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Multiple Existence Results of Nontrivial Solutions for a Class of Second-Order Partial Difference Equations

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Abstract: In this paper, we consider the existence and multiplicity of nontrivial solutions for discrete elliptic Dirichlet problems $\Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1) = -f((i,j),u(i,j)), (i,j) \in \Omega, u(i,0) = u(i,T_2+1) = 0\, i \in \mathbb{Z}(1,T_1)$, $u(0,j) = u(T_1+1,j) = 0\, j \in \mathbb{Z}(1,T_2)$, which have a symmetric structure. When the nonlinearity $f(\cdot,u)$ is resonant at both zero and infinity, we construct a variational functional on a suitable function space and turn the problem of finding nontrivial solutions of discrete elliptic Dirichlet problems to seeking nontrivial critical points of the corresponding functional. We establish a series of results based on the existence of one, two or five nontrivial solutions under reasonable assumptions. Our results depend on the Morse theory and local linking.

Keywords: partial difference equation; local linking; Morse theory; nontrivial solution

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1. Introduction

Let $\mathbb{N}$ and $\mathbb{Z}$ denote sets of all natural numbers and integers, respectively. For integers s, t with $s \leq t$, denote the discrete interval $\mathbb{Z}(s,t) := \{s, s+1, \cdots, t\}$. Given integers T_1, $T_2 \geq 2$, write $\Omega := \mathbb{Z}(1,T_1) \times \mathbb{Z}(1,T_2)$, We are interested in the existence of nontrivial solutions for the following nonlinear second-order partial difference equation

$$\Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1) = -f((i,j),u(i,j)), \qquad (i,j) \in \Omega, \tag{1}$$

subject to Dirichlet boundary conditions

$$u(i,0) = u(i,T_2+1) = 0 \quad i \in \mathbb{Z}(1,T_1), \qquad u(0,j) = u(T_1+1,j) = 0 \quad j \in \mathbb{Z}(1,T_2), \tag{2}$$

where Δ is the forward difference operator and $\Delta_1 u(i,j) = u(i+1,j) - u(i,j)$, $\Delta_2 u(i,j) = u(i,j+1) - u(i,j)$, $\Delta^2 u(i,j) = \Delta(\Delta u(i,j))$. $f((i,j),\cdot) \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies $f((i,j),0) = 0$. Obviously, $u = 0$ is a trivial solution to Problems (1) and (2). Meanwhile, we are interested in nontrivial solutions to Problems (1) and (2).

During the past decades, difference equations have been used extensively in various fields, for example, refs. [1,2] apply difference equations to establish some epidemic models. At the same time, many rich results have been obtained, here mention a few, refs. [3–7] give results on periodical solutions, sign-changing solutions, positive solutions and heteroclinic solutions for difference equations. With the rapid development of modern technology, partial difference equations, which involve two or more variables, have been widely applied in quantum mechanics, image processing, life sciences and other fields [8]. As a result, many scholars have turned their attention to studying partial difference equations and have achieved excellent results for these equations as well. For example, refs. [9–12] presented results on the existence and multiplicity of nontrivial solutions for second-order partial difference equations and [13–15] studied discrete Kirchhoff type problems via critical point theory.

Equation (1), a nonlinear second-order partial difference equation, with the addition of the Dirichlet boundary conditions of Equation (2), can be regarded as the discrete analogue of

$$\begin{cases} -\Delta u = f(u), & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

which has a long history of study and has captured extensive research interests. Among various techniques applied in the numerous obtained results, we find that the Morse theory is a powerful instrument to deal with the problem of the existence of solutions for both differential equations and difference equations. For example, refs. [16–18] established multiple existence results by using the Morse theory for Equation (3). Additionally, via the Morse theory, refs. [19] produced results based on three nontrivial solutions and [20] obtained four nontrivial solutions to Problems (1) and (2).

As it is well-known, Equation (1) is regarded as a discretization of Equation (3). It not only assists in the numerical simulation of Equation (3), but also has wide applications [8]. Consequently, it is a meaningful job to study Problems (1) and (2) to establish results based on the existence of one, two or five nontrivial solutions via the Morse theory.

We organize this paper as follows: we establish the variational functional of Problem (1) and (2) and display preliminaries in Section 2. Our main results and their corresponding proofs are provided in Section 3. Finally, we give a conclusion in Section 4.

2. Variational Structure and Some Auxiliary Results

Let E be a T_1T_2-dimensional Euclidean space equipped with the usual inner product $(\cdot,\cdot)$ and norm $|\cdot|$. Denote

$$\begin{aligned} S =&\{u : \mathbb{Z}(0, T_1+1) \times \mathbb{Z}(0, T_2+1) \to \mathbb{R} \quad \text{such that} \quad u(i, 0) = u(i, T_2+1) = 0, \\ & i \in \mathbb{Z}(0, T_1+1) \quad \text{and} \quad u(0, j) = u(T_1+1, j) = 0, \quad j \in \mathbb{Z}(0, T_2+1)\}. \end{aligned}$$

Define the inner product $\langle\cdot,\cdot\rangle$ on S as

$$\langle u, v\rangle = \sum_{i=1}^{T_1+1}\sum_{j=1}^{T_2} \Delta_1 u(i-1, j)\Delta_1 v(i-1, j) + \sum_{i=1}^{T_1}\sum_{j=1}^{T_2+1} \Delta_2 u(i, j-1)\Delta_2 v(i, j-1), \qquad \forall u, v \in S.$$

Then, as [19] or [20], the induced norm $\|\cdot\|$ is

$$\|u\| = \sqrt{\langle u, u\rangle} = \left(\sum_{i=1}^{T_1+1}\sum_{j=1}^{T_2} |\Delta_1 u(i-1, j)|^2 + \sum_{i=1}^{T_1}\sum_{j=1}^{T_2+1} |\Delta_2 u(i, j-1)|^2\right)^{\frac{1}{2}}, \quad \forall u \in S.$$

Thus, S is a Hilbert space and isomorphic to E. Here and hereafter, we take $u \in S$ as an extension of $u \in E$ when necessary.

