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On the Functions of Marcinkiewicz Integrals along Surfaces of Revolution on Product Domains via Extrapolation

Hussain Al-Qassem [1] and Mohammed Ali [2,*]

[1] Mathematics Program, Department of Mathematics, Statistics and Physics, College of Arts and Sciences, Qatar University, Doha 2713, Qatar; husseink@qu.edu.qa
[2] Department of Mathematics and Statistics, Faculty of Science and Arts, Jordan University of Science and Technology, P.O. Box 3030, Irbid 22110, Jordan
* Correspondence: myali@just.edu.jo

Abstract: In this paper, we establish certain L^p bounds for several classes of rough Marcinkiewicz integrals over surfaces of revolution on product spaces. By using these bounds and using an extrapolation argument, we obtain the L^p boundedness of these Marcinkiewicz integrals under very weak conditions on the kernel functions. Our results represent natural extensions and improvements of several known results on Marcinkiewicz integrals.

Keywords: rough integrals; surfaces of revolution; product domains; Marcinkiewicz integrals; extrapolation

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1. Introduction

Throughout this article, let $\kappa \geq 2$ ($\kappa = m$ or n) and $\mathbb{R}^\kappa$ be the Euclidean space of dimension κ. Additionally, let $\mathbb{S}^{\kappa-1}$ be the unit sphere in $\mathbb{R}^\kappa$ equipped with the normalized Lebesgue surface measure $d\mu_\kappa(\cdot) \equiv d\mu$.

For $\tau_1 = \alpha_1 + i\beta_1, \tau_2 = \alpha_2 + i\beta_2$ $(\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ *with* $\alpha_1, \alpha_2 > 0)$, we let:

$$K_{\Omega,h}(v,u) = \frac{\Omega(v,u)h(|v|,|u|)}{|v|^{m-\tau_1}|u|^{n-\tau_2}},$$

where h is a measurable function defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and Ω is a measurable function defined on $\mathbb{R}^m \times \mathbb{R}^n$, integrable over $\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}$, and satisfies the following:

$$\Omega(rv, su) = \Omega(v,u), \ \forall r,s > 0, \tag{1}$$

$$\int_{\mathbb{S}^{m-1}} \Omega(v,.)d\mu(v) = \int_{\mathbb{S}^{n-1}} \Omega(.,u)d\mu(u) = 0. \tag{2}$$

For a suitable mapping $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, the parametric Marcinkiewicz integral operator $\mathcal{M}_{\Omega,\Phi,h}$ along the surface of revolution $\Gamma_\Phi(x,y) = (x, y, \Phi(|x|, |y|))$ is defined, initially for $f \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$, by:

$$\mathcal{M}_{\Omega,\Phi,h}(f)(x,y,w) = \left(\iint_{\mathbb{R}_+ \times \mathbb{R}_+} |H_{r,s}(f)(x,y,w)|^2 \frac{drds}{rs} \right)^{1/2}, \tag{3}$$

where:

$$H_{r,s}(f)(x,y,w) = \frac{1}{r^{\tau_1}s^{\tau_2}} \int_{|u|\leq s} \int_{|v|\leq r} f(x-v, y-u, w-\Phi(|v|,|u|))K_{\Omega,h}(v,u)dvdu.$$

We remark that the Marcinkiewicz operator is a natural generalization of the Marcinkiewicz operator $\mathcal{M}^{\phi}_{\Omega,h}$ along the surface of revolution $\Gamma_{\phi}(x) = (x, \phi(|x|))$ in the one parameter setting, which is given by:

$$\mathcal{M}^{\phi}_{\Omega,h}(f)(x, x_{m+1}) = \left(\int_{\mathbb{R}_+}\left|\frac{1}{r^{\tau_1}}\int_{|v|\leq r} f(x-v, x_{m+1}-\phi(|v|))\frac{\Omega(v)h(|v|)}{|v|^{m-\tau_1}}dv\right|^2\frac{dr}{r}\right)^{1/2}. \quad (4)$$

The study of the L^p boundedeness of the operator $\mathcal{M}^{\phi}_{\Omega,h}$ under various conditions on h, Ω and ϕ has attracted the attention of many authors. For instance, the integral operator $\mathcal{M}^{\phi}_{\Omega,h}$ was initiated by the author of [1] whenever $\phi(t) = t$ and $h \equiv 1$. Precisely, he proved the L^p $(1 < p \leq 2)$ boundedness of $\mathcal{M}^{\phi}_{\Omega,1}$ provided that $\Omega \in Lip_{\nu}(\mathbb{S}^{m-1})$ for some $\nu \in (0,1]$. Thereafter, the study of the operator $\mathcal{M}^{\phi}_{\Omega,h}$ has been studied by many researchers. For a sample of known results relevant to our study, the readers are referred to consult [2–9].

