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On (α, p) -Cyclic Contractions and Related Fixed Point Theorems

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Abstract: Lipschitz mapping appears inevitably in many branches of mathematics, especially in functional analysis, and leads to the study of new results in metric fixed point theory. Goebel and Sims (resp. Goebel and Japon-Pineda) introduced a class of the Lipschitz mappings termed as (α, p) -Lipschitz mappings and studied not only the modified form of the Lipschitz condition, but also the behavior of a finite number of their iterates. The purpose of this paper is to discuss the various types of (α, p) -contractions with cyclic representation that extend the results due to Banach, Kannan, and Chatterjea. Moreover, based on such types of contractions and the property of symmetry, we obtain some related fixed-point results in the setting of metric spaces. Some examples are studied to illustrate the validity of our obtained results. As an application of our results, we establish the existence of the solution to a class of Fredholm integral equations.

Keywords: fixed point; (α, p) -cyclic contraction; (α, p) -Kannan-type contraction; (α, p) -Chatterjea-type contraction



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1. Introduction and Preliminaries

Let $(\mathfrak{M}, \rho)$ be a metric space and $\mathcal{F} : \mathfrak{M} \rightarrow \mathfrak{M}$ be a contraction mapping, that is,

$$\rho(\mathcal{F}v, \mathcal{F}w) \leq \theta \rho(v, w),$$

for all $v, w \in \mathfrak{M}$, where $\theta \in [0, 1)$. The fixed-point theorem, generally known as the Banach contraction principle, appeared in its explicit form in 1922 in his thesis ([1]). This principle states that every contraction mapping $\mathcal{F}$ on a complete metric space $(\mathfrak{M}, \rho)$ has a unique fixed point. Since then, because of its simplicity and usefulness, it has become a popular tool for solving the existence of integral and differential equations involved in many branches of sciences, engineering, and social sciences. Subsequently, a large number of extensions or generalizations of the Banach contraction principle appeared in the works of Boyd and Wong [2], Edelstein [3], Matkowski [4], Meir and Keeler [5], Rakotch [6], Reich [7], etc. For more details related to the recent development of metric fixed point theory and its possible application, one can refer to Almezal et al. [8], Joshi and Bose [9], and Hammad and Manuel [10,11].

There are more varieties of modification of the Lipschitz condition based on the 6(six) distances, say $\rho(v, w)$, $\rho(\mathcal{F}v, \mathcal{F}w)$, $\rho(v, \mathcal{F}v)$, $\rho(w, \mathcal{F}w)$, $\rho(v, \mathcal{F}w)$, and $\rho(w, \mathcal{F}v)$ for all $v, w \in \mathfrak{M}$. Lipschitz condition is an inequality between two of the distances, namely $\rho(v, w)$ and $\rho(\mathcal{F}v, \mathcal{F}w)$, which implies the behavior of continuity or uniform continuity of the mapping. Many researchers have proposed and studied other inequalities involving some or all of the above six distances. However, these conditions do not imply the continuity of

the mapping under consideration. On the other hand, Kannan- [12] and Chatterjea- [13] type contractions are also interesting to establish the existence of fixed-point problems. A mapping $\mathcal{F} : \mathfrak{M} \rightarrow \mathfrak{M}$ is called a Kannan-type contraction if there exists $\theta \in [0, \frac{1}{2})$ such that:

$$\rho(\mathcal{F}v, \mathcal{F}w) \leq \theta[\rho(v, \mathcal{F}v) + \rho(w, \mathcal{F}w)],$$

for all $v, w \in \mathfrak{M}$. Additionally, we say that $\mathcal{F} : \mathfrak{M} \rightarrow \mathfrak{M}$ is a Chatterjea-type contraction if there exists $\theta \in [0, \frac{1}{2})$ such that:

$$\rho(\mathcal{F}v, \mathcal{F}w) \leq \theta[\rho(v, \mathcal{F}w) + \rho(w, \mathcal{F}v)],$$

for all $v, w \in \mathfrak{M}$.

Goebel and Pineda [14], and Goebel and Sims [15] introduced a class mapping called an (α, p) -Lipschitz mapping.

Consider $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, where $\alpha_i \in \mathbb{R}$ with $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i \geq 0$ for all i and $\alpha_1, \alpha_n > 0$; then, we say that α is a multi-index and the number n is referred to as the length of the multi-index α .

Definition 1 ([14,15]). Consider $(\mathfrak{M}, \rho)$ to be a metric space and $\mathcal{C}$ a nonempty subset of $\mathfrak{M}$. A mapping $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}$ is called (α, p) -Lipschitzian for the constant $\theta \geq 0$ if there exists a multi-index α with length n and $p \in [1, \infty)$ such that:

$$\sum_{i=1}^n \alpha_i \rho^p(\mathcal{F}^i v, \mathcal{F}^i w) \leq \theta \rho^p(v, w), \quad (1)$$

for all $v, w \in \mathcal{C}$.

The smallest constant θ for which (1) holds is called the (α, p) -Lipschitz constant for the mapping $\mathcal{F}$. We shall call $\mathcal{F}$ an (α, p) -contraction or (α, p) -nonexpansive if (1) is satisfied with $\theta < 1$ or $\theta = 1$, respectively. In a similar way, when $p = 1$, we call $\mathcal{F}$ an α -contraction or α -nonexpansive if (1) is satisfied with $\theta < 1$ or $\theta = 1$, respectively. If α is not explicitly specified when $p = 1$ in (1), then $\mathcal{F}$ is called mean Lipschitzian mapping.

Example 1 ([16]). Let $\mathfrak{M} = [0, \infty)$ endow the usual metric. Define a mapping $\mathcal{F}$ on $\mathfrak{M}$ as $\mathcal{F}v = v + a$, $a > 0$ translation function. For $p \geq 1$ and $\alpha \in (0, 1)$, we obtain:

$$\alpha_1 |\mathcal{F}v - \mathcal{F}w|^p + \alpha_2 |\mathcal{F}^2 v - \mathcal{F}^2 w|^p = |v - w|^p,$$

for all $v, w \in \mathfrak{M}$. This shows that $\mathcal{F}$ is an (α, p) -nonexpansive with multi-index α of length $n = 2$.

For more details about the (α, p) -Lipschitz mapping, one can refer to Goebel and Sims [15], Goebel and Japon-Pineda [14], and Łukasz [17].

In 2003, Kirk et al. [18] introduced the notion of cyclic mapping and generalized the Banach contraction principle.

Definition 2 ([18]). Let $\mathcal{F} : \mathfrak{M} \rightarrow \mathfrak{M}$ be an operator on a nonempty set $\mathfrak{M}$. Then, $\mathfrak{M} = \cup_{i=1}^m \mathcal{A}_i$ is called a cyclic representation of $\mathfrak{M}$ with respect to $\mathcal{F}$ if:

- (i) $\mathcal{A}_i, i = 1, 2, \dots, m$ are nonempty subsets and $\mathcal{A}_{m+1} = \mathcal{A}_1$;
- (ii) $\mathcal{F}(\mathcal{A}_1) \subseteq \mathcal{A}_2, \dots, \mathcal{F}(\mathcal{A}_{m-1}) \subseteq \mathcal{A}_m, \mathcal{F}(\mathcal{A}_m) \subseteq \mathcal{A}_1$.

Definition 3 ([18]). A mapping $\mathcal{F} : \mathfrak{M} \rightarrow \mathfrak{M}$ is called a cyclic contraction if:

- (i) $\cup_{i=1}^m \mathcal{A}_i$ is a cyclic representation of $\mathfrak{M}$ with respect to $\mathcal{F}$;
- (ii) there exists $\theta \in [0, 1)$ such that:

$$\rho(\mathcal{F}v, \mathcal{F}w) \leq \theta \rho(v, w),$$

for all $v \in \mathcal{A}_i$, $w \in \mathcal{A}_{i+1}$, where $1 \leq i \leq m$.

We denote $\text{Fix}(\mathcal{F}) = \{\omega \in \mathfrak{M} : \mathcal{F}\omega = \omega\}$ as the set of fixed points of the mapping $\mathcal{F}$ on $\mathfrak{M}$.

Theorem 1 ([18]). Let $(\mathfrak{M}, \rho)$ be a complete metric space and $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$ be non-empty closed subsets of $\mathfrak{M}$ such that $\mathfrak{N} = \cup_{i=1}^m \mathcal{A}_i$. Let $\mathcal{F} : \mathfrak{N} \rightarrow \mathfrak{N}$ be a cyclic contraction; then, $\text{Fix}(\mathcal{F}) \subseteq \cap_{i=1}^m \mathcal{A}_i$ is singleton.

Example 2. Let $\mathfrak{M} = \mathbb{R}$ with usual metric and $\mathcal{A}_1 = [0, \frac{\pi}{2}]$, $\mathcal{A}_2 = [-\frac{\pi}{2}, 0]$ be subsets of $\mathfrak{M}$. Define $\mathcal{F} : \mathcal{A}_1 \cup \mathcal{A}_2 \rightarrow \mathcal{A}_1 \cup \mathcal{A}_2$ by $\mathcal{F}v = -\frac{1}{9} \sin v$. Then $\mathcal{F}(\mathcal{A}_1) \subseteq \mathcal{A}_2$ and $\mathcal{F}(\mathcal{A}_2) \subseteq \mathcal{A}_1$. It shows that $\mathfrak{M} = \cup \mathcal{A}_i$, $i = 1, 2$ is a cyclic representation of $\mathfrak{M}$ with respect to $\mathcal{F}$. Moreover:

$$\rho(\mathcal{F}v, \mathcal{F}w) = \frac{1}{9} |-\sin v + \sin w| \leq \frac{1}{9} |v - w|,$$

for all $v \in \mathcal{A}_1$, $w \in \mathcal{A}_2$. As such, $\mathcal{F}$ satisfies all the conditions of Theorem 1 and, hence, $\text{Fix}(\mathcal{F}) = \{0\} \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$.

For more results related to cyclic contraction, Kannan- and Chatterjea-type cyclic contractions in various spaces, one may refer to Aydi and Karapınar [19], Aydi et al. [20], Chandok [21], Chandok and Postolache [22], Karapınar and Nashine [23], Petric and Zlatanov [24], and references therein.

