0)	(1,1)	(1,0)
$\mathfrak{z}_4$	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)

Table 1. *Cont.*

$\left(\mathbf{F_{tb}}, \mathbf{G_{tb}}, \tilde{A}\right)$	$\left(\bar{2},\bar{4}\right)$	$\left(\bar{2},\bar{5}\right)$	$\left(\bar{2},\bar{6}\right)$	$\left(\bar{2},\bar{7}\right)$	$\left(\bar{3},\bar{0}\right)$	$\left(\bar{3},\bar{1}\right)$	$\left(\bar{3},\bar{2}\right)$	$\left(\bar{3},\bar{3}\right)$	$\left(\bar{3},\bar{4}\right)$	$\left(\bar{3},\bar{5}\right)$
$\mathfrak{z}_1$	$(0,1)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,1)$	$(0,0)$
$\mathfrak{z}_2$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,1)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$
$\mathfrak{z}_3$	$(1,1)$	$(1,0)$	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,1)$	$(0,0)$
$\mathfrak{z}_4$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$
$\left(\mathbf{F_{tb}}, \mathbf{G_{tb}}, \tilde{A}\right)$	$\left(\bar{3},\bar{6}\right)$	$\left(\bar{3},\bar{7}\right)$	$\left(\bar{4},\bar{0}\right)$	$\left(\bar{4},\bar{1}\right)$	$\left(\bar{4},\bar{2}\right)$	$\left(\bar{4},\bar{3}\right)$	$\left(\bar{4},\bar{4}\right)$	$\left(\bar{4},\bar{5}\right)$	$\left(\bar{4},\bar{6}\right)$	$\left(\bar{4},\bar{7}\right)$
$\mathfrak{z}_1$	$(0,0)$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,0)$	$(0,0)$
$\mathfrak{z}_2$	$(1,0)$	$(1,0)$	$(1,1)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$
$\mathfrak{z}_3$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$	$(1,1)$	$(1,0)$	$(1,1)$	$(1,0)$	$(1,1)$	$(1,0)$
$\mathfrak{z}_4$	$(1,1)$	$(1,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$
$\left(\mathbf{F_{tb}}, \mathbf{G_{tb}}, \tilde{A}\right)$	$\left(\bar{5},\bar{0}\right)$	$\left(\bar{5},\bar{1}\right)$	$\left(\bar{5},\bar{2}\right)$	$\left(\bar{5},\bar{3}\right)$	$\left(\bar{5},\bar{4}\right)$	$\left(\bar{5},\bar{5}\right)$	$\left(\bar{5},\bar{6}\right)$	$\left(\bar{5},\bar{7}\right)$		
$\mathfrak{z}_1$	$(0,1)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,0)$	$(0,0)$		
$\mathfrak{z}_2$	$(1,1)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$		
$\mathfrak{z}_3$	$(0,1)$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,1)$	$(0,0)$		
$\mathfrak{z}_4$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$		

Definition 8. *Let* $\left(\mathrm{F_{tb-1}}, \mathrm{G_{tb-1}}, \tilde{A}\right)$ *and* $\left(\mathrm{F_{tb}}, \mathrm{G_{tb}}, \mathfrak{B}\right)$ *be two TBSRs. Then,* $\left(\mathrm{F_{tb-1}}, \mathrm{G_{tb-1}}, \tilde{A}\right)$ *is called the TBSR subset of* $\left(\mathrm{F_{tb-2}}, \mathrm{G_{tb-2}}, \mathfrak{B}\right)$ *if*

3. $\tilde{A} \subseteq \mathfrak{B}$
4. $\mathrm{F_{tb-1}}(\mathfrak{z}) \subseteq \mathrm{F_{tb-2}}(\mathfrak{z})$ and $\mathrm{G_{tb-1}}(\mathfrak{z}) \supseteq \mathrm{G_{tb-2}}(\mathfrak{z})$ for all $\mathfrak{z} \in \tilde{A}$.

Example 2. *Let* $X = Z$ *and* $\gamma = Z_{12}$ *be two distinct rings and* $U = X \cup \gamma$. *Also, assume that* $\tilde{A} = \{\mathfrak{z}_1, \mathfrak{z}_2\}, \mathfrak{B} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3\}$.

Now define

$$\mathrm{F_{tb-1}}(\mathfrak{z}_1) = 2Z, \mathrm{F_{tb-1}}(\mathfrak{z}_2) = 6Z$$

And

$$\mathrm{F_{tb-2}}(\mathfrak{z}_1) = Z, \mathrm{F_{tb-2}}(\mathfrak{z}_2) = 3Z \text{ and } \mathrm{F_{tb-2}}(\mathfrak{z}_3) = 4Z.$$

Also, define

$$\mathrm{G_{tb-1}}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\right\}, \mathrm{G_{tb-1}}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{3}, \bar{6}, \bar{9}\right\} \text{ and}$$

$$\mathrm{G_{tb-2}}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{4}, \bar{8}\right\}, \mathrm{G_{tb-2}}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{6}\right\} \text{ and } \mathrm{G_{tb-2}}(\mathfrak{z}_3) = \left\{\bar{0}\right\}.$$

Then, we can note that

1. $\tilde{A} \subseteq \mathfrak{B}$
2. $\mathrm{F_{tb-1}}(\mathfrak{z}) \subseteq \mathrm{F_{tb-2}}(\mathfrak{z})$ *and* $\mathrm{G_{tb-1}}(\mathfrak{z}) \supseteq \mathrm{G_{tb-2}}(\mathfrak{z})$ for all $\mathfrak{z} \in \tilde{A}$.

Hence, both conditions are satisfied.

Remark 1. *It is not necessary that if* $\tilde{A} \subseteq \mathfrak{B}$ *then* $\left(\mathrm{F_{tb-1}}, \mathrm{G_{tb-1}}, \tilde{A}\right)$ *is called the TBSR subset of* $\left(\mathrm{F_{tb-2}}, \mathrm{G_{tb-2}}, \mathfrak{B}\right)$. *It holds only if* $\mathrm{F_{tb-1}}(\mathfrak{z}) \subseteq \mathrm{F_{tb-2}}(\mathfrak{z})$ *and* $\mathrm{G_{tb-1}}(\mathfrak{z}) \supseteq \mathrm{G_{tb-2}}(\mathfrak{z})$ *for all* $\mathfrak{z} \in \tilde{A}$.

3.1. AND Product for Two TBSRs

Definition 9. *Let us assume that* $(F_{tb-1}, G_{tb-1}, \tilde{A})$*and* $(F_{tb-2}, G_{tb-2}, \mathfrak{B})$*be two TBSRs. Then, AND product is denoted and defined by*

$$\begin{aligned}(F_{tb-1}, G_{tb-1}, \tilde{A}) \quad & \wedge (F_{tb-2}, G_{tb-2}, \mathfrak{B}) \\ & = \{((\mathfrak{z}, \mathfrak{z}), F_{tb-1}(\mathfrak{z}) \bigcap F_{tb-2}(\mathfrak{z}), G_{tb-1}(\mathfrak{z}) \bigcup G_{tb-2}(\mathfrak{z}))\text{where}(\mathfrak{z}, \mathfrak{z}) \\ & \in \tilde{A} \times \mathfrak{B}\}.\end{aligned}$$

Theorem 1. *The AND product of two TBSRs* $(F_{tb-1}, G_{tb-1}, \tilde{A})$ *and* $(F_{tb-2}, G_{tb-2}, \mathfrak{B})$ *is a TBSR , if* $G_{tb-1}(\mathfrak{z})$ *is a subring of* $G_{tb-2}(\mathfrak{z})$ *or* $G_{tb-2}(\mathfrak{z})$ *is a subring of* $G_{tb-1}(\mathfrak{z})$ *for all* $(\mathfrak{z}, \mathfrak{z}) \in \tilde{A} \times \mathfrak{B}$.

Proof.
Case 1: Assume that $G_{tb-1}(\mathfrak{z})$ is a subring of $G_{tb-2}(\mathfrak{z})$ or $G_{tb-2}(\mathfrak{z})$ is a subring of $G_{tb-1}(\mathfrak{z})$, then in either case, $G_{tb-1}(\mathfrak{z}) \cup G_{tb-2}(\mathfrak{z})$ is a subring for all $(\mathfrak{z}, \mathfrak{z}) \in \tilde{A} \times \mathfrak{B}$.
Case 2: Now we can see that $F_{tb-1}(\mathfrak{z}) \bigcap F_{tb-2}(\mathfrak{z})$ for all $(\mathfrak{z}, \mathfrak{z}) \in \tilde{A} \times \mathfrak{B}$ is always a subring because the intersection of any number of subrings is always a subring. Hence, in either case, the AND product is a TBSR. □

Example 3. *Let* $X = Z$ *and* $\mathrm{ɣ} = Z_{12}$ *be two distinct rings and* $U = X \cup \mathrm{ɣ}$. *Also, assume that* $\tilde{A} = \{\mathfrak{z}_1, \mathfrak{z}_2\}, \mathfrak{B} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3\}$.

