symmetry

MDPI

Article

Applying Generalized Type-II Hybrid Censored Samples on Generalized and q-Generalized Extreme Value Distributions under Linear Normalization

Rasha Abd El-Wahab Attwa [1,†] and Taha Radwan [2,3,*,†]

[1] Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt; Rasha_atwa75@hotmail.com
[2] Department of Management Information Systems, College of Business Administration, Qassim University, Buraydah 51452, Saudi Arabia
[3] Department of Mathematics and Statistics, Faculty of Management Technology and Information Systems, Port Said University, Port Said 42511, Egypt
* Correspondence: t.radwan@qu.edu.sa
† These authors contributed equally to this work.

Abstract: The generalized extreme value (GEV) distributions have wide applications for describing a variety of random events, such as those that occur during specific survival, financial, or reliability investigations. Also, the q-analogues of GEV distributions, called (q-GEVs), are characterized by their ability to provide more flexibility for modeling, which is due to the influence of the q parameter. In this study, we estimated the parameters of generalized and q-generalized extreme value distributions under linear normalization, called GEVL and q-GEVL, respectively. These parameters were estimated using the maximum likelihood estimator method and are based on the generalized type-II hybrid censored sample (G-Type-II HCS). The confidence intervals for these parameters were evaluated. Also, Shannon entropy was estimated for GEVL and q-GEVL distributions. The accuracy of these parameters and the performance of estimators were demonstrated through a real-life example and a simulation study.

Keywords: mathematical model; statistical model; GEVL; q-GEVP; MLE; confidence interval; entropy

Citation: Attwa, R.A.E.-W.; Radwan, T. Applying Generalized Type-II Hybrid Censored Samples on Generalized and q-Generalized Extreme Value Distributions under Linear Normalization. *Symmetry* **2023**, *15*, 1869. https://doi.org/10.3390/sym15101869

Academic Editors: Arne Johannssen, Nataliya Chukhrova and Quanxin Zhu

Received: 10 August 2023
Revised: 27 September 2023
Accepted: 29 September 2023
Published: 5 October 2023

1. Introduction

Asymmetrical models, such as Gumbel, logistic, Weibull, and generalized extreme value (GEV) distributions, have been widely used to describe a variety of random events, such as those that may arise during specific survival, financial, or reliability investigations. The Gumbel probability distribution is used to analyze and model the behavior of random phenomena in many fields, such as engineering, business, biology, management, sports, and economics [1]. We can find many examples of the Gumbel probability distribution, also known as the double exponential probability distribution, in [2–7].

In order to boost the flexibility of modeling, Provost et al. [8] created q-analogues of the generalized extreme value (q-GEVs) and Gumbel distributions.

A hybrid censoring technique, which combines Type I and Type II censoring schemes, has been proposed for adjustable efficiency levels or termination times [9]. In order to clarify the concept of the censored sample (G-Type-II HCS), we propose the following experiment.

Consider a life-testing experiment that begins with ν identical units undergoing a lifetime test. Let $x_1, x_2, \ldots, x_\nu$ be the results of lifetimes from distributions with CDF $(F(x))$ and PDF $(f(x))$. Let $a \in 1, 2, \ldots, \nu$ be an integer an integer and suppose $T_1 < T_2 \in (0, \infty)$ are time points. We have three cases as follows:

1. If the a^{th} failure occurs before the time point T_1, the experiment will be terminated at this time.

2. If the a^{th} failure occurs between the time points T_1 and T_2, then the experiment will be terminated at the time of the failure, x_a.
3. If the a^{th} failure occurs after the time point T_2, the experiment will be terminated at time point T_2.

This type of censoring, while aiming for a minimum number of failures, a, guarantees that the experiment will be completed by time T_2. Therefore, T_2, known as the absolute maximum time of the experiment, is not exceeded [10].

The maximum time for the experiment is fixed using the G-Type-II HCS is T_2, and this is an advantage from an experimental point of view. One of the following cases are observed using G-Type-II hybrid censoring sample:

Case I: $\{x_{1:\nu} < x_{2:\nu} < \ldots < x_{a:\nu} < \ldots < x_{\nu_1} \leq T_1\}$, where $x_{a:\nu} < T_1$,
Case II: $\{x_{1:\nu} < x_{2:\nu} < \ldots < T_1 < \ldots < x_{a:\nu} \leq T_2\}$, where $T_1 < x_{a:\nu} < T_2$,
Case III: $\{x_{1;\nu} < x_{2:\nu} < \ldots < T_{1:\nu} < \ldots < x_{\nu_2} \leq T_2\}$, where $x_{a:\nu} > T_2$.

Note that ν_1 and ν_2 are the number of observed failures up to time points T_1 and T_2, respectively. Then, for the G-Type-II HCS, the likelihood functions for the three different cases described above are as follows:

Case I

$$\frac{\nu!}{(\nu-\nu_1)!}\prod_{i=1}^{\nu_1} f(x_{i:\nu})[S(T_1)]^{\nu-\nu_1}; \nu_1 = a, a+1, \ldots, \ or\ \nu,$$

Case II

$$\frac{\nu!}{(\nu-a)!}\prod_{i=1}^{a} f(x_{i:r})[S(x_a)]^{\nu-a},$$

Case III

$$\frac{\nu!}{(\nu-\nu_2)!}\prod_{i=1}^{n_2} f(x_{i:\nu})[S(T_2)]^{\nu-\nu_2}; \nu_2 = 0, 1, 2, \ldots, or (a-1),$$

where $S(x)$ is the survival function.

Entropy was initially developed by Clausius et al. [11] in the context of information theory. He created a new route for the advancement of thermodynamics by using the idea of entropy to represent the second rule of thermodynamics quantitatively. This notion was continued by Shannon [12], and ever since then it has been used in a variety of domains, including economics and image and signal processing. On entropy estimation for various distributions, several papers have been provided. The entropy of the Weibull distribution with progressive censoring was studied by Naif and Malyk [13]. The entropy of the Rayleigh distribution based on the doubly generalized G-Type-II HCS was evaluated by Cho et al. [14]. Cho et al. [15] estimated the entropy of Weibull distribution using a generalized progressively censored sample. Ahmad [16] constructed estimators for entropy function of the Fréchet distribution based on the extended type I hybrid censored samples. The estimators for entropy function of the Lomax distribution with extended type I hybrid censored samples were developed by Mahmoud et al. [17].

In this study, we constructed maximum likelihood estimation to evaluate the parameters of the family of GEVL and q-GEVL distributions using the G-Type-II HCS scheme, to ensure applicability to Shannon entropy. Also, the confidence intervals for the parameters of GEVL and q-GEVL distributions were determined. Section 2 presents the GEVL and q-GEVL distributions and their respective entropy functions. The purpose of this section is to identify these distributions and to provide a detailed description of their entropy functions. In Section 3, we obtain the maximum likelihood estimation for the parameters of GEVL based on the G-Type-II HCS scheme. Also, the simulation of this procedure and calculation of the Shannon entropy are described. In Section 4, we evaluate the maximum likelihood estimation for the parameters of q-GEVL based on the G-Type-II HCS scheme. Also, the simulation of this procedure and calculation of the Shannon entropy are described. In Section 5, the confidence intervals for the parameters of GEVL and q-GEVL are determined. After that, the Conclusion Section (Section 6) is presented.

