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Lightlike Hypersurfaces of Almost Productlike Semi-Riemannian Manifolds

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Abstract: The main purpose of this paper is to investigate lightlike hypersurfaces of almost productlike semi-Riemannian manifolds. For this purpose, screen-semi-invariant, screen-invariant, radical-anti-invariant, and radical-invariant lightlike hypersurfaces of almost productlike semi-Riemannian manifolds are introduced and some examples of these classifications are presented. Furthermore, various characterizations dealing screen semi-invariant lightlike hypersurfaces are obtained.

Keywords: lightlike hypersurface; almost productlike manifold; semi-Riemannian manifold

MSC: 53C40; 53C42; 53C05

1. Introduction

In 2011, K. Takano [1] developed a new perspective on statistical structures on the basis of Hermitian manifolds, and he introduced the notion of Hermitian-like manifolds as follows:

Let $(\widetilde{M}, \widetilde{g})$ be a semi-Riemannian manifold equipped with almost complex structures J and J^* of tensor types $(1,1)$ satisfying

$$\widetilde{g}(JX, Y) = -\widetilde{g}(X, J^*Y) \tag{1}$$

for any tangent vector fields $X, Y \in \Gamma(T\widetilde{M})$. Then, $(\widetilde{M}, \widetilde{g}, J)$ is called a Hermite-like manifold. A Hermite-like manifold becomes a Hermitian manifold when $J = J^*$.

Considering the definition of K. Takano, a new geometric model has emerged that can be considered as a generalization of Hermitian geometry. Hermitian-like manifolds have recently been a very interesting research topic, and the geometry of these manifolds is still being studied in [2–7] etc.

Apart from Hermitian geometry, the theory of Riemannian product manifolds includes important physical and geometrical applications. There exist remarkable applications of Riemannian product manifolds in Kaluza–Klein theory, brane theory, and gauge theory in [8–13] etc. Moreover, some of the latest connected studies on Lorentzian manifolds can be seen in [14–21].

Inspired by the definition of Hermitian-like manifolds, the notion of productlike manifolds could also be introduced. From this point of view, the authors introduced almost productlike Riemannian manifolds and investigated hypersurfaces of these manifolds in [22].

In the present paper, lightlike hypersurfaces of almost productlike semi-Riemannian manifolds are examined, and some special classifications of these hypersurfaces are introduced.

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2. Preliminaries

Let $(\widetilde{M}, \widetilde{g})$ be an $(n+2)$—dimensional semi-Riemannian manifold and (M, g) be a lightlike hypersurface of $(\widetilde{M}, \widetilde{g})$. Then, the radical space $\mathrm{Rad}(T_pM)$ at each point $p \in M$ is a 1-dimensional subspace and it is defined by

$$\mathrm{Rad}(T_pM) = \{\xi \in T_pM : g_p(\xi, X) = 0, \ \forall X \in T_pM\}. \tag{2}$$

A complementary vector bundle of the radical distribution Rad TM is denoted by $\mathrm{S}(TM)$, and there exists the following orthogonal direct sum:

$$TM = \mathrm{Rad}(TM) \oplus_{orth} \mathrm{S}(TM), \tag{3}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Here, $S(TM)$ is a semi-Riemannian distribution and it is called a screen distribution of (M, g). Since $\mathrm{Rad}(TM)$ is a degenerate sub-bundle of TM, there exists a local section N of (M, g) satisfying

$$\widetilde{g}(N, N) = 0 \quad \text{and} \quad \widetilde{g}(\xi, N) = 1, \tag{4}$$

where $\xi \in \mathrm{Rad}(TM)$. The set $\mathrm{ltr}(TM)$ spanned by N is called the lightlike transversal bundle. $\mathrm{S}(TM)$ is not unique; thus, a lightlike hypersurface is usually denoted by the triplet $(M, g, \mathrm{S}(TM))$.

Let us denote the Levi-Civita connection of $(\widetilde{M}, \widetilde{g})$ by $\widetilde{\nabla}^0$. Then, the Gaussian and Weingarten-type formulas for (M, g) are given by

$$\widetilde{\nabla}^0_X Y = \nabla^0_X Y + B^0(X, Y)N, \tag{5}$$

and

$$\widetilde{\nabla}^0_X N = -A^0_N X + \tau^0(X)N \tag{6}$$

for any $X, Y \in \Gamma(TM)$, where ∇^0 is the induced linear connection on $\Gamma(TM)$, $B^0(X, Y)$ is the coefficient of the second fundamental form, A^0_N is the shape operator, τ^0 is a 1—form on $\Gamma(TM)$. ∇^0 is not a Riemannian connection, B^0 is symmetric and it vanishes on the radical distribution $\mathrm{Rad}(TM)$, and A^0_N is not self-adjoint [23,24].

A lightlike hypersurface $(M, g, \mathrm{S}(TM))$ is called totally geodesic if $B^0 = 0$. If there exists a function λ on M satisfying

$$B^0(X, Y) = \lambda \, g(X, Y) \tag{7}$$

for any $X, Y \in \Gamma(TM)$; then, the hypersurface is called totally umbilical [25].