Now define

$$F_{tb-1}(\mathfrak{z}_1) = 2Z, F_{tb-1}(\mathfrak{z}_2) = 6Z$$

$$\text{And } F_{tb-2}(\mathfrak{z}_1) = Z, F_{tb-2}(\mathfrak{z}_2) = 3Z \text{ and } F_{tb-2}(\mathfrak{z}_3) = 4Z.$$

Also, define

$$G_{tb-1}(\mathfrak{z}_1) = \left\{\bar{0}\right\}, G_{tb-1}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{6}\right\} \text{ and}$$
$$G_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\right\}, G_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{3}, \bar{6}, \bar{9}\right\} \text{ and } G_{tb-2}(\mathfrak{z}_3) = Z_{12}$$

Now as AND operation is defined as follows:

$$\begin{aligned}(F_{tb-1}, G_{tb-1}, \tilde{A}) \quad & \wedge (F_{tb-2}, G_{tb-2}, \mathfrak{B}) \\ & = \{((\mathfrak{z}, \mathfrak{z}), F_{tb-1}(\mathfrak{z}) \bigcap F_{tb-2}(\mathfrak{z}), G_{tb-1}(\mathfrak{z}) \bigcup G_{tb-2}(\mathfrak{z}))\text{where}(\mathfrak{z}, \mathfrak{z}) \\ & \in \tilde{A} \times \mathfrak{B}\}.\end{aligned}$$

Now $\tilde{A} \times \mathfrak{B} = \{(\mathfrak{z}_1, \mathfrak{z}_2), (\mathfrak{z}_1, \mathfrak{z}_2), (\mathfrak{z}_1, \mathfrak{z}_3), (\mathfrak{z}_2, \mathfrak{z}_1), (\mathfrak{z}_2, \mathfrak{z}_2), (\mathfrak{z}_2, \mathfrak{z}_3)\}$. *Then*

$$F_{tb-1}(\mathfrak{z}_1) \bigcap F_{tb-2}(\mathfrak{z}_1) = 2Z \bigcap Z = 2Z; F_{tb-1}(\mathfrak{z}_1) \bigcap F_{tb-2}(\mathfrak{z}_2) = 2Z \bigcap 3Z = 6Z;$$

$$F_{tb-1}(\mathfrak{z}_1) \bigcap F_{tb-2}(\mathfrak{z}_3) = 2Z \bigcap 4Z = 4Z$$

Similarly,

$$F_{tb-1}(\mathfrak{z}_2) \bigcap F_{tb-2}(\mathfrak{z}_1) = 6Z; F_{tb-1}(\mathfrak{z}_2) \bigcap F_{tb-2}(\mathfrak{z}_2) = 6Z;$$

$$F_{tb-1}(\mathfrak{z}_2) \bigcap F_{tb-2}(\mathfrak{z}_3) = 12Z$$

Also,

$$G_{tb-1}(\mathfrak{z}_1)\bigcup G_{tb-2}(\mathfrak{z}_1)=\left\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\right\};G_{tb-1}(\mathfrak{z}_1)\bigcup G_{tb-2}(\mathfrak{z}_2)=\left\{\bar{0},\bar{3},\bar{6},\bar{9}\right\};G_{tb-1}(\mathfrak{z}_1)\bigcup G_{tb-2}(\mathfrak{z}_3)=Z_{12}$$

Similarly,

$$G_{tb-1}(\mathfrak{z}_2)\bigcup G_{tb-2}(\mathfrak{z}_1)=\left\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\right\};G_{tb-1}(\mathfrak{z}_2)\bigcup G_{tb-2}(\mathfrak{z}_2)=\left\{\bar{0},\bar{3},\bar{6},\bar{9}\right\};G_{tb-1}(\mathfrak{z}_2)\bigcup G_{tb-2}(\mathfrak{z}_3)=Z_{12}$$

So

$$(F_{tb-1},G_{tb-1},\tilde{A})\bigwedge(F_{tb-2},G_{tb-2},\mathfrak{B})=\left\{\begin{array}{c}\left((\mathfrak{z}_1,\mathfrak{z}_1);2Z;\left\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\right\}\right),\ \left((\mathfrak{z}_1,\mathfrak{z}_2);6Z;\left\{\bar{0},\bar{3},\bar{6},\bar{9}\right\}\right),\\ ((\mathfrak{z}_1,\mathfrak{z}_3);4Z;Z_{12}),\ \left((\mathfrak{z}_2,\mathfrak{z}_1);6Z;\left\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\right\}\right),\\ \left((\mathfrak{z}_2,\mathfrak{z}_2);6Z;\left\{\bar{0},\bar{3},\bar{6},\bar{9}\right\}\right),\ ((\mathfrak{z}_2,\mathfrak{z}_3);12Z;Z_{12})\end{array}\right\}.$$

Hence, we can observe that the AND product is a TBSR.

Remark 2. *The AND product of two TBSRs is not a TBSR in general.*

Example 4. *Let* $X=Z$ *and* $\gamma=Z_{12}$ *be two distinct rings and* $U=X\cup\gamma$. *Also, assume that* $\tilde{A}=\{\mathfrak{z}_1,\mathfrak{z}_2\},\mathfrak{B}=\{\mathfrak{z}_1,\mathfrak{z}_2,\mathfrak{z}_3\}$.

Now define

$$F_{tb-1}(\mathfrak{z}_1)=2Z,F_{tb-1}(\mathfrak{z}_2)=6Z$$
$$\text{And }F_{tb-2}(\mathfrak{z}_1)=Z,F_{tb-2}(\mathfrak{z}_2)=3Z\text{ and }F_{tb-2}(\mathfrak{z}_3)=4Z.$$

Also, define

$$G_{tb-1}(\mathfrak{z}_1)=\left\{\bar{0},\bar{4},\bar{8}\right\},G_{tb-1}(\mathfrak{z}_2)=\left\{\bar{0},\bar{3},\bar{6}\right\}\text{ and}$$

$$G_{tb-2}(\mathfrak{z}_1)=\left\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\right\},G_{tb-2}(\mathfrak{z}_2)=\left\{\bar{0},\bar{6}\right\}\text{ and }G_{tb-2}(\mathfrak{z}_3)=\left\{\bar{0}\right\}$$

Now

$$F_{tb-1}(\mathfrak{z}_1)\bigcap F_{tb-2}(\mathfrak{z}_1)=2Z;F_{tb-1}(\mathfrak{z}_1)\bigcap F_{tb-2}(\mathfrak{z}_2)=6Z;$$

$$F_{tb-1}(\mathfrak{z}_1)\bigcap F_{tb-2}(\mathfrak{z}_3)=4Z$$

Similarly,

$$F_{tb-1}(\mathfrak{z}_2)\bigcap F_{tb-2}(\mathfrak{z}_1)=6Z;F_{tb-1}(\mathfrak{z}_2)\bigcap F_{tb-2}(\mathfrak{z}_2)=6Z;$$

$$F_{tb-1}(\mathfrak{z}_2)\bigcap F_{tb-2}(\mathfrak{z}_3)=12Z$$

Also,

$$G_{tb-1}(\mathfrak{z}_1)\bigcup G_{tb-2}(\mathfrak{z}_1)=\left\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\right\};G_{tb-1}(\mathfrak{z}_1)\bigcup G_{tb-2}(\mathfrak{z}_2)=\left\{\bar{0},\bar{4},\bar{6},\bar{8}\right\};$$

$$G_{tb-1}(\mathfrak{z}_1)\bigcup G_{tb-2}(\mathfrak{z}_3)=\left\{\bar{0},\bar{4},\bar{8}\right\}$$

And

$$G_{tb-1}(\mathfrak{z}_2)\bigcup G_{tb-2}(\mathfrak{z}_1)=\left\{\bar{0},\bar{2},\bar{3},\bar{4},\bar{6},\bar{8},\bar{9},\overline{10}\right\};G_{tb-1}(\mathfrak{z}_1)\bigcup G_{tb-2}(\mathfrak{z}_2)=\left\{\bar{0},\bar{3},\bar{6},\bar{9}\right\};$$

$$G_{tb-1}(\mathfrak{z}_1)\bigcup G_{tb-2}(\mathfrak{z}_3)=\left\{\bar{0},\bar{3},\bar{6},\bar{9}\right\}$$

$$\left(F_{tb-1},G_{tb-1},\tilde{A}\right)\bigwedge\left(F_{tb-2},G_{tb-2},\mathfrak{B}\right)=\left\{\begin{array}{c}\left((\mathfrak{z}_1,\mathfrak{z}_1);2Z;\left\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\right\}\right),\ \left((\mathfrak{z}_1,\mathfrak{z}_2);6Z;\left\{\bar{0},\bar{4},\bar{6},\bar{8}\right\}\right),\\ \left((\mathfrak{z}_1,\mathfrak{z}_3);4Z;\left\{\bar{0},\bar{4},\bar{8}\right\}\right),\ \left((\mathfrak{z}_2,\mathfrak{z}_1);6Z;\left\{\bar{0},\bar{2},\bar{3},\bar{4},\bar{6},\bar{8},\bar{9},\overline{10}\right\}\right),\\ \left((\mathfrak{z}_2,\mathfrak{z}_2);6Z;\left\{\bar{0},\bar{3},\bar{6},\bar{9}\right\}\right),\ \left((\mathfrak{z}_2,\mathfrak{z}_3);12Z;\left\{\bar{0},\bar{3},\bar{6},\bar{9}\right\}\right)\end{array}\right\}.$$

We can note that $G_{tb-1}(\mathfrak{z}_1)\cup G_{tb-2}(\mathfrak{z}_2)=\left\{\bar{0},\bar{4},\bar{6},\bar{8}\right\}$ *that is not a subring of* Z_{12}.

3.2. OR Product of Two TBSRs

Definition 10. *Let us assume that* $\left(F_{tb-1},G_{tb-1},\tilde{A}\right)$ *and* $(F_{tb-2},G_{tb-2},\mathfrak{B})$ *be two TBSRs. Then, OR product is denoted and defined by*

$$\left(F_{tb-1},G_{tb-1},\tilde{A}\right)\vee\left(F_{tb-2},G_{tb-2},\mathfrak{B}\right)=\left\{\left((\mathfrak{z},\mathfrak{z}),F_{tb-1}(\mathfrak{z})\cup F_{tb-2}(\mathfrak{z}),G_{tb-1}(\mathfrak{z})\cap G_{tb-2}(\mathfrak{z})\right)\middle|(\mathfrak{z},\mathfrak{z})\in\tilde{A}\times\mathfrak{B}\right\}.$$

Theorem 2. *The OR product of two TBSRs* $\left(F_{tb-1},G_{tb-1},\tilde{A}\right)$ *and* $(F_{tb-2},G_{tb-2},\mathfrak{B})$ *is a TBSR, if* $F_{tb-1}(\mathfrak{z})$ *is a subring of* $F_{tb-2}(\mathfrak{z})$ *or* $F_{tb-2}(\mathfrak{z})$ *is a subring of* $F_{tb-1}(\mathfrak{z})$ *for all* $(\mathfrak{z},\mathfrak{z})\in\tilde{A}\times\mathfrak{B}$.