Our main focus in this paper is the operator $\mathcal{M}_{\Omega,\Phi,h}$. When $\Phi \equiv 0$ and $\tau_1 = 1 = \tau_2$, we denote the operator $\mathcal{M}_{\Omega,\Phi,h}$ by $\mathcal{M}_{\Omega,h}$. In addition, when $h \equiv 1$, then $\mathcal{M}_{\Omega,h}$ reduces to the classical Marcinkiewicz integral on product domains, which is denoted by $\mathcal{M}_{\Omega}$. The investigation of the L^p boundedness of the operator $\mathcal{M}_{\Omega}$ was initiated in [10], in which the author proved the L^2 boundedness of $\mathcal{M}_{\Omega}$ under the condition $\Omega \in L(\log L)^2(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$. Subsequently, the L^p boundedness of $\mathcal{M}_{\Omega}$ has attracted the attention of many authors. For instance, in [11], the authors proved the L^p $(1 < p < \infty)$ boundedness of $\mathcal{M}_{\Omega}$ if $\Omega \in L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$. Further, they pointed out that by adapting a similar argument as that used in [12] to the product space setting, the assumption $\Omega \in L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ is optimal in the sense that if we replace it by any weaker condition $\Omega \in L(\log L)^{\varepsilon}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ with $0 < \varepsilon < 1$, then $\mathcal{M}_{\Omega}$ may lose the L^2 boundedness. On the other hand, under the assumption Ω belongs to $B^{(0,0)}_q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ with $q > 1$, it was proved in [13] that $\mathcal{M}_{\Omega}$ is of type (p, p) for all $p \in (1,\infty)$ and that the condition $\Omega \in B^{(0,0)}_q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ is optimal in the sense that we cannot replace it by $\Omega \in B^{(0,\varepsilon)}_q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ with $\varepsilon \in (-1, 0)$ so that $\mathcal{M}_{\Omega}$ is bounded on $L^2(\mathbb{R}^m \times \mathbb{R}^n)$. Here, $B^{(0,\nu)}_q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ is a special class of block spaces introduced in [14]. Later on, the authors of [15] employed Yano's extrapolation argument [16] to establish the L^p boundedness of $\mathcal{M}_{\Omega,h}$ for all $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, provided that Ω belongs to either $L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ or to $B^{(0,0)}_q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ and $h \in \Delta_{\gamma}(\mathbb{R}_+ \times \mathbb{R}_+)$ for some $\gamma > 1$, where $\Delta_{\gamma}(\mathbb{R}_+ \times \mathbb{R}_+)$ (for $\gamma > 1$) denotes the collection of measurable functions h such that:

$$\|h\|_{\Delta_{\gamma}(\mathbb{R}_+\times\mathbb{R}_+)} = \sup_{j,k\in\mathbb{Z}}\left(\int_{2^j}^{2^{j+1}}\int_{2^k}^{2^{k+1}}|h(r,s)|^{\gamma}\frac{drds}{rs}\right)^{1/\gamma} < \infty.$$

For a sample of past studies, as well as more information about the applications and development of the operator $\mathcal{M}_{\Omega}$, we refer the readers to see [11,13,17–22] and the references therein.

By the work done in these cited papers, many mathematicians have been motivated to study the Marcinkiewicz operator along surfaces of revolution on product spaces of the form:

$$\mathcal{M}^{\phi,\psi}_{\Omega,h}(f)(x, x_{m+1}, y, y_{n+1}) = \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|H^{\phi,\psi}_{r,s}(f)\right|^2\frac{drds}{rs}\right)^{1/2}, \quad (5)$$

where:

$$H^{\phi,\psi}_{r,s}(f) = \frac{1}{r^{\tau_1}s^{\tau_2}}\int_{|u|\leq s}\int_{|v|\leq r} f(x-v, x_{m+1}-\phi(|v|), y-u, y_{n+1}-\psi(|u|))K_{\Omega,h}(v,u)dvdu.$$

The L^p boundedness of the operator $\mathcal{M}^{\phi,\psi}_{\Omega,h}$ under different conditions on the functions ϕ, ψ, Ω, and h was discussed by many authors (one can consult [19,23–26]).

Very recently, in [27], the authors studied the L^p boundedness of the singular integral operators $\mathcal{T}_{\Omega,\Phi,h}$ along surfaces of revolution on product domains, which is defined by:

$$\mathcal{T}_{\Omega,\Phi,h}(f)(x,y,w) = \iint_{\mathbb{R}^m\times\mathbb{R}^n} f(x-v, y-u, w-\Phi(|v|,|u|))\frac{\Omega(v,u)h(|v|,|u|)}{|v|^m|u|^n}dvdu, \quad (6)$$

where $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a suitable mapping. Under various conditions on Φ, the authors proved the L^p boundedness of $\mathcal{T}_{\Omega,\Phi,h}$ if Ω belongs to either $L(\log L)^2(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ or to $B_q^{(1,0)}(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$.

In light of the results in [24] regarding the boundedness of Marcinkiewicz operator $\mathcal{M}^{\phi,\psi}_{\Omega,h}$ and of the results in [27] regarding the boundedness of singular integral $\mathcal{T}_{\Omega,\Phi,h}$, a question arises naturally, which is the following:

Question: Under the same conditions as those imposed on Φ in [27], is the operator $\mathcal{M}_{\Omega,\Phi,h}$ bounded whenever $h \in \Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\gamma > 1$ and $\Omega \in L(\log L)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}) \cup B_q^{(0,0)}(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ with $q > 1$?

In this article, we shall answer this question in affirmative. Indeed, we have the following:

Theorem 1. *Let $\Phi \in C^1(\mathbb{R}_+\times\mathbb{R}_+)$ such that for any fixed $t, \ell > 0$, we have $\Psi_{1,t}(.) = \Phi(t,.)$, $\Psi_{2,\ell}(.) = \Phi(.,\ell)$ are in $C^2(\mathbb{R}_+)$, increasing and convex functions with $\Psi_{1,t}(0) = \Psi_{2,\ell}(0) = 0$. Suppose that $h \in \Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\gamma > 1$ and $\Omega \in L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $q \in (1,2]$. Then, there is a constant C_p such that:*

$$\|\mathcal{M}_{\Omega,\Phi,h}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \le C_p\frac{\gamma}{(q-1)(\gamma-1)}\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \quad (7)$$

for all $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.