2. The Family of GEVL and q-GEVL Distributions

The limit of the cumulative density function (CDF) L_ϕ is described by the extremal types theorem as having one type of three types [18]. The three types, which are together grouped in the family below, are frequently referred to as the Gumbel, Fréchet, and Weibull types:

$$L_\phi(x;\alpha,\beta,\phi) = \begin{cases} \exp\{-(1+\phi(\frac{x-\alpha}{\beta}))^{\frac{-1}{\phi}}\}, & \phi \neq 0, \\ \exp\{-\exp(-\frac{x-\alpha}{\beta})\}, & \phi \to 0, \end{cases} \tag{1}$$

and the probability density function (PDF) l_ϕ can be given by:

$$l_\phi(x;\alpha,\beta,\phi) = \begin{cases} \frac{1}{\beta}\exp\{-(1+\phi(\frac{x-\alpha}{\beta}))^{\frac{-1}{\phi}}\} \times (1+\phi(\frac{x-\alpha}{\beta}))^{\frac{-1}{\phi}-1}, & \phi \neq 0, \\ \frac{1}{\beta}\exp\{-\exp(-(\frac{x-\alpha}{\beta}))\} \times \exp(-(\frac{x-\alpha}{\beta})), & \phi \to 0, \end{cases} \tag{2}$$

where α is a location parameter, β is a positive scale parameter, ϕ is the shape parameter, and the values of x are defined by:

$$x \in \begin{cases} (\alpha - \frac{\beta}{\phi}, \infty), & \phi > 0, \\ (-\infty, \infty), & \phi \to 0, \\ (-\infty, \alpha - \frac{\beta}{\phi}), & \phi < 0. \end{cases}$$

The distribution in Equation (1) is known as a generalized extreme value (GEV) distribution under linear normalization. We denote it by $GEVL(x;\alpha,\beta,\phi)$. The Gumbel probability distribution in Equations (1) and (2) as $\phi \to 0$ is used to analyze and model the behavior of random phenomena in many fields. Bashir et al. [19] examined and contrasted three estimation methods used to approximate the parameter values for simulated observations taken from the GEVL distribution. Figure 1 refers to the cumulative distribution and density function of GEVL distribution for $\phi \to 0$.

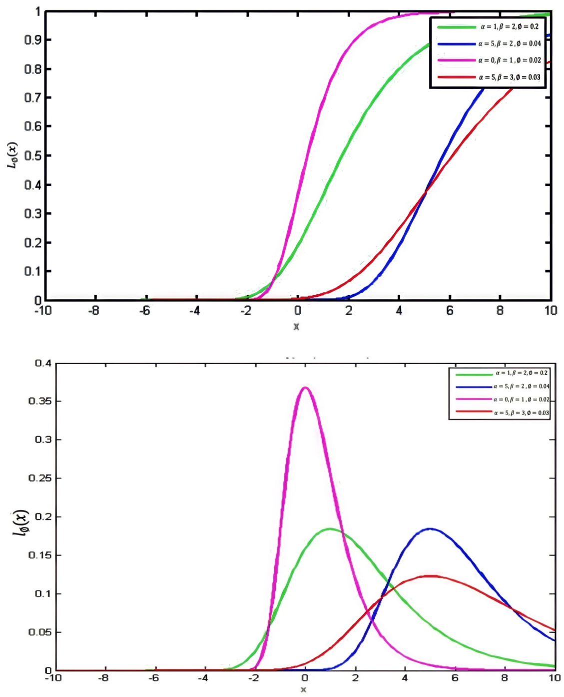

Figure 1. The cumulative distribution and density function of GEVL distribution for $\phi \to 0$.

Provost et al. [8] proposed the q-GEVL distribution and q-Gumbel distribution (obtained by letting $\phi \to 0$ in the q-GEVL model), and the corresponding distributions are provided by:

$$Ł(x;\alpha,\beta,\phi,q) = \begin{cases} [1+q(\phi(\frac{x-\alpha}{\beta})+1)^{-\frac{1}{\phi}}]^{-\frac{1}{q}}, & \phi \neq 0,\ q \neq 0 \\ (1+qe^{-(\frac{x-\alpha}{\beta})})^{-\frac{1}{q}}, & \phi \to 0,\ q \neq 0 \end{cases} \tag{3}$$

and

$$ł(x;\alpha,\beta,\phi,q) = \begin{cases} \frac{1}{\beta}(1+\phi(\frac{x-\alpha}{\beta}))^{\frac{-1}{\phi}-1} \times [1+q(\phi(\frac{x-\alpha}{\beta})+1)^{-\frac{1}{\phi}}]^{-\frac{1}{q}-1}, & \phi \neq 0,\ q \neq 0 \\ (1+qe^{-(\frac{x-\alpha}{\beta})})^{-\frac{1}{q}-1}\frac{1}{\beta}e^{-(\frac{x-\alpha}{\beta})}, & \phi \to 0,\ q \neq 0, \end{cases} \tag{4}$$

where the values of x can be determined by:

$$x \in \begin{cases} (-\infty,\infty) & \phi \to 0, \quad q > 0, \\ (\frac{\frac{\alpha}{\beta}+\ln(-q)}{\beta},\infty) & \phi \to 0, \quad q < 0. \end{cases}$$

Figure 2 refers to the cumulative distribution and density function of q-GEVL distribution for $\phi \to 0$.

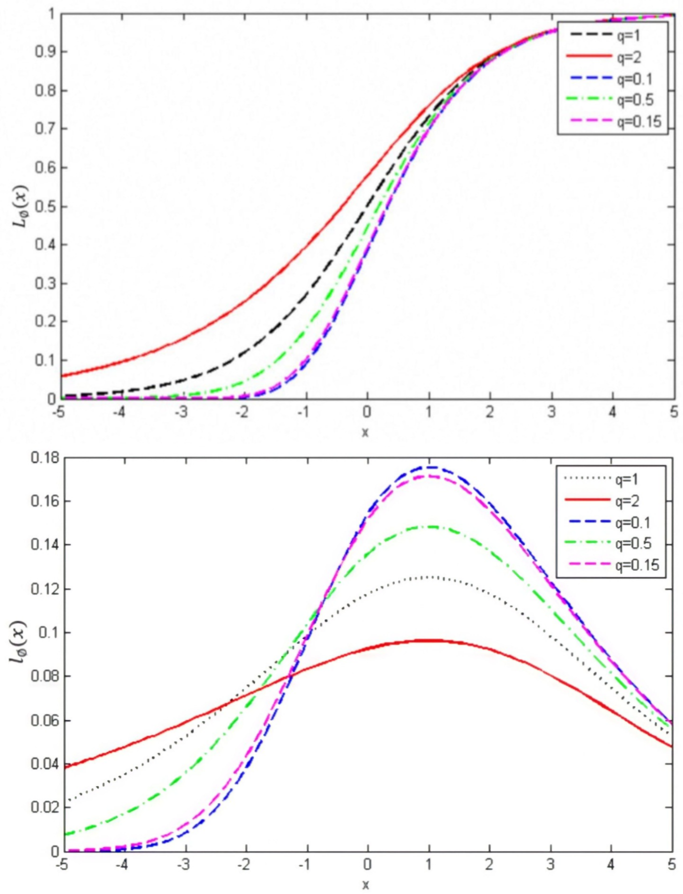

Figure 2. The cumulative distribution and density function of q-GEVL distribution for $\phi \to 0$.

