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On the Laplacian and Signless Laplacian Characteristic Polynomials of a Digraph

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Abstract: Let D be a digraph with n vertices and a arcs. The Laplacian and the signless Laplacian matrices of D are, respectively, defined as $L(D) = Deg^+(D) - A(D)$ and $Q(D) = Deg^+(D) + A(D)$, where $A(D)$ represents the adjacency matrix and $Deg^+(D)$ represents the diagonal matrix whose diagonal elements are the out-degrees of the vertices in D. We derive a combinatorial representation regarding the first few coefficients of the (signless) Laplacian characteristic polynomial of D. We provide concrete directed motifs to highlight some applications and implications of our results. The paper is concluded with digraph examples demonstrating detailed calculations.

Keywords: digraphs; adjacency matrix (spectrum); (signless) Laplacian matrix (spectrum); (signless) Laplacian coefficients

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1. Introduction

We consider a digraph $D = (V, \Gamma)$ of order n, where the vertex set is given by $V = \{v_1, v_2, \ldots, v_n\}$ and $|V| = n$. The set of arcs is denoted by Γ, which contains ordered pairs of distinct vertices. Throughout this paper, digraphs mean directed graphs without loops or multiple arcs. Two vertices, u and v, of D are adjacent when they are linked via an arc $(u, v) \in \Gamma$, or $(v, u) \in \Gamma$, and they are called doubly adjacent when $\{(u, v), (v, u)\} \in \Gamma$. For any vertex v_i, the set

$$\overleftrightarrow{N_i} = \{v_j \in V : \{(v_i, v_j), (v_j, v_i)\} \in \Gamma\}$$

contains all vertices that are doubly adjacent of v_i.

A sequence of vertices $W : u = u_0, u_1, \ldots, u_l = v$ is called a walk W, which has length l from vertex u to vertex v. Here, (u_{k-1}, u_k) is an arc in D for $1 \le k \le l$. W is a closed walk if $u = v$. If each pair of different vertices u, v in D admits a walk from u to v and a walk from v to u, then D is strongly connected. Let $c_2^{(i)}$ be the number of closed walks having length 2 originated from v_i. In the digraph D, the sequence $(c_2^{(1)}, c_2^{(2)}, \ldots, c_2^{(n)})$ is a closed walk sequence of length 2.

A digraph D is weakly connected or connected when the undirected version G of D is connected.

A digraph D is called symmetric if (u, v) and $(v, u) \in \Gamma$, for all $u, v \in V$. $G \to \overleftrightarrow{G}$ forms a one-to-one mapping between simple graphs and symmetric digraphs, where $\overleftrightarrow{G}$ and G share the same vertices and each edge uv of G is mapped to symmetric arcs (u, v) and (v, u). Clearly, a graph corresponds to a symmetric digraph under this mapping.

We define the adjacency matrix $A(D)$ as a $n \times n$ binary matrix with rows and columns indexed by the vertices. The element a_{ij} takes $a_{ij} = 1$, if $(v_i, v_j) \in \Gamma$ and $a_{ij} = 0$ otherwise. We refer the readers to [1,2] for some recent works on the spectral properties of $A(D)$. Denote by $Deg^+(D) = diag(d_1^+, d_2^+, \ldots, d_n^+)$ the diagonal matrix of out-degrees. The matrices $L(D) = Deg^+(D) - A(D)$ and $Q(D) = Deg^+(D) + A(D)$ are known as the

Laplacian and the signless Laplacian matrix of the digraph D. We call the eigenvalues of the matrices $L(D)$ and $Q(D)$ the Laplacian and the signless Laplacian eigenvalues of D, respectively. Let $P_L(D,x)$ and $P_Q(D,x)$ be the characteristic polynomials of the matrices $L(D)$ and $Q(D)$, respectively. They are the Laplacian and the signless Laplacian characteristic polynomial of D. Some recent results on the spectral properties of $L(D)$, $Q(D)$ and related results can be found in, e.g., [3–12].

Two graphs that are non-isomorphic are co-spectral if they have the same spectrum relevant to a given graph matrix. Apparently, two isomorphic graphs have the same spectrum relevant to a given graph matrix. If two graphs share the same spectrum relevant to a given matrix, is it true that they are isomorphic? This is one of the most investigated and difficult problems in spectral theory of graphs and digraphs. It yields the following problem.

Problem 1. *With respect to a given graph (digraph) matrix, which graphs (digraphs) are determined by their spectra?*

A graph G is determined by its spectrum relevant to a given graph matrix if there is no other graph sharing the same spectrum as G relevant to the graph matrix. When two graphs admit the same spectrum relevant to a given graph matrix, they share the same characteristic polynomial as well as the coefficients of the characteristic polynomial. This observation prompted many researchers to examine carefully the coefficients of characteristic polynomial relevant to a given graph matrix. Apparently, if two graphs differ in at least one coefficient of their characteristic polynomial relevant to a given graph matrix, they cannot be co-spectral regarding the graph matrix. Many results have been reported in the literature on this problem. Sachs [13] and Mowshowitz [14] established the coefficients of the adjacency characteristic polynomial of an arbitrary digraph. It is revealed that the coefficients of the adjacency characteristic polynomial of a tree count matchings. The formulas for the first four coefficients of the Laplacian characteristic polynomial of a graph were derived in [15]. Cvetković et al. [16] presented the formulas for the first three coefficients of the signless Laplacian characteristic polynomial of a graph. Guo et. al [17] provided combinatorial expression for the first few coefficient of the (normalized) Laplacian and the signless Laplacian characteristic polynomials of a graph. Based upon the fifth coefficient of the adjacency characteristic polynomial of trees, Lepović et al. [18] showed that no starlike trees are co-spectral. The coefficients of the Laplacian characteristic polynomial were employed [19] to reveal the Laplacian spectra of three-rose graphs. Some recent results can be found in, e.g., [20–22].

The issue of spectral determination in digraphs related to adjacency matrices has also been considered. Some families of digraphs are characterized through adjacency spectra [23]. Recently, some researchers [11] studied the spectral determination problem for oriented graphs relevant to a generalized skew matrix.