Proof.
Case 1: Assume that $F_{tb-1}(\mathfrak{z})$ is a subring of $F_{tb-2}(\mathfrak{z})$ or $F_{tb-2}(\mathfrak{z})$ is a subring of $F_{tb-1}(\mathfrak{z})$, then in either case, $F_{tb-1}(\mathfrak{z})\cup F_{tb-2}(\mathfrak{z})$ is a subring for all $(\mathfrak{z},\mathfrak{z})\in\tilde{A}\times\mathfrak{B}$.
Case 2: Now we can see that $G_{tb-1}(\mathfrak{z})\bigcap G_{tb-2}(\mathfrak{z})$ for all $(\mathfrak{z},\mathfrak{z})\in\tilde{A}\times\mathfrak{B}$ is always a subring because the intersection of any number of subrings is always a subring. Hence, in either case the OR product is a TBSR. □

Example 5. *Let* $X=Z$ *and* $\mathrm{Y}=Z_{12}$ *be two distinct rings and* $U=X\cup\mathrm{Y}$. *Also, assume that* $\tilde{A}=\{\mathfrak{z}_1,\mathfrak{z}_2\},\mathfrak{B}=\{\mathfrak{z}_1,\mathfrak{z}_2,\mathfrak{z}_3\}$.

Now define

$$F_{tb-1}(\mathfrak{z}_1)=2Z,F_{tb-1}(\mathfrak{z}_2)=6Z$$
$$\text{And } F_{tb-2}(\mathfrak{z}_1)=Z,F_{tb-2}(\mathfrak{z}_2)=2Z \text{ and } F_{tb-2}(\mathfrak{z}_3)=12Z.$$

Also, define

$$G_{tb-1}(\mathfrak{z}_1)=\left\{\bar{0}\right\},G_{tb-1}(\mathfrak{z}_2)=\left\{\bar{0},\bar{6}\right\}\text{ and}$$
$$G_{tb-2}(\mathfrak{z}_1)=\left\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\right\},G_{tb-2}(\mathfrak{z}_2)=\left\{\bar{0},\bar{3},\bar{6},\bar{9}\right\}\text{ and }G_{tb-2}(\mathfrak{z}_3)=\left\{\bar{0},\bar{4},\bar{8}\right\}$$

Now OR operation is defined as follows

$$\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right) \vee \left(F_{tb-2}, G_{tb-2}, \mathfrak{B}\right) = \left\{\left((\mathfrak{z},\mathfrak{z}), F_{tb-1}(\mathfrak{z}) \cup F_{tb-2}(\mathfrak{z}), G_{tb-1}(\mathfrak{z}) \cap G_{tb-2}(\mathfrak{z})\right) \middle| (\mathfrak{z},\mathfrak{z}) \in \tilde{A} \times \mathfrak{B}\right\}$$

Now $\tilde{A} \times \mathfrak{B} = \{(\mathfrak{z}_1,\mathfrak{z}_2), (\mathfrak{z}_1,\mathfrak{z}_2), (\mathfrak{z}_1,\mathfrak{z}_3), (\mathfrak{z}_2,\mathfrak{z}_1), (\mathfrak{z}_2,\mathfrak{z}_2), (\mathfrak{z}_2,\mathfrak{z}_3)\}$. *Then*

$$F_{tb-1}(\mathfrak{z}_1) \cup F_{tb-2}(\mathfrak{z}_1) = Z; F_{tb-1}(\mathfrak{z}_1) \cup F_{tb-2}(\mathfrak{z}_2) = 2Z;$$

$$F_{tb-1}(\mathfrak{z}_1) \cup F_{tb-2}(\mathfrak{z}_3) = 2Z$$

Similarly,

$$F_{tb-1}(\mathfrak{z}_2) \cup F_{tb-2}(\mathfrak{z}_1) = Z; F_{tb-1}(\mathfrak{z}_2) \cup F_{tb-2}(\mathfrak{z}_2) = 2Z;$$

$$F_{tb-1}(\mathfrak{z}_2) \cup F_{tb-2}(\mathfrak{z}_3) = 6Z$$

Also,

$$G_{tb-1}(\mathfrak{z}_1) \cap G_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}\right\}; G_{tb-1}(\mathfrak{z}_1) \cap G_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}\right\}; G_{tb-1}(\mathfrak{z}_1) \cap G_{tb-2}(\mathfrak{z}_3) = \left\{\bar{0}\right\}$$

Similarly,

$$G_{tb-1}(\mathfrak{z}_2) \cap G_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{6}\right\}; G_{tb-1}(\mathfrak{z}_2) \cap G_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{6}\right\}; G_{tb-1}(\mathfrak{z}_2) \cap G_{tb-2}(\mathfrak{z}_3) = \left\{\bar{0}\right\}$$

So

$$\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right) \cup \left(F_{tb-2}, G_{tb-2}, \mathfrak{B}\right) = \left\{\begin{array}{c} \left((\mathfrak{z}_1,\mathfrak{z}_1); Z; \left\{\bar{0}\right\}\right), \left((\mathfrak{z}_1,\mathfrak{z}_2); 2Z; \left\{\bar{0}\right\}\right), \\ \left((\mathfrak{z}_1,\mathfrak{z}_3); 2Z; \left\{\bar{0}\right\}\right), \left((\mathfrak{z}_2,\mathfrak{z}_1); Z; \left\{\bar{0}, \bar{6}\right\}\right), \\ \left((\mathfrak{z}_2,\mathfrak{z}_2); 2Z; \left\{\bar{0}, \bar{6}\right\}\right), \left((\mathfrak{z}_2,\mathfrak{z}_3); 6Z; \left\{\bar{0}\right\}\right) \end{array}\right\}.$$

Hence, we can observe that the OR product is a TBSR.

Remark 3. *The OR product of two TBSRs is not a TBSR in general.*

Example 6. *Let* $X = Z$*and* $\gamma = Z_{12}$ *be two distinct rings* $U = X \cup \gamma$. *Also, assume that* $\tilde{A} = \{\mathfrak{z}_1, \mathfrak{z}_2\}, \mathfrak{B} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3\}$.

Now define

$$F_{tb-1}(\mathfrak{z}_1) = 2Z, F_{tb-1}(\mathfrak{z}_2) = 6Z$$
$$\text{And } F_{tb-2}(\mathfrak{z}_1) = Z, F_{tb-2}(\mathfrak{z}_2) = 3Z \text{ and } F_{tb-2}(\mathfrak{z}_3) = 12Z.$$

Also, define

$$G_{tb-1}(\mathfrak{z}_1) = \left\{\bar{0}\right\}, G_{tb-1}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{6}\right\} \text{ and}$$

$$G_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\right\}, G_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{3}, \bar{6}, \bar{9}\right\} \text{ and } G_{tb-2}(\mathfrak{z}_3) = \left\{\bar{0}, \bar{4}, \bar{8}\right\}$$

Now

$$F_{tb-1}(\mathfrak{z}_1) \cup F_{tb-2}(\mathfrak{z}_1) = Z; F_{tb-1}(\mathfrak{z}_1) \cup F_{tb-2}(\mathfrak{z}_2) = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \ldots\};$$

$$F_{tb-1}(\mathfrak{z}_1) \cup F_{tb-2}(\mathfrak{z}_3) = 2Z$$

Similarly,

$$F_{tb-1}(\mathfrak{z}_2) \cup F_{tb-2}(\mathfrak{z}_1) = Z; F_{tb-1}(\mathfrak{z}_2) \cup F_{tb-2}(\mathfrak{z}_2) = 3Z;$$

$$F_{tb-1}(\mathfrak{z}_2) \cup F_{tb-2}(\mathfrak{z}_3) = 6z$$

Also,

$$G_{tb-1}(\mathfrak{z}_1) \cap G_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}\right\}; G_{tb-1}(\mathfrak{z}_1) \cap G_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}\right\}; G_{tb-1}(\mathfrak{z}_1) \cap G_{tb-2}(\mathfrak{z}_3) = \left\{\bar{0}\right\}$$

Similarly,

$$G_{tb-1}(\mathfrak{z}_2) \cap G_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{6}\right\}; G_{tb-1}(\mathfrak{z}_2) \cap G_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{6}\right\}; G_{tb-1}(\mathfrak{z}_2) \cap G_{tb-2}(\mathfrak{z}_3) = \left\{\bar{0}\right\}$$

So

$$\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right) \cup \left(F_{tb-2}, G_{tb-2}, \mathfrak{B}\right) = \left\{ \begin{array}{c} \left((\mathfrak{z}_1, \mathfrak{z}_1); Z; \left\{\bar{0}\right\}\right), \left((\mathfrak{z}_1, \mathfrak{z}_2); \{0, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \ldots\}; \left\{\bar{0}\right\}\right), \\ \left((\mathfrak{z}_1, \mathfrak{z}_3); 2Z; \left\{\bar{0}\right\}\right), \left((\mathfrak{z}_2, \mathfrak{z}_1); Z; \left\{\bar{0}, \bar{6}\right\}\right), \\ \left((\mathfrak{z}_2, \mathfrak{z}_2); 3Z; \left\{\bar{0}, \bar{6}\right\}\right), \left((\mathfrak{z}_2, \mathfrak{z}_3); 6Z; \left\{\bar{0}\right\}\right) \end{array} \right\}.$$

Hence, we can observe that $F_{tb-1}(\mathfrak{z}_1) \cup F_{tb-2}(\mathfrak{z}_2) = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \ldots\}$ *is not a subring of* Z*. Hence, the OR product of two TBSRs need not be a TBSR.*

3.3. Extended Union of TBSRs

In this part, we have to discuss the basic definition of an extended union for TBSRs. Moreover, we have to elaborate on the theorems and some remarks related to this theory.