The differential entropy is a measure of uncertainty and is defined as follows:

Let X be an absolutely continuous random variable with probability density function (PDF) $f(x)$. It is written as:

$$H(X) = -\int_S f(x) \log f(x) dx \tag{5}$$

The expectancy of a random variable $(-\log f(X))$ is a statistic that has recently gained the interest of investigators.

The Shannon entropy of GEVL family is well known as:

$$H(X) = \log \beta + (\phi + 1)\gamma + 1 \tag{6}$$

The Shannon entropy of each type in Equation (6) is evaluated by Ravi and Saeb [20]. On the other hand, Eliwa, et al. [21], evaluated the Shannon entropy of $q - GEVL$ family as follows:

$$H(X) = \log \beta + (\phi + 1)\gamma + (1 + q)[1 - \sum_{n=2}^{\infty} (-1)^{n+1} q^{n-1} \Gamma(n - 1)], \tag{7}$$

where γ is the Euler–Macheronic constant.

3. Maximum Likelihood Estimation for the Family of GEVL Based on G-Type-II HCS

3.1. Estimation of GEVL Parameters through G-Type-II HCS

In order to estimate the parameters of the family of generalized extreme distribution under linear normalization, whose cumulative function and density function as in Equations (1) and (2) is based on the G-Type-II HCS, we suppose that ν_1 and ν_2 denote the number of failures that occur by time points T_1 and T_2, respectively. Then, based on the three forms of the G-Type-II HCS, the likelihood function will take one of the following forms:

Case I

$$L_I(\alpha, \beta, \phi) = \frac{\nu!}{(\nu - \nu_1)!} \prod_{i=1}^{\nu_1} \frac{1}{\beta} e^{-[1+\phi \frac{x_i - \alpha}{\beta}]^{-\frac{1}{\phi}}} [1 + \phi \frac{x_i - \alpha}{\beta}]^{-\frac{1}{\phi} - 1} [1 - e^{-[1+\phi \frac{T_1 - \alpha}{\beta}]^{-\frac{1}{\phi}}}]^{\nu - \nu_1},$$

Case II

$$L_{II}(\alpha, \beta, \phi) = \frac{\nu!}{(\nu - a)!} \prod_{i=1}^{a} \frac{1}{\beta} e^{-[1+\phi \frac{x_i - \alpha}{\beta}]^{-\frac{1}{\phi}}} [1 + \phi \frac{x_i - \alpha}{\beta}]^{-\frac{1}{\phi} - 1} [1 - e^{-[1+\phi \frac{x_r - \alpha}{\beta}]^{-\frac{1}{\phi}}}]^{\nu - a},$$

Case III

$$L_{III}(\alpha, \beta, \phi) = \frac{\nu!}{(\nu - \nu_2)!} \prod_{i=1}^{\nu_2} \frac{1}{\beta} e^{-[1+\phi \frac{x_i - \alpha}{\beta}]^{-\frac{1}{\phi}}} [1 + \phi \frac{x_i - \alpha}{\beta}]^{-\frac{1}{\phi} - 1} [1 - e^{-[1+\phi \frac{T_2 - \alpha}{\beta}]^{-\frac{1}{\phi}}}]^{\nu - \nu_2}.$$

The log likelihood functions are:

Case I

$$\ell_I(\alpha, \beta, \phi) = E_1 - \nu_1 \log \beta - \sum_{i=1}^{\nu_1} [1 + \phi \frac{x_i - \alpha}{\beta}]^{-\frac{1}{\phi}} - (\frac{1}{\phi} + 1) \sum_{i=1}^{\nu_1} \log [1 + \phi \frac{x_i - \alpha}{\beta}]$$

$$+ (\nu - \nu_1) \log [1 - e^{-[1+\phi \frac{T_1 - \alpha}{\beta}]^{-\frac{1}{\phi}}}]],$$

Case II

$$\ell_{II}(\alpha, \beta, \phi) = E_2 - a \log \beta - \sum_{i=1}^{a} [1 + \phi \frac{x_i - \alpha}{\beta}]^{-\frac{1}{\phi}} - (\frac{1}{\phi} + 1) \sum_{i=1}^{r} \log [1 + \phi \frac{x_i - \alpha}{\beta}]$$

$$+(\nu - a)\log[1 - e^{-[1+\phi\frac{x_a-\alpha}{\beta}]^{-\frac{1}{\phi}}}]],$$

Case III

$$\ell_{III}(\alpha,\beta,\phi) = E_3 - \nu_2\log\beta - \sum_{i=1}^{\nu_2}[1+\phi\frac{x_i-\alpha}{\beta}]^{-\frac{1}{\phi}} - (\frac{1}{\phi}+1)\sum_{i=1}^{\nu_2}\log[1+\phi\frac{x_i-\alpha}{\beta}]$$

$$+(\nu - \nu_2)\log[1 - e^{-[1+\phi\frac{T_2-\alpha}{\beta}]^{-\frac{1}{\phi}}}]],$$

where E_1, E_2, and E_3 are normalizing constants that do not depend on the parameters. We can rewrite cases I, II, and III as a single formula as follows:

$$\ell(\alpha,\beta,\phi) = E - d\log\beta - \sum_{i=1}^{d}[1+\phi\frac{x_i-\alpha}{\beta}]^{-\frac{1}{\phi}} - (\frac{1}{\phi}+1)\sum_{i=1}^{d}\log[1+\phi\frac{x_i-\alpha}{\beta}]$$

$$-(\nu - d)\log[1 - e^{-[1+\phi\frac{s-\alpha}{\beta}]^{-\frac{1}{\phi}}}]],$$

where $E = E_1, d = \nu_1 \; and \; s = T_1$ for case I; $E = E_2, d = a \; and \; s = x_a$ for case II; and $E = E_3, d = \nu_2 \; and \; s = T_2$ for case III.