With motivation from the above works, we aim to investigate the coefficients of the characteristic polynomial of the (signless) Laplacian matrices of a digraph. We obtain algebraic expressions for the first few coefficients of the (signless) Laplacian characteristic polynomials of a digraph.

Given a graph $G=(V,E)$ with the vertex set $V(G)=\{x_1,x_2,\ldots,x_n\}$, let G_i be the graph obtained from G by scrapping the ith vertex x_i and its incident edges. The graph G is reconstructible if it can be determined (up to isomorphism) by all vertex-deleted graphs G_i. Ulam's reconstruction conjecture states that every graph with $n>3$ vertices is reconstructible. Many variations of this problem have been considered, and one of them is the spectral reconstruction problem. It claims that "a graph G is spectrally reconstructible if it can be determined up to isomorphism by the adjacency characteristic polynomial of G and the adjacency characteristic polynomials of its vertex deleted subgraphs G_i". Our study of coefficients of the characteristic polynomial of the (signless) Laplacian matrices of a digraph is also motivated by this line of research.

2. Laplacian Coefficients of D

In this section, we establish an algebraic representation for the first five Laplacian coefficients of a digraph.

To determine the coefficients of a Laplacian characteristic polynomial of a digraph D, we need the following Lemma, which can be found in [24].

Lemma 1. *Let $B = (b_{ij})$ be a $n \times n$ matrix having the characteristic polynomial*

$$\Phi(B, y) = det(yI - B) = y^n + \sum_{i=1}^{n} c_i(G) y^{n-i}.$$

Let $s_k = tr(B^k)$ be the trace of B^k. Then, for the coefficients of $\Phi(B, y)$, we have the following:

$$c_1 = -s_1 \text{ and } kc_k = -s_k - c_1 s_{k-1} - c_2 s_{k-2} - \ldots - a_{k-1} c_1, \quad k = 2, 3, \ldots, n. \tag{1}$$

The following interesting result about the trace of the product of matrices can be found in [24].

Lemma 2. *For $n \times n$ matrices A and B, we have $tr(AB) = tr(BA)$.*

The following result is the main result of this section and shows how the first five coefficients of the Laplacian characteristic polynomial of a digraph D can be expressed in terms of the structure of the digraph.

Theorem 1. *Let D be a connected digraph of order $n \geq 3$ with a arcs having vertex out-degrees $d_1^+ \geq d_2^+ \geq \cdots \geq d_n^+$. Let $P_L(D, x) = x^n + \sum_{i=1}^{n} a_i x^{n-i}$ be the Laplacian characteristic polynomial of D. Then,*

(1) $a_1 = -a$,

(2) $a_2 = \frac{a^2}{2} - \frac{1}{2} \sum_{i=1}^{n} \left((d_i^+)^2 + c_2^{(i)} \right)$,

(3) $a_3 = \frac{1}{3} \sum_{i=1}^{n} n(\mathbb{C}_3^i) - \sum_{i=1}^{n} c_2^{(i)} d_i^+ - \frac{1}{3} \sum_{i=1}^{n} (d_i^+)^3 + \frac{a}{2} \sum_{i=1}^{n} \left((d_i^+)^2 + c_2^{(i)} \right) - \frac{a^3}{6}$,

(4) *and*

$$\begin{aligned} a_4 = {} & \sum_{i=1}^{n} d_i^+ n(\mathbb{C}_3^i) - \frac{1}{4} \sum_{i=1}^{n} (d_i^+)^4 - \sum_{i=1}^{n} (d_i^+)^2 c_2^{(i)} - \frac{1}{2} \sum_{i=1}^{n} d_i^+ M_i - \frac{(4a+3)}{12} \sum_{i=1}^{n} n(\mathbb{C}_3^i) \\ & - \frac{1}{4} \sum_{i=1}^{n} \Big(n(P_3^1(v_i v_j v_k v_j v_i)) + n(P_3^2(v_i v_j v_i v_k v_i)) \Big) \\ & - \frac{1}{4} \sum_{i=1}^{n} c_2^{(i)} + \frac{a}{3} \sum_{i=1}^{n} (d_i^+)^3 + a \sum_{i=1}^{n} d_i^+ c_2^{(i)} \\ & - \frac{a^2}{4} \sum_{i=1}^{n} \left((d_i^+)^2 + c_2^{(i)} \right) + \frac{1}{8} \Big(\sum_{i=1}^{n} ((d_i^+)^2 + c_2^{(i)}) \Big)^2 + \frac{a^4}{24}. \end{aligned}$$

Here, $c_2^{(i)}$ is the number of closed walks of length 2 at the vertex v_i, $n(\mathbb{C}_3^i)$ is the number of directed cycles of length 3 at the vertex v_i, $n(\mathbb{C}_4^i)$ is the number of directed cycles of length 4 at the vertex v_i, $n(P_3^1(v_i v_j v_k v_j v_i))$ is the number of paths on 3 vertices v_i, v_j, v_k at the vertex v_i with arcs $(v_i, v_j), (v_j, v_k), (v_k, v_j), (v_j, v_i)$, $n(P_3^2(v_i v_j v_i v_k v_i))$ is the number of paths on 3 vertices v_i, v_j, v_k at the vertex v_i with arcs $(v_i, v_j), (v_j, v_i), (v_i, v_k), (v_k, v_i)$ and $M_i = \sum_{(v_i, v_j), (v_j, v_i) \in E(D)} d_i^+$ is the sum of the out-degrees of the vertices which are both out-neighbor and the in-neighbor of the vertex v_i.