Let $\mathcal{F}$ be a self-mapping on a metric space $(\mathfrak{M}, \rho)$. For a given $\epsilon > 0$, $v_0 \in \mathfrak{M}$ is said to be an ϵ -fixed point of $\mathcal{F}$ on $\mathfrak{M}$ if $\rho(v_0, \mathcal{F}v_0) < \epsilon$. Every fixed point is an ϵ -fixed point, but the converse is not true. We say that $\mathcal{F}$ has the approximate fixed point property (AFPP) if for all $\epsilon > 0$, $\mathcal{F}$ has an ϵ -fixed point. Details can be checked from Berinde [25], Kohlenbach and Leustean [26], Miandaragh et al. [27], Reich and Zaslavski [28], etc.

Example 3 ([29]). Let us consider $\mathfrak{M} = [0, \infty)$ with a usual metric ρ and $\mathcal{F} : \mathfrak{M} \rightarrow \mathfrak{M}$ be given by $\mathcal{F}v = v + \frac{1}{v+1} \forall v \in \mathfrak{M}$. Setting $0 < \epsilon < 1$ and taking $v_0 \in \mathfrak{M}$ such that $v_0 > \frac{1-\epsilon}{\epsilon}$, we have:

$$\rho(\mathcal{F}v_0, v_0) = |\mathcal{F}v_0 - v_0| = \left| \frac{1}{v_0 + 1} \right| < \epsilon.$$

This shows that $\mathcal{F}$ has an ϵ -fixed point, but $\text{Fix}(\mathcal{F}) = \emptyset$.

In this paper, we introduce various types of (α, p) -contraction with cyclic representation viz., (α, p) -cyclic contraction, and (α, p) -Kannan (resp. Chatterjea)-type cyclic contraction and obtain some related fixed-point results of such types of contractions in the setting of metric space.

2. Main Results

Let $(\mathfrak{M}, \rho)$ be a metric space and $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$ be nonempty subsets of $\mathfrak{M}$ such that $\mathfrak{N} = \cup_{i=1}^m \mathcal{A}_i$.

Definition 4. A mapping $\mathcal{F} : \mathfrak{N} \rightarrow \mathfrak{N}$ is called an (α, p) -cyclic Lipschitzian if:

- (i) $\cup_{i=1}^m \mathcal{A}_i$ is a cyclic representation of $\mathfrak{N}$ with respect to $\mathcal{F}$;
- (ii) there exists $\theta \geq 0$ for some multi-index α of length n and $p \in [1, \infty)$, satisfying the inequality:

$$\sum_{i=1}^n \alpha_i \rho^p(\mathcal{F}^i v, \mathcal{F}^i w) \leq \theta \rho^p(v, w), \quad (2)$$

for all $v \in \mathcal{A}_i$, $w \in \mathcal{A}_{i+1}$, $1 \leq i \leq m$.

The smallest constant θ for which (2) holds is called an (α, p) -cyclic Lipschitz constant for the mapping $\mathcal{F}$. Additionally, the mapping $\mathcal{F}$ is called an (α, p) -cyclic contraction or an (α, p) -cyclic non-expansive accordingly, as $\theta < 1$ or $\theta = 1$. Moreover, if α is not explicitly specified when $p = 1$ in (2), then $\mathcal{F}$ is called a mean cyclic contraction or mean cyclic non-expansive mapping accordingly, as $\theta < 1$ or $\theta = 1$. If $\mathfrak{M} = \mathcal{A}_1$, then (α, p) -cyclic contraction becomes (α, p) -contraction. Further, if $p = 1$ in (2), then we call $\mathcal{F}$ an α -cyclic contraction.

Definition 5. A mapping $\mathcal{F} : \mathfrak{N} \rightarrow \mathfrak{N}$ is called an (α, p) -Kannan-type cyclic contraction if:

- (i) $\cup_{i=1}^m \mathcal{A}_i$ is a cyclic representation of $\mathfrak{N}$ with respect to $\mathcal{F}$;
- (ii) there exists $\theta \in [0, \frac{1}{2})$ for some multi-index α of length n and $p \in [1, \infty)$, satisfying the inequality:

$$\sum_{i=1}^n \alpha_i \rho^p(\mathcal{F}^i v, \mathcal{F}^i w) \leq \theta [\rho^p(v, \mathcal{F}v) + \rho^p(w, \mathcal{F}w)], \quad (3)$$

for all $v \in \mathcal{A}_i, w \in \mathcal{A}_{i+1}, 1 \leq i \leq m$.

Definition 6. A mapping $\mathcal{F} : \mathfrak{N} \rightarrow \mathfrak{N}$ is called an (α, p) -Chatterjea-type cyclic contraction if:

- (i) $\cup_{i=1}^m \mathcal{A}_i$ is a cyclic representation of $\mathfrak{N}$ with respect to $\mathcal{F}$;
- (ii) there exists $\theta \in [0, \frac{1}{2})$ for some multi-index α of length n and $p \in [1, \infty)$, satisfying the inequality:

$$\sum_{i=1}^n \alpha_i \rho^p(\mathcal{F}^i v, \mathcal{F}^i w) \leq \theta [\rho^p(v, \mathcal{F}w) + \rho^p(w, \mathcal{F}v)], \quad (4)$$

for all $v \in \mathcal{A}_i, w \in \mathcal{A}_{i+1}, 1 \leq i \leq m$.

Example 4. Let $\mathfrak{M} = [0, 1]$ with the usual metric $\rho(v, w) = |v - w|$. Consider $\mathcal{A}_1 = [\frac{5}{8}, 1]$ and $\mathcal{A}_2 = [\frac{1}{2}, 1]$ as two closed subsets of $\mathfrak{M}$ such that $\mathfrak{N} = \mathcal{A}_1 \cup \mathcal{A}_2 = [\frac{1}{2}, 1]$. Let $\mathcal{F} : \mathfrak{N} \rightarrow \mathfrak{N}$ be a function defined by $\mathcal{F}v = \frac{1+v^2}{2}$, for all $v \in \mathfrak{N}$. It is easy to check that $\mathcal{F}\mathcal{A}_1 \subseteq \mathcal{A}_2, \mathcal{F}\mathcal{A}_2 \subseteq \mathcal{A}_1$.

This shows that $\mathfrak{N} = \mathcal{A}_1 \cup \mathcal{A}_2$ is a cyclic representation of $\mathfrak{N}$ with respect to $\mathcal{F}$.

Additionally, we have: $\mathcal{F}^2 v = \frac{5+2v^2+v^4}{8}$ for all $v \in \mathfrak{N}$. Now, for all $v \in \mathcal{A}_1$ and $w \in \mathcal{A}_2$, we have:

$$|\mathcal{F}v - \mathcal{F}w| = \frac{1}{2}|v^2 - w^2| = \frac{v+w}{2}|v-w| \leq |v-w|.$$

Also, we have:

$$|\mathcal{F}^2 v - \mathcal{F}^2 w| \leq \frac{1}{4}|v^2 - w^2| + \frac{1}{8}|v^4 - w^4| \leq |v-w|.$$

For $p \geq 1$, we obtain:

$$\alpha_1 \rho^p(\mathcal{F}v, \mathcal{F}w) + \alpha_2 \rho^p(\mathcal{F}^2 v, \mathcal{F}^2 w) \leq \rho^p(v, w).$$

This shows that $\mathcal{F}$ is (α, p) -cyclic nonexpansive mapping with multi-index α of length $n = 2$.

For more examples on (α, p) -cyclic contraction and (α, p) -Kannan (resp. Chatterjea)-type cyclic contraction, we refer to Examples 5–7.

Throughout this section, for the sake of simplicity, we present our results for multi-index α of length $n = 2$.

Theorem 2. Let $(\mathfrak{M}, \rho)$ be a complete metric space and $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$ be non-empty closed subsets of $\mathfrak{M}$ such that $\mathfrak{N} = \cup_{i=1}^m \mathcal{A}_i$. Let $\mathcal{F} : \mathfrak{N} \rightarrow \mathfrak{N}$ be an (α, p) -cyclic contraction of multi-index α of length $n = 2$ such that $\theta < \gamma = \min\{\alpha_1, \alpha_2\}$. Then, $\text{Fix}(\mathcal{F}) \subseteq \cap_{i=1}^m \mathcal{A}_i$ is singleton.

Proof. Let $v_0 \in \mathfrak{N}$. Then, there exists $i_0 \in \{1, 2, \dots, m\}$ such that $v_0 \in \mathcal{A}_{i_0}$ and $v_1 = \mathcal{F}v_0 \in \mathcal{A}_{i_0+1}$. Similarly, there exists $v_2 \in \mathfrak{N}$ such that $v_2 = \mathcal{F}v_1 = \mathcal{F}^2v_0 \in \mathcal{A}_{i_0+2}$. Thus, we define a sequence $\{v_n\}$ given by $v_{n+1} = \mathcal{F}v_n = \mathcal{F}^{n+1}v_0 \in \mathcal{A}_{i_0+n}$, for all $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $v_{n_0+1} = v_{n_0}$, then $\mathcal{F}v_{n_0} = v_{n_0}$ i.e., $v_{n_0} \in \text{Fix}(\mathcal{F})$. Suppose that $v_{n+1} \neq v_n$, for all $n \geq 0$. Setting $s = \max\{\rho(v_0, \mathcal{F}v_0), \rho(\mathcal{F}v_0, \mathcal{F}^2v_0)\}$, we have $\rho(v_0, \mathcal{F}v_0) \leq s$ and $\rho(\mathcal{F}v_0, \mathcal{F}^2v_0) \leq s$.

Now, we discuss the following cases.