The corresponding log likelihood equations are:

$$\frac{\partial\ell}{\partial\alpha} = -\frac{1}{\beta}\sum_{i=1}^{d}A^{-(\frac{1}{\phi}+1)} + \sum_{i=1}^{d}\frac{\phi+1}{\beta}A1^{-1} + \frac{(n-d)A1^{-(\frac{1}{\phi}+1)}e^{A1^{-\frac{1}{\phi}}}}{\beta[1-e^{A1^{-\frac{1}{\phi}}}]} = 0$$

$$\frac{\partial\ell}{\partial\beta} = \frac{-d}{\beta} + \frac{\phi+1}{\beta^2}\sum_{i=1}^{d}\frac{(x_i-\alpha)}{A} - \frac{1}{\beta^2}\sum_{i=1}^{d}(x_i-\alpha)Y^{-(\frac{1}{\phi}+1)}\phi$$

$$-\frac{(n-d)(s-\alpha)A1^{-(\frac{1}{\phi}+1)}e^{-A1^{-\frac{1}{\phi}}}}{\beta^2(1-e^{A1^{-\frac{1}{\phi}}})} = 0$$

$$\frac{\partial\ell}{\partial\phi} = \sum_{i=1}^{d}\frac{A^{-\frac{1}{\phi}}\log(A)}{\phi^2} + \frac{1}{\beta\phi}\sum_{i=1}^{d}\frac{(x_i-\alpha)}{A^{-(\frac{1}{\phi})}} + \frac{1}{\phi^2}\sum_{i=1}^{d}\log A - \frac{1}{\beta}\sum_{i=1}^{d}\frac{(\frac{1}{\phi}+1)(x_i-\alpha)}{A}$$

$$-\frac{(\nu-d)A1^{-\frac{1}{\phi}}e^{-A1^{-\frac{1}{\phi}}}\log(A1)}{\phi^2(1-A1^{-\frac{1}{\phi}})} + \frac{(s-\alpha)A1^{-(\frac{1}{\phi}+1)}}{\beta\phi} = 0, \tag{8}$$

where

$$A = [1+\phi\frac{x_i-\alpha}{\beta}],$$

$$A1 = [1+\frac{\phi(s-\alpha)}{\beta}].$$

The systems specified by Equation (8) yields the maximum likelihood estimates for the parameters of the family of GEVL which follow the G-Type-II HCS. Since this equation cannot be solved analytically, the Newton–Raphson technique will be applied.

3.2. Simulation Study

A simulation study was used to demonstrate the performance of the estimators produced in the preceding section. We used the family of GEVL based on the G-Type-II HCS with $\alpha = 6$, $\beta = 1$ and $\phi = 0.5$ to simulate a small random sample of size $\nu = 20$:

$$4, 5.1555, 5.3180, 5.4521, 5.5765, 5.6986, 5.8227, 5.9520, 6.0894, 6.2382, 6.4022,$$

$$6.5867, 6.7983, 7.0472, 7.3488, 7.7288, 8.2339, 8.9611, 10.1616, 12.8308$$

We applied these data to the G-Type-II HCS by solving the nonlinear systems that are specified in Equation (8) and using the Newton–Raphson technique, and MATLAB (Version 2021) was used for estimation. Then, we used Equation (6) to calculate entropy. The maximum likelihood estimations (MLEs) of the parameters and entropy of the GEVL with the G-Type-II HCS were yielded by the proposed values of T_1, T_2, and a in each case as shown in Table 1.

Table 1. MLEs for α, β, ϕ and estimated entropy of GEVL with the G-Type-II HCS.

	T_1	T_2	a	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\phi}$	$\hat{H}$
Case I	6	7	3	6.1	1.1699	−0.442	1.479
Case II	6	8	9	5.946	0.694	−0.8091	0.7449
Case III	7	9	18	5.2156	0.9455	0.3014	1.6951

4. Maximum Likelihood Estimation for the Family of q-GEVL Based on G-Type-II HCS

4.1. Estimation of q-GEVL Parameters through G-Type-II HCS

In order to estimate the parameters of the family of q-GEVL, whose cumulative function and density function as in Equations (3) and (4) is based on the G-Type-II HCS, we used ν_1 and ν_2 denoted the number of failures that occurred by time points T_1 and T_2, respectively. Then, based on the three forms of the G-Type-II HCS, the likelihood function will take one of the following forms:

Case I

$$L_I(\alpha, \beta, \phi, q) = \frac{\nu!}{(\nu-\nu_1)!} \prod_{i=1}^{\nu_1} \frac{1}{\beta}(1+\phi(\frac{x_i-\alpha}{\beta}))^{\frac{-1}{\phi}-1} \times [1+q(\phi(\frac{x_i-\alpha}{\beta})+1)^{-\frac{1}{\phi}}]^{-\frac{1}{q}-1}$$

$$[1-\{1+q(\phi(\frac{T_1-\alpha}{\beta})+1)^{-\frac{1}{\phi}}\}^{-\frac{1}{q}}]^{\nu-\nu_1},$$

Case II

$$L_{II}(\alpha, \beta, \phi, q) = \frac{\nu!}{(\nu-a)!} \prod_{i=1}^{a} \frac{1}{\beta}(1+\phi(\frac{x_i-\alpha}{\beta}))^{\frac{-1}{\phi}-1} \times [1+q(\phi(\frac{x_i-\alpha}{\beta})+1)^{-\frac{1}{\phi}}]^{-\frac{1}{q}-1}$$

$$[1-\{1+q(\phi(\frac{x_a-\alpha}{\beta})+1)^{-\frac{1}{\phi}}\}^{-\frac{1}{q}}]^{\nu-a},$$

Case III

$$L_{III}(\alpha, \beta, \phi, q) = \frac{\nu!}{(\nu-\nu_2)!} \prod_{i=1}^{\nu_2} \frac{1}{\beta}(1+\phi(\frac{x_i-\alpha}{\beta}))^{\frac{-1}{\phi}-1} \times [1+q(\phi(\frac{x_i-\alpha}{\beta})+1)^{-\frac{1}{\phi}}]^{-\frac{1}{q}-1}$$

$$[1-\{1+q(\phi(\frac{T_2-\alpha}{\beta})+1)^{-\frac{1}{\phi}}\}^{-\frac{1}{q}}]^{\nu-\nu_2}.$$

The log likelihood functions are:

Case I

$$\ell_I(\alpha, \beta, \phi, q) = E_1 - \nu_1 \log\beta - (1+\frac{1}{\phi}) \sum_{i=1}^{\nu_1} \log(1+\phi(\frac{x_i-\alpha}{\beta})$$

$$-(\frac{1}{q}+1)\sum_{i=1}^{\nu_1}\log[1+q(1+\phi\frac{x_i-\alpha}{\beta})-\frac{1}{\phi}]$$
$$+(\nu-\nu_1)\log[1-[1+q(1+\phi\frac{T_1-\alpha}{\beta})-\frac{1}{\phi}]-\frac{1}{q}],$$

Case II

$$\ell_{II}(\alpha,\beta,\phi,q)=E_2-a\log\beta-(1+\frac{1}{\phi})\sum_{i=1}^{a}\log(1+\phi(\frac{x_i-\alpha}{\beta})$$
$$-(\frac{1}{q}+1)\sum_{i=1}^{a}\log[1+q(1+\phi\frac{x_i-\alpha}{\beta})-\frac{1}{\phi}]$$
$$+(\nu-a)\log[1-[1+q(1+\phi\frac{x_a-\alpha}{\beta})^{-}\frac{1}{\phi}]-\frac{1}{q}],$$