Consider the functional $J : S \to \mathbb{R}$ expressed in the following form as

$$\begin{aligned} J(u) =& \frac{1}{2}\sum_{i=1}^{T_1+1}\sum_{j=1}^{T_2} |\Delta_1 u(i-1, j)|^2 + \frac{1}{2}\sum_{i=1}^{T_1}\sum_{j=1}^{T_2+1} |\Delta_2 u(i, j-1)|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i, j), u(i, j)) \\ =& \frac{1}{2}\|u\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i, j), u(i, j)), \qquad \forall u \in S, \end{aligned} \tag{4}$$

where $F((i, j), u) = \int_0^u f((i, j), \tau)d\tau$ for each $(i, j) \in \Omega$. Note that $f((i, j), u)$ is continuously differentiable with respect to u. It is clear that $J \in C^2(S, \mathbb{R})$ and solutions to Problems (1) and (2) are precisely critical points of $J(u)$. Moreover, for any $u, v \in S$, when using Dirichlet boundary conditions, a direct computation shows that the Fréchet derivative of J is

$$\begin{aligned} \langle J'(u), v\rangle =& \sum_{i=1}^{T_1+1}\sum_{j=1}^{T_2} \Delta_1 u(i-1,j)\Delta_1 v(i-1,j) + \sum_{i=1}^{T_1}\sum_{j=1}^{T_2+1} \Delta_2 u(i,j-1)\Delta_2 v(i,j-1) \\ &- \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} f((i,j),u(i,j))v(i,j) \\ =& -\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\{\Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1) + f((i,j),u(i,j))\}v(i,j). \end{aligned} \tag{5}$$

Let the discrete Laplacian be denoted by Ξ, where $\Xi u(i,j) = \Delta_1^2 u(i-1,j) + \Delta_2^2 u(i,j-1)$. Given a $T_1T_2 \times T_1T_2$ matrix D as

$$D = \begin{pmatrix} L & -I_{T_1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -I_{T_1} & L & -I_{T_1} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -I_{T_1} & L & -I_{T_1} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_{T_1} & L & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & L & -I_{T_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -I_{T_1} & L & -I_{T_1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -I_{T_1} & L & -I_{T_1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -I_{T_1} & L \end{pmatrix}_{T_1T_2 \times T_1T_2},$$

where I_{T_1} is a $T_1 \times T_1$ identity matrix and

$$L = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 4 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 4 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 4 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 4 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 4 \end{pmatrix}_{T_1 \times T_2}.$$

The eigenvalues of matrix D are the same as the Dirichlet eigenvalues of $-\Xi$ on Ω. According to [10,13], D is a positive definite symmetric matrix and $-\Xi$ is invertible and distinct. The Dirichlet eigenvalues of $-\Xi$ on Ω can be rearranged in the form of $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_{T_1T_2}$. Let $\phi_k = (\phi_k(1), \phi_k(2), \cdots, \phi_k(T_1T_2))^{tr}$, $k \in [1, T_1T_2]$ be an eigenvector corresponding to the eigenvalue λ_k, which yields

$$S = W^- \oplus W^0 \oplus W^+,$$

where $W^- = \mathrm{span}\{\phi_1, \cdots, \phi_{k-1}\}$, $W^0 = \mathrm{span}\{\phi_k\}$ and $W^+ = (W^- \oplus W^0)^\perp$.

For later use, we define another norm of Euclidean space E as

$$\|u\|_2 = \left(\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} |u(i,j)|^2\right)^{\frac{1}{2}}, \qquad u \in E.$$

Then, for any $u \in S$, it holds that

$$\lambda_1 \|u\|_2^2 \le \|u\|^2 \le \lambda_{T_1T_2}\|u\|_2^2. \tag{6}$$

Particularly,

$$\begin{aligned} \lambda_{k+1}\|u\|_2^2 \le \|u\|^2 \le \lambda_{T_1T_2}\|u\|_2^2, \qquad & u \in W^+, \\ \lambda_1\|u\|_2^2 \le \|u\|^2 \le \lambda_{k-1}\|u\|_2^2, \qquad & u \in W^-. \end{aligned} \tag{7}$$

In the following paragraphs, we state some collected results which will be used later in this paper.

We can say that the functional J satisfies the Palais–Smale condition (PS) if any sequence $\{u_n\} \subseteq S$, satisfying $|J(u_n)| \le M$, $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence [21]. Notice that if (PS) is satisfied, then the weaker Cerami condition (C) is satisfied. Moreover, the deformation condition (D) is also satisfied [21,22].

Now, we recall some basic results based on the Morse theory and we can refer to [17,23–25] for more detail.

Definition 1. *Based on [23,24], denote U be a neighborhood of u_0 and u_0 is an isolated critical group of J with $J(u_0) = c \in \mathbb{R}$. Then, the group*

$$C_q(J, u_0) := H_q(J^c \cap U, J^c \cap U \backslash u_0), \quad q \in \mathbb{Z},$$

is called the q-th critical group of J at u_0. Let $\kappa = \{u \in S | J'(u) = 0\}$. For all $a \in \mathbb{R}$, each critical point of J is greater than a and $J \in C^2(S, \mathbb{R})$ satisfies (D). Then, the group

$$C_q(J, \infty) := H_q(S, J^a), \quad q \in \mathbb{Z},$$

is called the q-th critical group of J at infinity.

To obtain some nontrivial critical points, we need the following auxiliary propositions.