Theorem 2. *Let Ω and h be given as in Theorem 1. Suppose that $\Phi(t,\ell) = \sum_{i=0}^{d_1}\sum_{j=0}^{d_2} a_{j,i}t^{\alpha_i}\ell^{\beta_j}$ with $\alpha_i, \beta_j > 0$ is a generalized polynomial on $\mathbb{R}^2$. Then, there is a constant C_p such that the estimate (7) holds for all $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.*

Theorem 3. *Let Ω and h be given as in Theorem 1. Suppose that $\Phi(t,\ell) = \phi(t)P(\ell)$, where $\phi(t)$ is in $C^2(\mathbb{R}_+)$, increasing and convex function with $\phi(0) = 0$, and P is a generalized polynomial given by $P(\ell) = \sum_{j=0}^{d_2} a_j\ell^{\beta_j}$ with $\beta_j > 0$. Then, there is a constant C_p such that the estimate (7) holds for all $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.*

Theorem 4. *Let Ω and h be given as in Theorem 1. Suppose that $\Phi(t,\ell) = \phi_1(t) + \phi_2(\ell)$, where $\phi_j(\cdot)$ ($j = 1,2$) is either a generalized polynomial or is in $C^2(\mathbb{R}_+)$, increasing and convex function with $\phi_j(0) = 0$. Then, there is a constant C_p such that the estimate (7) holds for all $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.*

By the conclusions from Theorems 1–4, along with the extrapolation argument found in [16,28], we obtain the following:

Theorem 5. *Let Ω satisfy the conditions (1) and (2). Suppose that h and Φ are given as in either Theorem 1, Theorem 2, Theorem 3, or Theorem 4.*

(i) If $\mho \in B_q^{(0,0)}(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $q > 1$, then the inequality:

$$\begin{aligned}&\left\|\mathcal{M}_{\Omega,\Phi,h}(f)\right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\\ \leq\ & C_p\|h\|_{\Delta_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\left(1+\|\Omega\|_{B_q^{(0,0)}(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\right)\end{aligned}$$

holds for all $|1/p-1/2|<\min\{1/2,1/\gamma'\}$;

(ii) *If* $\Omega\in L(\log L)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$, *then the inequality:*

$$\begin{aligned}&\left\|\mathcal{M}_{\Omega,\Phi,h}(f)\right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\\ \leq\ & C_p\|h\|_{\Delta_\mu(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\left(1+\|\Omega\|_{L(logL)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\right)\end{aligned}$$

holds for all $|1/p-1/2|<\min\{1/2,1/\gamma'\}$.

Remark 1. *The conditions on* Ω *in Theorem 5 are optimal. In fact, they are the weakest conditions in their particular classes (see [11,13]).*

Remark 2. *For the special cases* $h\equiv 1$ *and* $\Phi\equiv 0$, *the authors of [22] confirmed the* L^p $(1<p<\infty)$ *boundedness of* $\mathcal{M}_{\Omega,\Phi,h}$ *whenever* $\Omega\in L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ *for some* $q>1$. *This result is extended in Theorem 5, in which* $\Omega\in L(\log L)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})\cup B_q^{(0,0)}(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})\supset L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$.

Remark 3. *For the special case* $h\in\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ *with* $\gamma>2$, *our results give the boundedness of* $\mathcal{M}_{\Omega,\Phi,h}$ *for all* $p\in(1,\infty)$, *which is the full range.*

Remark 4. *For the special case* $\Phi\equiv 0$, *Theorem 5 shows that* $\mathcal{M}_{\Omega,\Phi,h}$ *is bounded on* $L^p(\mathbb{R}^m\times\mathbb{R}^n)$ *for all* $|1/p-1/2|<\min\{1/2,1/\gamma'\}$, *which is the result established in [15]. Hence, our results essentially improve the main results in [15].*

Remark 5. *The surfaces of revolutions* $\Gamma_\Phi(x,y)=(x,y,\Phi(|x|,|y|))$ *considered in Theorems 1–5 cover several important natural classical surfaces. For instance, our theorems allow surfaces of the type* Γ_Φ *with* $\Phi(t,s)=s^2t^2(e^{-1/s}+e^{-1/t}),(s,t>0)$, $\Phi(t,s)=t^\alpha s^\beta$ *with* $\alpha,\beta>0$; $\Phi(t,s)=P(s,t)$ *is a polynomial,* $\Phi(t,s)=\phi_1(t)\phi_2(s)$, *where each* $\phi_i\in C^2(\mathbb{R}_+)$ *is a convex increasing function with* $\phi_i(0)=0$.

Henceforward, the constant C denotes a positive real constant which may not necessarily be the same at each occurrence, but is independent of all the essential variables.

2. Preliminary Lemmas

We devote this section to introducing some notations and establishing some auxiliary lemmas. For $\theta\geq 2$ and a suitable mapping $\Phi(r,s)$ on $\mathbb{R}_+\times\mathbb{R}_+$, we define the family of measures $\{\lambda_{\Omega,\Phi,h,r,s}:=\lambda_{r,s}:r,s\in\mathbb{R}_+\}$ and its concerning maximal operators λ_h^* and $\mathrm{M}_{h,\theta}$ on $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}$ by:

$$\iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} f d\lambda_{r,s} = \frac{1}{r^{\tau_1}s^{\tau_2}}\int_{1/2s\leq|u|\leq s}\int_{1/2r\leq|v|\leq r} f(v,u,\Phi(|v|,|u|))K_{\Omega,h}(v,u)dvdu,$$

$$\lambda_h^*(f)(x,y,w) = \sup_{r,s\in\mathbb{R}_+}||\lambda_{r,s}|*f(x,y,w)|,$$

and:

$$\mathrm{M}_{h,\theta}(f)(x,y,w) = \sup_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}||\lambda_{r,s}|*f(x,y,w)|\frac{drds}{rs},$$

where $|\lambda_{r,s}|$ is defined in the same way as $\lambda_{r,s}$ but with replacing Ωh by $|\Omega h|$.