Proof. Let $s_k = tr(L(D)^k)$ be the trace of the matrix $L(D)^k$, where $L(D) = Deg^+(D) - A(D)$ is the Laplacian matrix of D and $k \in \mathbb{N}$. We have, $s_1 = tr(L(D)) = \sum_{i=1}^{n} d_i^+ = a$, the number of arcs. This shows by using Lemma 1 that $a_1 = -s_1 = -a$. By a simple calculation, it can be seen that

$$L(D)^2 = \Big(Deg^+(D) - A(D)\Big)^2 = (Deg^+(D))^2 - Deg^+(D)A(D) - A(D)Deg^+(D) + A(D)^2. \quad (2)$$

Clearly, $tr(Deg^+(D)A(D)) = tr(A(D)Deg^+(D))$ by Lemma 2. It is easy to establish that each of the diagonal entries of $Deg^+(D)A(D)$ are zero, therefore we obtain

$$tr(Deg^+(D)A(D)) = tr(A(D)Deg^+(D)) = 0.$$

Additionally, $tr((Deg^+(D))^2) = \sum_{t=1}^{n}(d_t^+)^2$ and $tr(A(D)^2) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} = \sum_{t=1}^{n} c_2^{(t)}$, where $c_2^{(t)}$ is the number of symmetric pair of arcs or the number of closed walks of length 2 at the vertex v_t. Therefore, from (2), we obtain

$$s_2 = tr(L(D)^2) = tr((Deg^+(D))^2) - 2tr(Deg^+(D)A(D)) + tr(A(D)^2) = \sum_{t=1}^{n}\Big((d_t^+)^2 + c_2^{(t)}\Big).$$

Using this together with Lemma 1, we arrive at

$$a_2 = \frac{1}{2}(-s_2 - a_1 s_1) = \frac{a^2}{2} - \frac{1}{2}\sum_{t=1}^{n}\Big((d_t^+)^2 + c_2^{(t)}\Big).$$

By binomial theorem with index 3, it can be seen that

$$\begin{aligned} L(D)^3 &= \Big(Deg^+(D) - A(D)\Big)^3 \\ &= Deg^+(D)^3 - Deg^+(D)^2A(D) - Deg^+(D)A(D)Deg^+(D) - A(D)Deg^+(D)^2 \\ &+ Deg^+(D)A(D)^2 + A(D)Deg^+(D)A(D) + A(D)^2Deg^+(D) - A(D)^3. \end{aligned} \quad (3)$$

Clearly, by Lemma 2, we have

$$tr((Deg^+(D))^2A(D) = tr(Deg^+(D)A(D)Deg^+(D)) = tr(A(D)Deg^+(D)^2)$$

and each of the diagonal entries of $Deg^+(D)^2A(D)$ are zero, it follows that

$$tr((Deg^+(D))^2A(D)) = tr(Deg^+(D)A(D)Deg^+(D)) = tr(A(D)(Deg^+(D))^2) = 0.$$

Additionally, $tr(Deg^+(D)A(D)^2) = tr(A(D)Deg^+(D)A(D)) = tr(A(D)^2Deg^+(D))$, by Lemma 2 and the t-th diagonal entry of $Deg^+(D)A(D)^2$ is $c_2^{(t)}d_t^+$, it follows that

$$tr(Deg^+(D)A(D)^2) = tr(A(D)Deg^+(D)A(D)) = tr(A(D)^2Deg^+(D)) = \sum_{t=1}^{n} c_2^{(t)}d_t^+.$$

Further, $tr((Deg^+(D))^3) = \sum_{t=1}^{n}(d_t^+)^3$ and

$$tr(A(D)^3) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij}a_{jk}a_{ki} = (\text{number of directed cycles of length 3 in D}) = \sum_{t=1}^{n} n(\mathbb{C}_3^t).$$

Therefore, from (3), we obtain

$$s_3 = tr(L(D)^3) = tr((Deg^+(D))^3) - 3tr((Deg^+(D))^2A(D) + 3tr(Deg^+(D)A(D)^2) - tr(A(D)^3)$$
$$= \sum_{t=1}^{n}(d_t^+)^3 + 3\sum_{t=1}^{n} c_2^{(t)} d_t^+ - \sum_{t=1}^{n} n(\mathbb{C}_3^t).$$

Using these observations together with Lemma 1, we arrive at

$$\begin{aligned} a_3 =& \frac{1}{3}(-s_3 - a_1 s_2 - a_2 s_1) \\ &= \frac{1}{3}\sum_{i=1}^{n} n(\mathbb{C}_3^i) - \sum_{i=1}^{n} c_2^{(i)} d_i^+ - \frac{1}{3}\sum_{i=1}^{n}(d_i^+)^3 + \frac{a}{2}\Big(\sum_{i=1}^{n}\Big((d_i^+)^2 + c_2^{(i)}\Big)\Big) - \frac{a^3}{6}. \end{aligned}$$

Lastly, by binomial theorem with index 4, we have

$$\begin{aligned} L(D)^4 =& \Big(Deg^+(D) - A(D)\Big)^4 \\ =& (Deg^+(D))^4 - (Deg^+(D))^3A(D) - (Deg^+(D))^2A(D)Deg^+(D) \\ & - Deg^+(D)A(D)(Deg^+(D))^2 \\ & + (Deg^+(D))^2A(D)^2 + Deg^+(D)A(D)^2Deg^+(D) + Deg^+(D)A(D)Deg^+(D)A(D) \\ & - Deg^+(D)A(D)^3 - A(D)(Deg^+(D))^3 + A(D)(Deg^+(D))^2A(D) \\ & + A(D)Deg^+(D)A(D)Deg^+(D) \\ & - A(D)Deg^+(D)A(D)^2 + A(D)^2(Deg^+(D))^2 \\ & - A(D)^2Deg^+(D)A(D)^2 - A(D)^3Deg^+(D) + A(D)^4. \end{aligned} \tag{4}$$

Using Lemma 2, we have

$$\begin{aligned} tr((Deg^+(D))^3A(D)) &= tr((Deg^+(D))^2A(D)Deg^+(D)) = tr(Deg^+(D)A(D)(Deg^+(D))^2) \\ &= tr(A(D)(Deg^+(D))^3) \end{aligned}$$

and the i-th diagonal entry of $(Deg^+(D))^3A(D)$ is zero, we get

$$\begin{aligned} tr((Deg^+(D))^3A(D)) &= tr((Deg^+(D))^2A(D)Deg^+(D)) = tr(Deg^+(D)A(D)(Deg^+(D))^2) \\ &= tr(A(D)(Deg^+(D))^3) = 0. \end{aligned}$$