Proposition 1. *Based on [23,24], let S be a Hilbert space, $J \in C^2(S, \mathbb{R})$. Suppose that u_0 is the isolated critical point of J with a limited Morse index $\mu(u_0)$ and zero nullity $\nu(u_0)$. Moreover, $J''(u_0)$ is a Fredholm operator. If u_0 is a local minimizer of J, then*

$$C_q(J, u_0) \cong \delta_{q,0}\mathbb{Z}, \qquad q \in \mathbb{Z}.$$

Proposition 2. *Based on [17], let $J \in C^2(S, \mathbb{R})$ satisfy (D). There hold*
(Q_1) J possesses a critical point u such that $C_q(J, u) \ncong 0$ if $C_q(J, \infty) \ncong 0$ for some q;
(Q_2) J admits a non-zero critical point if 0 is the isolated critical point of J and $C_q(J, \infty) \ncong C_q(J, 0)$ for some q.

To compute the critical group at infinity and 0, Propositions 3 and 4, respectively, are necessary.

Proposition 3. *Based on [25,26], assume $S = W_\infty^- \oplus (W_\infty^-)^\perp$. Let J be bounded from below by $(W_\infty^-)^\perp$ and $J(u) \to -\infty$ as $\|u\| \to \infty$ with $u \in W_\infty^-$. Then,*

$$C_k(J, \infty) \ncong 0, \qquad k = \dim W_\infty^- < \infty.$$

Proposition 4. *Based on [16], let 0 be an isolated critical point of J with a Morse index μ_0 and zero nullity ν_0. If J has a local linking at the 0 subject to $S = W_0^- \oplus W_0^+$, $m = \dim W_0^- < \infty$; that is, there exists $\rho > 0$ such that*

$$\begin{aligned} J(u) \le 0, \qquad & u \in W_0^-, \qquad \|u\| \le \rho, \\ J(u) \ge 0, \qquad & u \in W_0^+, \qquad \|u\| \le \rho. \end{aligned}$$

Then

$$C_q(J, 0) \cong \delta_{q,m}\mathbb{Z}, \qquad q \in \mathbb{Z}$$

if $m = \mu_0$ *or* $m = \mu_0 + \nu_0$.

In our detailed proofs, the following Mountain Pass Lemma is also needed.

Proposition 5. *Based on [24], let S be a real Banach space and $J \in C^1(S, \mathbb{R})$ satisfy (PS). Further, if $J(0) = 0$ and*
(Q_3) *there exists constants ρ, $a > 0$ such that $J|_{\partial B_\rho} \geq a$;*
(Q_4) *there exists $e \in S \backslash B_\rho$ such that $J(e) \leq 0$.*
Then, J possesses a critical value $c \geq a$ given by

$$c = \inf_{h \in \Gamma} \sup_{u \in [0,1]} J(h(u)),$$

where

$$\Gamma = \{h \in C([0,1], S) | h(0) = 0, h(1) = e\}.$$

3. Main Results and Proofs

In this section, we state our main results and present the associated proofs at length. Denote

$$\lambda_m = \lim_{|u| \to 0} \frac{f((i,j),u)}{u}, \qquad \forall (i,j) \in \Omega, \tag{8}$$

$$\lambda_k = \lim_{|u| \to \infty} \frac{f((i,j),u)}{u}, \qquad \forall (i,j) \in \Omega, \tag{9}$$

and

$$g((i,j),u) = f((i,j),u) - \lambda_k u, \qquad g_0((i,j),u) = f((i,j),u) - \lambda_m u,$$

where $G((i,j),u) = \int_0^u g((i,j),\tau) d\tau$, $G_0((i,j),u) = \int_0^u g_0((i,j),\tau) d\tau$. For any $(i,j) \in \Omega$, assume that:
$(\mathbf{I_0^\pm})$ there exists some $\delta > 0$ such that $\pm G_0((i,j),u) \geq 0$ as $|u(i,j)| \leq \delta$;
$(\mathbf{I_1})$ there exists $u_1 > 0$ and $u_2 < 0$ such that $f((i,j),u_1) = f((i,j),u_2) = 0$;
$(\mathbf{I_2})$ there exists constants $A, B > 0$ and $0 < r < 1$ such that $|g((i,j),u)| \leq A|u(i,j)|^r + B$;
$(\mathbf{I_3})$ $\liminf_{\|v\| \to \infty, v \in W^0} \frac{1}{\|v\|^{2r}} \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} G((i,j),u) \geq \frac{4\beta^2}{\alpha}$;
$(\mathbf{I_4})$ $\limsup_{\|v\| \to \infty, v \in W^0} \frac{1}{\|v\|^{2r}} \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} G((i,j),u) \leq -\frac{4\beta^2}{\alpha}$,
where $\alpha = \min\{1 - \frac{\lambda_k}{\lambda_{k+1}}, \frac{\lambda_k}{\lambda_{k-1}} - 1\}$ and $\beta = A(T_1 T_2)^{\frac{1-r}{2}} \lambda_1^{-\frac{1+r}{2}}$.

Our main results are as follows:

Theorem 1. *Let $(\mathbf{I_1})$ and $(\mathbf{I_2})$ hold. Then, Problems (1) and (2) possess at least five nontrivial solutions if one of the following conditions is fulfilled:*
(1) $(\mathbf{I_3})$, $(\mathbf{I_0^+})$, *$k, m \geq 2$ and $k \neq m$;*
(2) $(\mathbf{I_3})$, $(\mathbf{I_0^-})$, *$k \geq 2$, $m > 2$ and $k \neq m - 1$;*
(3) $(\mathbf{I_4})$, $(\mathbf{I_0^+})$, *$k > 2$, $m \geq 2$ and $k \neq m - 1$;*
(4) $(\mathbf{I_4})$, $(\mathbf{I_0^-})$, *$k, m > 2$ and $k \neq m$.*

Theorem 2. *Suppose $(\mathbf{I_1})$, $(\mathbf{I_2})$ and $(\mathbf{I_3})[(\mathbf{I_4})]$ are satisfied. Moreover, if one of the following conditions is met:*
(1) $(\mathbf{I_0^+})$ *with $m \neq k[m \neq k-1]$;* **(2)** $(\mathbf{I_0^-})$ *with $m \neq k+1[m \neq k]$.*
Then, Problems (1) and (2) have at least one nontrivial solution.