Suppose that the set $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame field on $\Gamma(S(TM))$. The hypersurface is called minimal if

$$\mathrm{trace}_{\mathrm{S}(TM)} B^0 = 0, \tag{8}$$

where $\mathrm{trace}_{\mathrm{S}(TM)}$ denotes the trace with respect to $S(TM)$ [26].

Now, we recall some basic facts on lightlike hypersurfaces of statistical semi-Riemannian manifolds.

Let $(\widetilde{M}, \widetilde{g})$ be a semi-Riemannian manifold, and $\widetilde{\nabla}$ be a torsion-free connection on $(\widetilde{M}, \widetilde{g})$. If the following relations are satisfied for any $X, Y, Z \in \Gamma(T\widetilde{M})$, then the triplet $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ is called a statistical semi-Riemannian manifold [27]:

$$Z\widetilde{g}(X, Y) = \widetilde{g}(\widetilde{\nabla}_Z X, Y) + \widetilde{g}(X, \widetilde{\nabla}^*_Z Y) \tag{9}$$

and

$$\widetilde{\nabla}^0_X Y = \frac{1}{2}(\widetilde{\nabla}_X Y + \widetilde{\nabla}^*_X Y). \tag{10}$$

Here, $\widetilde{\nabla}^0$ is the Levi-Civita connection and $\widetilde{\nabla}^*$ is called the dual connection of $\widetilde{\nabla}$.

Denote the Riemannian curvature tensor fields with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ by $\widetilde{R}$ and $\widetilde{R}^*$ respectively. Then, there exists the following relation between $\widetilde{R}$ and $\widetilde{R}^*$ for any $X, Y, Z, W \in \Gamma(T\widetilde{M})$ [28]:

$$\widetilde{g}(\widetilde{R}^*(X,Y)Z,W) = -\widetilde{g}(Z,\widetilde{R}(X,Y)W). \tag{11}$$

Now, let $(M, g, S(TM))$ be a lightlike hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$. The Gaussian and Weingarten-type formulas with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{12}$$
$$\widetilde{\nabla}_X N = -A^*_N X + \tau^*(X)N \tag{13}$$

and

$$\widetilde{\nabla}^*_X Y = \nabla^*_X Y + B^*(X,Y)N, \tag{14}$$
$$\widetilde{\nabla}^*_X N = -A_N X + \tau(X)N \tag{15}$$

for any $X, Y \in \Gamma(TM)$, respectively. Here, $\nabla_X Y$, $\nabla^*_X Y$, $A_N X$, $A^*_N X \in \Gamma(TM)$, ∇ and ∇^* are the induced connections on M, B and B^* are the second fundamental forms, A_N and A^*_N are the shape operators with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, respectively. Any lightlike hypersurface of a statistical manifold does not need to be a statistical manifold with respect to the induced connections ∇ and ∇^* [29].

Let us indicate the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ by P. For any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}(TM))$, we have

$$\nabla_X PY = \overline{\nabla}_X PY + C(X,PY)\xi, \tag{16}$$
$$\nabla_X \xi = -\overline{A}_\xi X - \tau(X)\xi, \tag{17}$$

where $\overline{\nabla}_X PY$ and $\overline{A}_\xi X$ belong to $\Gamma(S(TM))$. Here, C is called the local second fundamental form of $S(TM)$ with respect to ∇, $\overline{\nabla}$ is the induced connection of ∇, and $\overline{A}$ is the local shape operator with respect to ∇.

In a similar way to (16) and (17), we can write the following relations on $\Gamma(TM)$ with respect to ∇^*:

$$\nabla^*_X PY = \overline{\nabla}^*_X PY + C^*(X,PY)\xi, \tag{18}$$
$$\nabla^*_X \xi = -\overline{A}^*_\xi X - \tau^*(X)\xi, \tag{19}$$

where $\overline{\nabla}^*_X PY$ and $\overline{A}^*_\xi X$ belong to $\Gamma(S(TM))$. Here C^* is called the local second fundamental form of $S(TM)$ with respect to ∇^*, $\overline{\nabla}^*$ is the induced connection of ∇^* and $\overline{A}^*$ is the local shape operator with respect to ∇^*. Using the fact that

$$\begin{aligned} 0 &= B^0(X,\xi) \\ &= \frac{1}{2}[B(X,\xi) + B^*(X,\xi)], \end{aligned}$$

we find

$$B(X,\xi) + B^*(X,\xi) = 0. \tag{20}$$

From (16) and (19), we also have [29]

$$C(X,PY) = g(A_N X, PY), \; C^*(X,PY) = g(A^*_N X, PY). \tag{21}$$

As a result of (20), we obtain that the second fundamental forms B and B^* do not vanish on $\mathrm{Rad}(TM)$. Additionally, we obtain

$$B(X,Y) = g(\overline{A}^*_\xi X, Y) + B^*(X,\xi)\widetilde{g}(Y,N), \tag{22}$$
$$B^*(X,Y) = g(\overline{A}_\xi X, Y) + B(X,\xi)\widetilde{g}(Y,N) \tag{23}$$

for any $X, Y \in \Gamma(TM)$ [30].

Now we recall some special submanifolds of statistical manifolds [31,32].