Additionally, by Lemma 2

$$\begin{aligned} tr((Deg^+(D))^2A(D)^2) &= tr(A(D)(Deg^+(D))^2A(D)) = tr(Deg^+(D)A(D)^2Deg^+(D)) \\ &= tr(A(D)^2(Deg^+(D))^2) \end{aligned}$$

and the i-th diagonal entry of $(Deg^+(D))^2A(D)^2$ is $(d_i^+)^2c_2^{(i)}$, we obtain

$$\begin{aligned} tr((Deg^+(D))^2A(D)^2) &= tr(A(D)(Deg^+(D))^2A(D)) = tr(Deg^+(D)A(D)^2Deg^+(D)) \\ &= tr(A(D)^2(Deg^+(D))^2) = \sum_{i}^{n}(d_i^+)^2c_2^{(i)}. \end{aligned}$$

Again by Lemma 2, we have

$$tr(Deg^+(D)A(D)Deg^+(D)A(D)) = tr(A(D)Deg^+(D)A(D)Deg^+(D))$$

and the i-th diagonal entry of $Deg^+(D)A(D)Deg^+(D)A(D)$ is

$$d_i^+ \sum_{k=1}^{n} d_k^+ a_{ik} a_{ki} = d_i^+ \sum_{(v_i,v_k),(v_k,v_i)\in E(D)} d_k^+ = d_i^+ M_i,$$

where M_i is the sum of the out-degrees of the vertices which are both out and in-neighbors of the vertex v_i. It follows that

$$tr(Deg^+(D)A(D)Deg^+(D)A(D)) = tr(A(D)Deg^+(D)A(D)Deg^+(D)) = \sum_{i=1}^{n} d_i^+ M_i.$$

Using Lemma 2, we have

$$\begin{aligned} tr(Deg^+(D)A(D)^3) &= tr(A(D)Deg^+(D)A(D)^2) \\ &= tr(A(D)^2 Deg^+(D)A(D)) = tr(A(D)^3 Deg^+(D)) \end{aligned}$$

and the i-th diagonal entry of $Deg^+(D)A(D)^3$ is

$$d_i^+ \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ij} a_{jk} a_{ki} = d_i^+ n(\mathbb{C}_3^i),$$

where $n(\mathbb{C}_3^i)$ is the number of directed 3-cycles at the vertex v_i. It follows that

$$\begin{aligned} tr(Deg^+(D)A(D)^3) &= tr(A(D)Deg^+(D)A(D)^2) = tr(A(D)^2 Deg^+(D)A(D)) \\ &= tr(A(D)^3 Deg^+(D)) = \sum_{i=1}^{n} d_i^+ n(\mathbb{C}_3^i). \end{aligned}$$

Further, $tr((Deg^+(D))^4) = \sum_{i=1}^{n} (d_i^+)^4$ and

$$\begin{aligned} tr(A(D)^4) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{ij} a_{jk} a_{kl} a_{li} \\ &= (\text{number of directed four cycles in D}) + \sum_{i=1}^{n} n(P_3^1(v_i v_j v_k v_j v_i)) \\ &+ \sum_{i=1}^{n} n(P_3^2(v_i v_j v_i v_k v_i)) + \sum_{i=1}^{n} c_2^{(i)} \\ &= \sum_{i=1}^{n} n(\mathbb{C}_4^i) + \sum_{i=1}^{n} \Big(n(P_3^1(v_i v_j v_k v_j v_i)) + n(P_3^2(v_i v_j v_i v_k v_i)) \Big) + \sum_{i=1}^{n} c_2^{(i)}, \end{aligned}$$

Now, from (4), we have

$$\begin{aligned} s_4 = tr(L(D)^4) &= tr((Deg^+(D))^4) - 4tr((Deg^+(D))^3 A(D)) + 4tr((Deg^+(D))^2 A(D)^2) \\ &+ 2tr(Deg^+(D)A(D)Deg^+(D)A(D)) - 4tr(Deg^+(D)A(D)^3) + tr(A(D)^4) \\ &= \sum_{i=1}^{n} (d_i^+)^4 - 4\sum_{i=1}^{n} (d_i^+)^2 c_2^{(i)} + 2\sum_{i=1}^{n} d_i^+ M_i - 4\sum_{i=1}^{n} d_i^+ n(\mathbb{C}_3^i) + \sum_{i=1}^{n} n(\mathbb{C}_3^i) \\ &+ \sum_{i=1}^{n} \Big(n(P_3^1(v_i v_j v_k v_j v_i)) + n(P_3^2(v_i v_j v_i v_k v_i)) \Big) + \sum_{i=1}^{n} c_2^{(i)}. \end{aligned}$$

Using these observations together with Lemma 1, we arrive at

$$
\begin{aligned}
a_4 &= \frac{1}{4}(-s_4 - a_1 s_3 - a_2 s_2 - a_3 s_1) \\
&= \sum_{i=1}^{n} d_i^+ n(\mathbb{C}_3^i) - \frac{1}{4}\sum_{i=1}^{n}(d_i^+)^4 - \sum_{i=1}^{n}(d_i^+)^2 c_2^{(i)} - \frac{1}{2}\sum_{i=1}^{n} d_i^+ M_i - \frac{(4a+3)}{12}\sum_{i=1}^{n} n(\mathbb{C}_3^i) \\
&- \frac{1}{4}\sum_{i=1}^{n}\Big(n(P_3^1(v_iv_jv_kv_jv_i)) + n(P_3^2(v_iv_jv_iv_kv_i)\Big) - \frac{1}{4}\sum_{i=1}^{n} c_2^{(i)} + \frac{a}{3}\sum_{i=1}^{n}(d_i^+)^3 + a\sum_{i=1}^{n} d_i^+ c_2^{(i)} \\
&- \frac{a^2}{4}\sum_{i=1}^{n}\Big((d_i^+)^2 + c_2^{(i)}\Big) + \frac{1}{8}\Big(\sum_{i=1}^{n}((d_i^+)^2 + c_2^{(i)})\Big)^2 + \frac{a^4}{24}.
\end{aligned}
$$

This completes the proof of the theorem. □

The k-th spectral moment of a digraph D is defined as the sum of the k-th powers of the adjacency eigenvalues of D. Likewise, the Laplacian spectral moments of D are defined as

$$
tr(L(D)^k) = \sum_{i=1}^{n} \mu_i(D)^k, \quad k = 1, 2, \ldots,
$$

where $\mu_1(D), \mu_2(D), \ldots, \mu_{n-1}(D), \mu_n(D)$ are the Laplacian eigenvalues of D.