Theorem 3. *Assume $(\mathbf{I_1})$, $(\mathbf{I_2})$ and $(\mathbf{I_4})[(\mathbf{I_3})]$ are true. Further, if $k = 1$ and either:*
(1) $(\mathbf{I_0^+})$ *with $m \geq 1[m > 1]$; or* **(2)** $(\mathbf{I_0^-})$ *with $m > 1[m \neq 2]$.*
Then, Problems (1) and (2) have at least two nontrivial solutions.

According to the propositions given in Section 2, (PS) is necessary. Therefore, first, we must verify that J satisfies (PS) at length.

Lemma 1. *If J satisfies $(\mathbf{I_2})$, $(\mathbf{I_3})$ or $(\mathbf{I_4})$, then J satisfies (PS).*

Proof. Suppose that $\{u_n\} \subseteq S$ and there exists a constant $M > 0$ such that

$$|J(u_n)| \leq M, \qquad J'(u_n) \to 0, \quad \text{as} \quad n \to \infty.$$

Since S is a T_1T_2-dimensional Hilbert space, it suffices to show that $\{u_n\}$ is bounded. Otherwise, we can assume that $\|u_n\| \to \infty$ as $n \to \infty$. Recall the expression of J; for any $(i,j) \in \Omega$, we have

$$\langle J'(u_n), \varphi \rangle = \langle u_n, \varphi \rangle - \lambda_k \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (u_n(i,j), \varphi(i,j)) - \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (g((i,j), u_n(i,j)), \varphi(i,j)).$$

Set $\varphi = w_n^+ \in W^+$, based on $(\mathbf{I_2})$, which yields

$$\begin{aligned} \alpha \|w_n^+\|^2 \leq & \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right) \|w_n^+\|^2 \leq \|w_n^+\|^2 - \lambda_k \|w_n^+\|_2^2 \\ = & \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (g((i,j), u_n(i,j)), w_n^+(i,j)) + \langle J'(u_n), w_n^+ \rangle \\ \leq & \|w_n^+\| + \sum_{i=1}^{T_1} \sum_{j=1}^{T_2} (A|u_n(i,j)|^r + B)|w_n^+(i,j)| \\ \leq & \|w_n^+\| + B\sqrt{T_1T_2}\|w_n^+\|_2 + A\|w_n^+\|_2 \|u_n\|_{2r}^r \\ \leq & \left(1 + \frac{B\sqrt{T_1T_2}}{\sqrt{\lambda_{k+1}}}\right) \|w_n^+\| + A(T_1T_2)^{\frac{1-r}{2}} \|u_n\|_2^r \|w_n^+\|_2 \\ \leq & c_1 \|w_n^+\| + \beta \|u_n\|^r \|w_n^+\|, \end{aligned} \tag{10}$$

where $c_1 := 1 + \frac{B\sqrt{T_1T_2}}{\sqrt{\lambda_{k+1}}}$. Thus,

$$\|w_n^+\|^2 \leq \frac{c_1}{\alpha} \|w_n^+\| + \frac{\beta}{\alpha} \|w_n^+\| \|u_n\|^r.$$

Further,

$$\|w_n^+\| \leq \frac{c_1}{\alpha} + \frac{\beta}{\alpha} \|u_n\|^r,$$

which implies that

$$\frac{\|w_n^+\|}{\|u_n\|} \to 0, \qquad \text{as} \quad n \to \infty. \tag{11}$$

Together with Equation (10), we have

$$\begin{aligned} \|w_n^+\|^2 - \lambda_k \|w_n^+\|_2^2 \leq & c_1 \left(\frac{c_1}{\alpha} + \frac{\beta}{\alpha} \|u_n\|^r\right) + \beta \|u_n\|^r \left(\frac{c_1}{\alpha} + \frac{\beta}{\alpha} \|u_n\|^r\right) \\ = & \frac{c_1^2}{\alpha} + \frac{2c_1\beta}{\alpha} \|u_n\|^r + \frac{\beta^2}{\alpha} \|u_n\|^{2r}. \end{aligned} \tag{12}$$

Take $\varphi = w_n^- \in W^-$, which is similar to Equation (10), to obtain

$$
\begin{aligned}
-\alpha\|w_n^-\|^2 \geq & \left(1-\frac{\lambda_k}{\lambda_{k-1}}\right)\|w_n^-\|^2 \geq \|w_n^-\|^2 - \lambda_k\|w_n^-\|_2^2 \\
= & \sum_{i=1}^{T_1}\sum_{j=1}^{T_2}(g((i,j),u_n(i,j)),w_n^-(i,j)) + \langle J'(u_n),w_n^-\rangle \\
\geq & -\|w_n^-\| - B\sqrt{T_1T_2}\|w_n^-\|_2 - A\|w_n^-\|_2\|u_n\|_{2r}^r \geq -c_2\|w_n^-\| - \beta\|w_n^-\|\|u_n\|^r,
\end{aligned} \tag{13}
$$

where $c_2 := 1 + \frac{B\sqrt{T_1T_2}}{\sqrt{\lambda_1}}$. Then,

$$
\alpha\|w_n^-\|^2 \leq c_2\|w_n^-\| + \beta\|w_n^-\|\|u_n\|^r,
$$

which means that $\|w_n^-\| \leq \frac{c_2}{\alpha} + \frac{\beta}{\alpha}\|u_n\|^r$, that is,

$$
\frac{\|w_n^-\|}{\|u_n\|} \to 0, \qquad \text{as} \quad n \to \infty. \tag{14}
$$