Case III

$$\ell_{III}(\alpha,\beta,\phi,q)=E_3-\nu_2\log\beta-(1+\frac{1}{\phi})\sum_{i=1}^{\nu_2}\log(1+\phi(\frac{x_i-\alpha}{\beta})$$
$$-(\frac{1}{q}+1)\sum_{i=1}^{\nu_2}\log[1+q(1+\phi\frac{x_i-\alpha}{\beta})-\frac{1}{\phi}]$$
$$+(\nu-\nu_2)\log[1-[1+q(1+\phi\frac{T_2-\alpha}{\beta})^{-}\frac{1}{\phi}]-\frac{1}{q}],$$

where E_1, E_2, and E_3 are normalizing constants that do not depend on the parameters. We can rewrite the cases I, II, and III as a single formula as follows:

$$\ell(\alpha,\beta,\phi,q)=E-d\log\beta-(1+\frac{1}{\phi})\sum_{i=1}^{d}\log(1+\phi(\frac{x_i-\alpha}{\beta})$$
$$-(\frac{1}{q}+1)\sum_{i=1}^{d}\log[1+q(1+\phi\frac{x_i-\alpha}{\beta})-\frac{1}{\phi}]$$
$$+(\nu-d)\log[1-[1+q(1+\phi\frac{s-\alpha}{\beta})-\frac{1}{\phi}]-\frac{1}{q}],$$

where $E=E_1, d=\nu_1$ *and* $s=T_1$ for case I; $E=E_2, d=a$ *and* $s=x_a$ for case II; and $E=E_3, d=\nu_2$ *and* $s=T_2$ for case III.

The corresponding log likelihood equations are:

$$\frac{\partial\ell}{\partial\alpha}=\sum_{i=1}^{d}\frac{\phi+1}{\beta A}-\frac{1+q}{\beta}\sum_{i=1}^{d}\frac{A^{-(\frac{1}{\phi}+1)}}{1+qA^{-\frac{1}{\phi}}}+\frac{(\nu-d)A1^{-(\frac{1}{\phi}+1)}(1+qA1^{-\frac{1}{\phi}})^{-(\frac{1}{q}+1)}}{\beta[1-(1+qA1^{-\frac{1}{\phi}})^{-\frac{1}{q}}]}=0$$

$$\frac{\partial\ell}{\partial\beta}=\frac{-d}{\beta}+\frac{\phi+1}{\beta^2}\sum_{i=1}^{d}\frac{(x_i-\alpha)}{A}-\frac{1+q}{\beta^2}\sum_{i=1}^{d}\frac{(x_i-\alpha)A^{-(\frac{1}{\phi}+1)}}{1+qA^{-\frac{1}{\phi}}}$$
$$-\frac{(\nu-d)(s-\alpha)A1^{-(\frac{1}{\phi}+1)}[1+qA1^{-\frac{1}{\phi}}]^{-(\frac{1}{q}+1)}}{\beta^2(1-(1+qA1)^{-\frac{1}{\phi}})^{-\frac{1}{q}})}=0$$

$$\frac{\partial\ell}{\partial\phi}=\sum_{i=1}^{d}\frac{\log(A)}{\phi^2}-\sum_{i=1}^{d}\frac{(\frac{1}{\phi})(x_i-\alpha)}{\beta A}$$
$$-\sum_{0}^{d}\frac{(1+q)}{(1+qA^{-\frac{1}{\phi}})}[\frac{A^{-\frac{1}{\phi}}\log A}{\phi^2}-\frac{(x_i-\alpha)A^{-(\frac{1}{\phi}+1)}}{\phi\beta}]$$

$$+\frac{(\nu-d)[1+qA1^{-\frac{1}{\phi}}]^{-(\frac{1}{q}+1)}}{[1-[1+qA1]^{-\frac{1}{\phi}}]^{-\frac{1}{q}}]}[\frac{A1^{-\frac{1}{\phi}}\log(A1)}{\phi^2}-\frac{(s-\alpha)A1^{-(\frac{1}{\phi}+1)}}{\phi\beta}]=0$$

$$\frac{\partial\ell}{\partial q}=\sum_{i=1}^{d}\frac{\log(A)}{\phi^2}-\sum_{i=1}^{d}\frac{(\frac{1}{\phi}+1)(x_i-\alpha)}{\beta A}-\sum_{i=1}^{d}\frac{(q+1)}{[1+qA^{-\frac{1}{\phi}}]}[\frac{A^{-\frac{1}{\phi}}\log(A)}{\phi^2}-\frac{(x_i-\alpha)A^{-(\frac{1}{\phi}+1)}}{\phi\beta}]$$

$$+\frac{(\nu-d)(1+qA1^{-\frac{1}{\phi}})^{-(\frac{1}{q}+1)}}{(1-(1+qA1^{-\frac{1}{\phi}})^{-\frac{1}{q}})}[\frac{A1^{-\frac{1}{\phi}}\log(A1)}{\phi^2}-\frac{(s-\alpha)A1^{-(\frac{1}{\phi}+1)}}{\phi\beta}], \quad (9)$$

where

$$A=[1+\phi\frac{x_i-\alpha}{\beta}],$$

$$A1=[1+\frac{\phi(s-\alpha)}{\beta}].$$

Systems that are specified by Equation (9) yield the maximum likelihood estimations for the parameters of the family of q-generalized extreme value distribution under linear normalization based on the G-Type-II HCS. Since this equation cannot be solved analytically, the Newton–Raphson technique will be applied.

4.2. Simulation Study

A simulation study was used to demonstrate the performance of the estimators produced in the preceding section. We used the family of q-GEVL distribution based on the G-Type-II HCS with $\alpha = 5$, $\beta = 1$, $\phi = 0.5$ and $q = 0.5$ to simulate a small random sample of size $\nu = 20$ based on the G-Type-II HCS:

$$0.5, 3, 3.7590, 3.9617, 4.1244, 4.2720, 4.4142, 4.5563, 4.7021, 4.8551, 5.0188, 5.1974,$$

$$5.3959, 5.6216, 5.8847, 6.2007, 6.5956, 7.1163, 7.8607, 9.0806, 11.7742$$

We applied these data to the G-Type-II HCS by solving the nonlinear systems that are specified in Equation (9) and using the Newton–Raphson technique, and MATLAB (Version 2021) was used for estimation. Then, we used Equation (6) to calculate entropy. The maximum likelihood estimates (MLEs) of the parameters and entropy of the q-GEVL are yielded by proposed values of T_1, T_2 and a in each case as shown in Table 2.

Table 2. MLEs for α, β, ϕ, q and estimated entropy of the q-GEVL with the G-Type-II HCS.