Definition 11. *Let* $\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right)$ *and* $\left(F_{tb-2}, G_{tb-2}, \mathfrak{B}\right)$ *be two TBSRs. Then, the extended union of two TBSRs is denoted and defined by*

$$\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right) \cup_{ext.} \left(F_{tb-2}, G_{tb-2}, \mathfrak{B}\right) = \left(F_{tb-3}, G_{tb-3}, €\right); € = \tilde{A} \cup \mathfrak{B} \text{ and}$$
$$F_{tb-3}(\mathfrak{z}) = \begin{cases} F_{tb-1}(\mathfrak{z}) & ; \text{if } \mathfrak{z} \in \tilde{A} - \mathfrak{B} \\ F_{tb-2}(\mathfrak{z}) & ; \text{if } \mathfrak{z} \in \mathfrak{B} - \tilde{A} \\ F_{tb-1}(\mathfrak{z}) \cup F_{tb-2}(\mathfrak{z}) & ; \text{if } \mathfrak{z} \in \tilde{A} \cap \mathfrak{B} \end{cases}$$
$$G_{tb-3}(\mathfrak{z}) = \begin{cases} G_{tb-1}(\mathfrak{z}) & ; \text{if } \mathfrak{z} \in \tilde{A} - \mathfrak{B} \\ G_{tb-2}(\mathfrak{z}) & ; \text{if } \mathfrak{z} \in \mathfrak{B} - \tilde{A} \\ G_{tb-1}(\mathfrak{z}) \cap G_{tb-2}(\mathfrak{z}) & ; \text{if } \mathfrak{z} \in \tilde{A} \cap \mathfrak{B} \end{cases}$$

Theorem 3. *The extended union of two TBSRs is a TBSR if* $F_{tb-1}(\mathfrak{z})$ *is a subring of* $F_{tb-2}(\mathfrak{z})$ *or* $F_{tb-2}(\mathfrak{z})$ *is a subring of* $F_{tb-1}(\mathfrak{z})$ *for all* $\mathfrak{z} \in \tilde{A} \cup \mathfrak{B}$.

Proof.

Case 1: Assume that $F_{tb-1}(\mathfrak{z})$ is a subring of $F_{tb-2}(\mathfrak{z})$ or $F_{tb-2}(\mathfrak{z})$ is a subring of $F_{tb-1}(\mathfrak{z})$ for all $\mathfrak{z} \in \tilde{A} \cup \mathfrak{B}$, then in both cases, $F_{tb-1}(\mathfrak{z}) \cup F_{tb-2}(\mathfrak{z})$ is a subring for all $\mathfrak{z} \in \tilde{A} \cup \mathfrak{B}$. Now if $\mathfrak{z} \in \tilde{A} - \mathfrak{B}$ or $\mathfrak{z} \in \mathfrak{B} - \tilde{A}$ then it is a trivial case.

Case 2: We can easily observe that $G_{tb-1}(\mathfrak{z}) \cap G_{tb-2}(\mathfrak{z})$ is a subring because the intersection of any number of subrings is always a subring. Hence, in either case, the extended union of two TBSRs is a TBSR. □

Example 7. *Let* $X = Z$ *and* $\gamma = Z_{12}$ *be two distinct rings and* $U = X \cup \gamma$. *Also, assume that* $\tilde{A} = \{\mathfrak{z}_1, \mathfrak{z}_2\}, \mathfrak{B} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3\}$.

Now define

$$F_{tb-1}(\mathfrak{z}_1) = 2Z, F_{tb-1}(\mathfrak{z}_2) = 6Z$$
$$\text{And } F_{tb-2}(\mathfrak{z}_1) = Z, F_{tb-2}(\mathfrak{z}_2) = 2Z \text{ and } F_{tb-2}(\mathfrak{z}_3) = 12Z.$$

Also, define

$$G_{tb-1}(\mathfrak{z}_1) = \left\{\bar{0}\right\}, G_{tb-1}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{6}\right\} \text{ and}$$
$$G_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\right\}, G_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{3}, \bar{6}, \bar{9}\right\} \text{ and } G_{tb-2}(\mathfrak{z}_3) = \left\{\bar{0}, \bar{4}, \bar{8}\right\}$$

Now as

$$(F_{tb-1}, G_{tb-1}, \tilde{A}) \cup_{e\mathfrak{z}t.} (F_{tb-2}, G_{tb-2}, \mathfrak{B}) = (F_{tb-3}, G_{tb-3}, €); € = \tilde{A} \cup \mathfrak{B} \text{ and}$$
$$F_{tb-3}(\mathfrak{z}) = \begin{cases} F_{tb-1}(\mathfrak{z}) & ; \text{ if } \mathfrak{z} \in \tilde{A} - \mathfrak{B} \\ F_{tb-2}(\mathfrak{z}) & ; \text{ if } \mathfrak{z} \in \mathfrak{B} - \tilde{A} \\ F_{tb-1}(\mathfrak{z}) \cup F_{tb-2}(\mathfrak{z}) & ; \text{ if } \mathfrak{z} \in \tilde{A} \cap \mathfrak{B} \end{cases}$$
$$G_{tb-3}(\mathfrak{z}) = \begin{cases} G_{tb-1}(\mathfrak{z}) & ; \text{ if } \mathfrak{z} \in \tilde{A} - \mathfrak{B} \\ G_{tb-2}(\mathfrak{z}) & ; \text{ if } \mathfrak{z} \in \mathfrak{B} - \tilde{A} \\ G_{tb-1}(\mathfrak{z}) \cap G_{tb-2}(\mathfrak{z}) & ; \text{ if } \mathfrak{z} \in \tilde{A} \cap \mathfrak{B} \end{cases}$$

So from the above observation, we can see that $\mathfrak{z}_1, \mathfrak{z}_2 \in \tilde{A} \cap \mathfrak{B}$ *and* $\mathfrak{z}_3 \in \mathfrak{B} - \tilde{A}$
Hence,

$$F_{tb-3}(\mathfrak{z}_1) = F_{tb-1}(\mathfrak{z}_1) \cup F_{tb-2}(\mathfrak{z}_1) = Z; F_{tb-3}(\mathfrak{z}_2) = F_{tb-1}(\mathfrak{z}_2) \cup F_{tb-2}(\mathfrak{z}_2) = 2Z; F_{tb-3}(\mathfrak{z}_3) = 12Z$$

And

$$G_{tb-3}(\mathfrak{z}_1) = G_{tb-1}(\mathfrak{z}_1) \cap G_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}\right\}; G_{tb-3}(\mathfrak{z}_2) = G_{tb-1}(\mathfrak{z}_2) \cap G_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{6}\right\}; G_{tb-3}(\mathfrak{z}_3) = \left\{\bar{0}, \bar{4}, \bar{8}\right\}$$

Hence, TBSR $(F_{tb-3}, G_{tb-3}, €)$ *is given by*

$$(F_{tb-3}, G_{tb-3}, €) = \left\{ \left(\mathfrak{z}_1, Z, \left\{\bar{0}\right\}\right), \left(\mathfrak{z}_2, 2Z, \left\{\bar{0}, \bar{6}\right\}\right), \left(\mathfrak{z}_3, 12Z, \left\{\bar{0}, \bar{4}, \bar{8}\right\}\right) \right\}$$

Remark 4. *The extended union of two TBSRs need not be a TBSR in general.*

Example 8. *Let* $X = Z$ *and* $\gamma = Z_{12}$ *be two distinct rings and* $U = X \cup \gamma$. *Also, assume that* $\tilde{A} = \{\mathfrak{z}_1, \mathfrak{z}_2\}, \mathfrak{B} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3\}$.

Now define

$$F_{tb-1}(\mathfrak{z}_1) = 4Z, F_{tb-1}(\mathfrak{z}_2) = 2Z$$
$$\text{And } F_{tb-2}(\mathfrak{z}_1) = Z, F_{tb-2}(\mathfrak{z}_2) = 3Z \text{ and } F_{tb-2}(\mathfrak{z}_3) = 4Z.$$

Also, define

$$G_{tb-1}(\mathfrak{z}_1) = \left\{\bar{0}\right\}, G_{tb-1}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{6}\right\} \text{ and}$$
$$G_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\right\}, G_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{3}, \bar{6}, \bar{9}\right\} \text{ and } G_{tb-2}(\mathfrak{z}_3) = \left\{\bar{0}, \bar{4}, \bar{8}\right\}$$

Now

$$\mathrm{F_{tb-3}}(\mathfrak{z}_1) = \mathrm{F_{tb-1}}(\mathfrak{z}_1) \cup \mathrm{F_{tb-2}}(\mathfrak{z}_1) = \mathrm{Z}; \mathrm{F_{tb-3}}(\mathfrak{z}_2) = \mathrm{F_{tb-1}}(\mathfrak{z}_2) \cup \mathrm{F_{tb-2}}(\mathfrak{z}_2) = \{0, \pm 2, \pm 3, \pm 4, \pm 9, \ldots\};$$
$$\mathrm{F_{tb-3}}(\mathfrak{z}_3) = \mathrm{F_{tb-2}}(\mathfrak{z}_3) = 4\mathrm{Z}$$

Also,

$$\mathrm{G_{tb-1}}(\mathfrak{z}_1) \cap \mathrm{G_{tb-2}}(\mathfrak{z}_1) = \left\{\bar{0}\right\}; \mathrm{G_{tb-1}}(\mathfrak{z}_2) \cap \mathrm{G_{tb-2}}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{6}\right\}; \mathrm{G_{tb-1}}(\mathfrak{z}_3) \cap \mathrm{G_{tb-2}}(\mathfrak{z}_3) = \left\{\bar{0}, \bar{4}, \bar{8}\right\}$$
$$= (\mathrm{F_{tb-3}}, \mathrm{G_{tb-3}}, €) = \left(\mathrm{F_{tb-1}}, \mathrm{G_{tb-1}}, \tilde{\mathrm{A}}\right) \cup_{\text{ext.}} (\mathrm{F_{tb-2}}, \mathrm{G_{tb-2}}, \mathfrak{B})$$
$$= \left\{ \left(\mathfrak{z}_1; \mathrm{Z}; \left\{\bar{0}\right\}\right), \left(\mathfrak{z}_2; \{0, \pm 2, \pm 3, \pm 4, \pm 6, \ldots.\}; \left\{\bar{0}, \bar{6}\right\}\right), \left(\mathfrak{z}_3; 4\mathrm{Z}; \left\{\bar{0}, \bar{4}, \bar{8}\right\}\right) \right\}.$$

Hence, we can observe that $\mathrm{F_{tb-1}}(\mathfrak{z}_1) \cup \mathrm{F_{tb-2}}(\mathfrak{z}_2) = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \ldots.\}$ *is not a subring of* Z. *Hence, the extended union need not be TBSR.*

3.4. The Extended Intersection of TBSRs

In this part, we have to discuss the basic definition of an extended intersection for TBSRs. Moreover, we have to elaborate on the theorems and some remarks related to this theory.