The next result presents the formula for the first four Laplacian spectral moments of D.

Corollary 1. *Let D be a connected digraph of order $n \geq 3$ with a arcs having vertex out-degrees $d_1^+ \geq d_2^+ \geq \cdots \geq d_n^+$. Then,*

(1) $\sum\limits_{i=1}^{n} \mu_i(D) = a,$

(2) $\sum\limits_{i=1}^{n} \mu_i(D)^2 = \sum\limits_{i=1}^{n}\Big((d_i^+)^2 + c_2^{(i)}\Big),$

(3) $\sum\limits_{i=1}^{n} \mu_i(D)^3 = \sum\limits_{i=1}^{n}(d_i^+)^3 + 3\sum\limits_{i=1}^{n} c_2^{(i)} d_i^+ - \sum\limits_{i=1}^{n} n(\mathbb{C}_3^i)$, *and*

(4)

$$
\begin{aligned}
\sum_{i=1}^{n}\mu_i(D)^4 &= \sum_{i=1}^{n}(d_i^+)^4 - 4\sum_{i=1}^{n}(d_i^+)^2 c_2^{(i)} + 2\sum_{i=1}^{n} d_i^+ M_i - 4\sum_{i=1}^{n} d_i^+ n(\mathbb{C}_3^i) + \sum_{i=1}^{n} n(\mathbb{C}_3^i) \\
&+ \sum_{i=1}^{n}\Big(n(P_3^1(v_iv_jv_kv_jv_i)) + n(P_3^2(v_iv_jv_iv_kv_i))\Big) + \sum_{i=1}^{n} c_2^{(i)}.
\end{aligned}
$$

where $c_2^{(i)}, n(\mathbb{C}_3^i), M_i, n(P_3^1(v_iv_jv_kv_jv_i))$ and $n(P_3^2(v_iv_jv_iv_kv_i))$ are defined in Theorem 1.

3. The Signless Laplacian Coefficients of D

In this section, we present algebraic representation for some of the signless Laplacian coefficients of D.

The next Theorem is the main result of this section and gives the first five coefficients of the signless Laplacian characteristic polynomial of a digraph D in terms of structure of the digraph.

Theorem 2. *Let D be a connected digraph of order $n \geq 3$ with a arcs having vertex out-degrees $d_1^+ \geq d_2^+ \geq \cdots \geq d_n^+$. Let $P_Q(D, y) = y^n + \sum\limits_{i=1}^{n} b_i y^{n-i}$ be the signless Laplacian characteristic polynomial of D. Then,*

(1) $b_1 = -a,$

(2) $\quad b_2 = \frac{a^2}{2} - \frac{1}{2}\sum_{i=1}^{n}\left((d_i^+)^2 + c_2^{(i)}\right)$,

(3) $\quad b_3 = -\frac{1}{3}\sum_{i=1}^{n} n(\mathbb{C}_3^i) - \sum_{i=1}^{n} c_2^{(i)} d_i^+ - \frac{1}{3}\sum_{i=1}^{n}(d_i^+)^3 + \frac{a}{2}\sum_{i=1}^{n}\left((d_i^+)^2 + c_2^{(i)}\right) - \frac{a^3}{6}$,

(4) *and*

$$\begin{aligned} b_4 = \; & -\sum_{i=1}^{n} d_i^+ n(\mathbb{C}_3^i) - \frac{1}{4}\sum_{i=1}^{n}(d_i^+)^4 - \sum_{i=1}^{n}(d_i^+)^2 c_2^{(i)} - \frac{1}{2}\sum_{i=1}^{n} d_i^+ M_i - \frac{(4a-3)}{12}\sum_{i=1}^{n} n(\mathbb{C}_3^i) \\ & -\frac{1}{4}\sum_{i=1}^{n}\left(n(P_3^1(v_iv_jv_kv_jv_i)) + n(P_3^2(v_iv_jv_iv_kv_i))\right) - \frac{1}{4}\sum_{i=1}^{n} c_2^{(i)} + \frac{a}{3}\sum_{i=1}^{n}(d_i^+)^3 + a\sum_{i=1}^{n} d_i^+ c_2^{(i)} \\ & -\frac{a^2}{4}\sum_{i=1}^{n}\left((d_i^+)^2 + c_2^{(i)}\right) + \frac{1}{8}\left(\sum_{i=1}^{n}((d_i^+)^2 + c_2^{(i)})\right)^2 + \frac{a^4}{24}. \end{aligned}$$

Here, $c_2^{(i)}$ is the number of closed walks of length 2 at the vertex v_i, $n(\mathbb{C}_3^i)$ is the number of directed cycles of length 3 at the vertex v_i, $n(\mathbb{C}_4^i)$ is the number of directed cycles of length 4 at the vertex v_i, $n(P_3^1(v_iv_jv_kv_jv_i))$ is the number of paths on vertices v_i, v_j, v_k at the vertex v_i with arcs $(v_i, v_j), (v_j, v_k), (v_k, v_j), (v_j, v_i)$, $P_3^2(v_i) = v_iv_jv_iv_kv_i$ is the number of paths on vertices v_i, v_j, v_k at the vertex v_i with arcs $(v_i, v_j), (v_j, v_i), (v_i, v_k), (v_k, v_i)$ and $M_i = \sum_{(v_i,v_j),(v_j,v_i)\in E(D)} d_i^+$ is the sum of the out-degrees of the vertices which are both out-neighbor and the in-neighbor of the vertex v_i.

Proof. Let $s_k = tr(Q(D)^k)$ be the trace of the matrix $Q(D)^k$, where $Q(D) = Deg^+(D) + A(D)$ is the signless Laplacian matrix of D and $k \in \mathbb{N}$. The rest of the proof follows by proceeding similar to Theorem 1 and so is omitted. □

The signless Laplacian spectral moments of D are defined as

$$tr(Q(D)^k) = \sum_{i=1}^{n} q_i(D)^k, \quad k = 1, 2, \ldots,$$

where $q_1(D), q_2(D), \ldots, q_{n-1}(D), q_n(D)$ are the signless Laplacian eigenvalues of D.

The following Corollary 2 follows from Theorem 2 and presents formula for the first four signless Laplacian spectral moments of D.