	T_1	T_2	a	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\phi}$	$\hat{q}$	$\hat{H}$
Case I	6	8	4	5.169	0.858	−0.0415	1.048	2.269
Case II	5	7	13	5.106	0.648	−0.1299	1.1601	1.95
Case III	5	8	20	5.1686	1.0794	0.0282	0.9075	2.494

5. Confidence Intervals for the Parameters of the Proposed Procedure

To estimate the approximation confidence intervals for the parameters of the GEVL and q-GEVL distributions based on the G-Type-II HCS, we need the observed information matrices of degrees 3×3 and 4×4. These matrices are denoted by $I(\Theta_1)$ and $I(\Theta_2)$, respectively, where $\Theta_1 = (\alpha,\ \beta,\ \phi)$ and $\Theta_2 = (\alpha,\ \beta,\ \phi,\ q)$. Then, the 3×3 total observed information matrix associated with the GEVL distribution is given by $I(\Theta_1)$, whereas their parameters are replaced by their MLEs where

$$I(\Theta_1)=\begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\phi}\\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\phi}\\ I_{\phi\alpha} & I_{\phi\beta} & I_{\phi\phi}\end{pmatrix}$$

with

$$I_{\alpha\alpha}=\frac{\partial^2\ell}{\partial\alpha^2}=-\frac{3(\phi+1)}{\beta^2}\sum_{i=1}^{d}A^{\frac{-1}{\phi}}+\sum_{i=1}^{d}\frac{2\phi(\phi+1)}{A^2\beta^2}-\frac{(n-d)}{\beta^2(1-e^{-A^{frac-1\phi}})}\left[\frac{e^{-2A1^{\frac{-1}{\phi}}}A1^{-2(1+\frac{1}{\phi})}}{(1-e^{-A1^{\frac{-1}{\phi}}})}\right.$$

$$\left.-A1^{-2(1+\frac{1}{\phi})}-(\phi+1)A1^{-(2+\frac{1}{\phi})}\right]$$

$$I_{\beta\beta}=\frac{\partial^2\ell}{\partial\beta^2}=\frac{4d}{\beta^2}-\sum_{i=1}^{d}\frac{(x_i-\alpha)}{\beta^3}\left[6A^{-(1+\frac{1}{\phi})}+\frac{2(\phi+1)(x_i-\alpha)}{A^2\beta}-\frac{A^{-(1+\frac{1}{\phi})}(\phi+1)(x_i-\alpha)}{\beta}\right.$$

$$\left.-\frac{4(\phi+1)}{A}\right]-\frac{(\nu-d)(s-\alpha)e^{-A1^{-\frac{1}{\phi}}}A1^{-(1+\frac{1}{\phi})}}{\beta^3(1-e^{-A1^{-\frac{1}{\phi}}})}\left[2+\frac{(s-\alpha)A1^{-(1+\frac{1}{\phi})}}{\beta}+\frac{(s-\alpha)e^{-A1^{-\frac{1}{\phi}}}A1^{-(1+\frac{1}{\phi})}}{\beta(1-e^{-A1^{-\frac{1}{\phi}}})}\right.$$

$$\left.-\frac{(\phi+1)(s-\alpha)}{\beta}\right]$$

$$I_{\phi\phi}=\frac{\partial^2\ell}{\partial\phi^2}=\sum_{i=1}^{d}\left[\frac{6A^{\frac{-1}{\phi}}}{\phi^3}-\frac{4\log(A)}{\phi^3}-\frac{6(x_i-\alpha)A^{-(1+\frac{1}{\phi})}}{\beta\phi^2}+\frac{4(x_i-\alpha)}{A\beta\phi^2}+\frac{2(x_i-\alpha)^2(\phi+1)}{\beta\phi A^2}\right.$$

$$+\frac{3(x_i-\alpha)A^{-(2+\frac{1}{\phi})}(\phi\alpha-\phi x_i+\phi^2\alpha+x_i\phi^2+A\beta\log(A))}{\beta^2\phi^3}$$

$$\left.-\frac{3\log(A)A^{-(1+\frac{1}{\phi})}(\phi\alpha-\phi x_i+A\beta\log(A))}{\beta\phi^4}\right]$$

$$-\frac{(\nu-d)e^{-A1^{-\frac{1}{\phi}}}}{\phi^4\beta^2(1-e^{-A1^{-\frac{1}{\phi}}})}\left[A^{-2(1+\frac{1}{\phi})}(\phi\alpha-\phi s+A1\beta\log^2 A1)\left(1+\frac{e^{-\frac{1}{\phi}}}{(1-e^{-A1^{-\frac{1}{\phi}}})}\right)\right.$$

$$-A^{-(2+\frac{1}{\phi})}(\phi^2(\alpha^2+\phi\alpha^2s^2+\phi s^2-2\phi\alpha s-2\alpha\phi s)+A1\beta(A1\beta\log^2(A1)-2\alpha\phi^2+2s\phi^2$$

$$\left.-2A1\beta^2\log(A1)-2\phi\alpha\log(A1)-2s\beta\log(A1)))\right]$$

and so on.

The 4×4 total observed information matrix associated with the q-distribution $(\phi\rightarrowtail 0)$ is given by $I(\Theta_2)$, where in the parameters are replaced by their MLEs where

$$I(\Theta_2)=\begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\phi} & I_{\alpha q}\\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\phi} & I_{\beta q}\\ I_{\phi\alpha} & I_{\phi\beta} & I_{\phi\phi} & I_{\phi q}\\ I_{q\alpha} & I_{q\beta} & I_{q\phi} & I_{qq}\end{pmatrix}$$

with

$$I_{\alpha\alpha}=\frac{\partial^2\ell^*}{\partial\alpha^2}=\sum_{i=1}^{d}\frac{3\phi(\phi+1)}{A^2\beta^2}+\sum_{i=1}^{d}\frac{2(1+q)A^{-(\frac{1}{\phi})}}{\beta^2(1+qA^{-\frac{1}{\phi}})}\left[\frac{qA^{-(\frac{1}{\phi})}}{(1+qA^{-\frac{1}{\phi}})}-(1+\phi)A^{-1}\right]$$

$$-\frac{(n\nu-d)A1^{-(\frac{1}{\phi})}(1+qA^{-\frac{1}{\phi}})^{-(\frac{1}{q})}}{\beta^2(1-(1+qA^{-\frac{1}{\phi}})^{-\frac{1}{q}})}\left[\frac{A1^{-(\frac{1}{\phi})}(1+qA^{-\frac{1}{\phi}})^{-(\frac{1}{q})}}{(1-(1+qA^{-\frac{1}{\phi}})^{-\frac{1}{q}})}\right.$$

$$\left.-(\phi+1)A1^{-1}-(q+1)A1^{-(\frac{1}{\phi}+1)}(1+qA1^{-\frac{1}{\phi}})^{-1}\right]$$

$$I_{\beta\beta} = \frac{\partial^2 \ell^*}{\partial \beta^2} = \frac{4d}{\beta^2} + \sum_{i=1}^{d} \frac{(x_i-\alpha)}{\beta^3}[\frac{3\phi(\phi+1)(x_i-\alpha)}{A^2\beta} - \frac{6(\phi+1)}{A\beta^3} + \frac{2qA^{-2(\frac{1}{\phi})}(1+q)(x_i-\alpha)}{\beta(1+qA^{-\frac{1}{\phi}})^2}$$

$$+\frac{4A^{-(\frac{1}{\phi}+2)}(1+q)(x_i-\alpha)}{\beta(1+qA^{-\frac{1}{\phi}})}] - \frac{(n-d)(s-\alpha)A1^{-(\frac{1}{\phi}+1)}(1+qA^{-\frac{1}{\phi}})^{-(\frac{1}{q}+1)}}{\beta^3(1-(1+qA1^{-\frac{1}{\phi}})^{-\frac{1}{q}})}$$

$$[2 - \frac{(s-\alpha)(1+qA^{-\frac{1}{\phi}})^{-(\frac{1}{q}+1)}A1^{-(\frac{1}{\phi}+1)}}{\beta(1-(1+qA1^{-\frac{1}{\phi}})^{-\frac{1}{q}})} - \frac{(s-\alpha)(\phi+1)A1^{-1}}{\beta}$$