Definition 12. *Let* $\left(\mathrm{F_{tb-1}}, \mathrm{G_{tb-1}}, \tilde{\mathrm{A}}\right)$ *and* $(\mathrm{F_{tb-2}}, \mathrm{G_{tb-2}}, \mathfrak{B})$ *be two TBSRs. Then, the extended intersection of two TBSRs is denoted and defined by*

$$\left(\mathrm{F_{tb-1}}, \mathrm{G_{tb-1}}, \tilde{\mathrm{A}}\right) \cap_{\text{ext.}} (\mathrm{F_{tb-2}}, \mathrm{G_{tb-2}}, \mathfrak{B}) = (\mathrm{F_{tb-3}}, \mathrm{G_{tb-3}}, €); € = \tilde{\mathrm{A}} \cup \mathfrak{B} \text{ and}$$
$$\mathrm{F_{tb-3}}(\mathfrak{z}) = \begin{cases} \mathrm{F_{tb-1}}(\mathfrak{z}) & ; \text{if } \mathfrak{z} \in \tilde{\mathrm{A}} - \mathfrak{B} \\ \mathrm{F_{tb-2}}(\mathfrak{z}) & ; \text{if } \mathfrak{z} \in \mathfrak{B} - \tilde{\mathrm{A}} \\ \mathrm{F_{tb-1}}(\mathfrak{z}) \cap \mathrm{F_{tb-2}}(\mathfrak{z}) & ; \text{if } \mathfrak{z} \in \tilde{\mathrm{A}} \cap \mathfrak{B} \end{cases}$$
$$\mathrm{G_{tb-3}}(\mathfrak{z}) = \begin{cases} \mathrm{G_{tb-1}}(\mathfrak{z}) & ; \text{if } \mathfrak{z} \in \tilde{\mathrm{A}} - \mathfrak{B} \\ \mathrm{G_{tb-2}}(\mathfrak{z}) & ; \text{if } \mathfrak{z} \in \mathfrak{B} - \tilde{\mathrm{A}} \\ \mathrm{G_{tb-1}}(\mathfrak{z}) \cup \mathrm{G_{tb-2}}(\mathfrak{z}) & ; \text{if } \mathfrak{z} \in \tilde{\mathrm{A}} \cap \mathfrak{B} \end{cases}$$

Theorem 4. *The extended intersection of two TBSRs is a TBSR if* $\mathrm{G_{tb-1}}(\mathfrak{z})$ *is a subring of* $\mathrm{G_{tb-2}}(\mathfrak{z})$ *or* $\mathrm{G_{tb-2}}(\mathfrak{z})$ *is a subring of* $\mathrm{G_{tb-1}}(\mathfrak{z})$ *for all* $\mathfrak{z} \in \tilde{\mathrm{A}} \cup \mathfrak{B}$.

Proof.

Case 1: Assume that $\mathrm{G_{tb-1}}(\mathfrak{z})$ is a subring of $\mathrm{G_{tb-2}}(\mathfrak{z})$ or $\mathrm{G_{tb-2}}(\mathfrak{z})$ is a subring of $\mathrm{G_{tb-1}}(\mathfrak{z})$ for all $\mathfrak{z} \in \tilde{\mathrm{A}} \cup \mathfrak{B}$, then in both cases, $\mathrm{G_{tb-1}}(\mathfrak{z}) \cup \mathrm{G_{tb-2}}(\mathfrak{z})$ is a subring for all $\mathfrak{z} \in \tilde{\mathrm{A}} \cup \mathfrak{B}$. Now if $\mathfrak{z} \in \tilde{\mathrm{A}} - \mathfrak{B}$ or $\mathfrak{z} \in \mathfrak{B} - \tilde{\mathrm{A}}$ then it is a trivial case.

Case 2: We can easily observe that $\mathrm{F_{tb-1}}(\mathfrak{z}) \cap \mathrm{F_{tb-2}}(\mathfrak{z})$ is a subring because the intersection of any number of subrings is always a subring. Hence, in either case, the extended intersection of two TBSRs is a TBSR. □

Example 9. *Let* X = Z and ɣ = Z_{12} *be two distinct rings and* U = X ∪ ɣ. *Also, assume that* $\tilde{\mathrm{A}} = \{\mathfrak{z}_1, \mathfrak{z}_2\}, \mathfrak{B} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3\}$.

Now define

$$\mathrm{F_{tb-1}}(\mathfrak{z}_1) = 2\mathrm{Z}, \mathrm{F_{tb-1}}(\mathfrak{z}_2) = 6\mathrm{Z}$$
$$\text{And } \mathrm{F_{tb-2}}(\mathfrak{z}_1) = \mathrm{Z}, \mathrm{F_{tb-2}}(\mathfrak{z}_2) = 3\mathrm{Z} \text{ and } \mathrm{F_{tb-2}}(\mathfrak{z}_3) = 4\mathrm{Z}.$$

Also, define

$$\mathrm{G_{tb-1}}(\mathfrak{z}_1) = \left\{\bar{0}\right\}, \mathrm{G_{tb-1}}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{6}\right\} \text{ and}$$

$$G_{tb-2}(\mathfrak{z}_1) = \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\right\}, G_{tb-2}(\mathfrak{z}_2) = \left\{\overline{0}, \overline{3}, \overline{6}, \overline{9}\right\} \text{ and } G_{tb-2}(\mathfrak{z}_3) = Z_{12}$$

Now as

$$\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right) \cap_{ext.} (F_{tb-2}, G_{tb-2}, \mathfrak{B}) = (F_{tb-3}, G_{tb-3}, €); € = \tilde{A} \cup \mathfrak{B} \text{ and}$$
$$F_{tb-3}(\mathfrak{z}) = \begin{cases} F_{tb-1}(\mathfrak{z}) & ; \text{ if } \mathfrak{z} \in \tilde{A} - \mathfrak{B} \\ F_{tb-2}(\mathfrak{z}) & ; \text{ if } \mathfrak{z} \in \mathfrak{B} - \tilde{A} \\ F_{tb-1}(\mathfrak{z}) \cap F_{tb-2}(\mathfrak{z}) & ; \text{ if } \mathfrak{z} \in \tilde{A} \cap \mathfrak{B} \end{cases}$$
$$G_{tb-3}(\mathfrak{z}) = \begin{cases} G_{tb-1}(\mathfrak{z}) & ; \text{ if } \mathfrak{z} \in \tilde{A} - \mathfrak{B} \\ G_{tb-2}(\mathfrak{z}) & ; \text{ if } \mathfrak{z} \in \mathfrak{B} - \tilde{A} \\ G_{tb-1}(\mathfrak{z}) \cup G_{tb-2}(\mathfrak{z}) & ; \text{ if } \mathfrak{z} \in \tilde{A} \cap \mathfrak{B} \end{cases}$$

So from the above observation, we can see that $\mathfrak{z}_1, \mathfrak{z}_2 \in \tilde{A} \cap \mathfrak{B}$ *and* $\mathfrak{z}_3 \in \mathfrak{B} - \tilde{A}$
Hence,

$$F_{tb-3}(\mathfrak{z}_1) = F_{tb-1}(\mathfrak{z}_1) \cap F_{tb-2}(\mathfrak{z}_1) = 2Z; F_{tb-3}(\mathfrak{z}_2) = F_{tb-1}(\mathfrak{z}_2) \cap F_{tb-2}(\mathfrak{z}_2) = 6Z; F_{tb-3}(\mathfrak{z}_3) = 4Z$$

And

$$G_{tb-3}(\mathfrak{z}_1) = G_{tb-1}(\mathfrak{z}_1) \cup G_{tb-2}(\mathfrak{z}_1) = \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\right\}; G_{tb-3}(\mathfrak{z}_2) = G_{tb-1}(\mathfrak{z}_2) \cup G_{tb-2}(\mathfrak{z}_2)$$
$$= \left\{\overline{0}, \overline{3}, \overline{6}, \overline{9}\right\}; G_{tb-3}(\mathfrak{z}_3) = Z_{12}$$

Hence, TBSR $(F_{tb-3}, G_{tb-3}, €)$ *is given by*

$$(F_{tb-3}, G_{tb-3}, €) = \left\{ \left(\mathfrak{z}_1, 2Z, \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\right\}\right), \left(\mathfrak{z}_2, 6Z, \left\{\overline{0}, \overline{3}, \overline{6}, \overline{9}\right\}\right), (\mathfrak{z}_3, 4Z, Z_{12}) \right\}$$

Remark 5. *The extended intersection of two TBSRs need not be a TBSR in general.*

Example 10. *Let* $X = Z$ *and* $\mathrm{Y} = Z_{12}$ *be two distinct rings and* $U = X \cup \mathrm{Y}$. *Also, assume that* $\tilde{A} = \{\mathfrak{z}_1, \mathfrak{z}_2\}, \mathfrak{B} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3\}$.