Definition 1. *Any lightlike hypersurface of a statistical semi-Riemannian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ is called*

(i) *totally geodesic with respect to $\widetilde{\nabla}$ if $B = 0$,*
(ii) *totally geodesic with respect to $\widetilde{\nabla}^*$ if $B^* = 0$,*
(iii) *totally tangentially umbilical with respect to $\widetilde{\nabla}$ if there exists a smooth function k such that $B(X,Y) = kg(X,Y)$ for any $X, Y \in \Gamma(TM)$,*
(iv) *totally tangentially umbilical with respect to $\widetilde{\nabla}^*$ if there exists a smooth function k^* such that $B^*(X,Y) = k^*g(X,Y)$ for any $X, Y \in \Gamma(TM)$,*
(v) *totally normally umbilical with respect to $\widetilde{\nabla}$ if there exists a smooth function k such that $A^*_N X = kX$ for any $X, Y \in \Gamma(TM)$*
(vi) *totally normally umbilical with respect to $\widetilde{\nabla}^*$ if there exists a smooth function k^* such that $A_N X = k^* X$ for any $X, Y \in \Gamma(TM)$.*

3. Almost Productlike Semi-Riemannian Manifolds and Their Lightlike Hypersurfaces

Let $\widetilde{M}$ be an $(n+2)$—dimensional smooth manifold equipped with a tensor field of type $(1,1)$ such that $F^2 = I_{n+2}$, where I_{n+2} denotes the identity map. Then, $(\widetilde{M}, F)$ is called an almost product manifold. If an almost product manifold $(\widetilde{M}, F)$ admits a (semi) Riemannian metric $\widetilde{g}$ satisfying

$$\widetilde{g}(FX, FY) = \widetilde{g}(X,Y) \tag{24}$$

for any $X, Y \in \Gamma(T\widetilde{M})$, then the triplet $(\widetilde{M}, \widetilde{g}, F)$ is called an almost product Riemannian manifold [33].

Inspiring the definition of Hermite-like manifolds of K. Takano [1], we can give the following definition:

Definition 2. *If a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is equipped with the almost product structure F that has another tensor field F^* of type $(1,1)$ satisfying*

$$\widetilde{g}(FX, Y) = \widetilde{g}(X, F^*Y) \tag{25}$$

for any $X, Y \in \Gamma(T\widetilde{M})$, then $\widetilde{M}$ is called an almost productlike semi-Riemannian manifold. An almost productlike semi-Riemannian manifold is denoted by the triplet $(\widetilde{M}, \widetilde{g}, F)$ throughout this study.

For any almost productlike semi-Riemannian manifold, we have $F^2 = (F^*)^2 = I_{n+2}$ and $(F^*)^* = F$. Putting F^*Y instead of Y in (25), we have

$$\widetilde{g}(FX, F^*Y) = \widetilde{g}(X,Y). \tag{26}$$

If $F^* = F$, then an almost productlike semi-Riemannian manifold becomes an almost product semi-Riemannian manifold.

Example 1. *Let* $(\widetilde{M}, \widetilde{g})$ *be a* 5*—dimensional semi-Riemannian manifold with a semi-Riemannian metric* $\widetilde{g}$*, which is given by*

$$\widetilde{g} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Define almost product structures

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$F^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \end{pmatrix}.$$

Then, Equation (25) is satisfied. Therefore, $(\widetilde{M}, \widetilde{g}, F)$ *is an almost productlike semi-Riemannian manifold.*

Example 2. *Let* $(\widetilde{M}, \widetilde{g})$ *be a* 4*—dimensional Lorentzian manifold with a Lorentzian metric* $\widetilde{g}$ *that is given by*

$$\widetilde{g} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define almost product structures

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$F^* = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, we find that $(\widetilde{M}, \widetilde{g}, F)$ *is an almost productlike Lorentzian manifold.*

The manifolds that are given in Examples 1 and 2 do not satisfy (24). Thus, these manifolds are not examples of almost product manifolds.

An almost productlike semi-Riemannian manifold is called an almost productlike statistical semi-Riemannian manifold if there exists a linear connection $\widetilde{\nabla}$ satisfying (9) and (10). An almost productlike statistical semi-Riemannian manifold is denoted by $(\widetilde{M}, \widetilde{g}, F, \widetilde{\nabla})$.

Proposition 1. *Let $(\widetilde{M}, \widetilde{g}, F, \widetilde{\nabla})$ be an almost productlike statistical semi-Riemannian manifold. For any $X, Y, Z \in \Gamma(T\widetilde{M})$, we have*

$$\widetilde{g}\Big((\widetilde{\nabla}_X F)Y, Z\Big) = \widetilde{g}(Y, (\widetilde{\nabla}^*_X F^*)Z). \tag{27}$$

Proof. The proof of the proposition is straightforward from (9) and (25). □

In view of (27), we immediately obtain the following proposition:

Proposition 2. *For any almost productlike statistical semi-Riemannian manifold $(\widetilde{M}, \widetilde{g}, F, \widetilde{\nabla})$, we have*

$$\widetilde{\nabla}_X F = 0 \Leftrightarrow \widetilde{\nabla}^*_X F^* = 0 \tag{28}$$

for any $X \in \Gamma(T\widetilde{M})$.