Furthermore,

$$
\begin{aligned}
\|w_n^-\|^2 - \lambda_k\|w_n^-\|_2^2 \leq & -\alpha\|w_n^-\|^2 \leq \alpha\|w_n^-\|^2 \\
\leq & c_2\left(\frac{c_2}{\alpha}+\frac{\beta}{\alpha}\|u_n\|^r\right) + \beta\|u_n\|^r\left(\frac{c_2}{\alpha}+\frac{\beta}{\alpha}\|u_n\|^r\right) \\
= & \frac{c_2^2}{\alpha} + \frac{2c_2\beta}{\alpha}\|u_n\|^r + \frac{\beta^2}{\alpha}\|u_n\|^{2r}.
\end{aligned} \tag{15}
$$

On the other hand, when we recall the expressions of c_1 and c_2, we obtain $c_2 > c_1 > 0$. Combining Equation (12) with Equation (15), it follows that

$$
\begin{aligned}
J(u_n) = & \frac{1}{2}(\|w_n\|^2 - \lambda_k\|w_n\|_2^2) - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} G((i,j),u_n(i,j)) \\
\leq & \frac{\beta^2}{\alpha}\|u_n\|^{2r} + \frac{c_2^2}{\alpha} + \frac{2c_2\beta}{\alpha}\|u_n\|^r - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} G((i,j),v_n(i,j)) \\
& - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2}[G((i,j),u_n(i,j)) - G((i,j),v_n(i,j))].
\end{aligned} \tag{16}
$$

Notice that $S = W^+ \oplus W^- \oplus W^0$; thus, Equations (11) and (14) indicate

$$
\frac{\|v_n\|}{\|u_n\|} \to 1, \qquad \text{as} \quad n \to \infty. \tag{17}
$$

Note that $(\mathbf{I_3})$ is valid. For any given $\varepsilon > 0$, there exists some $R > 0$ such that

$$
-\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} G((i,j),v_n(i,j)) \leq (-4+\varepsilon)\frac{\beta^2}{\alpha}\|v_n\|^{2r}, \qquad v_n \in W^0 \quad \text{with} \quad \|v_n\| \geq R. \tag{18}
$$

Owing to the Mean Value Theorem, it holds that

$$\begin{aligned}
&\left|\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}[G((i,j),u_n(i,j))-G((i,j),v_n(i,j))]\right| \\
=&\left|\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}w_n(i,j)\int_0^1 g((i,j),v_n(i,j)+tw_n(i,j))dt\right| \\
\leq&\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}\left|w_n(i,j)\int_0^1[A|v_n(i,j)+tw_n(i,j)|^r|+B]dt\right| \\
\leq&\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}A[|v_n(i,j)|^r|w_n(i,j)|+|w_n(i,j)|^{1+r}]+\frac{B\sqrt{T_1T_2}}{\sqrt{\lambda_1}}\|w_n\| \\
\leq& A\|v_n\|_{2r}^r\|w_n\|_2+\beta\|w_n\|^{1+r}+\frac{B\sqrt{T_1T_2}}{\sqrt{\lambda_1}}\|w_n\| \\
\leq&\beta\|v_n\|^r\left(\frac{2c_2}{\alpha}+\frac{2\beta}{\alpha}\|u_n\|^r\right)+\beta\|u_n\|^{1+r}+\frac{B\sqrt{T_1T_2}}{\sqrt{\lambda_1}}\left(\frac{2c_2}{\alpha}+\frac{2\beta}{\alpha}\|u_n\|^r\right).
\end{aligned} \tag{19}$$

Therefore, Equations (18) and (19) lead to

$$\begin{aligned}
J(u_n)\leq&\frac{\beta^2}{\alpha}\left[\|u_n\|^{2r}+2\|u_n\|^r\|v_n\|^r+(-4+\varepsilon)\|v_n\|^{2r}\right]+\beta\|u_n\|^{1+r} \\
&+\frac{2c_2\beta}{\alpha}\|u_n\|^r+c_3+\frac{2\beta c_2}{\alpha}\|v_n\|^r \\
=&\frac{\beta^2}{\alpha}\|u_n\|^{2r}[1+\frac{2\|v_n\|^r}{\|u_n\|^r}-\frac{(4-\varepsilon)\|v_n\|^{2r}}{\|u_n\|^{2r}}+\frac{\alpha}{\beta\|u_n\|^{1-r}}+\frac{2c_2}{\beta\|u_n\|^r} \\
&+\frac{c_3\alpha}{\beta^2\|u_n\|^{2r}}+\frac{2c_2\|v_n\|^r}{\beta\|u_n\|^{2r}}],
\end{aligned} \tag{20}$$

where $c_3 := \frac{2B\sqrt{T_1T_2}c_2}{\alpha\sqrt{\lambda_1}}+\frac{2\beta c_2}{\alpha}+\frac{c_2^2}{\alpha}$. Since ε is arbitrary and $0<r<1$, Equation (20) implies that $J(u_n)\to-\infty$, as $n\to\infty$, which is contradictory. Therefore, $\{u_n\}$ is bounded, and this completes the proof. $\square$

To show J is coercive, we present the following two lemmas.

Lemma 2. *Let* $(\mathbf{I_2})$ *be true. Then, for any* $u\in W^+$, $J(u)\to+\infty$ *as* $\|u\|\to\infty$.

Proof. For any $u\in W^+$, it holds that

$$\begin{aligned}
J(u)=&\frac{1}{2}(\|u\|^2-\lambda_k\|u\|_2^2)-\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}G((i,j),u(i,j)) \\
\geq&\frac{\alpha}{2}\|u\|^2-\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}[G((i,j),u(i,j))-G((i,j),0)]-\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}G((i,j),0) \\
=&\frac{\alpha}{2}\|u\|^2-\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}u(i,j)\int_0^1|g((i,j),tu(i,j))|dt-c_4 \\
\geq&\frac{\alpha}{2}\|u\|^2-\beta\|u\|^{1+r}-\frac{B\sqrt{T_1T_2}}{\sqrt{\lambda_{k+1}}}\|u\|-c_4.
\end{aligned} \tag{21}$$