Case-(i): When $2\theta < \alpha_1$ and setting $h^p = \frac{\theta}{\gamma} < 1$. Taking $v = v_0$ and $w = \mathcal{F}v_0$ in (2), we obtain:

$$\begin{aligned}\alpha_1 \rho^p(\mathcal{F}v_0, \mathcal{F}^2v_0) &\leq \alpha_1 \rho^p(\mathcal{F}v_0, \mathcal{F}^2v_0) + \alpha_2 \rho^p(\mathcal{F}^2v_0, \mathcal{F}^3v_0) \\ &\leq \theta \rho^p(v_0, \mathcal{F}v_0) \leq \theta s^p \\ \Rightarrow \rho^p(\mathcal{F}v_0, \mathcal{F}^2v_0) &\leq \frac{\theta}{\alpha_1} s^p \leq \frac{\theta}{\gamma} s^p.\end{aligned}$$

It follows that:

$$\rho(\mathcal{F}v_0, \mathcal{F}^2v_0) \leq hs.$$

Again, taking $v = \mathcal{F}v_0$ and $w = \mathcal{F}^2v_0$ in (2), we obtain:

$$\begin{aligned}\alpha_1 \rho^p(\mathcal{F}^2v_0, \mathcal{F}^3v_0) &\leq \alpha_1 \rho^p(\mathcal{F}^2v_0, \mathcal{F}^3v_0) + \alpha_2 \rho^p(\mathcal{F}^3v_0, \mathcal{F}^4v_0) \\ &\leq \theta \rho^p(\mathcal{F}v_0, \mathcal{F}^2v_0) = \theta (hs)^p \\ \Rightarrow \rho^p(\mathcal{F}^2v_0, \mathcal{F}^3v_0) &\leq \frac{\theta}{\gamma} (hs)^p = (h^2s)^p.\end{aligned}$$

Therefore, $\rho(\mathcal{F}^2v_0, \mathcal{F}^3v_0) \leq h^2s$. And from (2), we obtain:

$$\begin{aligned}\alpha_1 \rho^p(\mathcal{F}^3v_0, \mathcal{F}^4v_0) &\leq \alpha_1 \rho^p(\mathcal{F}^3v_0, \mathcal{F}^4v_0) + \alpha_2 \rho^p(\mathcal{F}^4v_0, \mathcal{F}^5v_0) \\ &\leq \theta \rho^p(\mathcal{F}^2v_0, \mathcal{F}^3v_0) = \theta (h^2s)^p \\ \Rightarrow \rho^p(\mathcal{F}^3v_0, \mathcal{F}^4v_0) &\leq \left(\frac{\theta}{\alpha_1}\right)^p (h^2s)^p \leq (h^3s)^p.\end{aligned}$$

Therefore, $\rho(\mathcal{F}^3v_0, \mathcal{F}^4v_0) \leq h^3s$.

Additionally, we have:

$$\begin{aligned}\alpha_1 \rho^p(\mathcal{F}^4v_0, \mathcal{F}^5v_0) &\leq \alpha_1 \rho^p(\mathcal{F}^4v_0, \mathcal{F}^5v_0) + \alpha_2 \rho^p(\mathcal{F}^5v_0, \mathcal{F}^6v_0) \\ &\leq \theta \rho^p(\mathcal{F}^3v_0, \mathcal{F}^4v_0) \\ \Rightarrow \rho^p(\mathcal{F}^4v_0, \mathcal{F}^5v_0) &\leq h^4s.\end{aligned}$$

Similarly, we obtain:

$$\rho(\mathcal{F}^n v_0, \mathcal{F}^{n+1} v_0) \leq h^n s, \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Therefore, $\rho(\mathcal{F}^n v_0, \mathcal{F}^{n+1} v_0) \rightarrow 0$ as $n \rightarrow +\infty$. We show that $\{v_n\}$ is a Cauchy sequence in $\mathfrak{N}$. For any $n_0 \geq 0$, such that $n \geq n_0$ and $p \geq 1$, we obtain:

$$\begin{aligned}\rho(\mathcal{F}^n v_0, \mathcal{F}^{n+p} v_0) &\leq \rho(\mathcal{F}^n v_0, \mathcal{F}^{n+1} v_0) + \rho(\mathcal{F}^{n+1} v_0, \mathcal{F}^{n+2} v_0) \\ &\quad + \dots + \rho(\mathcal{F}^{n+p-1} v_0, \mathcal{F}^{n+p} v_0) \\ &\leq h^n (1 + h + h^2 + \dots + h^{p-1}) s \\ &\leq \frac{h^n}{1-h} s.\end{aligned}$$

Taking $n \rightarrow +\infty$, $\rho(\mathcal{F}^n v_0, \mathcal{F}^{n+p} v_0) \rightarrow 0$, we obtain $\{v_n\}$ as a Cauchy sequence in $\mathfrak{N} = \cup_{i=1}^m \mathcal{A}_i$.

Case-(ii): When $\theta < \alpha_2$ and setting $h^p = \frac{\theta}{\gamma} < 1$. Taking $v = v_0$ and $w = \mathcal{F}v_0$ in (2), we have:

$$\begin{aligned}\alpha_2 \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) &\leq \alpha_1 \rho^p(\mathcal{F} v_0, \mathcal{F}^2 v_0) + \alpha_2 \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) \\ &\leq \theta \rho^p(v_0, \mathcal{F} v_0) = \theta s^p \\ \Rightarrow \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) &\leq \frac{\theta}{\alpha_2} s^p \leq \frac{\theta}{\gamma} s^p = (hs)^p.\end{aligned}$$

Therefore, we obtain:

$$\rho(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) \leq hs.$$

Again, taking $v = \mathcal{F}v_0$ and $w = \mathcal{F}^2 v_0$ in (2), we obtain:

$$\begin{aligned}\alpha_2 \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) &\leq \alpha_1 \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) + \alpha_2 \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) \\ &\leq \theta \rho^p(\mathcal{F} v_0, \mathcal{F}^2 v_0) = \theta s^p \\ \Rightarrow \rho(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) &\leq hs.\end{aligned}$$

And:

$$\begin{aligned}\alpha_2 \rho^p(\mathcal{F}^4 v_0, \mathcal{F}^5 v_0) &\leq \alpha_1 \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) + \alpha_2 \rho^p(\mathcal{F}^4 v_0, \mathcal{F}^5 v_0) \\ &\leq \theta \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) \\ \Rightarrow \rho(\mathcal{F}^4 v_0, \mathcal{F}^5 v_0) &\leq h^2 s.\end{aligned}$$

Additionally, we have:

$$\rho(\mathcal{F}^5 v_0, \mathcal{F}^6 v_0) \leq h^2 s.$$

Moreover, we obtain:

$$\begin{aligned}\alpha_2 \rho^p(\mathcal{F}^6 v_0, \mathcal{F}^7 v_0) &\leq \alpha_1 \rho^p(\mathcal{F}^5 v_0, \mathcal{F}^6 v_0) + \alpha_2 \rho^p(\mathcal{F}^6 v_0, \mathcal{F}^7 v_0) \\ &\leq \theta \rho^p(\mathcal{F}^4 v_0, \mathcal{F}^5 v_0) \\ \Rightarrow \rho(\mathcal{F}^6 v_0, \mathcal{F}^7 v_0) &\leq h^3 s.\end{aligned}$$

Further, we obtain:

$$\begin{aligned}\alpha_2 \rho^p(\mathcal{F}^7 v_0, \mathcal{F}^8 v_0) &\leq \alpha_1 \rho^p(\mathcal{F}^6 v_0, \mathcal{F}^7 v_0) + \alpha_2 \rho^p(\mathcal{F}^7 v_0, \mathcal{F}^8 v_0) \\ &\leq \theta \rho^p(\mathcal{F}^5 v_0, \mathcal{F}^6 v_0) \\ \Rightarrow \rho(\mathcal{F}^7 v_0, \mathcal{F}^8 v_0) &\leq h^3 s.\end{aligned}$$

Following the same steps as in above, we obtain:

$$\rho(\mathcal{F}^m v_0, \mathcal{F}^{m+1} v_0) \leq h^l s,$$

whenever $m = 2l$ or $m = 2l + 1$. Therefore, $\rho(\mathcal{F}^m v_0, \mathcal{F}^{m+1} v_0) \rightarrow 0$ as $m \rightarrow +\infty$. To show the sequence $\{v_n\}$ is a Cauchy in $\mathfrak{N}$, the following two subcases arise. Choose non-zero positive integers m, n with $m < n = m + q$.

Subcase-(a): For $m = 2l$ with $l, q \geq 1$, then:

$$\begin{aligned}\rho(\mathcal{F}^m v_0, \mathcal{F}^{m+q} v_0) &= \rho(\mathcal{F}^{2l} v_0, \mathcal{F}^{2l+q} v_0) \\ &\leq \rho(\mathcal{F}^{2l} v_0, \mathcal{F}^{2l+1} v_0) + \rho(\mathcal{F}^{2l+1} v_0, \mathcal{F}^{2l+2} v_0) \\ &\quad + \rho(\mathcal{F}^{2l+2} v_0, \mathcal{F}^{2l+3} v_0) + \rho(\mathcal{F}^{2l+3} v_0, \mathcal{F}^{2l+4} v_0) + \dots \\ &\quad + \rho(\mathcal{F}^{2l+q-2} v_0, \mathcal{F}^{2l+q-1} v_0) + \rho(\mathcal{F}^{2l+q-1} v_0, \mathcal{F}^{2l+q} v_0) \\ &\leq h^l s + h^l s + h^{l+1} s + h^{l+1} s + \dots \\ &\leq 2h^l (1 + h + h^2 + h^3 + \dots) s \\ &\leq 2h^l \frac{1}{(1-h)} s.\end{aligned}$$

Subcase-(b): Similarly, for $m = 2l + 1$ with $l, q \geq 1$, we obtain:

$$\begin{aligned}\rho(\mathcal{F}^m v_0, \mathcal{F}^{m+q} v_0) &= \rho(\mathcal{F}^{2l+1} v_0, \mathcal{F}^{2l+q+1} v_0) \\ &\leq \rho(\mathcal{F}^{2l+1} v_0, \mathcal{F}^{2l+2} v_0) + \rho(\mathcal{F}^{2l+2} v_0, \mathcal{F}^{2l+3} v_0) \\ &\quad + \rho(\mathcal{F}^{2l+3} v_0, \mathcal{F}^{2l+4} v_0) + \rho(\mathcal{F}^{2l+4} v_0, \mathcal{F}^{2l+5} v_0) + \dots \\ &\quad + \rho(\mathcal{F}^{2l+q-1} v_0, \mathcal{F}^{2l+q} v_0) + \rho(\mathcal{F}^{2l+q} v_0, \mathcal{F}^{2l+q+1} v_0) \\ &\leq h^l s + h^{l+1} s + h^{l+1} s + h^{l+2} s + \dots \\ &\leq 2h^l (1 + h + h^2 + h^3 + \dots) s \\ &\leq 2h^l \frac{1}{(1-h)} s.\end{aligned}$$

Taking $l \rightarrow +\infty$ in all sub-cases, since $h < 1$, we obtain: $\rho(\mathcal{F}^m v_0, \mathcal{F}^n v_0) \rightarrow 0$. Therefore, $\{v_n\}$ is a Cauchy sequence in $\mathfrak{N}$. Thus, in all cases, the sequence $\{v_n\}$ is a Cauchy sequence in $\mathfrak{N}$. Since $\mathfrak{N} = \cup_{i=1}^m \mathcal{A}_i$ is closed in $\mathfrak{M}$, $\mathfrak{N}$ is also complete and there exists a point $z \in \mathfrak{N}$ such that $v_n = \mathcal{F}^n v_0 \rightarrow z$ as $n \rightarrow +\infty$. Further, as $\mathfrak{N} = \cup_{i=1}^m \mathcal{A}_i$ is a cyclic representation of $\mathfrak{N}$ with respect to $\mathcal{F}$, the sequence $\{v_n\}$ has an infinite number of terms in $\mathcal{A}_i$, for all $i \in \{1, 2, \dots, m\}$. It follows that $z \in \cap_{i=1}^m \mathcal{A}_i$, so $\cap_{i=1}^m \mathcal{A}_i \neq \emptyset$.