An almost productlike statistical semi-Riemannian manifold $(\widetilde{M}, \widetilde{g}, F, \widetilde{\nabla})$ is called a locally productlike statistical semi-Riemannian manifold if Equation (28) is satisfied for any $X \in \Gamma(T\widetilde{M})$ [22].

Definition 3. *Let $(M, g, \mathrm{S}(TM))$ be a lightlike hypersurface of $(\widetilde{M}, \widetilde{g}, F)$. Then, the lightlike hypersurface $(M, g, \mathrm{S}(TM))$ is called*

(i) *a screen-semi-invariant lightlike hypersurface if $F(\mathrm{Rad}(TM))$ and $F(\mathrm{ltr}(TM))$ belong to $\mathrm{S}(TM)$,*
(ii) *a screen-invariant lightlike hypersurface if $F(\mathrm{S}(TM))$ belongs to $\mathrm{S}(TM)$,*
(iii) *a radical-anti-invariant lightlike hypersurface if $F(\mathrm{Rad}(TM))$ belongs to $\mathrm{ltr}\,(TM)$,*
(iv) *a radical-invariant lightlike hypersurface if $F(\mathrm{Rad}(TM))$ belongs to $\mathrm{Rad}(TM)$.*

The following statements occur:

(i) If $(M, g, \mathrm{S}(TM))$ is a screen-semi-invariant lightlike hypersurface, then $F^*(\mathrm{Rad}(TM))$ and $F^*(\mathrm{ltr}(TM))$ belong to $\mathrm{S}(TM)$.
(ii) If $(M, g, \mathrm{S}(TM))$ is a screen-invariant lightlike hypersurface, then $F^*(\mathrm{S}(TM))$ belongs to $\mathrm{S}(TM)$.
(iii) If $(M, g, \mathrm{S}(TM))$ is a radical-anti-invariant lightlike hypersurface, then $F^*(\mathrm{Rad}(TM))$ belongs to $\mathrm{ltr}\,(TM)$.
(iv) If $(M, g, \mathrm{S}(TM))$ is a radical-invariant lightlike hypersurface, then $F^*(\mathrm{Rad}(TM))$ belongs to $\mathrm{Rad}(TM)$.

Remark 1. *Semi-invariant submanifolds of almost product Riemannian manifolds were introduced by B. Şahin and M. Atçeken in [34], screen-semi-invariant lightlike hypersurfaces were studied by M. Atçeken and E. Kılıç in [35], radical-anti-invariant lightlike submanifolds were studied by E. Kılıç and B. Şahin in [36], and radical-invariant lightlike hypersurfaces were investigated by E. Kılıç and O. Bahadır in [37]. Special classifications that are presented in Definition 3 are introduced inspired by these studies.*

Now, we present some examples:

Example 3. *Let $(\widetilde{M}, \widetilde{g}, F)$ be an almost productlike semi-Riemannian manifold of Example 1. Consider a hypersurface of $(\widetilde{M}, \widetilde{g}, F)$ that is defined by*

$$M_1 = \{(x_1, x_2, x_3, x_4, x_5) : x_1 = x_3\}.$$

Then, the induced degenerate metric on M_1 *is*

$$g_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

With a straightforward computation, we have

$$\begin{aligned} \mathrm{Rad}\,(TM_1) &= \mathrm{Span}\{\xi = \partial_1 + \partial_3\}, \\ \mathrm{S}(TM_1) &= \mathrm{Span}\{e_1 = \partial_2, e_2 = \partial_4, e_3 = \partial_5\} \end{aligned}$$

and

$$\mathrm{ltr}\,(TM_1) = \mathrm{Span}\left\{N = \frac{1}{2}(-\partial_1 + \partial_3)\right\},$$

where $\{\partial_1, \partial_2, \partial_3, \partial_4, \partial_5\}$ *is the natural frame field on* $\Gamma(T\widetilde{M})$*. Therefore, we obtain*

$$F\xi = \partial_2 + \partial_5, \;\; F^{\star}\xi = \partial_1 + \frac{1}{2}\partial_3$$

and

$$FN = \frac{1}{2}(-\partial_2 + \partial_5), \; F^{\star}N = \frac{1}{4}(-2\partial_2 + \partial_3),$$

which imply that $(M_1, g_1, \mathrm{S}(TM_1))$ *is a screen semi-invariant lightlike hypersurface of* $(\widetilde{M}, \widetilde{g}, F)$.

Example 4. *Let* $(\widetilde{M}, \widetilde{g}, F)$ *be an almost productlike Lorentzian manifold of Example 2. Consider a hypersurface M that is defined by*

$$M_1 = \{(x_1, x_2, x_3, x_4) : x_1 = x_2\}.$$

Then, the induced degenerate metric on M is

$$g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

With a straightforward computation, we obtain

$$\begin{aligned} \mathrm{Rad}(TM) &= \mathrm{Span}\{\xi = \partial_1 + \partial_2\}, \\ \mathrm{S}(TM_1) &= \mathrm{Span}\{e_1 = \partial_3, e_2 = \partial_4\}, \\ \mathrm{ltr}(TM) &= \mathrm{Span}\{N = \frac{1}{4}(-\partial_1 + \partial_2)\}, \end{aligned}$$

where $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ *is the natural frame field on* $\Gamma(T\widetilde{M})$.