Now define

$$F_{tb-1}(\mathfrak{z}_1) = 2Z, F_{tb-1}(\mathfrak{z}_2) = 6Z$$
$$\text{And } F_{tb-2}(\mathfrak{z}_1) = Z, F_{tb-2}(\mathfrak{z}_2) = 3Z \text{ and } F_{tb-2}(\mathfrak{z}_3) = 4Z.$$

Also, define

$$G_{tb-1}(\mathfrak{z}_1) = \left\{\overline{0}, \overline{4}, \overline{8}\right\}, G_{tb-1}(\mathfrak{z}_2) = \left\{\overline{0}, \overline{3}, \overline{6}, \overline{9}\right\} \text{ and}$$

$$G_{tb-2}(\mathfrak{z}_1) = \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\right\}, G_{tb-2}(\mathfrak{z}_2) = \left\{\overline{0}, \overline{4}, \overline{8}\right\} \text{ and } G_{tb-2}(\mathfrak{z}_3) = \left\{\overline{0}, \overline{6}\right\}$$

Now

$$\begin{gathered} F_{tb-3}(\mathfrak{z}_1) = F_{tb-1}(\mathfrak{z}_1) \cap F_{tb-2}(\mathfrak{z}_1) = 2Z; F_{tb-3}(\mathfrak{z}_2) = F_{tb-1}(\mathfrak{z}_2) \cap F_{tb-2}(\mathfrak{z}_2) = 6Z; \\ F_{tb-3}(\mathfrak{z}_3) = F_{tb-2}(\mathfrak{z}_3) = 4Z \end{gathered}.$$

Also,

$$G_{tb-3}(\mathfrak{z}_1) = G_{tb-1}(\mathfrak{z}_1) \cup G_{tb-2}(\mathfrak{z}_1) = \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\right\}; G_{tb-3}(\mathfrak{z}_2) = G_{tb-1}(\mathfrak{z}_2) \cup G_{tb-2}(\mathfrak{z}_2)$$
$$= \left\{\overline{0}, \overline{3}, \overline{4}, \overline{6}, \overline{8}, \overline{9}\right\}; G_{tb-3}(\mathfrak{z}_3) = G_{tb-1}(\mathfrak{z}_3) \cup G_{tb-2}(\mathfrak{z}_3) = \left\{\overline{0}, \overline{6}\right\}$$
$$= (F_{tb-3}, G_{tb-3}, €) = \left(F_{tb-1}, G_{tb-1}, \tilde{A}\right) \cap_{ext.} (F_{tb-2}, G_{tb-2}, \mathfrak{B})$$
$$= \left\{\left(\mathfrak{z}_1; 2Z; \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\right\}\right), \left(\mathfrak{z}_2; 6Z; \left\{\overline{0}, \overline{3}, \overline{4}, \overline{6}, \overline{8}, \overline{9}\right\}\right), \left(\mathfrak{z}_3; 4Z; \left\{\overline{0}, \overline{6}\right\}\right)\right\}.$$

Hence, we can observe that $G_{tb-1}(\mathfrak{z}_1) \cup G_{tb-2}(\mathfrak{z}_2) = \left\{\overline{0}, \overline{3}, \overline{4}, \overline{6}, \overline{8}, \overline{9}\right\}$ *is not a subring of* Z_{12}. *Hence, extended intersections need not be TBSR.*

3.5. Restricted Union of TBSRs

In this part, we have to discuss the basic definition of a restricted union for TBSRs. Moreover, we have to elaborate on the theorems and some remarks related to this theory.

Definition 13. *Let* $\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right)$ *and* $(F_{tb-2}, G_{tb-2}, \mathfrak{B})$ *be two TBSRs. Then, restricted union of two TBSRs is denoted and defined by*

$$\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right) \cup_{res.} (F_{tb-2}, G_{tb-2}, \mathfrak{B}) = \{\mathfrak{z}, F_{tb-1}(\mathfrak{z}) \cup F_{tb-2}(\mathfrak{z}), G_{tb-1}(\mathfrak{z}) \cap G_{tb-2}(\mathfrak{z}) \text{ for all } \mathfrak{z} \in \tilde{A} \cap \mathfrak{B}\}$$

Theorem 5. *The restricted union of two TBSRs is a TBSR if* $F_{tb-1}(\mathfrak{z})$ *is a subring of* $F_{tb-2}(\mathfrak{z})$ *or* $F_{tb-2}(\mathfrak{z})$ *is a subring of* $F_{tb-1}(\mathfrak{z})$ *for all* $\mathfrak{z} \in \tilde{A} \cap \mathfrak{B}$.

Proof.
Case 1: Assume that $F_{tb-1}(\mathfrak{z})$ is a subring of $F_{tb-2}(\mathfrak{z})$ or $F_{tb-2}(\mathfrak{z})$ is a subring of $F_{tb-1}(\mathfrak{z})$ for all $\mathfrak{z} \in \tilde{A} \cap \mathfrak{B}$, then in both cases, $F_{tb-1}(\mathfrak{z}) \cup F_{tb-2}(\mathfrak{z})$ is a subring for all $\mathfrak{z} \in \tilde{A} \cap \mathfrak{B}$.
Case 2: We can easily observe that $G_{tb-1}(\mathfrak{z}) \cap G_{tb-2}(\mathfrak{z})$ is a subring because the intersection of any number of subrings is always a subring. Hence, in either case, the restricted union of two TBSRs is a TBSR. □

Example 11. *Let* $X = Z$ *and* $\text{ɣ} = Z_{12}$ *be two distinct rings and* $U = X \cup \text{ɣ}$. *Also, assume that* $\tilde{A} = \{\mathfrak{z}_1, \mathfrak{z}_2\}, \mathfrak{B} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3\}$.

Now define

$$F_{tb-1}(\mathfrak{z}_1) = 2Z, F_{tb-1}(\mathfrak{z}_2) = 6Z$$
$$\text{And } F_{tb-2}(\mathfrak{z}_1) = Z, F_{tb-2}(\mathfrak{z}_2) = 3Z \text{ and } F_{tb-2}(\mathfrak{z}_3) = 4Z.$$

Also, define

$$G_{tb-1}(\mathfrak{z}_1) = \left\{\overline{0}, \overline{4}, \overline{8}\right\}, G_{tb-1}(\mathfrak{z}_2) = \left\{\overline{0}, \overline{6}\right\} \text{ and}$$
$$G_{tb-2}(\mathfrak{z}_1) = \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\right\}, G_{tb-2}(\mathfrak{z}_2) = \left\{\overline{0}, \overline{3}, \overline{6}, \overline{9}\right\} \text{ and } G_{tb-2}(\mathfrak{z}_3) = \left\{\overline{0}\right\}$$

Now as

$$\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right) \cup_{res.} (F_{tb-2}, G_{tb-2}, \mathfrak{B})$$
$$= \{\mathfrak{z}, F_{tb-1}(\mathfrak{z}) \cup F_{tb-2}(\mathfrak{z}), G_{tb-1}(\mathfrak{z}) \cap G_{tb-2}(\mathfrak{z}) \text{ for all } \mathfrak{z} \in \tilde{A} \cap \mathfrak{B}\}$$

So from the above observation, we can see that $\mathfrak{z}_1, \mathfrak{z}_2 \in \tilde{A} \cap \mathfrak{B}$
Hence,

$$F_{tb-1}(\mathfrak{z}_1) \cup F_{tb-2}(\mathfrak{z}_1) = Z; F_{tb-1}(\mathfrak{z}_2) \cup F_{tb-2}(\mathfrak{z}_2) = 3Z$$

And

$$G_{tb-1}(\mathfrak{z}_1) \cap G_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{4}, \bar{8}\right\}; G_{tb-1}(\mathfrak{z}_2) \cap G_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{6}\right\}$$

Hence, for all $\mathfrak{z}_1, \mathfrak{z}_2 \in \tilde{A} \cap \mathfrak{B}$ *, the restricted union is given by*

$$\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right) \cup_{res.} (F_{tb-2}, G_{tb-2}, \mathfrak{B}) = \left\{\left(\mathfrak{z}_1, Z, \left\{\bar{0}, \bar{4}, \bar{8}\right\}\right), \left(\mathfrak{z}_2, 3Z, \left\{\bar{0}, \bar{6}\right\}\right)\right\}$$

Remark 6. *The restricted union of two TBSRs need not be a TBSR in general.*

Example 12. *Let* $X = Z$ *and* $\text{ɣ} = Z_{12}$ *be two distinct rings* $U = X \cup \text{ɣ}$*. Also, assume that* $\tilde{A} = \{\mathfrak{z}_1, \mathfrak{z}_2\}, \mathfrak{B} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3\}$.