$$-\frac{(q+1)(s-\alpha)A1^{-(\frac{1}{\phi}+1)}(1+qA^{-\frac{1}{\phi}})^{-1}}{\beta}]$$

$$I_{\phi\phi} = \frac{\partial^2 \ell^*}{\partial \phi^2} = 2(q+1)\sum_{i=1}^{d}[\frac{F_x 1}{A^2\beta^2\phi^4(q+A^{\frac{1}{\phi}})} + \frac{q(\phi\alpha - \phi x_i + A\beta\log(A))^2}{A^2\beta^2\phi^4(q+A^{(\frac{1}{\phi})})^2}]$$

$$+\frac{(\nu-d)A^{-(2+\frac{1}{\phi})}}{\beta^2\phi^4(1-(1+qA1^{-\frac{1}{\phi}})^{-\frac{1}{q}})(1+qA1^{-\frac{1}{\phi}})^{-(1+\frac{1}{q})}}$$

$$[F_s 1 - \frac{(\phi\alpha - \phi s + A1\beta\log(A1))^2}{(1-(1+qA1^{-\frac{1}{\phi}})^{-\frac{1}{q}})(1+qA1^{-\frac{1}{\phi}})^{-(1+\frac{1}{q})}} - \frac{(q+1)A1^{-\frac{1}{\phi}(\phi\alpha-\phi s+A1\beta\log(A1))^2}}{(1+qA1^{-\frac{1}{\phi}})^{-1}}]$$

where

$$F_x 1 = \phi^2\alpha^2 + \phi^3\alpha^2 + \phi^2 x_i^2 + \phi^3 x_i^2 - 2\phi^2\alpha x_i - 2\phi^3\alpha x_i$$

$$+A^2\beta^2\log(A)^2 - A\beta\phi^2\alpha + 2A\beta\phi^2 x_i - 2A^2\beta^2\phi\log(A) + 2\beta\phi\alpha\log(A) - 2A\beta\phi x_i\log(A),$$

$$F_s 1 = \phi^2\alpha^2 + \phi^3\alpha^2 + \phi^2 s^2 + \phi^3 s^2 - 2\phi^2\alpha s\phi^3\alpha x_i$$

$$+A1^2\beta^2\log(A1)^2 - A1\beta\phi^2\alpha + 2A1\beta\phi^2 s$$

$$-2A1^2\beta^2\phi\log(A1) + 2\beta\phi\alpha\log(A1) - 2A\beta\phi s\log(A),$$

and so on.

Under standard regularity conditions, $(\Theta_1 - \hat{\Theta}_1)$ asymptotically follows the multivariate normal distribution $N_3(o, -I(\hat{\Theta}_1)^{-1})$, and the asymptotic distribution of $(\Theta_2 - \hat{\Theta}_2)$ is $N_4(o, -I(\hat{\Theta}_2)^{-1})$. These distributions can be utilized to construct the approximation confidence intervals for the model parameters.

Thus, denoting for example the total observed information matrix evaluated at $\hat{\Theta}_i$, that is $-I(\hat{\Theta}_i), i = 1,2$ by $-\hat{I}$, one would have the following approximate $100(1-\alpha)\%$ confidence intervals for the parameters of the q-GEVP distributions:

$$\hat{\alpha} \pm z_{\frac{\alpha}{2}}\sqrt{(-\hat{I}^{-1})_{\alpha\alpha}}, \qquad \hat{\beta} \pm z_{\frac{\alpha}{2}}\sqrt{(-\hat{I}^{-1})_{\beta\beta}},$$

$$\hat{\phi} \pm z_{\frac{\alpha}{2}}\sqrt{(-\hat{I}^{-1})_{\phi\phi}}, \qquad \hat{q} \pm z_{\frac{\alpha}{2}}\sqrt{(-\hat{I}^{-1})_{qq}},$$

where $z_{\frac{\alpha}{2}}$ denotes the $100(1-\frac{\alpha}{2})^{th}$ percentile of the standard normal distribution.

Real-Life Example

The following genuine dataset, which was provided by Cooray and Ananda [22], shows the stress–rupture life of Kevlar 49/epoxy strands when they are continuously compressed at a 90 percent stress level until they all rupture:

$$0.01, 0.08, 0.09, 0.09, 0.10, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 1.00, 0.06,$$
$$1.34, 0.10, 1.45, 1.50, 1.51, 0.63, 0.72, 0.99, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 4.20, 4.69, 7.89, 0.07, 0.07,$$
$$0.36, 0.38, 0.40, 0.65, 0.67, 0.68, 0.79, 0.80, 0.80, 0.83, 0.72, 0.42, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24,$$
$$1.01, 1.02, 1.03, 0.72, 0.73, 0.79, 0.85, 0.90, 0.92, 0.95, 1.05, 0.11, 0.24, 0.29, 0.34, 0.35, 1.10, 1.10, 1.11,$$
$$1.15, 1.18, 1.20, 1.29, 1.31, 0.11, 0.01, 0.02, 1.40, 1.43, 1.33.$$

The basic statistics for the dataset are illustrated in Table 3.

Table 3. Basic statistics.

Mean	Median	Variance	Standard Deviation	Minimum	Maximum	Range	Quantiles
0.613753	0.68	1.16632	1.079963	0.01	7.89	7.88	(0.155, 0.68, 1.105)

Using the K-S, Akaike information criterion (AIC), corrected AIC (AICC) and Bayesian information criterion (BIC) methods for testing the goodness of fit of the data quality (for more information, see [23] and [24]), we note from Table 4 that the presence of the new parameters (q) has created an inconvenience during the application.

Table 4 refers to the result of these methods (the goodness-of-fit tests) and the MLEs for the given data.

Table 4. The goodness-of-fit methods and MEL estimators of the given data.

	Goodness of Fit				Parameters			
Distributions	**K-S***	**AIC**	**AICC**	**BIC**	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\phi}$	$\hat{q}$
GEVL	0.1162	175.097	175.585	184.961	0.5194	0.4648	0.27648	-
q-GEVL	0.11202	533.345	534.433	543.808	0.5221	0.4650	0.2516	3.1×10^{-6}

* Critical value at 0.01 = 0.1894

We applied these data to the G-Type-II HCS by solving the nonlinear systems that are specified in Equation (9) and using the Newton–Raphson technique, and MATLAB (Version 2021) was used for estimation. Then, we used Equation (6) to evaluate the entropy. The maximum likelihood estimations (MLEs) of the parameters of GEVL and q-GEVL are yielded by proposed values of T_1, T_2, and a in each case as shown in Table 5:

Table 5. MLEs for α, β, ϕ, and q of the GEVL and q-GEVL with the G-Type-II HCS.

				GEVL			q-GEVL			
	T_1	T_2	a	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\phi}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\phi}$	$\hat{q}$
Case I	0.14	0.2	21	0.0416	0.0108	−0.0516	0.1116	0.0789	−0.157	−0.3034
Case II	0.19	0.7	38	0.5044	0.0101	0.0096	0.2756	0.2868	−0.066	−0.2474
Case III	0.8	3	85	1.1063	0.0101	0.0009	0.8644	0.0424	0.0404	0.0389

Also, the confidence intervals for the parameters are determined for the GEVL distribution in Table 6.