We prove that $\text{Fix}(\mathcal{F}) \neq \emptyset$. Taking $z \in \mathcal{A}_i$, $\mathcal{F}z \in \mathcal{A}_{i+1}$ for $i \in \{1, 2, \dots, m\}$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ with $v_{n_k} \in \mathcal{A}_{i-1}$. From (2), we obtain:

$$\begin{aligned}\alpha_1 \rho^p(v_{n_{k+1}}, \mathcal{F}z) + \alpha_2 \rho^p(v_{n_{k+2}}, \mathcal{F}^2 z) &= \alpha_1 \rho^p(\mathcal{F}v_{n_k}, \mathcal{F}z) + \alpha_2 \rho^p(\mathcal{F}^2 v_{n_k}, \mathcal{F}^2 z) \\ &\leq \theta \rho^p(v_{n_k}, z).\end{aligned}$$

Taking the limit as $n \rightarrow +\infty$, we obtain:

$$\alpha_1 \rho^p(z, \mathcal{F}z) + \alpha_2 \rho^p(z, \mathcal{F}^2 z) \leq 0.$$

This implies that $\rho(z, \mathcal{F}z) = 0$ and $\rho(z, \mathcal{F}^2 z) = 0$. Therefore, $\mathcal{F}(z) = z$ and, hence, $\text{Fix}(\mathcal{F}) \neq \emptyset$. Now, we prove that $\text{Fix}(\mathcal{F})$ is singleton. Let $z^* \in \cap_{i=1}^m \mathcal{A}_i$ be another element of $\text{Fix}(\mathcal{F})$ such that $z \neq z^*$. Again, from (2), we obtain:

$$\begin{aligned}\alpha_1 \rho^p(\mathcal{F}z, \mathcal{F}z^*) &\leq \alpha_1 \rho^p(\mathcal{F}z, \mathcal{F}z^*) + \alpha_2 \rho^p(\mathcal{F}^2 z, \mathcal{F}^2 z^*) \leq \theta \rho^p(z, z^*) \\ \Rightarrow \rho^p(z, z^*) &\leq \frac{\theta}{\alpha_1} \rho^p(z, z^*) \leq \frac{\theta}{\gamma} \rho^p(z, z^*) \Rightarrow (1 - \frac{\theta}{\gamma}) \rho^p(z, z^*) \leq 0\end{aligned}$$

leading to $1 \leq \frac{\theta}{\gamma}$, which is a contradiction. Hence, $\text{Fix}(\mathcal{F})$ is singleton, being a subset of $\cap_{i=1}^m \mathcal{A}_i$. $\square$

Example 5. Let $\mathfrak{M} = [0, 1]$ with the usual metric $\rho(v, w) = |v - w|$. Suppose $\mathcal{A}_1 = [0, 1/2]$ and $\mathcal{A}_2 = [0, 1]$, then $\mathfrak{M} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_2 = [0, 1/2]$. Define $\mathcal{F} : \mathfrak{M} \rightarrow \mathfrak{M}$ as $\mathcal{F}v = \frac{1-v^2}{2}$, for all $v \in \mathfrak{M}$. Then, we obtain $\mathcal{F}^2v = \frac{3+2v^2-v^4}{8}$. Additionally, we obtain $\mathcal{F}(\mathcal{A}_1) \subseteq \mathcal{A}_2, \mathcal{F}(\mathcal{A}_2) \subseteq \mathcal{A}_1$ and, hence, $\mathfrak{M} = \cup \mathcal{A}_i$, where $i = 1, 2$ is a cyclic representation of $\mathfrak{M}$ with respect to $\mathcal{F}$.

For all $v \in \mathcal{A}_1, w \in \mathcal{A}_2$ setting with $p = 2$, we obtain:

$$|\mathcal{F}v - \mathcal{F}w|^2 = \frac{1}{4}|v^2 - w^2|^2 \leq \frac{9}{16}|v - w|^2$$

and:

$$|\mathcal{F}^2v - \mathcal{F}^2w|^2 = \left| \frac{1}{8}(v^4 - w^4) + \frac{1}{4}(w^2 - v^2) \right|^2 \leq \frac{801}{2048}|v - w|^2.$$

Now, setting $\alpha_1 = \alpha_2 = \frac{1}{2}$, we obtain:

$$\alpha_1 \rho^2(\mathcal{F}v, \mathcal{F}w) + \alpha_2 \rho^2(\mathcal{F}^2v, \mathcal{F}^2w) \leq \theta \rho^2(v, w),$$

for all $v \in \mathcal{A}_1, w \in \mathcal{A}_2$, where $\theta = \frac{1953}{4096} = 0.4768066406$ and $\gamma = \min\{\alpha_1, \alpha_2\} = \frac{1}{2} = 0.5$. This shows that $\mathcal{F}$ is an $(\alpha, 2)$ -cyclic contraction with multi-index α of length $n = 2$ such that $\theta < \gamma$. Thus, all the conditions of Theorem 2 are satisfied and, hence, $\text{Fix}(\mathcal{F}) = \{\sqrt{2} - 1\} \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$ is singleton.

Corollary 1 ([16]). Let $(\mathfrak{M}, \rho)$ be a metric space and $\mathcal{F} : \mathfrak{M} \rightarrow \mathfrak{M}$ be a (α, p) -contraction of the multi-index α of length $n = 2$ such that $\theta + \alpha_1 < 1$. Then, $\mathcal{F}$ has the AFPP. Further, if $(\mathfrak{M}, \rho)$ is a complete metric space, then $\text{Fix}(\mathcal{F})$ is singleton.

Proof. Taking $\alpha = (\alpha_1, \alpha_2)$ as a multi-index of length $n = 2$, then $\alpha_1 + \alpha_2 = 1$ implies that $\alpha_2 = 1 - \alpha_1$, since, $\theta + \alpha_1 < 1$, that is to say $\frac{\theta}{1-\alpha_1} = \frac{\theta}{\alpha_2} \leq \frac{\theta}{\gamma} < 1$. Consequently, $\theta < \gamma$, where $\gamma = \min\{\alpha_1, \alpha_2\}$. Setting $\mathfrak{M} = \mathcal{A}_1 = \mathcal{A}_1$, then $\mathcal{F}(\mathcal{A}_1) \subseteq \mathcal{A}_2$ and $\mathcal{F}(\mathcal{A}_2) \subseteq \mathcal{A}_1$. Therefore, $\mathcal{F}$ is a (α, p) -cyclic contraction of multi-index α of length $n = 2$. Thus, all the conditions of Theorem 2 are satisfied and, hence, $\text{Fix}(\mathcal{F})$ is singleton. $\square$

Theorem 3. Let $(\mathfrak{M}, \rho)$ be a complete metric space and $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$ be non-empty closed subsets of $\mathfrak{M}$ such that $\mathfrak{N} = \cup_{i=1}^m \mathcal{A}_i$. Let $\mathcal{F} : \mathfrak{N} \rightarrow \mathfrak{N}$ be an (α, p) -Kannan-type cyclic contraction of multi-index α of length $n = 2$ such that $\theta < \gamma = \frac{1}{2} \min\{\alpha_1, \alpha_2\}$. Then, $\text{Fix}(\mathcal{F}) \subseteq \cap_{i=1}^m \mathcal{A}_i$ is singleton.

Proof. As in the above Theorem 2, we define a sequence, $\{v_n\}$, given by $v_{n+1} = \mathcal{F}v_n = \mathcal{F}^{n+1}v_0 \in \mathcal{A}_{i_0+n}$, for all $n \geq 0$. Suppose that $v_{n+1} \neq v_n$ for all $n \geq 0$. Now, we show that $\{\rho(v_n, v_{n+1})\} = \{\rho(\mathcal{F}^n v_0, \mathcal{F}^{n+1} v_0)\}$ is a monotone decreasing sequence of positive real numbers.

$$\begin{aligned} \alpha_1 \rho^p(\mathcal{F}v_n, \mathcal{F}v_{n+1}) + \alpha_2 \rho^p(\mathcal{F}^2v_n, \mathcal{F}^2v_{n+1}) &\leq \theta [\rho^p(v_n, \mathcal{F}v_n) + \rho^p(v_{n+1}, \mathcal{F}v_{n+1})] \\ \Rightarrow \alpha_1 \rho^p(v_{n+1}, v_{n+2}) + \alpha_2 \rho^p(v_{n+2}, v_{n+3}) &\leq \theta [\rho^p(v_n, v_{n+1}) + \rho^p(v_{n+1}, v_{n+2})]. \end{aligned}$$

Therefore:

$$\begin{aligned} \alpha_1 \rho^p(v_{n+1}, v_{n+2}) &\leq \alpha_1 \rho^p(v_{n+1}, v_{n+2}) + \alpha_2 \rho^p(v_{n+2}, v_{n+3}) \\ &\leq \theta [\rho^p(v_n, v_{n+1}) + \rho^p(v_{n+1}, v_{n+2})]. \end{aligned}$$

If $\rho(v_n, v_{n+1}) < \rho(v_{n+1}, v_{n+2})$, then:

$$\alpha_1 \rho^p(v_{n+1}, v_{n+2}) \leq 2\theta \rho^p(v_{n+1}, v_{n+2}) \Rightarrow 1 \leq \frac{2\theta}{\alpha_1} = \frac{\theta}{\alpha_1/2} \leq \frac{\theta}{\gamma}.$$