Note, $0<r<1$ and $\alpha,\beta>0$; then, Equation (21) implies that

$$J(u)\to+\infty,\qquad \text{as}\quad \|u\|\to\infty.$$

Thus, this proof is finished. □

Lemma 3. *If J satisfies $(\mathbf{I_2})$ and $(\mathbf{I_3})$, then for each $u \in W^- \oplus W^0$, $J(u) \to -\infty$ as $\|u\| \to \infty$.*

Proof. For any $u = w^- + v \in W^- \oplus W^0$, we have

$$\begin{aligned}
J(u) =& \frac{1}{2}(\|u\|^2 - \lambda_k \|u\|_2^2) - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} G((i,j), u(i,j)) \\
\leq& -\frac{\alpha}{2}\|w^-\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2}[G((i,j),u(i,j)) - G((i,j),v(i,j))] - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} G((i,j),v(i,j)) \\
=& -\frac{\alpha}{2}\|w^-\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} w^-(i,j)\int_0^1 |g((i,j),v(i,j)+tw^-(i,j))|dt \\
& - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} G((i,j),v(i,j)).
\end{aligned}$$

On one hand,

$$\begin{aligned}
& \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} w^-(i,j)\int_0^1 |g((i,j),v(i,j)+tw^-(i,j))|dt \\
& \leq A\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}|v(i,j)+tw^-(i,j)|^r|w^-(i,j)| + B\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}|w^-(i,j)| \\
& \leq A\sum_{i=1}^{T_1}\sum_{j=1}^{T_2}|v(i,j)|^r|w^-(i,j)| + A\|w^-\|_{r+1}^{r+1} + \frac{B\sqrt{T_1T_2}}{\sqrt{\lambda_1}}\|w^-\| \\
& \leq \frac{B\sqrt{T_1T_2}}{\sqrt{\lambda_1}}\|w^-\| + \beta\|w^-\|^{r+1} + \beta\|v\|^r\|w^-\|.
\end{aligned} \tag{22}$$

On the other hand, due to $(\mathbf{I_3})$, there exists some $R > 0$ when given $\varepsilon > 0$ such that

$$\sum_{i=1}^{T_1}\sum_{j=1}^{T_2} G((i,j),v(i,j)) \geq \left(\frac{4\beta^2}{\alpha} - \varepsilon\right)\|v\|^{2r}, \qquad v \in W^0, \quad \|v\| \geq R. \tag{23}$$

Combining Equations (22) with (23), it yields that

$$\begin{aligned}
J(u) \leq& -\frac{\alpha}{2}\|w^-\|^2 - \left(\frac{4\beta^2}{\alpha} - \varepsilon\right)\|v\|^{2r} + \frac{B\sqrt{T_1T_2}}{\sqrt{\lambda_1}}\|w^-\| + \beta\|w^-\|^{1+r} + \beta\|v\|^r\|w^-\| \\
=& -\frac{\alpha}{4}\left(\|w^-\| - \frac{2\beta}{\alpha}\|v\|^r\right)^2 - \frac{\alpha}{4}\|w\|^2 + \beta\|w^-\|^{1+r} + \frac{B\sqrt{T_1T_2}}{\sqrt{\lambda_1}}\|w^-\| \\
& - \left(\frac{3\beta^2}{\alpha} - \varepsilon\right)\|v\|^{2r} \to -\infty, \qquad \text{as} \quad \|u\| \to \infty.
\end{aligned}$$

This completes the proof. □

In the same manner, as with Lemmas 2 and 3, we present the following lemmas.

Lemma 4. *Let $(\mathbf{I_2})$ and $(\mathbf{I_4})$ be valid. Then, for each $u \in W^0 \oplus W^+$, $J(u) \to +\infty$ as $\|u\| \to \infty$.*

Lemma 5. *If J satisfies $(\mathbf{I_2})$, then for any $u \in W^-$, $J(u) \to -\infty$ as $\|u\| \to \infty$.*

Before displaying detailed proofs of our main results, we must prove that J has a local linking at 0.

Lemma 6. *Let Equation (8) and* $(\mathbf{I_0^+})$ *(or* $(\mathbf{I_0^-})$*) hold. Then, J has a local linking at 0 with respect to*

$$S = W_0^- \oplus (W_0^-)^{\perp},$$

where $W_0^- = span\{\phi_1, \cdots, \phi_m\}$ *(or* $W_0^- = span\{\phi_1, \cdots, \phi_{m-1}\}$*).*

Proof. Suppose that $(\mathbf{I_0^+})$ is satisfied. Thus, there exists $\delta > 0$ such that $|u(i,j)| \le \delta$, $\|u\| \le \delta\sqrt{T_1 T_2 \lambda_{T_1 T_2}}$ and

$$F((i,j),u) \ge \frac{1}{2}\lambda_m u^2.$$

For $u \in W_0^-$ with $0 < \|u\| \le \delta\sqrt{T_1 T_2 \lambda_{T_1 T_2}}$, we have

$$J(u) = \frac{1}{2}\|u\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} F((i,j),u(i,j)) \le \frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda_m\|u\|_2^2 = 0. \tag{24}$$

Moreover, Equation (8) means that

$$\lim_{u\to 0}\frac{2F((i,j),u)}{u^2} = \lim_{u\to 0}\frac{f((i,j),u)}{u} = \lambda_m.$$

Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\left|\frac{2F((i,j),u)}{u^2} - \lambda_m\right| < \varepsilon$ for $0 < |u(i,j)| < \delta$. Namely, $\lambda_m - \varepsilon < \frac{2F((i,j),u)}{u^2} < \lambda_m + \varepsilon$. Thus,

$$\frac{1}{2}(\lambda_m - \varepsilon)u^2 < F((i,j),u) < \frac{1}{2}(\lambda_m + \varepsilon)u^2.$$

For $u \in (W_0^-)^{\perp}$ with $0 < \|u\| < \delta\sqrt{T_1 T_2 \lambda_{T_1 T_2}}$, we have

$$J(u) \ge \frac{1}{2}\|u\|^2 - \frac{1}{2}(\lambda_m + \varepsilon)\|u\|_2^2 \ge \frac{1}{2}\left(1 - \frac{\lambda_m + \varepsilon}{\lambda_{m+1}}\right)\|u\|^2. \tag{25}$$

Set $\varepsilon < \lambda_{m+1} - \lambda_m$, then $J(u) > 0$. Obviously, $J(0) = 0$. Therefore, Equations (24) and (25) guarantee that J has a local linking at 0. □

Now, it is time for us to provide the detailed proofs of Theorems 1–3 via the Morse theory.