Lemma 1. *Let $\Omega \in L^q(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ with $1 < q \leq 2$ and satisfy the conditions (1) and (2). Suppose that $\Phi \in C^1(\mathbb{R}_+ \times \mathbb{R}_+)$. For $r, s > 0$, let:*

$$\mathcal{G}(t,\ell) \quad = \quad \iint_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} e^{-i\{trv\cdot\xi+\ell su\cdot\zeta+\eta\Phi(r,s)\}}\,\Omega(v,u)d\mu(v)d\mu(u).$$

Then, there are constants $C > 0$ and δ with $0 < \delta < \frac{1}{2q'}$ such that for $(\xi, \zeta, \eta) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}_+$, we have:

$$\int_{1/2}^{1}\int_{1/2}^{1} |\mathcal{G}(t,\ell)|^2 \frac{dtd\ell}{t\ell} \quad \leq \quad C\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{m-1})}|r\xi|^{\pm\frac{\delta}{q'}}|s\zeta|^{\pm\frac{\delta}{q'}}, \tag{8}$$

where $a^{\pm b} = \min\{a^b, a^{-b}\}$.

Proof. By Schwartz inequality, we get:

$$\begin{aligned}\int_{1/2}^{1}\int_{1/2}^{1} |\mathcal{G}(t,\ell)|^2 \frac{dtd\ell}{t\ell} \quad &\leq \quad C\int_{\mathbb{S}^{n-1}}\Big(\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{m-1}} \mathcal{F}(\xi,v,x) \\ &\times \quad \Omega(v,u)\overline{\Omega(x,u)}d\mu(v)d\mu(x)\Big)d\mu(u),\end{aligned}$$

where $\mathcal{F}(\xi, v, x) = \int_1^2 e^{-i\frac{t}{2}r\xi\cdot(v-x)}\frac{dt}{t}$. Let $\rho = \xi/\mid\xi\mid$. Then, by Van der Corput's lemma, we get:

$$\mid \mathcal{F}(\xi,v,x) \mid \quad \leq \quad C \mid r\xi\cdot(v-x) \mid^{-1} \leq C \mid r\xi \mid^{-1} \mid \rho\cdot(v-x) \mid^{-1},$$

with which, when combined with the trivial estimate $\mid \mathcal{F}(\xi, v, x) \mid \leq C$, we can deduce that:

$$\mid \mathcal{F}(\xi,v,x) \mid \leq C \mid r\xi \mid^{-\delta} \mid \rho\cdot(v-x) \mid^{-\delta}, \tag{9}$$

where $0 < \delta < 1$. Hence, by Hölder's inequality, we obtain:

$$\begin{aligned}\int_{1/2}^{1}\int_{1/2}^{1} |\mathcal{G}(t,\ell)|^2 \frac{dtd\ell}{t\ell} \quad &\leq \quad C|r\xi|^{-\frac{\delta}{q'}}\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \\ &\times \quad \Big(\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{m-1}} |\rho\cdot(v-x)|^{-\delta q'}d\mu(v)d\mu(x)\Big)^{1/q'}.\end{aligned}$$

By choosing δ so that $0 < \delta < \frac{1}{2q'}$, we see that the last integral is finite. Thus,

$$\int_{1/2}^{1}\int_{1/2}^{1} |\mathcal{G}(t,\ell)|^2 \frac{dtd\ell}{t\ell} \quad \leq \quad C\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}|r\xi|^{-\frac{\delta}{q'}}. \tag{10}$$

Similarly, we have:

$$\int_{1/2}^{1}\int_{1/2}^{1} |\mathcal{G}(t,\ell)|^2 \frac{dtd\ell}{t\ell} \quad \leq \quad C\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}|s\zeta|^{-\frac{\delta}{q'}}. \tag{11}$$

Additionally, by the conditions (1) and (2) and a simple change of variable, we have:

$$\begin{aligned}\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(t,\ell)|^2\frac{dtd\ell}{t\ell} \le\ & C\int_{1/2}^{1}\int_{1/2}^{1}\left(\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}|e^{-itr\xi\cdot v}-1||\Omega(v,u)|d\mu(v)d\mu(u)\right)^2\frac{dtd\ell}{t\ell}\\ \le\ & C\|\Omega\|^2_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}|r\xi|^2.\end{aligned}$$

By combining the last estimate with the trivial estimate $\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(t,\ell)|^2\frac{dtd\ell}{t\ell} \le C\|\Omega\|^2_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}$, we get:

$$\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(t,\ell)|^2\frac{dtd\ell}{t\ell} \le C\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}|r\xi|^{\frac{\delta}{q'}}. \tag{12}$$

Similarly, we have:

$$\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(t,\ell)|^2\frac{dtd\ell}{t\ell} \le C\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}|s\zeta|^{\frac{\delta}{q'}}. \tag{13}$$

Therefore, by combining the estimates (10)–(13), we get (8), which ends the proof of this lemma. □

Lemma 2. *Suppose that $\Omega \in L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ with $q>1$ satisfies the conditions (1) and (2), $h\in\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ with $\gamma>1$, $\Phi\in C^1(\mathbb{R}_+\times\mathbb{R}_+)$, and $\theta\ge 2$. Then, there is a real number $C>0$ such that the estimates:*

$$\|\lambda_{r,s}\| \le C\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}, \tag{14}$$

$$\begin{aligned}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right|^2\frac{drds}{rs} \le\ & C\ln^2(\theta)\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\\ \times\ & \left|\theta^k\xi\right|^{\pm\frac{2\delta}{q'\epsilon'}}\left|\theta^j\zeta\right|^{\pm\frac{2\delta}{q'\epsilon'}}\end{aligned} \tag{15}$$

hold for all $j,k\in\mathbb{Z}$, where δ is the same as in Lemma 1, $\epsilon=\max\{2,\gamma'\}$ and $\|\lambda_{r,s}\|$ indicates the total variation of $\lambda_{r,s}$.