This is a contradiction and, hence, $\rho(v_{n+1}, v_{n+2}) \leq \rho(v_n, v_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, $\{\rho(v_n, v_{n+1})\} = \{\rho(\mathcal{F}^n v_0, \mathcal{F}^{n+1} v_0)\}$ is a monotone decreasing sequence of positive real numbers. Setting:

$$s = \rho(v_0, \mathcal{F}v_0) \geq \dots \geq \rho^p(\mathcal{F}^n v_0, \mathcal{F}^{n+1} v_0) \geq \rho^p(\mathcal{F}^{n+1} v_0, \mathcal{F}^{n+2} v_0) \geq \dots$$

Now, as in Theorem 2, it arises the following cases:

Case-(i): When $2\theta < \alpha_1$ and setting $h^p = \frac{\theta}{\gamma} < 1$. Taking $v = v_0$ and $w = \mathcal{F}v_0$ in (3), we obtain:

$$\begin{aligned} \alpha_1 \rho^p(\mathcal{F}v_0, \mathcal{F}^2 v_0) &\leq \alpha_1 \rho^p(\mathcal{F}v_0, \mathcal{F}^2 v_0) + \alpha_2 \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) \\ &\leq \theta[\rho^p(v_0, \mathcal{F}v_0) + \rho^p(\mathcal{F}v_0, \mathcal{F}^2 v_0)] \leq 2\theta s^p \\ \Rightarrow \rho^p(\mathcal{F}v_0, \mathcal{F}^2 v_0) &\leq \frac{2\theta}{\alpha_1} s^p = \frac{\theta}{\alpha_1/2} s^p \leq \frac{\theta}{\gamma} s^p = (hs)^p \Rightarrow \rho(\mathcal{F}v_0, \mathcal{F}^2 v_0) \leq hs. \end{aligned}$$

Again, taking $v = \mathcal{F}v_0$ and $w = \mathcal{F}^2 v_0$ in (3), we obtain:

$$\begin{aligned} \alpha_1 \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) &\leq \alpha_1 \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) + \alpha_2 \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) \\ &\leq \theta[\rho^p(\mathcal{F}v_0, \mathcal{F}^2 v_0) + \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0)] \\ &\leq 2\theta \rho^p(\mathcal{F}v_0, \mathcal{F}^2 v_0) = 2\theta (hs)^p \\ \Rightarrow \rho(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) &\leq \frac{\theta}{\alpha_1/2} (hs)^p \leq \frac{\theta}{\gamma} (hs)^p = h^2 s. \end{aligned}$$

Additionally, from (3), we obtain:

$$\begin{aligned} \alpha_1 \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) &\leq \alpha_1 \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) + \alpha_2 \rho^p(\mathcal{F}^4 v_0, \mathcal{F}^5 v_0) \\ &\leq \theta[\rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) + \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0)] \\ &\leq 2\theta \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) = 2\theta (h^2 s)^p \\ \Rightarrow \rho(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) &\leq h^3 s. \end{aligned}$$

Similarly, we obtain:

$$\rho(\mathcal{F}^4 v_0, \mathcal{F}^5 v_0) \leq h^4 s.$$

In general, we obtain:

$$\rho(\mathcal{F}^n v_0, \mathcal{F}^{n+1} v_0) \leq h^n s, \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Therefore, $\rho(\mathcal{F}^n v_0, \mathcal{F}^{n+1} v_0) \rightarrow 0$ as $n \rightarrow +\infty$.

Case-(ii): When $2\theta < \alpha_2$ and setting $h^p = \frac{\theta}{\gamma} < 1$. Taking $v = v_0$ and $w = \mathcal{F}v_0$ in (3), we have:

$$\begin{aligned} \alpha_2 \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) &\leq \alpha_1 \rho^p(\mathcal{F}v_0, \mathcal{F}^2 v_0) + \alpha_2 \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) \\ &\leq \theta[\rho^p(v_0, \mathcal{F}v_0) + \rho^p(\mathcal{F}v_0, \mathcal{F}^2 v_0)] \leq 2\theta s^p \\ \Rightarrow \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) &\leq \frac{2\theta}{\alpha_2} s^p = \frac{\theta}{\alpha_2/2} s^p \leq \frac{\theta}{\gamma} s^p = (hs)^p. \end{aligned}$$

Therefore, we obtain:

$$\rho(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) \leq hs.$$

Again, taking $v = \mathcal{F}v_0$ and $w = \mathcal{F}^2v_0$ in (3), we obtain:

$$\begin{aligned}\alpha_2\rho^p(\mathcal{F}^3v_0, \mathcal{F}^4v_0) &\leq \alpha_1\rho^p(\mathcal{F}^2v_0, \mathcal{F}^3v_0) + \alpha_2\rho^p(\mathcal{F}^3v_0, \mathcal{F}^4v_0) \\ &\leq \theta[\rho^p(\mathcal{F}v_0, \mathcal{F}^2v_0) + \rho^p(\mathcal{F}^2v_0, \mathcal{F}^3v_0)] \leq 2\theta s^p \\ \Rightarrow \rho^p(\mathcal{F}^3v_0, \mathcal{F}^4v_0) &\leq \frac{\theta}{\alpha_2/2}s^p \leq \frac{\theta}{\gamma}s^p \Rightarrow \rho(\mathcal{F}^3v_0, \mathcal{F}^4v_0) \leq hs.\end{aligned}$$

And:

$$\begin{aligned}\alpha_2\rho^p(\mathcal{F}^4v_0, \mathcal{F}^5v_0) &\leq \alpha_1\rho^p(\mathcal{F}^3v_0, \mathcal{F}^4v_0) + \alpha_2\rho^p(\mathcal{F}^4v_0, \mathcal{F}^5v_0) \\ &\leq \theta[\rho^p(\mathcal{F}^2v_0, \mathcal{F}^3v_0) + \rho^p(\mathcal{F}^3v_0, \mathcal{F}^4v_0)] \leq 2\theta s^p \\ \Rightarrow \rho(\mathcal{F}^4v_0, \mathcal{F}^5v_0) &\leq h^2s.\end{aligned}$$

Additionally, we have:

$$\rho(\mathcal{F}^5v_0, \mathcal{F}^6v_0) \leq h^2s.$$

Further, we obtain:

$$\rho(\mathcal{F}^6v_0, \mathcal{F}^7v_0) \leq h^3s.$$

Additionally, we obtain:

$$\rho(\mathcal{F}^7v_0, \mathcal{F}^8v_0) \leq h^3s.$$

In general, we obtain:

$$\rho(\mathcal{F}^m v_0, \mathcal{F}^{m+1} v_0) \leq h^l s,$$

whenever $m = 2l$ or $m = 2l + 1$. Therefore, $\rho(\mathcal{F}^m v_0, \mathcal{F}^{m+1} v_0) \rightarrow 0$ as $m \rightarrow +\infty$.

Following the same steps as in Theorem 2, one can show in all cases that the sequence $\{v_n\}$ is a Cauchy in $\mathfrak{N}$. Since $\mathfrak{N} = \bigcup_{i=1}^m \mathcal{A}_i$ is closed in $\mathfrak{M}$, then $\mathfrak{N}$ is also complete and there exists a point $z \in \mathfrak{N}$ such that $v_n = \mathcal{F}^n v_0 \rightarrow z$ as $n \rightarrow +\infty$. Further, as $\mathfrak{N} = \bigcup_{i=1}^m \mathcal{A}_i$ is a cyclic representation of $\mathfrak{N}$ with respect to $\mathcal{F}$, the sequence $\{v_n\}$ has an infinite number of terms in $\mathcal{A}_i$ for all $i \in \{1, 2, \dots, m\}$. It follows that $z \in \bigcap_{i=1}^m \mathcal{A}_i$, so $\bigcap_{i=1}^m \mathcal{A}_i \neq \emptyset$.

We prove that $\text{Fix}(\mathcal{F}) \neq \emptyset$. Taking $z \in \mathcal{A}_i$, $\mathcal{F}z \in \mathcal{A}_{i+1}$ for $i \in \{1, 2, \dots, m\}$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ with $v_{n_k} \in \mathcal{A}_{n_k}$ and $\mathcal{F}v_{n_k} \in \mathcal{A}_{n_k+1}$. From (3), we obtain:

$$\begin{aligned}\alpha_1\rho^p(v_{n_{k+1}}, \mathcal{F}z) &\leq \alpha_1\rho^p(v_{n_{k+1}}, \mathcal{F}z) + \alpha_2\rho^p(v_{n_{k+2}}, \mathcal{F}^2z) \\ &= \alpha_1\rho^p(\mathcal{F}v_{n_k}, \mathcal{F}z) + \alpha_2\rho^p(\mathcal{F}^2v_{n_k}, \mathcal{F}^2z) \\ &\leq \theta[\rho^p(v_{n_k}, \mathcal{F}v_{n_k}) + \rho^p(z, \mathcal{F}z)] \\ &= \theta[\rho^p(v_{n_k}, v_{n_k+1}) + \rho^p(z, \mathcal{F}z)].\end{aligned}$$

Taking the limit as $n \rightarrow +\infty$, we obtain:

$$\begin{aligned}\rho^p(z, \mathcal{F}z) &\leq \frac{\theta}{\alpha_1}\rho^p(z, \mathcal{F}z) \leq \frac{\theta}{\gamma}\rho^p(z, \mathcal{F}z) = h^p\rho^p(z, \mathcal{F}z) \\ \Rightarrow (1-h)\rho(z, \mathcal{F}z) &\leq 0.\end{aligned}$$

It follows that $(1-h)\rho(z, \mathcal{F}z) \leq 0$ and, hence, $\rho(z, \mathcal{F}z) = 0$. Therefore, $\text{Fix}(\mathcal{F}) \neq \emptyset$. Finally, we prove that $\text{Fix}(\mathcal{F})$ is singleton. Let $z^* \in \bigcap_{i=1}^m \mathcal{A}_i$ be another element of $\text{Fix}(\mathcal{F})$ such that $z \neq z^*$. Now, from (3), we obtain:

$$\begin{aligned}\alpha_1\rho^p(\mathcal{F}z, \mathcal{F}z^*) + \alpha_2\rho^p(\mathcal{F}^2z, \mathcal{F}^2z^*) &\leq \theta[\rho^p(z, \mathcal{F}z) + \rho^p(z^*, \mathcal{F}z^*)] \\ \Rightarrow (\alpha_1 + \alpha_2)\rho^p(z, z^*) &\leq 0.\end{aligned}$$

It follows that $\alpha_1 + \alpha_2 = 0$. This is a contradiction and, hence, $\text{Fix}(\mathcal{F})$ is singleton. $\square$

Example 6. Let $\mathfrak{M} = [0, 1]$ with the usual metric $\rho(v, w) = |v - w|$. Consider $\mathcal{A}_1 = [0, \frac{1}{2}]$ and $\mathcal{A}_2 = [\frac{1}{5}, 1]$ be two closed subsets of $\mathfrak{M}$ such that $\mathfrak{M} = \mathcal{A}_1 \cup \mathcal{A}_2 = [0, 1]$ and $\mathcal{A}_1 \cap \mathcal{A}_2 = [\frac{1}{5}, \frac{1}{2}]$. Let $\mathcal{F} : \mathfrak{M} \rightarrow \mathfrak{M}$ be a function defined by:

$$\mathcal{F}v = \begin{cases} \frac{1}{5}, & 0 \leq v \leq \frac{1}{2} \\ \frac{1}{5}(1-v), & \frac{1}{2} < v \leq 1. \end{cases}$$

Obviously, $\mathcal{F}(\mathcal{A}_1) = \{\frac{1}{5}\} \subseteq \mathcal{A}_2 = [\frac{1}{5}, 1]$ and $\mathcal{F}(\mathcal{A}_2) = [0, \frac{1}{10}) \cup \{\frac{1}{5}\} \subseteq \mathcal{A}_1 = [0, \frac{1}{2}]$. This shows that $\mathfrak{M} = \mathcal{A}_1 \cup \mathcal{A}_2$ is a cyclic representation of $\mathfrak{M}$ with respect to $\mathcal{F}$.