Proof of Theorem 1. For brevity, here, we only prove case (1) at length, as proofs of the other cases are similar and, thus, omitted. Clearly, $J(0) = 0$ and Lemma 2 guarantee that J is bounded from below by $(W_\infty^-)^{\perp} := W^+$. Further, Lemma 3 shows that $J(u) \to -\infty$ as $\|u\| \to \infty$ for any $u \in W_\infty^- := W^- \oplus W^0$. Therefore, Proposition 3 ensures that

$$C_{\mu_\infty+\nu_\infty}(J,\infty) = C_k(J,\infty) \ncong 0, \tag{26}$$

where $\mu_\infty = \dim W^-$, $\nu_\infty = \dim W^0$. Obviously, 0 is an isolated critical point. If Equation (8) is valid, then 0 is degenerate with $\mu_0 = \dim W_0^-$, $\nu_0 = \dim span\{\phi_m\}$. Thus, Lemma 6 guarantees J has a local linking at $u = 0$. Moreover, Proposition 4 indicates that

$$C_q(J,0) \cong \delta_{q,m}\mathbb{Z}, \qquad q \in \mathbb{Z}, \tag{27}$$

where $m = \mu_0 + \nu_0$. Consider $m \ne k$; then,

$$C_q(J,\infty) \ncong C_q(J,0),$$

if $q = \mu_\infty + \nu_\infty$. Lemma 1 proves that J satisfies (PS), which leads to J satisfying (D). Then, Proposition 2 implies that there exists some $u^* \ne 0$ such that

$$C_{\mu_\infty+\nu_\infty}(J,u^*) \ncong 0. \tag{28}$$

Since there exists some $u_1 > 0$ such that $f((i,j), u_1) = 0$, we intend to find the local minimizer of J. For each $(i,j) \in \Omega$, define

$$\widetilde{J}(u) = \frac{1}{2}\|u\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} \widetilde{F}((i,j), u(i,j)), \qquad u \in S,$$

where $\widetilde{F}((i,j), u) = \int_0^u \widetilde{f}((i,j), \tau)d\tau$, and

$$\widetilde{f}((i,j), u) = \begin{cases} f((i,j), u), & u \in [0, u_1], \\ 0, & u < 0 \quad \text{or} \quad u > u_1, \end{cases}$$

Therefore, $\widetilde{J}$ is continuous and coercive. Moreover, $\widetilde{J}$ is bounded from below and satisfies (PS). Thus, there exists a minimizer $\widetilde{u}_0^+$ of $\widetilde{J}$. By maximum principle, we can obtain $\widetilde{u}_0^+ = 0$ or $0 < \widetilde{u}_0^+(i,j) < u_1$ for any $(i,j) \in \Omega$. Furthermore, Equation (8) means that 0 is not a minimizer. In the sequence, $\widetilde{u}_0^+ \neq 0$ is a local minimizer of $\widetilde{J}$. Further, $\widetilde{u}_0^+ > 0$ is a local minimizer of J, which means that $\widetilde{u}_0^+$ is nondegenerate. Therefore, $\widetilde{u}_0^+$ is an isolated critical point of J, which leads to $J''(u_0^+)$ as a Fredholm operator with a finite Morse index and zero nullity. Due to Proposition 1, we can find that

$$C_q(J, \widetilde{u}_0^+) \cong \delta_{q,0}\mathbb{Z}, \qquad q \in \mathbb{Z}. \tag{29}$$

For the case that there exists some $u_2 < 0$ such that $f((i,j), u_2) = 0$, repeating the above steps shows that $\widetilde{u}_0^- < 0$ is a local minimizer of J and

$$C_q(J, \widetilde{u}_0^-) \cong \delta_{q,0}\mathbb{Z}, \qquad q \in \mathbb{Z}. \tag{30}$$

Now, we denote $\widehat{F}((i,j), v) = \int_0^v \widehat{f}((i,j), \tau)d\tau$, where

$$\widehat{f}((i,j), v) = f((i,j), v + \widetilde{u}_0^+) - f((i,j), \widetilde{u}_0^+), \qquad (i,j) \in \Omega, \qquad v \in S.$$

The corresponding functional is then given by

$$\widehat{J}(v) = \frac{1}{2}\|v\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} \widehat{F}((i,j), v(i,j)), \qquad (i,j) \in \Omega, \qquad v \in S.$$

If v is a nontrivial critical point of $\widehat{J}$, then $v + \widetilde{u}_0^+$ is a nontrivial critical point of J satisfying

$$C_q(\widehat{J}, v) = C_q(J, v + \widetilde{u}_0^+), \qquad q \in \mathbb{Z}.$$