Proof. It is clear that the estimate (14) is obtained by the definition of $\lambda_{r,s}$. Thanks to Hölder's inequality, we have:

$$\begin{aligned}\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right| \le\ & C\int_{\frac{1}{2}s}^{s}\int_{\frac{1}{2}r}^{r}|h(t,\ell)|\left|\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}e^{-i\{tv\cdot\xi+\ell u\cdot\zeta+\Phi(t,\ell)\eta\}}\right.\\ \times\ & \left.\Omega(v,u)d\mu(v)d\mu(u)\right|\frac{dtd\ell}{t\ell}\\ \le\ & C\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\left(\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(t,\ell)|^{\gamma'}\frac{dtd\ell}{t\ell}\right)^{1/\gamma'}.\end{aligned}$$

For the case $\gamma\in(1,2]$, we can deduce that:

$$\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right| \le \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|\Omega\|^{(1-2/\gamma')}_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\left(\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(t,\ell)|^2\frac{dtd\ell}{t\ell}\right)^{1/\gamma'}.$$

However, for the case $\gamma > 2$, by using Hölder's inequality, we get:

$$\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right| \leq \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\left(\int\limits_{1/2}^{1}\int\limits_{1/2}^{1}|\mathcal{G}(t,\ell)|^2\frac{dtd\ell}{t\ell}\right)^{1/2}.$$

Therefore, for either case of γ, we have:

$$\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right| \leq C\|\Omega\|_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}^{(\epsilon-2)/\gamma'}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\left(\int_{1/2}^{1}\int_{1/2}^{1}|\mathcal{G}(t,\ell)|^2\frac{dtd\ell}{t\ell}\right)^{1/\epsilon'},$$

where $\epsilon = \max\{2,\gamma'\}$. Hence, Lemma 1 leads to:

$$\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right|^2 \leq C\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}^2\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}^2|r\xi|^{\pm\frac{2\delta}{\epsilon'q'}}|s\zeta|^{\pm\frac{2\delta}{\epsilon'q'}}.$$

As $\theta^k \leq r \leq \theta^{k+1}$ and $\theta^j \leq s \leq \theta^{j+1}$, we get:

$$\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right|^2 \leq C\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}^2\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}^2\left|\theta^k\xi\right|^{\pm\frac{2\delta}{\epsilon'q'}}\left|\theta^j\zeta\right|^{\pm\frac{2\delta}{\epsilon'q'}}. \tag{16}$$

Consequently,

$$\begin{aligned}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\hat{\lambda}_{r,s}(\xi,\zeta,\eta)\right|^2\frac{drds}{rs} &\leq C\ln^2(\theta)\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}^2\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}^2\\ &\times \left|\theta^k\xi\right|^{\pm\frac{2\delta}{q'\epsilon'}}\left|\theta^j\zeta\right|^{\pm\frac{2\delta}{q'\epsilon'}}.\end{aligned}$$

The proof is complete. □

The following lemmas play a key role in proving our main results.

Lemma 3. *Let $h \in \Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ with $\gamma > 1$ and $\Omega \in L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $1 < q \leq 2$. Assume that $\Phi \in C^1(\mathbb{R}_+\times\mathbb{R}_+)$ such that for any fixed $t,\ell > 0$, we have $\Psi_{1,t}(.) = \Phi(t,.)$, $\Psi_{2,\ell}(.) = \Phi(.,\ell)$ are in $C^2(\mathbb{R}_+)$, increasing and convex functions with $\Psi_{1,t}(0) = \Psi_{2,\ell}(0) = 0$. Then, for $f \in L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ with $p \in (\gamma',\infty)$, there exists $C_p > 0$ such that:*

$$\|\lambda_h^*(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq C_p\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \tag{17}$$

and:

$$\|\mathrm{M}_{h,\theta}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq C_p\ln^2(\theta)\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}. \tag{18}$$

Proof. Thanks to Hölder's inequality, we get:

$$\begin{aligned}||\lambda_{r,s}|*f(x,y,w)| \leq{}& C\|\Omega\|_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}^{1/\gamma'}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\left(\frac{1}{rs}\int\limits_{s/2}^{s}\int\limits_{r/2}^{r}\int_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}|\Omega(v,u)|\right.\\ &\times \left.|f(x-tv,y-\ell u,w-\Phi(t,\ell))|^{\gamma'}d\mu(v)d\mu(u)dtd\ell\right)^{1/\gamma'}.\end{aligned}$$

Hence, by Minkowski's inequality for integrals and Lemma 2.4 in [27], we can deduce:

$$\begin{aligned}\|\lambda_h^*(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} &\leq C\|\Omega\|_{L^1(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}^{1/\gamma'}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\left(\|\sigma_\Phi^*(|f|^{\gamma'})\|_{L^{(p/\gamma')}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\right)^{1/\gamma'}\\ &\leq C\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})},\end{aligned}$$

where:

$$\iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} f d\sigma_{r,s} = \frac{1}{r^{\tau_1}s^{\tau_2}} \int_{1/2s\le|u|\le s} \int_{1/2r\le|v|\le r} f(v,u,\Phi(|v|,|u|)) \frac{\Omega(v,u)}{|v|^{m-\tau_1}|u|^{n-\tau_2}} dvdu$$

and:

$$\sigma^*_\Phi(f)(x,y,w) = \sup_{r,s\in\mathbb{R}_+} ||\sigma_{r,s}| * f(x,y,w)|.$$

It is easy to see that the inequality (18) can be obtained from the inequality (17). □

Similarly, by Lemmas 2.5–2.7 in [27], we get, respectively, the following results.