Table 6. The confidence intervals (CIs) for the parameters for the GEVL distribution.

	T_1	T_2	a	$CI(\hat{\alpha})$	$CI(\hat{\beta})$	$CI(\hat{\varphi})$
Case I	0.14	0.2	21	$[-0.2147, 0.2979]$	$[-0.07, 0.0915]$	$[-1.3825, 1.2793]$
Case II	0.19	0.7	38	$[-0.1838, 1.1926]$	$[-0.0095, 0.0297]$	$[-0.009, 0.001]$
Case III	0.8	3	85	$[-0.2615, 2.4741]$	$[-0.0095, 0.0297]$	$[-0.0001, 0.001]$

6. Conclusions

In this study, we estimated the parameters of GEVL and q-GEVL distributions based on the generalized type-II hybrid censored sample (G-Type-II HCS). We estimated these parameters using the maximum likelihood method. The obtained results have been used for estimation Shannon entropy for these distributions. Also, the confidence intervals for these parameters were computed. The simulation system served as an illustration for the investigation. Additionally, it was used to model a real-life example, and, after verifying that the data fit with the suggested distributions, an estimation of the parameters was carried out. For an example of GEVL distribution, confidence interval computation was performed.

Author Contributions: Conceptualization, R.A.E.-W.A. and T.R.; methodology, R.A.E.-W.A. and T.R.; software, R.A.E.-W.A. and T.R.; validation, R.A.E.-W.A. and T.R.; formal analysis, R.A.E.-W.A. and T.R.; investigation, R.A.E.-W.A. and T.R.; resources, R.A.E.-W.A. and T.R.; data curation, R.A.E.-W.A. and T.R.; writing—original draft preparation, R.A.E.-W.A. and T.R.; writing—review and editing, R.A.E.-W.A. and T.R.; visualization, R.A.E.-W.A. and T.R.; supervision, R.A.E.-W.A. and T.R.; project administration, R.A.E.-W.A. and T.R.; funding acquisition, T.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: Researchers would like to thank the Deanship of Scientific Research, Qassim University, for funding the publication of this project.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Mbah, A.; Ahsanullah, M. Some Characterizations of the Power Function Distribution Based on Lower Generalized Order Statistics. *Pak. J. Stat.* **2007**, *23*, 139–146.
2. Luo, C.W.; Zhu, J. Estimates of the Parameters of the Gumbel Distribution and their Application to Analysis of Water Level Data. *Chin. J. Appl. Probab. Statist.* **2005**, *21*, 169–175. (In Chinese)
3. Coles, S.G. *An Introduction to Statistical Modeling of Extreme Values*; Springer: New York, NY, USA; John van Voorst: London, UK, 2001.
4. Gumbel, E.J. *Statistics of Extremes*; Columbia University Press: New York, NY, USA, 1958.
5. Hosking, J.R.M.; Wallis, J.R.; Wood, E.F. Estimation of the Generalized Extreme Value Distribution by the Method of Probability Weighted Moments. *Technometrics* **1985**, *27*, 251–261. [CrossRef]
6. Barakat, H.M.; Nigm, E.S.M.; Khaled, O.M. *Statistical Techniques for Modelling Extreme Value Data and Related Applications*; Cambridge Scholars Publishing: Newcastle upon Tyne, UK, 2019.
7. Barakat, H.M.; Nigm, E.M.; Khaled, O.M. Extreme value modeling under power normalization. *Appl. Math. Model.* **2013**, *37*, 10162–10169. [CrossRef]
8. Provost, S.B.; Saboor, A.; Cordeiro, G.M.; Mansoor, M. On the q-generalized extreme value distribution. *REVSTAT-Stat. J.* **2018**, *16*, 45–70.
9. Epstein, B. Truncated life tests in the exponential case. *Ann. Math. Stat.* **1954**, *25*, 555–564 [CrossRef]
10. Balakrishnan, N.; Kundu, D. Hybrid censoring: Models, inferential results and applications. *Comput. Stat. Data Anal.* **2013**, *57*, 166–209. [CrossRef]
11. Clausius, R.; Hirst, T.A. *The Mechanical Theory of Heat: With Its Application to the Steam Engine and to the Physical Properties of Bodies 1*; John van Voorst: London, UK, 1867.
12. Shannon, C.E. A mathematical theory of communication. *Bell Syst. Tech. J.* **1948**, *27*, 379–423. [CrossRef]
13. Naif, A.; Malyk, I.V. A Generalization of Binomial Exponential-2 Distribution: Copula, Properties and Applications. *Symmetry* **2020**, *12*, 1338. [CrossRef]

14. Cho, Y.; Sun, H.; Lee, K. An Estimation of the entropy for a Rayleigh distribution based on doubly- generalized Type-II hybrid censored samples. *Entropy* **2014**, *16*, 3655–3669. [CrossRef]
15. Cho, Y.; Sun, H.; Lee, K. Estimating the entropy of a Weibull distribution under generalized progressive hybrid censoring. *Entropy* **2015**, *17*, 102–122. [CrossRef]
16. Ahmad, M.A.M. An estimation of the entropy for a fréchet distribution based on generalized hybrid censored samples. *Int. J. Sci. Basic Appl. Res.* **2019**, *48*, 226–236.
17. Mahmoud, R.M.; Ahmad, M.A.M.; Mohammed, B.S.K. Estimating the entropy of a Lomax distribution under generalized type-I hybrid censoring. *J. Univ. Shanghai Sci. Technol.* **2021**, *23*, 470–481.
18. De Haan, L.; Ferreira, A. *Extreme Value Theory: An Introduction*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2007.
19. Bashir, A.A.; Mohd, A.A.; Kamarulzaman, I.; Noratiqah, M.A. Extreme Value Distributions: An Overview of Estimation and Simulation. *J. Probab. Stat.* **2022** , *2022*, 5449751. [CrossRef]
20. Ravi, S.; Saeb, A. A note on entropies of l-max stable, p-max stable, generalized Pareto and generalized log-Pareto distributions. *ProbStat Forum.* **2012**, *5*, 62–79.
21. Eliwa, M.S.; Zaid, E.A.; El-Morshedy, M. ON q-generalized extreme values under power normalization with properties, estimation methods and applications to COVID-19 data. *REVSTAT-Stat. J.* **2022**, *accepted*. Available online: https://revstat.ine.pt/index.php/REVSTAT/article/view/456 (accessed on 1 April 2022).
22. Cooray, K.; Ananda, M. A Generalization of the Half-Normal Distribution with Application to Lifetime Data. *Commun. Stat. Theory Methods* **2008**, *37*, 1323–1337. [CrossRef]
23. Sugiura, N. Further analysis of the data by Akaike's information criterion and the finite corrections. *Commun. Stat. Theory Methods* **1978**, *7*, 1326. [CrossRef]
24. Stoica, P.; Selen, Y. Model-order selection: A review of information criterion rules. *IEEE Signal Process. Mag.* **2004**, *21*, 36–47. [CrossRef]