Corollary 2. *Let D be a connected digraph of order $n \geq 3$ with a arcs having vertex out-degrees $d_1^+ \geq d_2^+ \geq \cdots \geq d_n^+$. Then,*

(1) $\quad \sum_{i=1}^{n} q_i(D) = a$,

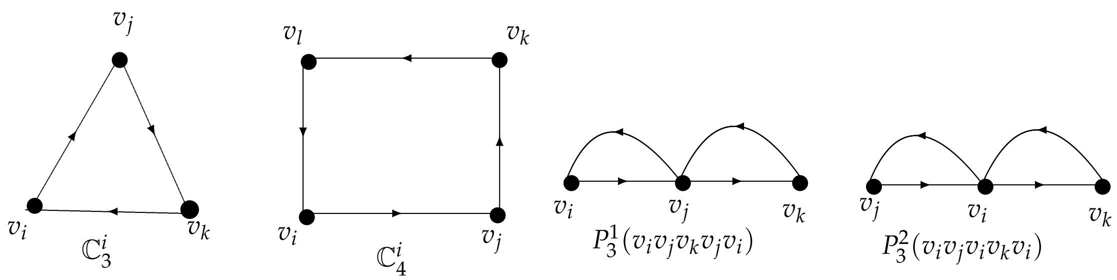
(2) $\quad \sum_{i=1}^{n} q_i(D)^2 = \sum_{i=1}^{n}\left((d_i^+)^2 + c_2^{(i)}\right)$,

(3) $\quad \sum_{i=1}^{n} q_i(D)^3 = \sum_{i=1}^{n}(d_i^+)^3 + 3\sum_{i=1}^{n} c_2^{(i)} d_i^+ + \sum_{i=1}^{n} n(\mathbb{C}_3^i)$, *and*

(4)

$$\begin{aligned} \sum_{i=1}^{n} q_i(D)^4 = & \sum_{i=1}^{n}(d_i^+)^4 + 4\sum_{i=1}^{n}(d_i^+)^2 c_2^{(i)} + 2\sum_{i=1}^{n} d_i^+ M_i + 4\sum_{i=1}^{n} d_i^+ n(\mathbb{C}_3^i) + \sum_{i=1}^{n} n(\mathbb{C}_3^i) \\ & + \sum_{i=1}^{n}\left(n(P_3^1(v_iv_jv_kv_jv_i)) + n(P_3^2(v_iv_jv_iv_kv_i))\right) + \sum_{i=1}^{n} c_2^{(i)}. \end{aligned}$$

Here, $c_2^{(i)}, n(\mathbb{C}_3^i), M_i, n(P_3^1(v_iv_jv_kv_jv_i))$ and $n(P_3^2(v_iv_jv_iv_kv_i))$ are defined in Theorem 1.



4. Examples

In this section, we consider some examples of the digraphs to highlight the applications of Theorems 1 and 2.

Example 1. *Let $\overrightarrow{P}_n = v_1v_2v_3\dots v_{n-1}v_n$ be the directed path on $n \geq 4$ vertices with $a = n-1$ arcs of the form (v_i, v_{i+1}), $i = 1, 2, \dots, n-1$. For the directed path $\overrightarrow{P}_n$, we have $d_i^+ = 1$, for $i \leq n-1$, $d_n^+ = 0$, $c_2^{(i)} = 0$, for $1 \leq i \leq n$, as there is no closed walk of length 2 at any vertex, $M_i = 0$, for $1 \leq i \leq n$. Additionally, $n(\mathbb{C}_3^i) = 0 = n(\mathbb{C}_4^i)$ as there is no triangle and 4-cycle at any vertex v_i in $\overrightarrow{P}_n$. Further, it is clear that there are no paths of the form $P_3^1(v_iv_jv_kv_jv_i)$ and $P_3^2(v_iv_jv_iv_kv_i)$ in $\overrightarrow{P}_n$. With this information, it follows that*

$$\begin{aligned}
a_1(\overrightarrow{P}_n) &= -(n-1), \quad a_2(\overrightarrow{P}_n) = \frac{(n-1)^2}{2} - \frac{(n-1)}{2}, \\
a_3(\overrightarrow{P}_n) &= -\frac{(n-1)^3}{6} + \frac{(n-1)^2}{2} - \frac{(n-1)}{3}, \\
a_4(\overrightarrow{P}_n) &= \frac{(n-1)^4}{24} - \frac{(n-1)^3}{4} + \frac{11(n-1)^2}{24} - \frac{(n-1)}{4}.
\end{aligned}$$

Let us add arc (v_2, v_1) in $\overrightarrow{P}_n$ and let H_1 be the resulting digraph. For H_1, it is clear that $a = n, d_1^+ = 1, d_2^+ = 2, d_n^+ = 0, d_i^+ = 1$ for $i \geq 3$, $c_2^{(1)} = 1 = c_2^{(2)}, c_2^{(i)} = 0$, for $i \geq 3$, $M_1 = 2, M_2 = 1$ and $M_i = 0$, for $2 \leq i \leq n$. Additionally, $n(\mathbb{C}_3^i) = 0 = n(\mathbb{C}_4^i)$ as there is no triangle and 4-cycle at any vertex v_i in H_1. Further, it is clear that there are no paths of the form $P_3^1(v_iv_jv_kv_jv_i)$ and $P_3^2(v_iv_jv_iv_kv_i)$ in H_1. With this information, it follows that

$$\begin{aligned}
a_1(H_1) &= -n, \quad a_2(H_1) = \frac{n^2}{2} - \frac{(n+4)}{2}, \\
a_3(H_1) &= -\frac{n^3}{6} + \frac{n(n+4)}{2} - \frac{n}{3} - 5, \\
a_4(H_1) &= \frac{n^4}{24} - \frac{n^2(n+4)}{4} + \frac{(n+4)^2}{8} + \frac{n(n+6)}{3} + \frac{11n}{4} - 11.
\end{aligned}$$