Lemma 4. *Let h and Ω be given as in Lemma 3. Assume that $\Phi(t,\ell) = \sum_{i=0}^{d_1}\sum_{j=0}^{d_2} a_{j,i} t^{\alpha_i}\ell^{\beta_j}$ with $\alpha_i, \beta_j > 0$ is a generalized polynomial on $\mathbb{R}^2$. Then, for $f \in L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ with $p\in(\gamma',\infty)$, there exists $C_p > 0$ such that:*

$$\|\lambda^*_h(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \le C_p\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}$$

and:

$$\|\mathrm{M}_{h,\theta}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \le C_p\ln^2(\theta)\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}.$$

Lemma 5. *Let h and Ω be given as in Lemma 3. Assume that $\Phi(t,\ell) = \phi(t)P(\ell)$, where $\phi(t)$ is in $C^2(\mathbb{R}_+)$, increasing and convex function with $\phi(0) = 0$, and P is a generalized polynomial given by $P(\ell) = \sum_{j=0}^{d_2} a_j\ell^{\beta_j}$ with $\beta_j > 0$. Then, for $f \in L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ with $p\in(\gamma',\infty)$, there exists $C_p > 0$ such that:*

$$\|\lambda^*_h(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \le C_p\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}$$

and:

$$\|\mathrm{M}_{h,\theta}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \le C_p\ln^2(\theta)\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}.$$

Lemma 6. *Let h and Ω be given as in Lemma 3. Assume that $\Phi(t,\ell) = \phi_1(t)+\phi_2(\ell)$, where $\phi_j(\cdot)$ $(j = 1,2)$ is either a generalized polynomial or is in $C^2(\mathbb{R}_+)$, increasing and convex function with $\phi_j(0) = 0$. Then, for $f \in L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ with $p\in(\gamma',\infty)$, there exists $C_p > 0$ such that:*

$$\|\lambda^*_h(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \le C_p\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}$$

and:

$$\|\mathrm{M}_{h,\theta}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \le C_p\ln^2(\theta)\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}.$$

Lemma 7. *Let $\theta\ge 2$, $h\in\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ with $\gamma > 1$, $\Omega\in L^q\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$ with $1 < q \le 2$, and Φ be given as in either Theorem 1, Theorem 2, Theorem 3, or Theorem 4. Then, for an arbitrary set of functions $\{\mathcal{F}_{k,j}(\cdot,\cdot,\cdot),\ j,k\in\mathbb{Z}\}$ defined on $\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}$, there exists a constant $C_p > 0$ such that the inequality:*

$$\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left| \lambda_{r,s} * \mathcal{F}_{j,k} \right|^2 \frac{drds}{rs} \right)^{1/2} \right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}$$
$$\leq C_p \ln^2(\theta) \|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left| \mathcal{F}_{j,k} \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \tag{19}$$

holds for all $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$.

Proof. We will follow a similar argument as in [20]. We point out here that we shall prove this lemma only whenever Φ is given as in Theorem 1, since the proofs for the other cases follow the same method, except that we invoke Lemmas 4–6 instead of invoking Lemma 3. Additionally, we shall prove this lemma only for the case $1 < \gamma \leq 2$, since $\Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+) \subseteq \Delta_2(\mathbb{R}_+ \times \mathbb{R}_+)$ for all $\gamma \geq 2$. In this case, we have $|1/p - 1/2| < 1/\gamma'$, which shows that $\frac{2\gamma}{3\gamma-2} < p < \frac{2\gamma}{2-\gamma}$. We need to consider two cases.

Case 1. $2 \leq p < \frac{2\gamma}{2-\gamma}$. By duality, there exists a non-negative function $X \in L^{(p/2)'}(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$ such that $\|X\|_{L^{(p/2)'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq 1$ and:

$$\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left| \lambda_{r,s} * \mathcal{F}_{j,k} \right|^2 \frac{drds}{rs} \right)^{1/2} \right\|^2_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}$$
$$= \iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} \sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left| \lambda_{r,s} * \mathcal{F}_{j,k}(x,y,w) \right|^2 \frac{drds}{rs} X(x,y,w)dxdydw.$$

By Schwartz's inequality, we have:

$$\begin{aligned} \left| \lambda_{r,s} * \mathcal{F}_{j,k}(x,y,w) \right|^2 &\leq C\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|^\gamma_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \Bigg(\int_{\frac{1}{2}s}^{s} \int_{\frac{1}{2}r}^{r} \iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}} \\ &\times \left| \mathcal{F}_{j,k}(x - tv, y - \ell u, w - \Phi(t,\ell)) \right|^2 \\ &\times |h(t,\ell)|^{2-\gamma} |\Omega(v,u)| d\sigma(v) d\sigma(u) \frac{dtd\ell}{t\ell} \Bigg). \end{aligned}$$

Hence, we have:

$$\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left| \lambda_{r,s} * \mathcal{F}_{j,k} \right|^2 \frac{drds}{rs} \right)^{1/2} \right\|^2_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq C\|h\|^\gamma_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}$$
$$\begin{aligned} &\times \|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbf{S}^{n-1})} \iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} \left(\sum_{j,k\in\mathbb{Z}} \left| \mathcal{F}_{j,k}(x,y,w) \right|^2 \right) \mathrm{M}_{|h|^{2-\gamma},\theta} \widetilde{X}(-x,-y,-w) dxdydw \\ &\leq C\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|^\gamma_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left\| \sum_{j,k\in\mathbb{Z}} \left| \mathcal{F}_{j,k} \right|^2 \right\|_{L^{(p/2)}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\ &\times \left\| \mathrm{M}_{|h|^{2-\gamma},\theta}(\widetilde{G}) \right\|_{L^{(p/2)'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}, \end{aligned}$$

where $\widetilde{X}(-x,-y,-w) = X(x,y,w)$. Notice that, since $h \in \Delta_\gamma(\mathbb{R}_+ \times \mathbb{R}_+)$, then we have $|h|^{2-\gamma} \in \Delta_{\frac{\gamma}{2-\gamma}}(\mathbb{R}_+ \times \mathbb{R}_+)$. Thus, by Lemma 3 and Hölder's inequality,

$$\begin{aligned}
&\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left| \lambda_{r,s} * \mathcal{F}_{j,k} \right|^2 \frac{drds}{rs} \right)^{1/2} \right\|^2_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\
\leq\ & C \ln^2(\theta) \|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left| \mathcal{F}_{j,k} \right|^2 \right)^{1/2} \right\|^2_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\
\times\ & \left\| \lambda^*_{|h|^{2-\mu}}(\widetilde{X}) \right\|_{L^{(p/2)'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\
\leq\ & C_p \ln^2(\theta) \|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left| \mathcal{F}_{j,k} \right|^2 \right)^{1/2} \right\|^2_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}.
\end{aligned}$$