Now define

$$F_{tb-1}(\mathfrak{z}_1) = 2Z, F_{tb-1}(\mathfrak{z}_2) = 3Z$$
$$\text{And } F_{tb-2}(\mathfrak{z}_1) = Z, F_{tb-2}(\mathfrak{z}_2) = 2Z \text{ and } F_{tb-2}(\mathfrak{z}_3) = 6Z.$$

Also, define

$$G_{tb-1}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{4}, \bar{8}\right\}, G_{tb-1}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{6}\right\} \text{ and}$$
$$G_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\right\}, G_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{3}, \bar{6}, \bar{9}\right\} \text{ and } G_{tb-2}(\mathfrak{z}_3) = \left\{\bar{0}, \bar{4}, \bar{8}\right\}$$

Now

$$F_{tb-1}(\mathfrak{z}_1) \cup F_{tb-2}(\mathfrak{z}_1) = Z; F_{tb-1}(\mathfrak{z}_2) \cup F_{tb-2}(\mathfrak{z}_2) = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \ldots\};$$

Also,

$$G_{tb-1}(\mathfrak{z}_1) \cap G_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{4}, \bar{8}\right\}; G_{tb-1}(\mathfrak{z}_2) \cap G_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{6}\right\}$$
$$\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right) \cup_{res.} (F_{tb-2}, G_{tb-2}, \mathfrak{B}) = \left\{\left(\mathfrak{z}_1; Z; \left\{\bar{0}, \bar{4}, \bar{8}\right\}\right), \left(\mathfrak{z}_2; \{0, \pm 2, \pm 3, \pm 4, \pm 6, \ldots.\}, \left\{\bar{0}, \bar{6}\right\}\right)\right\}.$$

Hence, we can observe that $F_{tb-1}(\mathfrak{z}_1) \cup F_{tb-2}(\mathfrak{z}_2) = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \ldots.\}$ *is not a subring of* Z*. Hence, restricted unions need not be TBSR.*

3.6. The Restricted Intersection of TBSRs

In this part, we have to discuss the basic definition of a restricted intersection for TBSRs. Moreover, we have to elaborate on the theorems and some remarks related to this theory.

Definition 14. *Let* $\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right)$ *and* $(F_{tb-2}, G_{tb-2}, \mathfrak{B})$ *be two TBSRs. Then, restricted intersection of two TBSRs is denoted and defined by*

$$\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right) \cap_{res.} (F_{tb-2}, G_{tb-2}, \mathfrak{B}) = \left\{(\mathfrak{z}, F_{tb-1}(\mathfrak{z}) \cap F_{tb-2}(\mathfrak{z}), G_{tb-1}(\mathfrak{z}) \cup G_{tb-2}(\mathfrak{z})) \text{ for all } \mathfrak{z} \in \tilde{A} \cap \mathfrak{B}\right\}$$

Theorem 6. *The restricted intersection of two TBSRs is a TBSR if* $G_{tb-1}(\mathfrak{z})$ *is a subring of* $G_{tb-2}(\mathfrak{z})$ *or* $G_{tb-2}(\mathfrak{z})$ *is a subring of* $G_{tb-1}(\mathfrak{z})$ *for all* $\mathfrak{z} \in \tilde{A} \cup \mathfrak{B}$.

Proof. Same as above. □

Example 13. *Let* $\mathrm{X} = \mathrm{Z}$ *and* $\mathrm{Y} = \mathrm{Z}_{12}$ *be two distinct rings and* $\mathrm{U} = \mathrm{X} \cup \mathrm{Y}$. *Also, assume that* $\tilde{\mathrm{A}} = \{\mathfrak{z}_1, \mathfrak{z}_2\}, \mathfrak{B} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3\}$.

Now define

$$\begin{aligned} &\mathrm{F}_{tb-1}(\mathfrak{z}_1) = 2\mathrm{Z}, \mathrm{F}_{tb-1}(\mathfrak{z}_2) = 6\mathrm{Z} \\ &\text{And } \mathrm{F}_{tb-2}(\mathfrak{z}_1) = \mathrm{Z}, \mathrm{F}_{tb-2}(\mathfrak{z}_2) = 3\mathrm{Z} \text{ and } \mathrm{F}_{tb-2}(\mathfrak{z}_3) = 4\mathrm{Z}. \end{aligned}$$

Also, define

$$\mathrm{G}_{tb-1}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{4}, \bar{8}\right\}, \mathrm{G}_{tb-1}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{6}\right\} \text{ and}$$

$$\mathrm{G}_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\right\}, \mathrm{G}_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{3}, \bar{6}, \bar{9}\right\} \text{ and } \mathrm{G}_{tb-2}(\mathfrak{z}_3) = \mathrm{Z}_{12}$$

Now as

$$\begin{aligned} &\left(\mathrm{F}_{tb-1}, \mathrm{G}_{tb-1}, \tilde{\mathrm{A}}\right) \cap_{\text{res.}} \left(\mathrm{F}_{tb-2}, \mathrm{G}_{tb-2}, \mathfrak{B}\right) \\ &\qquad = \left\{(\mathfrak{z}, \mathrm{F}_{tb-1}(\mathfrak{z}) \cap \mathrm{F}_{tb-2}(\mathfrak{z}), \mathrm{G}_{tb-1}(\mathfrak{z}) \cup \mathrm{G}_{tb-2}(\mathfrak{z})) \text{ for all } \mathfrak{z} \in \tilde{\mathrm{A}} \cap \mathfrak{B}\right\} \end{aligned}$$

So from the above observation, we can see that $\mathfrak{z}_1, \mathfrak{z}_2 \in \tilde{\mathrm{A}} \cap \mathfrak{B}$. *So*

$$\mathrm{F}_{tb-1}(\mathfrak{z}_1) \cap \mathrm{F}_{tb-2}(\mathfrak{z}_1) = 2\mathrm{Z}; \mathrm{F}_{tb-3}(\mathfrak{z}_2) = \mathrm{F}_{tb-1}(\mathfrak{z}_2) \cap \mathrm{F}_{tb-2}(\mathfrak{z}_2) = 6\mathrm{Z}$$

And

$$\mathrm{G}_{tb-1}(\mathfrak{z}_1) \cup \mathrm{G}_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\right\}; \mathrm{G}_{tb-1}(\mathfrak{z}_2) \cup \mathrm{G}_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{3}, \bar{6}, \bar{9}\right\}$$

Hence,

$$\left(\mathrm{F}_{tb-1}, \mathrm{G}_{tb-1}, \tilde{\mathrm{A}}\right) \cap_{\text{res.}} \left(\mathrm{F}_{tb-2}, \mathrm{G}_{tb-2}, \mathfrak{B}\right) = \left\{\left(\mathfrak{z}_1, 2\mathrm{Z}, \left\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\right\}\right), \left(\mathfrak{z}_2, 6\mathrm{Z}, \left\{\bar{0}, \bar{3}, \bar{6}, \bar{9}\right\}\right)\right\}$$

Remark 7. *The restricted intersection of two TBSRs need not be a TBSR in general.*

Example 14. *Let* $\mathrm{X} = \mathrm{Z}$ *and* $\mathrm{Y} = \mathrm{Z}_{12}$ *be two distinct rings and* $\mathrm{U} = \mathrm{X} \cup \mathrm{Y}$. *Also, assume that* $\tilde{\mathrm{A}} = \{\mathfrak{z}_1, \mathfrak{z}_2\}, \mathfrak{B} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3\}$.

Now define

$$\begin{aligned} &\mathrm{F}_{tb-1}(\mathfrak{z}_1) = 2\mathrm{Z}, \mathrm{F}_{tb-1}(\mathfrak{z}_2) = 6\mathrm{Z} \\ &\text{And } \mathrm{F}_{tb-2}(\mathfrak{z}_1) = \mathrm{Z}, \mathrm{F}_{tb-2}(\mathfrak{z}_2) = 3\mathrm{Z} \text{ and } \mathrm{F}_{tb-2}(\mathfrak{z}_3) = 4\mathrm{Z}. \end{aligned}$$

Also, define

$$\mathrm{G}_{tb-1}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{4}, \bar{8}\right\}, \mathrm{G}_{tb-1}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{3}, \bar{6}, \bar{9}\right\} \text{ and}$$

$$\mathrm{G}_{tb-2}(\mathfrak{z}_1) = \left\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\right\}, \mathrm{G}_{tb-2}(\mathfrak{z}_2) = \left\{\bar{0}, \bar{4}, \bar{8}\right\} \text{ and } \mathrm{G}_{tb-2}(\mathfrak{z}_3) = \left\{\bar{0}, \bar{6}\right\}$$

Now

$$\mathrm{F}_{tb-1}(\mathfrak{z}_1) \cap \mathrm{F}_{tb-2}(\mathfrak{z}_1) = 2\mathrm{Z}; \mathrm{F}_{tb-3}(\mathfrak{z}_2) = \mathrm{F}_{tb-1}(\mathfrak{z}_2) \cap \mathrm{F}_{tb-2}(\mathfrak{z}_2) = 6\mathrm{Z}$$

Also,

$$G_{tb-1}(\mathfrak{z}_1) \cup G_{tb-2}(\mathfrak{z}_1) = \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\right\}; G_{tb-1}(\mathfrak{z}_2) \cup G_{tb-2}(\mathfrak{z}_2) = \left\{\overline{0}, \overline{3}, \overline{4}, \overline{6}, \overline{8}, \overline{9}\right\}$$

$$\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right) \cap_{res.} \left(F_{tb-2}, G_{tb-2}, \mathfrak{B}\right) = \left\{\left(\mathfrak{z}_1; 2\mathfrak{z}; \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\right\}\right), \left(\mathfrak{z}_2; 6\mathfrak{z}; \left\{\overline{0}, \overline{3}, \overline{4}, \overline{6}, \overline{8}, \overline{9}\right\}\right)\right\}.$$

Hence, we can observe that $G_{tb-1}(\mathfrak{z}_1) \cup G_{tb-2}(\mathfrak{z}_2) = \left\{\overline{0}, \overline{3}, \overline{4}, \overline{6}, \overline{8}, \overline{9}\right\}$ *is not a subring of* Z_{12}. *Hence, restricted intersections need not be TBSR.*

4. Decision-Making Approach Based on the Developed Theory

For the selection of optimal results, we have to define such kind of mathematical forms that can help us choose the best result from the set of given alternatives. For this, first of all, we will discuss the score function.

Definition 15. *Let* $\tilde{A} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3, \ldots, \mathfrak{z}_q\}$ *for* $1 \leq i \leq q$, $X = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_m\}$ *for* $1 \leq j \leq m$, $\gamma = \{\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n\}$ *for* $1 \leq k \leq n$ *and* $\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right)$ *be TBSRs. aThen, score function is given by*

$$\mathrm{Scor}_i = \mathfrak{O}_i - \mathfrak{G}_i$$

where $\mathfrak{O}_i = \sum_{j,k} \mathfrak{P}^{*}_{ijk}$ *and* $\mathfrak{G}_i = \sum_{j,k} \mathfrak{P}^{\boxdot}_{ijk}$.