Let us add another arc (v_3, v_2) in H_1 and let H_2 be the resulting digraph. For H_2, it is clear that $a = n+1, d_1^+ = 1, d_2^+ = d_3^+ = 2, d_n^+ = 0, d_i^+ = 1$ for $i \geq 4$, $c_2^{(1)} = 1 = c_2^{(3)}, c_2^{(2)} = 2c_2^{(i)} = 0$, for $i \geq 4$, $M_1 = M_3 = 2, M_2 = 3$ and $M_i = 0$, for $4 \leq i \leq n$. Also, $n(\mathbb{C}_3^i) = 0 = n(\mathbb{C}_4^i)$ as there is no triangle and 4-cycle at any vertex v_i in H_2. Further, there are paths $P_3^1(v_1v_2v_3v_2v_1)$, $P_3^2(v_2v_3v_2v_1v_2)$ and $P_3^1(v_3v_2v_1v_2v_3)$ at the vertices v_1, v_2 and v_3 in H_2. With this information, it follows that

$$\begin{aligned}
a_1(H_2) &= -(n+1), \quad a_2(H_2) = \frac{(n+1)^2}{2} - \frac{(n+9)}{2}, \\
a_3(H_2) &= -\frac{(n+1)^3}{6} + \frac{(n+1)(n+9)}{2} - \frac{(n+13)}{3} - 7, \\
a_4(H_2) &= \frac{(n+1)^4}{24} - \frac{(n+1)^2(n+9)}{4} + \frac{(n+9)^2}{8} + \frac{(n+1)(n+13)}{3} + \frac{27n}{4} - 21.
\end{aligned}$$

Example 2. *Let $\overrightarrow{C}_n = v_1v_2v_3\dots v_{n-1}v_nv_1$ be the directed cycle on $n \geq 5$ vertices with $a = n$ arcs of the form $(v_i, v_{i+1}), (v_n, v_1)$, $i = 1, 2, \dots, n-1$. For the directed cycle $\overrightarrow{C}_n$, we have $d_i^+ = 1$, for $1 \leq i \leq n$, $c_2^{(i)} = 0, M_i = 0$ for $1 \leq i \leq n$, as there is no closed walk of length 2 at any vertex. Additionally, $n(\mathbb{C}_3^i) = 0 = n(\mathbb{C}_4^i)$ as there is no triangle and 4-cycle at any vertex v_i in $\overrightarrow{P}_n$. Further, it is clear that there are no paths of the form $P_3^1(v_iv_jv_kv_jv_i)$ and $P_3^2(v_iv_jv_iv_kv_i)$ in $\overrightarrow{C}_n$. With this information, it follows that*

$$\begin{aligned} a_1(\overrightarrow{C}_n) &= -n, \quad a_2(\overrightarrow{C}_n) = \frac{n^2}{2} - \frac{n}{2}, \\ a_3(\overrightarrow{C}_n) &= -\frac{n^3}{6} + \frac{n^2}{2} - \frac{n}{3}, \\ a_4(\overrightarrow{C}_n) &= \frac{n^4}{24} - \frac{n^3}{4} + \frac{11n^2}{24} - \frac{n}{4}. \end{aligned}$$

Let us add arc (v_2, v_1) in $\overrightarrow{C}_n$ and let K_1 be the resulting digraph. For K_1, it is clear that $a = n+1, d_1^+ = 1, d_2^+ = 2, d_i^+ = 1$, for $i \geq 3$, $c_2^{(1)} = c_2^{(2)} = 1, c_2^{(i)} = 0, M_1 = 2, M_2 = 1$, $M_i = 0$ for $i \geq 3$. Additionally, $n(\mathbb{C}_3^i) = 0 = n(\mathbb{C}_4^i)$ as there is no triangle and 4-cycle at any vertex v_i in K_1. Further, it is clear that there are no paths of the form $P_3^1(v_iv_jv_kv_jv_i)$ and $P_3^2(v_iv_jv_iv_kv_i)$ in K_1. With this information, it follows that

$$\begin{aligned} a_1(K_1) &= -(n+1), \quad a_2(K_1) = \frac{(n+1)^2}{2} - \frac{(n+5)}{2}, \\ a_3(K_1) &= -\frac{(n+1)^3}{6} + \frac{(n+1)(n+5)}{2} - \frac{n}{3} - \frac{16}{3}, \\ a_4(K_1) &= \frac{(n+1)^4}{24} - \frac{(n+1)^2(n+5)}{4} + \frac{(n+1)(n+7)}{3} + \frac{(n+1)^2}{8} + \frac{11n-13}{4}. \end{aligned}$$

Similar to Laplacian coefficients a_1, a_2, a_3 and a_4 we can obtain the signless Laplacian coefficients b_1, b_2, b_3 and b_4 of the digraphs $\overrightarrow{P}_n, H_1, \overrightarrow{C}_n$ and K_1.

5. Conclusions

From Theorem 1, we arrive at the following conclusion about the Laplacian spectral determination of digraphs.

Theorem 3. *If digraphs D_1 and D_2 are Laplacian co-spectral, then (i) D_1 and D_2 have the same order;*
(ii) D_1 and D_2 have the same number of arcs;
(iii) the quantity $\sum\limits_{i=1}^{n} \left((d_i^+)^2 + c_2^{(i)}\right)$ is same for D_1 and D_2;
(iv) the quantity $\sum\limits_{i=1}^{n} (d_i^+)^3 + 3\sum\limits_{i=1}^{n} c_2^{(i)} d_i^+ - \sum\limits_{i=1}^{n} n(\mathbb{C}_3^i)$ is the same for D_1 and D_2;
(v) the quantity $\sum\limits_{i=1}^{n} (d_i^+)^4 - 4\sum\limits_{i=1}^{n} (d_i^+)^2 c_2^{(i)} + 2\sum\limits_{i=1}^{n} d_i^+ M_i - 4\sum\limits_{i=1}^{n} d_i^+ n(\mathbb{C}_3^i) + \sum\limits_{i=1}^{n} n(\mathbb{C}_3^i)$ $+ \sum\limits_{i=1}^{n} \left(n(P_3^1(v_iv_jv_kv_jv_i)) + n(P_3^2(v_iv_jv_iv_kv_i))\right) + \sum\limits_{i=1}^{n} c_2^{(i)}$ is the same for D_1 and D_2.

From Theorem 2, we arrive at the following conclusion about the signless Laplacian spectral determination of digraphs.