Case 2. $\frac{2\gamma}{3\gamma-2} < p < 2$. By duality, there exists a collection of functions $\mathrm{Y} = \mathrm{Y}_{j,k}(x,y,w,r,s)$ defined on $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ such that:

$$\left\| \left\| \|\mathrm{Y}_{j,k}\|_{L^2([\theta^k,\theta^{k+1}]\times[\theta^j,\theta^{j+1}],\frac{drds}{rs})} \right\|_{l^2} \right\|_{L^{p'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq 1$$

and:

$$\begin{aligned}
&\left\| \left(\sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left| \lambda_{r,s} * \mathcal{F}_{j,k} \right|^2 \frac{drds}{rs} \right)^{1/2} \right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\
=\ & \iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} \sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left(\lambda_{r,s} * \mathcal{F}_{j,k}(x,y,w) \right) \mathrm{Y}_{j,k}(x,y,w,r,s) \frac{drds}{rs} dxdydw \\
\leq\ & C_p \ln(\theta) \left\| (\Theta(\mathrm{Y}))^{1/2} \right\|_{L^{p'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \left\| \left(\sum_{j,k\in\mathbb{Z}} \left| \mathcal{F}_{j,k} \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}, \qquad (20)
\end{aligned}$$

where:

$$\Theta(\mathrm{Y})(x,y,w) = \sum_{j,k\in\mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left| \lambda_{r,s} * \mathrm{Y}_{j,k}(x,y,w,r,s) \right|^2 \frac{drds}{rs}.$$

Thanks to the duality, we can deduce that a function $W \in L^{(p'/2)'}(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$ exists such that $\|W\|_{L^{(p'/2)'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq 1$ and:

$$
\begin{aligned}
&\left\|(\Theta(\Upsilon))^{1/2}\right\|^2_{L^{p'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\\
=\ & \sum_{j,k\in\mathbb{Z}}\iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\lambda_{r,s}*\Upsilon_{j,k}(x,y,w,r,s)\right|^2\frac{drds}{rs}W(x,y,w)dxdydw\\
\leq\ & C\|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^{\gamma}_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}\left\|\lambda^*{}_{|h|^{2-\gamma}}(W)\right\|_{L^{(p'/2)'}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\\
\times\ & \left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\Upsilon_{j,k}(\cdot,\cdot,r,s)\right|^2\frac{drds}{rs}\right)\right\|_{L^{(p'/2)}(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\\
\leq\ & C\|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})}\|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)}
\end{aligned}
$$

which shows, along with (20), that the inequality (19) holds for the case $\frac{2\gamma}{3\gamma-2}<p<2$. This finishes the proof of this lemma. □

3. Proof of Main Theorems

Let us first prove Theorem 1. Assume that $h\in\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\gamma>1$, $\Omega\in L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $1<q\leq 2$ and $\theta=2^{q'\gamma'}$. It is clear that Minkowski's inequality leads to:

$$
\begin{aligned}
\mathcal{M}_{\Omega,\Phi,h}(f)(x,y,w) &= \Bigg(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\Bigg|\sum_{j,k=0}^{\infty}\frac{1}{s^{\tau_1}r^{\tau_2}}\int_{2^{-j-1}s<|u|\leq 2^{-j}s}\int_{2^{-k-1}r<|v|\leq 2^{-k}r}K_{\Omega,h}(v,u)\\
&\times\ f(x-v,y-u,w-\Phi(|v|,|u|))dvdu\Bigg|^2\frac{drds}{rs}\Bigg)^{1/2}\\
&\leq \sum_{j,k=0}^{\infty}\Bigg(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\Bigg|\frac{1}{s^{\tau_1}r^{\tau_2}}\int_{2^{-j-1}s<|u|\leq 2^{-j}s}\int_{2^{-k-1}r<|v|\leq 2^{-k}r}K_{\Omega,h}(v,u)\\
&\times\ f(x-v,y-u,w-\Phi(|v|,|u|))dvdu\Bigg|^2\frac{drds}{rs}\Bigg)^{1/2}\\
&\leq \frac{2^{\alpha_1+\alpha_2}}{(2^{\alpha_1}-1)(2^{\alpha_2}-1)}\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}|\lambda_{r,s}*f(x,y,w)|^2\frac{drds}{rs}\right)^{1/2}. \qquad (21)
\end{aligned}
$$

For $j\in\mathbb{Z}$, choose a set of smooth partition of unity $\{\Gamma_j\}$ defined on $(0,\infty)$ and adapted to the interval $[\theta^{-1-j},\theta^{1-j}]\equiv\mathcal{I}_j$ with the following properties:

$$
\begin{aligned}
\Gamma_j &\in C^\infty,\ \ 0\leq\Gamma_j\leq 1,\ \ \sum_{j\in\mathbb{Z}}\Gamma_j(t)=1,\\
\mathrm{supp}\ (\Gamma_j) &\subseteq \mathcal{I}_j\ \ \textit{and}\ \ \left|\frac{d^\mu\Gamma_j(t)}{dt^\mu}\right|\leq\frac{C_\mu}{t^\mu},
\end{aligned}
$$

where C_μ is independent of the lacunary sequence $\{\theta^j;j\in\mathbb{Z}\}$.