Therefore, we have

$$F\xi = \partial_1 + \partial_2, \;\; F^{*}\xi = -\partial_1 - \partial_2,$$

$$FN = \tfrac{1}{4}(\partial_1 - \partial_2), \;\; F^{*}N = \tfrac{1}{4}(-\partial_1 + \partial_2),$$

which imply that $(M, g, \mathrm{S}(TM))$ *is a radical-invariant lightlike hypersurface of* $(\widetilde{M}, \widetilde{g}, F)$.

4. Screen Semi-Invariant Lightlike Hypersurfaces

In this section, we investigate screen semi-invariant lightlike hypersurfaces on almost productlike semi-Riemannian manifolds.

Let $(M, g, S(TM))$ be a screen semi-invariant lightlike hypersurface of an almost productlike semi-Riemannian manifold $(\widetilde{M}, \widetilde{g}, F)$. In this case, we can put

$$FN = \rho, \; F^*N = \rho^*, \; F\xi = \mu \; \text{ and } \; F^*\xi = \mu^*, \tag{29}$$

where ρ, ρ^*, μ and μ^* belong to $\Gamma(S(TM))$. For any vector field $X \in \Gamma(S(TM))$, we write

$$FX = \varphi X + w^*(X)\xi + \eta^*(X)N \tag{30}$$

and

$$F^*X = \varphi^* X + w(X)\xi + \eta(X)N, \tag{31}$$

where φ and φ^* are projections on $\Gamma(TM)$ onto $\Gamma(S(TM))$, w, w^*, η and η^* are 1—forms defined by

$$w(X) = g(X, \rho), \;\; w^*(X) = g(X, \rho^*) \tag{32}$$

and

$$\eta(X) = g(X, \mu), \;\; \eta^*(X) = g(X, \mu^*) \tag{33}$$

for any $X \in \Gamma(TM)$.

Using the above facts, we give the following lemma:

Lemma 1. *Let $(M, g, S(TM))$ be an $(n+1)$—dimensional screen semi-invariant lightlike hypersurface of $(\widetilde{M}, \widetilde{g}, F)$. Then, we have*

$$\eta^*(\varphi X) = 0 \quad \text{and} \quad \eta(\varphi^* X) = 0 \tag{34}$$

for any $X \in \Gamma(TM)$. In particular, the equations

$$w^*(\varphi X) = 0 \quad \text{and} \quad w(\varphi^* X) = 0 \tag{35}$$

are satisfied for any $X \in \Gamma(S(TM))$.

Proof. From (29) and (30), we derive

$$\begin{aligned} X &= F(\varphi X) + w^*(X)F\xi + \eta^*(X)FN \\ &= \varphi^2 X + w^*(\varphi X)\xi + \eta^*(\varphi X)N + w^*(X)\mu + \eta^*(X)\rho \end{aligned} \tag{36}$$

for any $X \in \Gamma(TM)$. Considering the tangential and transversal parts of last equation, we obtain

$$\eta^*(\varphi X) = 0.$$

If we choose that X belongs to $\Gamma(S(TM))$, then we easily obtain from (36) that

$$w^*(\varphi X) = 0.$$

Using the fact that $(F^*)^2 = I_{n+2}$ and with a similar argument as that in the proof of (36), we have

$$X = (\varphi^*)^2 X + w(\varphi^* X)\xi + \eta(\varphi^* X)N + w(X)\mu^* + \eta(X)\rho^*, \tag{37}$$

which implies that

$$\eta(\varphi^* X) = 0$$

is satisfied for any $X \in \Gamma(TM)$. If we choose that X belongs to $\Gamma(\mathrm{S}(TM))$, then we obtain from (37) that

$$w(\varphi^* X) = 0.$$

This completes the proof of the lemma. □

Considering (36) and (37), we have the following lemma:

Lemma 2. *For any screen semi-invariant lightlike hypersurface $(M, g, \mathrm{S}(TM))$ of $(\widetilde{M}, \widetilde{g}, F)$, we have the following relations for any $X \in \Gamma(TM)$:*

$$\varphi^2 X = PX - w^*(X)\mu - \eta^*(X)\rho, \tag{38}$$

$$(\varphi^*)^2 X = PX - w(X)\mu^* - \eta(X)\rho^* \tag{39}$$

and

$$w^*(\varphi X) = w(\varphi^* X). \tag{40}$$

Lemma 3. *Let $(M, g, \mathrm{S}(TM))$ be a screen-semi-invariant lightlike hypersurface of $(\widetilde{M}, \widetilde{g}, F)$. Then, we have the following relations:*

$$g(\varphi X, Y) + \eta^*(X)\widetilde{g}(Y, N) = g(X, \varphi^* Y) + \eta(Y)\widetilde{g}(X, N) \tag{41}$$

and

$$g(\varphi X, \varphi^* Y) = g(X, Y) - w^*(X)\eta(Y) + \eta^*(Y)w(X) \tag{42}$$

for any $X, Y \in \Gamma(TM)$. In particular, the relation

$$g(\varphi X, Y) = g(X, \varphi^* Y) \tag{43}$$

is satisfied for any $X, Y \in \Gamma(\mathrm{S}(TM))$.