Now, for $v \in [0, \frac{1}{2}]$ and $w \in (\frac{1}{2}, 1]$, we obtain:

$$|\mathcal{F}v - \mathcal{F}w| = \left| \frac{1}{5} - \frac{1}{5}(1-w) \right| = \left| \frac{1}{5}w \right|$$

and:

$$|\mathcal{F}^2v - \mathcal{F}^2w| = \left| \frac{1}{5} - \frac{1}{5} \right| = 0.$$

For all $v \in [0, \frac{1}{2}]$ and $w \in (\frac{1}{2}, 1]$, taking with $p = 1$, $\alpha_1 = \alpha_2 = \frac{1}{2}$, we obtain:

$$\alpha_1\rho(\mathcal{F}v, \mathcal{F}w) + \alpha_2\rho(\mathcal{F}^2v, \mathcal{F}^2w) = \frac{1}{2} \left(\left| \frac{1}{5} - \frac{1}{5}(1-w) \right| + \left| \frac{1}{5} - \frac{1}{5} \right| \right) = \frac{1}{2} \left| \frac{1}{5}w \right| = \frac{1}{5} \left| \frac{w}{2} \right|.$$

Additionally, we have:

$$\frac{6w-1}{5} - \frac{w}{2} = \frac{7w-2}{10} > 0, \text{ for all } w \in (\frac{1}{2}, 1].$$

Therefore:

$$\begin{aligned} \alpha_1\rho(\mathcal{F}v, \mathcal{F}w) + \alpha_2\rho(\mathcal{F}^2v, \mathcal{F}^2w) &= \frac{1}{5} \left| \frac{w}{2} \right| \\ &\leq \frac{1}{5} \left[|v - \mathcal{F}v| + \left| \frac{6w-1}{5} \right| \right] \\ &= \frac{1}{7} \left[|v - \mathcal{F}v| + \left| w - \frac{1}{5}(1-w) \right| \right] \\ &= \theta [\rho(v, \mathcal{F}v) + \rho(w, \mathcal{F}w)], \end{aligned}$$

for all $v \in [0, \frac{1}{2}]$ and $w \in (\frac{1}{2}, 1]$, where $\theta = \frac{1}{5}$. This shows that $\mathcal{F}$ is $(\alpha, 1)$ -Kannan-type cyclic contraction of multi-index α of length $n = 2$ with $\theta = \frac{1}{5} < \gamma = \frac{1}{2} \min\{\alpha_1, \alpha_2\} = \frac{1}{4}$. Thus, all the conditions of Theorem 3 are satisfied and, hence, $\text{Fix}(\mathcal{F}) = \{\frac{1}{5}\} \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$.

Corollary 2. Let $(\mathfrak{M}, \rho)$ be a complete metric space and $\mathcal{F} : \mathfrak{M} \rightarrow \mathfrak{M}$ be an (α, p) -Kannan-type cyclic contraction of multi-index α of length $n = 2$ such that $\theta < \gamma = \frac{1}{2} \min\{\alpha_1, \alpha_2\}$. Then, $\text{Fix}(\mathcal{F}) \subseteq \cap_{i=1}^m \mathcal{A}_i$ is singleton.

Theorem 4. Let $(\mathfrak{M}, \rho)$ be a complete metric space and $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$ be non-empty closed subsets of $\mathfrak{M}$ such that $\mathfrak{M} = \cup_{i=1}^m \mathcal{A}_i$. Let $\mathcal{F} : \mathfrak{M} \rightarrow \mathfrak{M}$ be an (α, p) -Chatterjea-type cyclic contraction of multi-index α of length $n = 2$ such that $\theta < \gamma = \frac{1}{2p} \min\{\alpha_1, \alpha_2\}$; then, $\text{Fix}(\mathcal{F}) \subseteq \cap_{i=1}^m \mathcal{A}_i$ is singleton.

Proof. As in Theorem 2, we define a sequence $\{v_n\}$ given by $v_{n+1} = \mathcal{F}v_n = \mathcal{F}^{n+1}v_0 \in \mathcal{A}_{i_0+n}$ for all $n \geq 0$. Suppose that $v_{n+1} \neq v_n$ for all $n \geq 0$. Now, we show that $\{\rho(v_n, v_{n+1})\}$ is a monotone decreasing sequence of positive real numbers.

$$\begin{aligned} \alpha_1\rho^p(\mathcal{F}v_n, \mathcal{F}v_{n+1}) + \alpha_2\rho^p(\mathcal{F}^2v_n, \mathcal{F}^2v_{n+1}) &\leq \theta [\rho^p(v_n, \mathcal{F}v_{n+1}) + \rho^p(v_{n+1}, \mathcal{F}v_n)] \\ \Rightarrow \alpha_1\rho^p(v_{n+1}, v_{n+2}) + \alpha_2\rho^p(v_{n+2}, v_{n+3}) &\leq \theta \rho^p(v_n, v_{n+2}). \end{aligned}$$

Therefore:

$$\begin{aligned}\alpha_1 \rho^p(v_{n+1}, v_{n+2}) &\leq \alpha_1 \rho^p(v_{n+1}, v_{n+2}) + \alpha_2 \rho^p(v_{n+2}, v_{n+3}) \\ &\leq \theta \rho^p(v_n, v_{n+2}) \\ \Rightarrow \rho^p(v_{n+1}, v_{n+2}) &\leq \frac{\theta}{\alpha_1} [\rho(v_n, v_{n+1}) + \rho(v_{n+1}, v_{n+2})]^p.\end{aligned}$$

If $\rho(v_n, v_{n+1}) < \rho(v_{n+1}, v_{n+2})$, then:

$$\rho^p(v_{n+1}, v_{n+2}) \leq \frac{2^p \theta}{\alpha_1} \rho^p(v_{n+1}, v_{n+2}) \Rightarrow 1 \leq \frac{2^p \theta}{\alpha_1} = \frac{\theta}{\alpha_1 / 2^p} \leq \frac{\theta}{\gamma}.$$

This is a contradiction and, hence, $\rho(v_{n+1}, v_{n+2}) \leq \rho(v_n, v_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, $\{\rho(v_n, v_{n+1})\} = \{\rho(\mathcal{F}^n v_0, \mathcal{F}^{n+1} v_0)\}$ is a monotone decreasing sequence of positive real numbers. Setting:

$$s = \rho(v_0, \mathcal{F} v_0) \geq \dots \geq \rho^p(\mathcal{F}^n v_0, \mathcal{F}^{n+1} v_0) \geq \rho^p(\mathcal{F}^{n+1} v_0, \mathcal{F}^{n+2} v_0) \geq \dots$$

Now, as in Theorem 2, the following cases arise.

Case-(i): When $2^p \theta < \alpha_1$ and setting $h^p = \frac{\theta}{\gamma} < 1$. Taking $v = v_0$ and $w = \mathcal{F} v_0$ in (3), we obtain:

$$\begin{aligned}\alpha_1 \rho^p(\mathcal{F} v_0, \mathcal{F}^2 v_0) &\leq \alpha_1 \rho^p(\mathcal{F} v_0, \mathcal{F}^2 v_0) + \alpha_2 \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) \\ &\leq \theta [\rho^p(v_0, \mathcal{F}^2 v_0) + \rho^p(\mathcal{F} v_0, \mathcal{F} v_0)] \\ &\leq \theta [\rho(v_0, \mathcal{F} v_0) + \rho(\mathcal{F} v_0, \mathcal{F}^2 v_0)]^p \\ \Rightarrow \rho^p(\mathcal{F} v_0, \mathcal{F}^2 v_0) &\leq \frac{2^p \theta}{\alpha_1} s^p = \frac{\theta}{\alpha_1 / 2^p} s^p \leq \frac{\theta}{\gamma} s^p = (hs)^p \Rightarrow \rho(\mathcal{F} v_0, \mathcal{F}^2 v_0) \leq hs.\end{aligned}$$

Again, taking $v = \mathcal{F} v_0$ and $w = \mathcal{F}^2 v_0$ in (3), we obtain:

$$\begin{aligned}\alpha_1 \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) &\leq \alpha_1 \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) + \alpha_2 \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) \\ &\leq \theta [\rho^p(\mathcal{F} v_0, \mathcal{F}^3 v_0) + \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^2 v_0)] \\ &\leq \theta [\rho(\mathcal{F} v_0, \mathcal{F}^2 v_0) + \rho(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0)]^p \\ \Rightarrow \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) &\leq \frac{\theta}{\alpha_1 / 2^p} (hs)^p \leq \frac{\theta}{\gamma} (hs)^p = (h^2 s)^p \\ \Rightarrow \rho(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) &\leq h^2 s.\end{aligned}$$

Additionally, from (3), we obtain:

$$\begin{aligned}\alpha_1 \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) &\leq \alpha_1 \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) + \alpha_2 \rho^p(\mathcal{F}^4 v_0, \mathcal{F}^5 v_0) \\ &\leq \theta [\rho^p(\mathcal{F}^2 v_0, \mathcal{F}^4 v_0) + \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^3 v_0)] \\ \Rightarrow \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) &\leq \frac{\theta}{\alpha_1 / 2^p} (h^2 s)^p \leq \frac{\theta}{\gamma} (h^2 s)^p = (h^3 s)^p \\ \Rightarrow \rho(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) &\leq h^3 s.\end{aligned}$$

Similarly, we obtain:

$$\rho(\mathcal{F}^4 v_0, \mathcal{F}^5 v_0) \leq h^4 s.$$

In general, we obtain:

$$\rho(\mathcal{F}^n v_0, \mathcal{F}^{n+1} v_0) \leq h^n s, \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Therefore, $\rho(\mathcal{F}^n v_0, \mathcal{F}^{n+1} v_0) \rightarrow 0$ as $n \rightarrow +\infty$. Continuing the same process as in Theorem 2, we can show that $\{v_n\}$ is a Cauchy sequence in $\mathfrak{N}$.