Define the multiplier operators $\{T_{j,k}\}$ on $\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}$ by $(\widehat{T_{j,k}(f)})(\xi,\zeta,\eta)=\Gamma_j(|\xi|)\Gamma_k(|\zeta|)\hat{f}(\xi,\zeta,\eta)$. Hence, for any $f\in C_0^\infty(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$, we have $f(x,y,w)=\sum_{j,k\in\mathbb{Z}}(T_{j+a_2,k+a_1}(f))(x,y,w)$, which shows, by Minkowski's inequality, that:

$$
\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}|\lambda_{r,s}*f(x,y,w)|^2\frac{drds}{rs}\right)^{1/2}\leq C\sum_{a_1,a_2\in\mathbb{Z}}\mathcal{A}_{a_2,a_1}(f)(x,y,w), \qquad (22)
$$

where:

$$\mathcal{A}_{a_2,a_1}(f)(x,y,w) = \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} |\mathcal{B}_{a_2,a_1}(f)(x,y,w,r,s)|^2 \frac{drds}{rs}\right)^{1/2},$$

$$\mathcal{B}_{a_2,a_1}(g)(x,y,w,r,s) = \sum_{j,k\in\mathbb{Z}} \lambda_{r,s} * T_{j+a_2,k+a_1} * f(x,y,w)\chi_{[\theta^k,\theta^{k+1})\times[\theta^j,\theta^{j+1})}(r,s).$$

Therefore, to prove Theorem 1, it suffices to prove that for any p satisfying $|1/2-1/p| < \min\{1/\gamma',1/2\}$, there exists $\varepsilon > 0$ such that:

$$\begin{aligned} &\|\mathcal{A}_{a_2,a_1}(f)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\ \leq\ & C_p \ln(\theta)\ 2^{-\frac{\varepsilon}{2}(|a_1|+|a_2|)} \|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}. \end{aligned} \tag{23}$$

Let us first estimate the L^2-norm for $\mathcal{A}_{a_2,a_1}(f)$. By Plancherel's Theorem, Fubini's Theorem, and Lemma 2, we can deduce:

$$\begin{aligned} &\|\mathcal{A}_{a_2,a_1}(f)\|^2_{L^2(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\ \leq\ & \sum_{j,k\in\mathbb{Z}} \iiint_{\mho_{j+a_2,k+a_1}} \left(\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}} |\hat{\lambda}_{r,s}(\xi,\zeta,\eta)|^2 \frac{drds}{rs}\right) \left|\hat{f}(\xi,\zeta,\eta)\right|^2 d\xi d\zeta d\eta \\ \leq\ & C_p \ln^2(\theta) \|\Omega\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \\ \times\ & \sum_{j,k\in\mathbb{Z}} \iiint_{\mho_{j+a_2,k+a_1}} \left|\theta^k\xi\right|^{\pm\frac{2\delta}{q'\epsilon'}} \left|\theta^j\zeta\right|^{\pm\frac{2\delta}{q'\epsilon'}} \left|\hat{f}(\xi,\zeta,\eta)\right|^2 d\xi d\zeta d\eta \\ \leq\ & C_p \ln^2(\theta)\, 2^{-\varepsilon(|a_1|+|a_2|)} \|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \sum_{j,k\in\mathbb{Z}} \iiint_{\mho_{j+a_2,k+a_1}} \left|\hat{f}(\xi,\zeta,\eta)\right|^2 d\xi d\zeta d\eta \\ \leq\ & C_p \ln^2(\theta)\, 2^{-\varepsilon(|a_1|+|a_2|)} \|\mho\|^2_{L^q(\mathbb{S}^{m-1}\times\mathbf{S}^{n-1})} \|h\|^2_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \|f\|^2_{L^2(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}, \end{aligned} \tag{24}$$

where $\mho_{j,k} = \left\{(\xi,\zeta,\eta) \in \mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R} : (|\xi|,|\zeta|) \in \mathcal{I}_j\times\mathcal{I}_k\right\}$ and $\varepsilon \in (0,1)$.

Next, we estimate the L^p-norm of $\mathcal{A}_{a_2,a_1}(f)$ as follows: by employing a similar argument as that used in [29], along with the Littlewood–Paley theory and Lemma 7, we get:

$$\begin{aligned} &\|\mathcal{A}_{a_2,a_1}(g)\|_{L^p(\mathbb{R}^{m+1}\times\mathbb{R}^{n+1})} \\ \leq\ & C\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left(\left|\lambda_{r,s} * T_{j+a_2,k+a_1} * f\right|\right)^2 \frac{drds}{rs}\right)^{1/2}\right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\ \leq\ & C_p \ln(\theta) \|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \left\|\left(\sum_{j,k\in\mathbb{Z}}\left|T_{j+a_2,k+a_1} * f\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \\ \leq\ & C_p \frac{\gamma}{(\gamma-1)(q-1)} \|\Omega\|_{L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})} \|h\|_{\Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)} \|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}. \end{aligned} \tag{25}$$

Consequently, by interpolating between (24) and (25), we obtain (23), which, in turn, finishes the proof of Theorem 1.

Finally, we can prove Theorems 2–4 by following the same above arguments. We have omitted the details. This completes the proofs of our theorems.

4. Conclusions

In this paper, we prove sharp L^p estimates of several classes of rough Marcinkiewicz integrals over surfaces of revolution on product spaces, that is, $h \in \Delta_\gamma(\mathbb{R}_+\times\mathbb{R}_+)$ for some $\gamma > 1$ and $\Omega \in L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1})$ for some $1 < q \leq 2$. Furthermore, we employed these estimates along with Yano's extrapolation argument to prove the L^p boundedness of the operator $\mathcal{M}_{\Omega,\Phi,h}$ under the conditions $h \in \Delta_\gamma(\mathbb{R}^+\times\mathbb{R}^+)$ for some $\gamma > 1$ and Ω belongs to either the space $L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ or to the space $B_q^{(0,0)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ for some $q > 1$. In fact, our results extend and improve several known results on Marcinkiewicz integrals such as the results in [11,13,15,22,30]. In future work, we aim to confirm that $\mathcal{M}_{\Omega,\Phi,h}$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$ for the full range of $p \in (1, \infty)$ and $h \in \Delta_\gamma(\mathbb{R}^+ \times \mathbb{R}^+)$ with $\gamma > 1$.

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