Proof. The proof is straightforward by using (30) and (31) in (25). □

5. Screen Semi-Invariant Lightlike Hypersurfaces of Locally Productlike Statistical Manifolds

In this section, we investigate screen semi-invariant lightlike hypersurfaces of a locally productlike statistical semi-Riemannian manifold $(\widetilde{M}, \widetilde{g}, F, \widetilde{\nabla})$.

Proposition 3. *Let $(M, g, \mathrm{S}(TM))$ be a screen semi-invariant lightlike hypersurface of $(\widetilde{M}, \widetilde{g}, F, \widetilde{\nabla})$. Then, the following relations are satisfied for any $X \in \Gamma(TM)$:*

$$\nabla_X \rho - \eta^*(\nabla_X \rho)\rho = -\varphi A_N^* X \tag{44}$$

and

$$\eta^*(\nabla_X \rho) = \tau^*(X). \tag{45}$$

Proof. Using the fact that $(\widetilde{M}, \widetilde{g}, F)$ is a locally productlike semi-Riemannian manifold, we write

$$\widetilde{\nabla}_X N = \widetilde{\nabla}_X F\rho = F\widetilde{\nabla}_X \rho \tag{46}$$

for any $X \in \Gamma(TM)$. Considering (13) and (46), we obtain

$$F\widetilde{\nabla}_X \rho = -A_N^* X + \tau^*(X)N. \tag{47}$$

If we put (12) on the left-hand side of (47), we have

$$F\nabla_X\rho + B(X,\rho)FN = -A^*_N X + \tau^*(X)N,$$

which is equivalent to

$$\varphi\nabla_X\rho + w^*(\nabla_X\rho)\xi + \eta^*(\nabla_X\rho)N + B(X,\rho)\rho = -A^*_N X + \tau^*(X)N. \quad (48)$$

Considering the transversal and tangential parts of (48), we have (45) and the following equality, respectively:

$$\varphi\nabla_X\rho + w^*(\nabla_X\rho)\xi + B(X,\rho)\rho = -A^*_N X \quad (49)$$

for any $X \in \Gamma(TM)$. Substituting $X = \rho$ in (30) and the using (29), we find $\varphi\rho = 0$. Applying φ on both sides of (49), we obtain

$$\varphi^2\nabla_X\rho + w^*(\nabla_X\rho)\varphi\xi = -\varphi A^*_N X. \quad (50)$$

In view of (29), (36) and (50), we obtain (44). This completes the proof. □

Now, we recall the following definitions of some useful vector fields:

Definition 4. *Let $(\widetilde{M},\widetilde{g})$ be a semi-Riemannian manifold and $\widetilde{\nabla}^{\widetilde{M}}$ be a linear connection on $(\widetilde{M},\widetilde{g})$. A vector field v on $\widetilde{M}$ is called a torse-forming field with respect to $\widetilde{\nabla}^{\widetilde{M}}$, if the following condition holds for any $X \in \Gamma(T\widetilde{M})$:*

$$\widetilde{\nabla}^{\widetilde{M}}_X v = \alpha X + \psi(X)v, \quad (51)$$

where ψ is a linear form, and α is a function [38]. A torse-forming vector field becomes

(i) a torqued vector field if $\psi(v) = 0$,
(ii) a concircular vector field if ψ vanishes identically;
(iii) a concurrent vector field if $\alpha = 1$ and $\psi = 0$;
(iv) a recurrent vector field if $\alpha = 0$ [39–46].

From Proposition 3 and Definition 4, we have the following corollaries:

Corollary 1. *Let $(M,g,\mathrm{S}(TM))$ be a screen semi-invariant lightlike hypersurface of $(\widetilde{M},\widetilde{g},F,\widetilde{\nabla})$. If $A^*_N = 0$, then there exists at least one vector field lying on $\Gamma(\mathrm{S}(TM))$, which is recurrent with respect to ∇.*

Corollary 2. *Let $(M,g,\mathrm{S}(TM))$ be a screen semi-invariant lightlike hypersurface. If ρ is a torse-forming vector field with respect to ∇, then $A^*_N X \neq 0$ for any $X \in \mathrm{S}(TM)$.*

Proof. Suppose that ρ is a torse-forming vector field with respect to ∇ and $A^*_N = 0$. Putting (51) in (44), we have

$$\alpha X + \psi(X)\rho - \eta^*(\nabla_X\rho)\rho = 0, \quad (52)$$

where α is a function, and ψ is a linear form. If we choose X and ρ to be linearly independent, then we obtain $\alpha = 0$. This contradicts the fact that ρ is a torse-forming vector field. Thus, $A^*_N X \neq 0$. □