Definition 16. *Let* $\tilde{A} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3, \ldots, \mathfrak{z}_q\}$ *for* $1 \leq i \leq q$, $X = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3, \ldots, m\}$ *for* $1 \leq j \leq m$, $\gamma = \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n\}$ *for* $1 \leq k \leq n$ *and* $\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right)$ *be TBSRs. Then,* $\mathfrak{z}_i (1 \leq i \leq q)$ *is called optimal if and only if* $\mathrm{Scor}_i > \mathrm{Scor}_{i^\circ}$ *for* $\left(i \neq i^\circ\right)$.

4.1. Algorithm 1

Let $\tilde{A} = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3, \ldots, \mathfrak{z}_q\}$ for $1 \leq i \leq q$ $X = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_m\}$ for $1 \leq j \leq m$ $\gamma = \{\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n\}$ for $1 \leq k \leq n$ and $\left(F_{tb-1}, G_{tb-1}, \tilde{A}\right)$ be TBSRs. The overall algorithm for choosing the optimal result is given by
Step 1: Collect the data in tabular form for TBSRs.
Step 2: Find out the score values $\mathrm{Scor}_1, \mathrm{Scor}_2, \mathrm{Scor}_3, \ldots, \mathrm{Scor}_q$.
Step 3: Find out the maximum score value as $\max_i \mathrm{Scor}_i = \mathrm{Scor}_w$
Step 4: Scor_w is the optimal value.

4.2. Numerical Example

An organization uses data mining as a method to find patterns in data. Both business intelligence and data science require it. An organization can utilize a variety of data mining approaches to transform raw information into actionable data. Different kinds of data mining techniques can help an organization transfer raw material into actionable data. These techniques are given as follows:

1. **Data cleaning and preparation**

Data preparation and cleansing are important processes in the data mining process. They entail converting unprocessed data into an analytically friendly format. Accurate and relevant data mining outcomes depend on high-quality data. Here, some essential methods and procedures for data preparation and cleaning in data mining are presented: (1) data collection; (2) data integration; (3) data cleaning; (4) data reduction; (5) data sampling, etc.

2. **Machine learning and artificial learning**

Data mining relies heavily on machine learning and artificial intelligence (AI) to extract useful patterns and insights from massive databases. In conclusion, machine learning and artificial intelligence are essential to the data mining approach because they offer the tools and methods required to identify patterns, forecast outcomes, and derive insightful information from sizable and complicated datasets. These technologies are still developing and are essential for data-driven decision-making in many different fields.

3. **Data warehousing**

In the area of handling information and data analysis, data warehousing and data mining are two ideas that are closely connected, although they have diverse uses within the data lifecycle. Data warehousing provides an organized and controlled collection of data as the basis for data mining, which then pulls useful knowledge and patterns from these data to help rational decision-making. They work well together to maximize the value of data assets in organizations.

4. **Statistical technique**

Data mining, which is the act of identifying patterns, correlations, and meaningful data derived from massive datasets, heavily relies on statistical approaches. These methods assist data analysts in deriving insightful findings from data and taking sensible actions. Here, some essential statistical methods used in data mining are presented: (1) descriptive statistics; (2) inferential statistics; (3) regression analysis; (4) sampling techniques; (5) Bayesian statistics, etc.

Assume that an organization "W" wants to select the best data mining technique from the set $\{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3, \mathfrak{z}_4\}$ where

$\mathfrak{z}_1$ = Data cleaning and preparation.
$\mathfrak{z}_2$ = Machine learning and artificial intelligence.
$\mathfrak{z}_3$ = Data warehousing.
$\mathfrak{z}_4$ = Statistical technique.

Assume that the expert provides his assessment in the form of TBSR is

$$(\mathrm{F_{tb}}, \mathrm{G_{tb}}, \tilde{\mathrm{A}}) = \left\{ \left(\mathfrak{z}_1, \left\{ \bar{0} \right\}, \left\{ \bar{0}, \bar{4} \right\} \right), \left(\mathfrak{z}_2, \left\{ \bar{0}, \bar{3} \right\}, \left\{ \bar{0} \right\} \right), \left(\mathfrak{z}_3, \left\{ \bar{0}, \bar{2}, \bar{4} \right\}, \left\{ \bar{0}, \bar{2}, \bar{4}, \bar{6} \right\} \right), (\mathfrak{z}_4, Z_6, Z_8) \right\} \quad (1)$$

as given in Example 1, and the numerical form of this expression is given in Table 1.

Now we can use the above algorithm 1 for the optimal result.

Step 1: The tabular form of the above data given in Equation (1) is summarized in Table 1.
Step 2: Now we find out the score values, and their result are given in Table 2.
Step 3: Find out the maximum score value as $\max_i \mathrm{Scor_i} = \mathrm{Scor_w} = 42$.
Step 4:$\mathfrak{z}_2$ = Machine learning and artificial intelligence is the optimal value.

Table 2. Score values.

$(\mathrm{F_{tb}}, \mathrm{G_{tb}}, \tilde{\mathrm{A}})$	$\mathfrak{O}_i$	$\mathfrak{G}_i$	$\mathrm{Scor_i} = \mathfrak{O}_i - \mathfrak{G}_i$
$\mathfrak{z}_1$	8	12	−4
$\mathfrak{z}_2$	48	6	42
$\mathfrak{z}_3$	24	24	0
$\mathfrak{z}_4$	16	48	−32

5. Discussion of Results

This section of the article is related to the novelty and significance of the results. Moreover, we have compared our work with existing notions to elaborate on the purpose of the delivered notions. The overall discussion is given as follows:

1. From the analysis of Table 1, we can notice that the data given in Table 1 are based on TBSRs. The notion of TBSR is a combination of TBSS and ring structure. We can notice that the idea of TBSS is closer to bipolarity than that of the existing notions introduced by Shabir and Naz [27] and Karaaslan [28]. Moreover, we can observe that TBSSs have the ability to discuss the two-sided aspects of a certain situation. So based on TBSS and ring theory, we have defined the notion of TBSR, and due to this reason, the delivered approach is superior.
2. We can notice that TBSR is based on two-sided aspects of a certain situation, like the effects and side effects of medicine. This means that when a decision-maker wants

to handle such kind of information, then we can say that TBSR is the only structure that can discuss it compared to soft rings [17,18], which can handle only one aspect. As the data given in Table 1 are based on TBSRs, we can observe from Table 3 that no results were found corresponding to soft rings [17,18] because these notions fail to handle such kinds of situations. The overall results are given in Tables 3 and 4.

3. When we analyze the structural properties of TBSRs and soft rings, we can see that TBSR is a more advanced structure than soft rings, and it provides more space for decision-makers. Decision-makers can take their data in the more advanced form of TBSRs. This means that we can say that there are fewer chances of a loss of data when we consider the TBSRs as compared to the notion of soft rings. Moreover characteristic analysis of these existing notions with delivered theory is given in Table 4.

Table 3. The overall comparison analysis of the developed approach.

Methods	Score Values	Ranking Results
Acar et al. [17] method	Cannot handle data	No result
Celik et al. [18] method	Cannot handle data	No result
Proposed work	$Scor_{\mathfrak{z}_1} = -4$, $Scor_{\mathfrak{z}_2} = 42$, $Scor_{\mathfrak{z}_3} = 0$ $Scor_{\mathfrak{z}_4} = -32$	$Scor_{\mathfrak{z}_2} > Scor_{\mathfrak{z}_3} > Scor_{\mathfrak{z}_1} > Scor_{\mathfrak{z}_4}$

Table 4. Characteristic analysis.

Methods	Consideration of Soft Structure	Consideration of T-Bipolar Soft Structure
Acar et al. [17] method	Yes	No
Celik et al. [18] method	Yes	No
Proposed work	Yes	Yes

6. Conclusions

T-BSS can consider two-sided aspects as well as parameterization tools that rank this structure as more advanced than that of the soft set. When the data are given in T-BSRs, it means that the data contain two-sided aspects of any situation. This means that the data cannot be handled by the structure of a soft ring. We can observe that the data given in Table 1 are based on T-BSRs and cannot be handled by soft rings because two-sided aspects have been used in the structure of T-BSRs. To handle such a situation in this article, we utilized the TBSS notion to the algebraic structure of rings to find out the notion of TBSR. We have proposed the definition of AND product and OR product for two TBSRs. We have delivered the notion of the extended intersection for two TBSRs. Additionally, we have discussed the notion of an extended union of two TBSRs. The notions of restricted union and restricted intersection have been elaborated in this article. Moreover, we have proved that some theorems relate to these ideas. The main effect of this developed theory is that whenever someone faces a decision-making situation where two-sided aspects are given, then the notion of T-BSR can be utilized to solve such decision-making problems. For this purpose, a decision-making algorithm is established to discuss the applications of these developed notions for the classification of data mining techniques. We can observe from Table 3 that the developed algorithm works, and we can utilize this algorithm to classify the data mining technique. The overall results are given by $Scor_{\mathfrak{z}_1} = -4, Scor_{\mathfrak{z}_2} = 42, Scor_{\mathfrak{z}_3} = 0$, and $Scor_{\mathfrak{z}_4} = -32$, and their ranking is given as $Scor_{\mathfrak{z}_2} > Scor_{\mathfrak{z}_3} > Scor_{\mathfrak{z}_1} > Scor_{\mathfrak{z}_4}$. So we can see that $\mathfrak{z}_2$ is the best alternative. This means that the notion of T-BSR can be utilized in decision-making scenarios as an application part of developed ideas.

In the future, we can extend these developed notions to the rough set theory [32,33], the theory of modules, and the ideals theory. Moreover, some more algebraic structures can be explored under the environment of delivered notions.

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