Theorem 4. *If digraphs D_1 and D_2 are signless Laplacian co-spectral, then (i) D_1 and D_2 have the same order;*
(ii) D_1 and D_2 have the same number of arcs;
(iii) the quantity $\sum\limits_{i=1}^{n} \left((d_i^+)^2 + c_2^{(i)}\right)$ is the same for D_1 and D_2;

(iv) the quantity $\sum_{i=1}^{n} n(\mathbb{C}_3^i) + 3\sum_{i=1}^{n} c_2^{(i)} d_i^+ + \sum_{i=1}^{n} (d_i^+)^3$ *is the same for* D_1 *and* D_2*;*

(v) the quantity $\sum_{i=1}^{n} (d_i^+)^4 - 4\sum_{i=1}^{n} (d_i^+)^2 c_2^{(i)} + 2\sum_{i=1}^{n} d_i^+ M_i - 4\sum_{i=1}^{n} d_i^+ n(\mathbb{C}_3^i) + \sum_{i=1}^{n} n(\mathbb{C}_3^i)$ $+ \sum_{i=1}^{n} \Big(n(P_3^1(v_i v_j v_k v_j v_i)) + n(P_3^2(v_i v_j v_i v_k v_i)) \Big) + \sum_{i=1}^{n} c_2^{(i)}$ *is the same for* D_1 *and* D_2*.*

From (ii) of Theorems 3 and 4, it is clear that if two digraphs D_1 and D_2 have different numbers of arcs, then these digraphs have different (signless) Laplacian eigenvalues.

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References

1. Bhat, M.A. Energy of weighted digraphs. *Discret. Appl. Math.* **2017**, *223*, 1–14. [CrossRef]
2. Ganie, H.A.; Carmona, J.R. An (increasing) sequence of lower bounds for the spectral radius and energy of digraphs. *Discret. Math.* **2023**, *346*, 113118. [CrossRef]
3. Alhevaz, A.; Baghipur, M.; Shang, Y. Merging the spectral theories of distance Estrada and distance signless Laplacian Estrada indices of graphs. *Mathematics* **2019**, *7*, 995. [CrossRef]
4. Baghipur, M.; Ghorbani, M.; Ganie, H.A.; Shang, Y. On the second-largest reciprocal distance signless Laplacian eigenvalue. *Mathematics* **2021**, *9*, 512. [CrossRef]
5. Chat, B. A.; Ganie, H.A.; Pirzada, S. Bounds for the skew Laplacian spectral radius of oriented graphs. *Carpathian J. Math.* **2019**, *35*, 31–40. [CrossRef]
6. Ganie, H.A. Bounds for the skew Laplacian(skew adjacency) spectral radius of a digraph. *Trans. Combin.* **2019**, *8*, 1–12.
7. Ganie, H.A.; Chat, B.A. Bounds for the energy of weighted graphs. *Discret. Appl. Math.* **2019**, *268*, 91–101. [CrossRef]
8. Ganie, H.A.; Shang, Y. On the spectral radius and energy of signless Laplacian matrix of digraphs. *Heliyon* **2022**, *8*, e09186. [CrossRef]
9. Lokesha, V.; Shanthakumari, Y.; Reddy, P.S.K. Skew-zagreb energy of directed graphs. *Proc. Jangjeon Math. Soc.* **2020**, *23*, 557–568.
10. Pirzada, S.; Ganie, H.A.; Chat, B.A. On the real or integral spectrum of digraphs. *Oper. Matrices* **2020**, *14*, 795–813. [CrossRef]
11. Qiu, L.; Wang, W.; Wang, W. Oriented graphs determined by their generalized skew spectrum. *Linear Algebra Appl.* **2021**, *622*, 316–332. [CrossRef]
12. Shang, Y. More on the normalized Laplacian Estrada index. *Appl. Anal. Discrete Math.* **2014**, *8*, 346–357. [CrossRef]
13. Sachs, H. Beziehungen zwischen den in einen Graphen enthalten Kreisenund seinem characterischen Polynom. *Publ. Math. Debr.* **1964**, *11*, 119–134. [CrossRef]
14. Mowshowitz, A. The characteristic polynomial of a graph. *J. Comb. Theory* **1972**, *12*, 177–193. [CrossRef]
15. Oliveira, C.S.; de Abreu, N.M.M.; Jurkiewicz, S. The characteristic polynomial of the Laplacian of graphs in (a, b)-linear classes. *Linear Algebra Appl.* **2002**, *356*, 113–121. [CrossRef]
16. Cvetković, D.; Rowlinson, P.; Simić, S.K. Signless Laplacians of finite graphs. *Linear Algebra Appl.* **2007**, *423*, 155–171. [CrossRef]
17. Guo, J.M.; Li, J.; Huang, P.; Shiu, W.C. Coefficients of the characteristic polynomial of the (signless, normalized) Laplacian of a graph. *Graphs Comb.* **2017**, *33*, 1155–1164. [CrossRef]
18. Lepović, M.; Gutman, I. No starlike trees are cospectral. *Discret. Math.* **2002**, *242*, 291–295. [CrossRef]
19. Wang, J.F.; Huang, Q.X.; Belardo, F. On the spectral characterizations of 3-rose graphs. *Util. Math.* **2013**, *91*, 33–46.
20. Ganie, H.A.; Ingole, A.; Deshmukh, U. On the coefficients of skew Laplacian characteristic polynomial of digraphs. *Discret. Math. Algorithms Appl.* **2022**, *14*, 2250131. [CrossRef]
21. Liu, X.; Liu, S. On the A_α-characteristic polynomial of a graph. *Linear Algebra Appl.* **2018**, *546*, 274–288. [CrossRef]
22. Oboudi, M.R.; Abdian, A.Z. Peacock graphs are determined by their Laplacian spectra. *Iran. J. Sci. Technol. Trans. Sci.* **2020**, *44*, 787–790. [CrossRef]

23. Lin, H.; Shu, J. A note on the spectral characterization of strongly connected bicyclic digraphs. *Linear Algebra Appl.* **2012**, *436*, 2524–2530. [CrossRef]
24. Prasolov, V.V. *Problems and Theorems in Linear Algebra (Translations of Mathematical Monographs)*; American Mathematical Society: Cambridge, UK, 1994.