Corollary 3. *Let $(M,g,\mathrm{S}(TM))$ be a screen semi-invariant lightlike hypersurface of $(\widetilde{M},\widetilde{g},F,\widetilde{\nabla})$. If ρ is a parallel vector field with respect to ∇, then the shape operator takes the following form:*

$$A^*_N X = -B(X,\rho)\rho$$

Proof. Putting $\nabla_X \rho = 0$ in (49), the proof is straightforward. □

Corollary 4. *Let $(M, g, S(TM))$ be a screen semi-Riemannian lightlike hypersurface. If ρ is a parallel with respect to ∇, then $(M, g, S(TM))$ is not totally normally umbilical with respect to $\widetilde{\nabla}$.*

Now, let us consider the following distributions defined on a lightlike hypersurface $(M, g, S(TM))$ of $(\widetilde{M}, \widetilde{g}, F, \widetilde{\nabla})$:

$$D_1 = \mathrm{Span}\{\mu, \mu^*\} \text{ and } D_2 = \mathrm{Span}\{\rho, \rho^*\}.$$

Then, there exists an $(n-4)$—dimensional semi-Riemannian distribution D in $S(TM)$, such that we can write

$$S(TM) = D \oplus_{orth} \{D_1 \oplus D_2\}.$$

Here, $\oplus$ denotes the direct sum that is not orthogonal. In this case, we clearly write from (3) and (4) that

$$TM = D \oplus_{orth} \{D_1 \oplus D_2\} \oplus_{orth} \mathrm{Rad}(TM)$$

and

$$T\widetilde{M} = D \oplus_{orth} \{D_1 \oplus D_2\} \oplus_{orth} \{\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)\}.$$

From the above facts, it is clear that D is invariant with respect to F and F^*.

Now, let us consider

$$\overline{D} = D \oplus_{orth} \mathrm{Rad}(TM) \oplus_{orth} F(\mathrm{Rad}(TM)) \oplus_{orth} F^*(\mathrm{Rad}(TM)).$$

Then, it is clear that the $\overline{D}$ is also invariant with respect to F and F^*.

Theorem 1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\widetilde{M}, \widetilde{g}, F, \widetilde{\nabla})$. Then, the following statements are equivalent:*

(i) *$\overline{D}$ is integrable with respect to ∇.*

(ii) *The relation*

$$B(X, fY) = B(Y, fX) \tag{53}$$

is satisfied for any $X, Y \in \Gamma(\overline{D})$.

(iii) *The relation*

$$g(\overline{A}^*_\xi X, fY) - g(\overline{A}^*_\xi Y, fX) = B(X, \xi)\widetilde{g}(fY, N) - B(Y, \xi)\widetilde{g}(fX, N) \tag{54}$$

is satisfied for any $X, Y \in \Gamma(\overline{D})$, where $fX = \varphi X + w^*(X)\xi$.

Proof. Using the fact that $(\widetilde{M}, \widetilde{g}, F, \widetilde{\nabla})$ is a locally productlike statistical semi-Riemannian manifold, we can write

$$\widetilde{\nabla}_X FY = F\widetilde{\nabla}_X Y \tag{55}$$

for any $X, Y \in \Gamma(\overline{D})$. Since Y is perpendicular to ρ and ρ^*, we have

$$\widetilde{\nabla}_X FY = \widetilde{\nabla}_X(\varphi Y + w^*(Y)\xi). \tag{56}$$

From (12) and (56), we obtain

$$\widetilde{\nabla}_X FY = \nabla_X \varphi Y + B(X, \varphi Y)N + X[w^*(Y)]\xi + w^*(Y)\nabla_X \xi + w^*(Y)B(X, \xi)N. \tag{57}$$

Considering also (12) and (30), we have

$$\begin{aligned} F\widetilde{\nabla}_X Y &= F\nabla_X Y + B(X,Y)FN \\ &= \varphi\nabla_X Y + w^*(\nabla_X Y)\xi + \eta^*(\nabla_X Y)N + B(X,Y)\rho. \end{aligned} \tag{58}$$

In view of (55), (57) and (58), we obtain

$$\begin{aligned} &\nabla_X \varphi Y + B(X,\varphi Y)N + X[w^*(Y)]\xi + w^*(Y)\nabla_X \xi \\ &+w^*(Y)B(X,\xi)N = \varphi\nabla_X Y + w^*(\nabla_X Y)\xi + \eta^*(\nabla_X Y)N + B(X,Y)\rho. \end{aligned} \tag{59}$$

Interchanging the roles of X and Y, we also have

$$\begin{aligned} &\nabla_Y \varphi X + B(Y,\varphi X)N + Y[w^*(X)]\xi + w^*(X)\nabla_Y \xi \\ &+w^*(X)B(Y,\xi)N = \varphi\nabla_Y X + w^*(\nabla_Y X)\xi + \eta^*(\nabla_Y X)N + B(Y,X)\rho. \end{aligned} \tag{60}$$

If we subtract (59) and (60) side to side and consider the transversal parts, we have

$$\eta^*(\nabla_X Y) - \eta^*(\nabla_Y X) = B(X,\varphi Y) - B(Y,\varphi X) + w^*(Y)B(X,\xi) - w^*(X)B(Y,\xi), \tag{61}$$

which is equivalent to

$$B(X,fY) - B(Y,fX) = \eta^*([X,Y]). \tag{62}$$

In light of (62), we see that equation

$$B(X,fY) = B(Y,fX)$$

is satisfied for any $X,Y \in \Gamma(\overline{D})$ if and only if $[X,Y] \in \Gamma(\overline{D})$. Therefore, $(i) \Leftrightarrow (ii)$.