Case-(ii): When $2^p \theta < \alpha_2$ and setting $h^p = \frac{\theta}{\gamma} < 1$. Taking $v = v_0$ and $w = \mathcal{F}v_0$ in (3), we have:

$$\begin{aligned}\alpha_2 \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) &\leq \alpha_1 \rho^p(\mathcal{F}v_0, \mathcal{F}^2 v_0) + \alpha_2 \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) \\ &\leq \theta[\rho^p(v_0, \mathcal{F}^2 v_0) + \rho^p(\mathcal{F}v_0, \mathcal{F}v_0)] \\ \Rightarrow \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) &\leq \frac{2^p \theta}{\alpha_2} s^p = \frac{\theta}{\alpha_2/2^p} s^p \leq \frac{\theta}{\gamma} s^p = (hs)^p \\ \Rightarrow \rho(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) &\leq hs.\end{aligned}$$

Again, taking $v = \mathcal{F}v_0$ and $w = \mathcal{F}^2 v_0$ in (3), we obtain:

$$\begin{aligned}\alpha_2 \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) &\leq \alpha_1 \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^3 v_0) + \alpha_2 \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) \\ &\leq \theta[\rho^p(\mathcal{F}v_0, \mathcal{F}^3 v_0) + \rho^p(\mathcal{F}^2 v_0, \mathcal{F}^2 v_0)] \\ \Rightarrow \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) &\leq \frac{\theta}{\alpha_2/2^p} s^p \leq \frac{\theta}{\gamma} s^p \Rightarrow \rho(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) \leq hs.\end{aligned}$$

And:

$$\begin{aligned}\alpha_2 \rho^p(\mathcal{F}^4 v_0, \mathcal{F}^5 v_0) &\leq \alpha_1 \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^4 v_0) + \alpha_2 \rho^p(\mathcal{F}^4 v_0, \mathcal{F}^5 v_0) \\ &\leq \theta[\rho^p(\mathcal{F}^2 v_0, \mathcal{F}^4 v_0) + \rho^p(\mathcal{F}^3 v_0, \mathcal{F}^3 v_0)] \\ &\leq 2^p \theta (hs)^p \\ \Rightarrow \rho^p(\mathcal{F}^4 v_0, \mathcal{F}^5 v_0) &\leq \frac{\theta}{\alpha_2/2^p} (hs)^p \leq \frac{\theta}{\gamma} (hs)^p = (h^2 s)^p \\ \Rightarrow \rho(\mathcal{F}^4 v_0, \mathcal{F}^5 v_0) &\leq h^2 s.\end{aligned}$$

Additionally, we have:

$$\rho(\mathcal{F}^5 v_0, \mathcal{F}^6 v_0) \leq h^2 s.$$

Further, we obtain:

$$\rho(\mathcal{F}^6 v_0, \mathcal{F}^7 v_0) \leq h^3 s.$$

Additionally, we obtain:

$$\rho(\mathcal{F}^7 v_0, \mathcal{F}^8 v_0) \leq h^3 s.$$

In general, we obtain:

$$\rho(\mathcal{F}^m v_0, \mathcal{F}^{m+1} v_0) \leq h^l s,$$

whenever $m = 2l$ or $m = 2l + 1$. Therefore, $\rho(\mathcal{F}^m v_0, \mathcal{F}^{m+1} v_0) \rightarrow 0$ as $m \rightarrow +\infty$. Following the same steps as in Theorem 2, one can show that the sequence $\{v_n\}$ is a Cauchy in $\mathfrak{N}$. Since $\mathfrak{N} = \cup_{i=1}^m \mathcal{A}_i$ is closed in $\mathfrak{M}$, then $\mathfrak{N}$ is also complete and there exists a point $z \in \mathfrak{N}$ such that $v_n = \mathcal{F}^n v_0 \rightarrow z$ as $n \rightarrow +\infty$. Further, as $\mathfrak{N} = \cup_{i=1}^m \mathcal{A}_i$ is a cyclic representation of $\mathfrak{N}$ with respect to $\mathcal{F}$, the sequence $\{v_n\}$ has an infinite number of terms in $\mathcal{A}_i$ for all $i \in \{1, 2, \dots, m\}$. It follows that $z \in \cap_{i=1}^m \mathcal{A}_i$, so $\cap_{i=1}^m \mathcal{A}_i \neq \emptyset$.

We prove that $\text{Fix}(\mathcal{F}) \neq \emptyset$. Taking $z \in \mathcal{A}_i$, $\mathcal{F}z \in \mathcal{A}_{i+1}$ for $i \in \{1, 2, \dots, m\}$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ with $v_{n_k} \in \mathcal{A}_{n_k}$ and $\mathcal{F}v_{n_k} \in \mathcal{A}_{n_k+1}$. From (3), we obtain:

$$\begin{aligned}\alpha_1 \rho^p(v_{n_{k+1}}, \mathcal{F}z) &\leq \alpha_1 \rho^p(v_{n_{k+1}}, \mathcal{F}z) + \alpha_2 \rho^p(v_{n_{k+2}}, \mathcal{F}^2z) \\ &= \alpha_1 \rho^p(\mathcal{F}v_{n_k}, \mathcal{F}z) + \alpha_2 \rho^p(\mathcal{F}^2v_{n_k}, \mathcal{F}^2z) \\ &\leq \theta[\rho^p(v_{n_k}, \mathcal{F}z) + \rho^p(z, \mathcal{F}v_{n_k})] \\ &= \theta[\rho^p(v_{n_k}, \mathcal{F}z) + \rho^p(z, v_{n_{k+1}})].\end{aligned}$$

Taking the limit as $n \rightarrow +\infty$, we obtain:

$$\begin{aligned}\rho^p(z, \mathcal{F}z) &\leq \frac{\theta}{\alpha_1} \rho^p(z, \mathcal{F}z) \leq \frac{\theta}{\gamma} \rho^p(z, \mathcal{F}z) = h^p \rho^p(z, \mathcal{F}z) \\ \Rightarrow (1 - h) \rho^p(z, \mathcal{F}z) &\leq 0.\end{aligned}$$

It follows that $\rho(z, \mathcal{F}z) = 0$ and, hence, $\text{Fix}(\mathcal{F}) \neq \emptyset$. Now, we prove that $\text{Fix}(\mathcal{F})$ is singleton. Let $z^* \in \cap_{i=1}^m \mathcal{A}_i$ be another element of $\text{Fix}(\mathcal{F})$ such that $z \neq z^*$. Again, from (3), we obtain:

$$\begin{aligned}\alpha_1 \rho^p(z, z^*) &= \alpha_1 \rho^p(\mathcal{F}z, \mathcal{F}z^*) \leq \alpha_1 \rho^p(\mathcal{F}z, \mathcal{F}z^*) + \alpha_2 \rho^p(\mathcal{F}^2z, \mathcal{F}^2z^*) \\ &\leq \theta[\rho^p(z, \mathcal{F}z^*) + \rho^p(z^*, \mathcal{F}z)] = 2\theta \rho^p(z, z^*) \\ \Rightarrow 1 &\leq \frac{2\theta}{\alpha_1} \leq \frac{2^p \theta}{\alpha_1} = \frac{\theta}{\alpha_1/2^p} \leq \frac{\theta}{\gamma}.\end{aligned}$$

This is a contradiction; hence, $\text{Fix}(\mathcal{F})$ is singleton, being a subset of $\cap_{i=1}^m \mathcal{A}_i$. $\square$

Example 7. Let $\mathfrak{M} = [0, 1]$ with the usual metric $\rho(v, w) = |v - w|$. Suppose $\mathcal{A}_1 = [0, \frac{1}{2}]$ and $\mathcal{A}_2 = [0, 1]$, then $\mathfrak{M} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_2 = [0, \frac{1}{2}]$. Define $\mathcal{F} : \mathfrak{M} \rightarrow \mathfrak{M}$ as $\mathcal{F}v = \frac{v^2}{5}$, for all $v \in \mathfrak{M}$. Then, we obtain $\mathcal{F}^2v = \frac{v^4}{125}$. Additionally, note that $\mathcal{F}(\mathcal{A}_1) = [0, \frac{1}{20}] \subseteq \mathcal{A}_2$, $\mathcal{F}(\mathcal{A}_2) = [0, \frac{1}{5}] \subseteq \mathcal{A}_1$ and, hence, $\mathfrak{M} = \cup \mathcal{A}_i$, where $i = 1, 2$ is a cyclic representation of $\mathfrak{M}$ with respect to $\mathcal{F}$. Now we have:

$$|\mathcal{F}v - \mathcal{F}w| = \frac{1}{5} |v^2 - w^2| = \frac{(v+w)}{5} |v - w| \leq \frac{3}{10} |v - w|$$

and:

$$\begin{aligned}|\mathcal{F}^2v - \mathcal{F}^2w| &= \left| \frac{v^4}{125} - \frac{w^4}{125} \right| = \frac{(v^2 + w^2)(v + w)}{125} |v - w| \\ &\leq \frac{3}{200} |v - w|,\end{aligned}$$

for all $v \in \mathcal{A}_1, w \in \mathcal{A}_2$. Additionally, one can check $\frac{1}{5}(v + w + 5) \geq 1$ for $v \in \mathcal{A}_1, w \in \mathcal{A}_2$, and:

$$\begin{aligned}|v - w| &\leq \left(\left| \frac{1}{5}(v + w + 5)(v - w) \right| \right) \\ &= \left| \frac{1}{5}(v^2 - w^2) + (v - w) \right| \\ &= \left(\left| \frac{1}{5}v^2 - w \right| + \left| v - \frac{1}{5}w^2 \right| \right) \\ &\leq \left(\left| \frac{1}{5}v^2 - w \right|^2 + \left| v - \frac{1}{5}w^2 \right|^2 \right).\end{aligned}$$

For $p = 1$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$, we obtain:

$$\begin{aligned}\alpha_1|\mathcal{F}v - \mathcal{F}w| + \alpha_2|\mathcal{F}^2v - \mathcal{F}^2w| &\leq \frac{1}{2}\left(\frac{3}{10} + \frac{3}{200}\right)|v - w| \\ &\leq \frac{63}{400}[|v - \mathcal{F}w| + |w - \mathcal{F}v|],\end{aligned}$$

where $\theta = \frac{63}{400} = 0.1575 < \gamma = \frac{1}{2} \min\{\alpha_1, \alpha_2\} = \frac{1}{4} = 0.25$ for all $v \in \mathcal{A}_1$, $w \in \mathcal{A}_2$. This shows that $\mathcal{F}$ is an $(\alpha, 1)$ -Chatterjea-type cyclic contraction of multi-index α of length $n = 2$. Thus, all the conditions of Theorem 4 are satisfied and, hence, $\text{Fix}(\mathcal{F}) = \{0\}$.