Moreover, for all $(i,j) \in \Omega$, define

$$\widehat{f}^+((i,j), v) = \begin{cases} \widehat{f}((i,j), v), & v \geq 0, \\ 0, & v < 0, \end{cases}$$

and construct the corresponding functional as

$$\widehat{J}^+(v) = \frac{1}{2}\|v\|^2 - \sum_{i=1}^{T_1}\sum_{j=1}^{T_2} \widehat{F}^+((i,j), v(i,j)), \qquad v \in S,$$

where $\widehat{F}^+((i,j), v) = \int_0^v \widehat{f}^+((i,j), \tau)d\tau$. It is easy to deduce that $\widehat{J}^+$ satisfies (PS). Since $\widetilde{u}_0^+$ is a local minimizer of J, this leads to $v = 0$ being a local minimizer of $\widehat{J}^+$. What is more, for $e \in \mathrm{span}\{\phi_1\}$, $\widehat{J}^+(te) \to -\infty$ as $t \to +\infty$. Then, Proposition 5 implies that $\widehat{J}^+$ possesses a critical point $v^+ > 0$ such that

$$C_q(\widehat{J}, v^+) \cong \delta_{q,1}\mathbb{Z}, \qquad q \in \mathbb{Z}.$$

As a result, $u^+ = v^+ + \widetilde{u}_0^+ > 0$ is a mountain pass point of J and

$$C_q(J, u^+) \cong \delta_{q,1}\mathbb{Z}, \qquad q \in \mathbb{Z}. \tag{31}$$

Similarly, $u^- = v^- + \widetilde{u}_0^- < 0$ is also a mountain pass point of J and

$$C_q(J, u^-) \cong \delta_{q,1}\mathbb{Z}, \qquad q \in \mathbb{Z}. \tag{32}$$

Consequently, $u^\pm$, $\widetilde{u}_0^\pm$ and u^* are nontrivial critical points of J, which implies that Problems (1) and (2) possesses at least five nontrivial solutions. The proof of Theorem 1 is achieved. □

Proof of Theorem 2. Lemma 2 ensures that J is coercive on $(W_\infty^-)^\perp := W^+$, that is, J is bounded from below by W^+. Moreover, Lemma 3 guarantees

$$J(u) \to -\infty, \qquad \text{as} \quad \|u\| \to \infty \quad \text{and} \quad u \in W_\infty^- := W^0 \oplus W^-.$$

Therefore, taking account of Proposition 3, we obtain $C_{\nu_\infty+\nu_\infty}(J, \infty) \ncong 0$. Since Lemma 1 ensures that J satisfies (PS), this leads to J satisfying (D). Then, Proposition 2 indicates that there exists some critical point u^* such that

$$C_{\nu_\infty+\nu_\infty}(J, u^*) \ncong 0. \tag{33}$$

Recall Equation (8), where $u = 0$ is a degenerate critical point of J with finite Morse index μ_0 and zero nullity ν_0. Next, we must verify $u^* \neq 0$.

Case $(\mathbf{1})$ Let $(\mathbf{I_0^+})$ be true. Since 0 is an isolated critical point of J, according to Lemma 6, J has a local linking at 0. Then, according to Proposition 4, this means that

$$C_q(J, 0) \cong \delta_{q,m}\mathbb{Z}, \qquad q \in \mathbb{Z}, \tag{34}$$

where $m = \mu_0 + \nu_0$. Notice that $m \neq k$ implies $\mu_\infty + \nu_\infty \neq \mu_0 + \nu_0$; therefore, we have

$$C_q(J, 0) \ncong C_q(J, u^*).$$

Thus, $u^* \neq 0$ is a nontrivial critical point of J.

Case $(\mathbf{2})$ Let 0 be an isolated critical point of J. If $(\mathbf{I_0^-})$ is valid, then Lemma 6 indicates that J has a local linking at 0, and according to Proposition 4, this means that

$$C_q(J, 0) \cong \delta_{q,m}\mathbb{Z}, \qquad q \in \mathbb{Z}. \tag{35}$$

where $m = \mu_0$. Since $m \neq k + 1$, it follows that

$$C_q(J, 0) \ncong C_q(J, u^*), \qquad q \in \mathbb{Z}.$$

Namely, $u^* \neq 0$ is a nontrivial critical point of J, and Problems (1) and (2) possesses at least one nontrivial solution. The proof of Theorem 2 is completed. □

Proof of Theorem 3. Based on Lemma 1, J satisfies (D). Combining Lemma 1 with Lemma 4, we obtain that J is bounded from below by $(W_\infty^-)^\perp := W^0 \oplus W^+$. Moreover, Lemma 5 gives

$$J(u) \to -\infty, \qquad \text{as} \quad \|u\| \to \infty, \quad \forall u \in W_\infty^- := W^-.$$

Since $k = 1$ and $W^- = \emptyset$, Proposition 3 ensures that $C_0(J, \infty) \ncong 0$. Hence, there exists some critical point u_0 of J such that $C_0(J, u_0) \ncong 0$. Therefore,

$$C_q(J, \infty) \cong \delta_{q,0}\mathbb{Z}, \quad C_q(J, u_0) \cong \delta_{q,0}\mathbb{Z}, \qquad q \in \mathbb{Z}.$$

Consequently, u_0 is a local minimizer of J. Moreover, based on Equation (8), we conclude that $u = 0$ is a degenerate critical point of J satisfying Equations (34) and (35) if $(\mathbf{I}_\mathbf{0}^+)[(\mathbf{I}_\mathbf{0}^-)]$ is valid. Note that $m \geq 1[m > 1]$, $u_0 \neq 0$. If the critical set $\kappa = \{u_0, 0\}$, then the Morse inequality can be expressed as

$$(-1)^0 + (-1)^m = (-1)^0,$$

where $m = \mu_0 + \nu_0[m = \nu_0]$. Of course, this is impossible. As a result, J must have at least another critical point u_1 differing from u_0 and 0. Thus, u_0 and u_1 are two nontrivial critical points of J, and we complete the proof of Theorem 3. $\square$

4. Conclusions

Due to their applications, discrete elliptic Dirichlet problems have been discussed extensively. In this paper, we considered multiple existence results of nontrivial solutions for the discrete elliptic Dirichlet problem by combining the variational technique with the Morse theory. First, we constructed a suitable variational function space and established the corresponding functional. Then, we achieved a series of results based on the existence of one, two or five nontrivial solutions under reasonable assumptions via the Morse theory and local linking. In our future work, we will search for characterized solutions, such as sign-changing solutions, signed solutions, and ground state solutions for partial difference equations subject to various boundary conditions by variational methods and critical point theory.

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