From (20), (22), and (62), we conclude that $(ii) \Leftrightarrow (iii)$. Hence, the proof is completed. □

From Theorem 1, we immediately obtain the following corollaries:

Corollary 5. *If $(M,g,\mathrm{S}(TM))$ is totally geodesic with respect to $\widetilde{\nabla}$, then the $\overline{D}$ is integrable with respect to ∇.*

Corollary 6. *If $(M,g,\mathrm{S}(TM))$ is totally tangentially umbilical with respect to $\widetilde{\nabla}$, then $\overline{D}$ is not integrable with respect to ∇.*

With similar arguments of Theorem 1, we also obtain the following theorem:

Theorem 2. *For any screen semi-invariant lightlike hypersurface $(M,g,\mathrm{S}(TM))$, the following statements are equivalent:*

(i) *$\overline{D}$ is integrable with respect to ∇^*.*

(ii) *The relation*

$$B^*(X,f^*Y) = B^*(Y,f^*X)$$

is satisfied for any $X,Y \in \Gamma(\overline{D})$.

(iii) *The relation*

$$g(\overline{A}_\xi X, f^*Y) - g(\overline{A}_\xi Y, f^*X) = B^*(X,\xi)\widetilde{g}(f^*Y,N) - B^*(Y,\xi)\widetilde{g}(f^*X,N)$$

is satisfied for any $X,Y \in \Gamma(\overline{D})$, where $f^*X = \varphi^*X + w(X)\xi$.

Definition 5. *Let $(M,g,\mathrm{S}(TM))$ be a lightlike hypersurface of $(\widetilde{M},\widetilde{g},F,\widetilde{\nabla})$.*

(i) *The hypersurface is called mixed geodesic with respect to $\widetilde{\nabla}$ if $B(X,Y) = 0$ for any $X \in \Gamma(\overline{D})$ and $Y \in \Gamma(D_2)$.*

(ii) *The hypersurface is called mixed geodesic with respect to* $\widetilde{\nabla}^*$ *if* $B^*(X,Y) = 0$ *for any* $X \in \Gamma(\overline{D})$ *and* $Y \in \Gamma(D_2)$.

Theorem 3. *A screen semi-invariant lightlike hypersurface* $(M, g, S(TM))$ *is mixed geodesic with respect to* $\widetilde{\nabla}$ *if and only if* $A^*_N X$ *is perpendicular to* D_1 *for any* $X \in \Gamma(\overline{D})$.

Proof. Suppose that $(M, g, S(TM))$ is mixed geodesic with respect to $\widetilde{\nabla}$. From (12) and (13), we have

$$\widetilde{\nabla}_X \rho = \nabla_X \rho + B(X, \rho)N, \tag{63}$$
$$F\widetilde{\nabla}_X N = F(-A^*_N X + \tau^*(X)N) = -f A^*_N X - \eta^*(A^*_N X)N + \tau^*(X)\rho, \tag{64}$$

respectively, for any $X \in \Gamma(\overline{D})$. Using the fact that $(\widetilde{M}, \widetilde{g}, F)$ is a locally product manifold, and using (63) and (64), we see that equality

$$0 = B(X, \rho) = -\eta^*(A^*_N X)$$

is satisfied. Thus, we find

$$g(A^*_N X, \mu^*) = 0 \tag{65}$$

for any $X \in \Gamma(\overline{D})$. In a similar way, from (12) and (13), we have

$$\widetilde{\nabla}_X \rho^* = \nabla_X \rho^* + B(X, \rho^*)N, \tag{66}$$
$$F^*\widetilde{\nabla}_X N = -f^* A^*_N X - \eta(A^*_N X)N + \tau^*(X)\rho^*, \tag{67}$$

respectively. In view of (66) and (67), we obtain that equality

$$0 = B(X, \rho^*) = -\eta(A^*_N X)$$

is satisfied for any $X \in \Gamma(\overline{D})$. Thus, we find

$$g(A^*_N X, \mu) = 0. \tag{68}$$

This fact implies that $A^*_N X$ is perpendicular to D_1 for any $X \in \Gamma(\overline{D})$.
The converse of the proof is straightforward. □

With similar arguments, we obtain the following theorem:

Theorem 4. *A screen semi-invariant lightlike hypersurface* $(M, g, S(TM))$ *is mixed geodesic with respect to* $\widetilde{\nabla}^\star$ *if and only if* $A_N X$ *is perpendicular to* D_1 *for any* $X \in \Gamma(\overline{D})$.

By applying methods in the proof of the Theorem 3.11 in [37], we state the following theorem:

Theorem 5. *Let* $(M, g, S(TM))$ *be a screen semi-invariant lightlike hypersurface of* $(\widetilde{M}, \widetilde{g}, F, \widetilde{\nabla})$. *Then, M is the locally product manifold of* $\overline{D}$ *and* D_2 *if and only if f and* f^* *are parallels on* ∇ *and* ∇^*, *respectively.*

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