Corollary 3. Let $(\mathfrak{M}, \rho)$ be a complete metric space and $\mathcal{F} : \mathfrak{M} \rightarrow \mathfrak{M}$ be an (α, p) -Chatterjea-type cyclic contraction of multi-index α of length $n = 2$ such that $\theta < \frac{1}{2^p} \min\{\alpha_1, \alpha_2\}$. Then, $\text{Fix}(\mathcal{F}) \neq \emptyset$ and $\text{Fix}(\mathcal{F}) \subseteq \cap_{i=1}^m \mathcal{A}_i$ is singleton.

3. Application

In this section, we apply our result to establish the existence of a solution for a nonlinear Fredholm integral equation. We also provide one numerical example to validate our result.

Let $\mathfrak{M} = C[a, b]$ be the set of all real continuous functions on $[a, b]$ and ρ be a metric on $\mathfrak{M}$ defined by

$$\rho(f, g) = |f - g| = \sup_{t \in [a, b]} |f(t) - g(t)|,$$

for all $f, g \in \mathfrak{M}$. Then, $(\mathfrak{M}, \rho)$ is a complete metric space. Consider a nonlinear Fredholm integral equation:

$$v(t) = q(t) + \beta \int_a^b \mathcal{K}(t, r, v(r)) dr,$$

where $t, r \in [a, b]$ and β is a constant. Assume that $\mathcal{K} : [a, b] \times [a, b] \times \mathfrak{M} \rightarrow \mathbb{R}$ and $q : [a, b] \rightarrow \mathbb{R}$ are continuous functions, where $\mathcal{K}(t, r, v(r))$ is the kernel of the equation and $q(t)$ is a given function.

Theorem 5. Consider $(\mathfrak{M}, \rho)$ to be a metric space with metric $\rho(f, g) = |f - g| = \sup_{t \in [a, b]} |f(t) - g(t)|$ for all $f, g \in \mathfrak{M}$ and $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$ to be non-empty closed subsets of $\mathfrak{M}$ such that $\mathfrak{N} = \cup_{i=1}^m \mathcal{A}_i$. Let $\mathcal{F} : \mathfrak{N} \rightarrow \mathfrak{N}$ be a nonlinear operator such that $\cup_{i=1}^m \mathcal{A}_i$ is a cyclic representation of $\mathfrak{N}$ with respect to $\mathcal{F}$, defined by

$$\mathcal{F}v(t) = q(t) + \beta \int_a^b \mathcal{K}(t, r, v(r)) dr, \quad (5)$$

for all $v \in \mathcal{A}_i$, where $i \in \{1, 2, \dots, m\}$. Suppose that the following assumption holds: there exists θ , $0 < \alpha < 1$, and $p \geq 1$ such that $\theta < \gamma = \frac{1}{2^p} \min\{\alpha, 1 - \alpha\}$, satisfying the inequality

$$\begin{aligned}\alpha|\mathcal{K}(t, r, v(r)) - \mathcal{K}(t, r, w(r))|^p + (1 - \alpha)|\mathcal{K}(t, r, \mathcal{F}v(r)) - \mathcal{K}(t, r, \mathcal{F}w(r))|^p \\ \leq \xi(t, r) \left\{ |v(r) - \mathcal{F}w(r)|^p + |w(r) - \mathcal{F}v(r)|^p \right\},\end{aligned}$$

for all $v \in \mathcal{A}_i$, $w \in \mathcal{A}_{i+1}$, where $(r, t) \in [a, b] \times [a, b]$ and $\xi : [a, b] \times [a, b] \rightarrow \mathbb{R}$ is a continuous function satisfying:

$$\sup_{t \in [a, b]} \int_a^b \xi(t, r) dr < \frac{\theta}{\beta^p (b - a)^{p-1}}.$$

Then, the Fredholm integral Equation (5) has a unique solution in $\cap_{i=1}^m \mathcal{A}_i$.

Proof. First, we show that $\mathcal{F}$ is an (α, p) -Chatterjea-type cyclic contraction of multi-index α of length $n = 2$ for $p \geq 1$. One can prove that $\mathcal{F}$ satisfies (4) for $p = 1$. Now, we show

that $\mathcal{F}$ satisfies (4) for $p > 1$. As $p > 1$, there exists $q > 1$ such that $\frac{1}{q} + \frac{1}{p} = 1$; then, using Holder's inequality, we have:

$$\begin{aligned}
 & \alpha \left| \mathcal{F}v(t) - \mathcal{F}w(t) \right|^p + (1 - \alpha) \left| \mathcal{F}^2v(t) - \mathcal{F}^2w(t) \right|^p \\
 &= \alpha \beta^p \left| \int_a^b \mathcal{K}(t, r, v(r)) dr - \int_a^b \mathcal{K}(t, r, w(r)) dr \right|^p \\
 &+ (1 - \alpha) \beta^p \left| \int_a^b \mathcal{K}(t, r, \mathcal{F}v(r)) dr - \int_a^b \mathcal{K}(t, r, \mathcal{F}w(r)) dr \right|^p \\
 &\leq \alpha \beta^p \left[\int_a^b \left| \mathcal{K}(t, r, v(r)) dr - \mathcal{K}(t, r, w(r)) \right| dr \right]^p \\
 &+ (1 - \alpha) \beta^p \left[\int_a^b \left| \mathcal{K}(t, r, \mathcal{F}v(r)) dr - \mathcal{K}(t, r, \mathcal{F}w(r)) \right| dr \right]^p \\
 &\leq \alpha \beta^p \left[\left\{ \int_a^b 1^q dr \right\}^{\frac{1}{q}} \left\{ \int_a^b \left| \mathcal{K}(t, r, v(r)) dr - \mathcal{K}(t, r, w(r)) \right|^p dr \right\}^{\frac{1}{p}} \right]^p \\
 &+ (1 - \alpha) \beta^p \left[\left\{ \int_a^b 1^q dr \right\}^{\frac{1}{q}} \left\{ \int_a^b \left| \mathcal{K}(t, r, \mathcal{F}v(r)) dr - \mathcal{K}(t, r, \mathcal{F}w(r)) \right|^p dr \right\}^{\frac{1}{p}} \right]^p \\
 &\leq \alpha \beta^p (b - a)^{\frac{p}{q}} \int_a^b \left| \mathcal{K}(t, r, v(r)) dr - \mathcal{K}(t, r, w(r)) \right|^p dr \\
 &+ (1 - \alpha) \beta^p (b - a)^{\frac{p}{q}} \int_a^b \left| \mathcal{K}(t, r, \mathcal{F}v(r)) dr - \mathcal{K}(t, r, \mathcal{F}w(r)) \right|^p dr \\
 &= (b - a)^{p-1} \beta^p \left[\int_a^b \left\{ \alpha \left| \mathcal{K}(t, r, v(r)) dr - \mathcal{K}(t, r, w(r)) \right|^p \right. \right. \\
 &\quad \left. \left. + (1 - \alpha) \left| \mathcal{K}(t, r, \mathcal{F}v(r)) dr - \mathcal{K}(t, r, \mathcal{F}w(r)) \right|^p \right\} dr \right] \\
 &\leq (b - a)^{p-1} \beta^p \left[\int_a^b \xi(t, r) \left\{ |v(r) - \mathcal{F}w(r)|^p + |w(r) - \mathcal{F}v(r)|^p \right\} dr \right].
 \end{aligned}$$

Taking supremum over the interval $[a, b]$ on both sides of the inequality, we obtain:

$$\begin{aligned}
 & \alpha \left| \mathcal{F}v - \mathcal{F}w \right|^p + (1 - \alpha) \left| \mathcal{F}^2v - \mathcal{F}^2w \right|^p \\
 &= \sup_{t \in [a, b]} \left[\alpha \left| \mathcal{F}v(t) - \mathcal{F}w(t) \right|^p + (1 - \alpha) \left| \mathcal{F}^2v(t) - \mathcal{F}^2w(t) \right|^p \right] \\
 &\leq (b - a)^{p-1} \beta^p \left(\sup_{t \in [a, b]} \int_a^b \xi(t, u) du \right) \left[|v - \mathcal{F}w|^p + |w - \mathcal{F}v|^p \right] \\
 &\leq \theta \left[|v - \mathcal{F}w|^p + |w - \mathcal{F}v|^p \right],
 \end{aligned}$$

where $\theta < \gamma = \frac{1}{2^p} \min\{\alpha, 1 - \alpha\}$ for all $v \in \mathcal{A}_i, w \in \mathcal{A}_{i+1}$. This shows that $\mathcal{F}$ satisfies the inequality (4), that is to say, $\mathcal{F}$ satisfies the (α, p) -Chatterjea-type cyclic contraction of multi-index α of length $n = 2$. Thus, all the conditions of Theorem 4 are satisfied and, hence, the integral operator $\mathcal{F}$ defined by (5) has a unique solution in $\cap_{i=1}^m \mathcal{A}_i$. $\square$

4. Conclusions

We introduced the notions of various types of (α, p) -cyclic contractions, viz. (α, p) -cyclic contraction, (α, p) -Kannan (resp. Chatterjea)-type cyclic contractions. Using these new types of contractions, we can extend the other fixed-point results in the existing literature.

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